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Path geometries and almost Grassmann structures

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Abstract.

Any path geometry, or projective equivalence class of sprays, on an n-dimensional manifold M is naturally associated with an almost Grassmann structure on a $2n$-dimensional fibre bundle over that manifold. The almost Grassmann structure has special properties when the sprays are isotropic, and when they are geodesic for some Finsler function.

§1. Introduction

In this paper we show that with any path geometry, or projective equivalence class of sprays, on an n-dimensional manifold M there is naturally associated an almost Grassmann structure on a certain $2n$-dimensional fibre bundle $\mathcal{T}^\circ M$ over M (and we might as well make it clear from the start that this is not the slit tangent bundle $T^\circ M$).

We use the name 'almost Grassmann' for a type of structure that has appeared in the literature under many different titles, including Segre structure [15] and Grassmannian structure [11]. An almost Grassmann structure of type (p, q) is a Cartan geometry modelled on the Grassmannian of p-dimensional subspaces of a $(p + q)$-dimensional real vector space; here we have $p = 2$, $q = n$. The major ingredient of a Cartan geometry is its Cartan connection theory. We construct the normal Cartan connection of our structure explicitly, and show that its curvature is completely determined by the projective invariants of the class of sprays. We show in particular that the torsion of the Cartan connection vanishes if and only if the sprays are isotropic, and we discuss in some detail the special properties of the almost Grassmann structure in this case.

One important projective class of sprays is furnished by the geodesic sprays of a Finsler space, when no rule of parametrization is imposed. We show that the almost Grassmann structure associated with a Finsler geodesic class of sprays is nicely aligned with the Hilbert 2-form, and

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conversely we give a new formulation of the conditions for a projective equivalence class of sprays to be the geodesic sprays of a Finsler function (or a pseudo-Finsler function, to be exact) in terms of the properties of its almost Grassmann structure.

The initial stimulus for this paper was in fact the recent appearance of two articles on geometrical structures associated with second-order ordinary differential equations. The first, *Torsion-free path geometries and integrable second order ODE systems* by D. A. Grossman [15], deals with a special class of systems of second-order ordinary differential equations, and shows that there is what the author calls a Segre structure naturally associated with any system of this class, defined on its path space. The second source for the present paper is *Three-dimensional Cauchy-Riemann structures and second-order ordinary differential equations* by P. Nurowski and G. A. J. Sparling [17] (see also [16]). Nurowski and Sparling deal with a single second-order ordinary differential equation, without restriction, and show how to associate with it a 4-dimensional conformal structure of split or neutral signature, a result analoguous to the construction of the Fefferman conformal structure in CR geometry.

It may not be immediately apparent how these topics are related to the subject of this paper as announced above. To explain the relationship we must first mention that both Grossman and Nurowski and Sparling deal with the geometry of second-order ordinary differential equations under so-called point transformations of variables, that is, transformations in which there is no requirement to keep the dependent variables distinct from the independent variable. Now the geometry of systems of second-order ordinary differential equations with m dependent variables under point transformations is easily seen to be equivalent to path geometry in dimension $m+1$, or in other words to the projective geometry of sprays on an $(m+1)$-dimensional manifold; we believe moreover that it is better to use the latter interpretation, not least because path geometry is comparatively well-developed (see [21] for a recent text covering the subject) and the geometrical invariants have been known since Douglas's classic paper of 1928 [12]. We should perhaps clarify here the difference between a path geometry and a path space; by the former we mean an n-dimensional manifold M whose slit tangent bundle $T^\circ M$ is equipped with a projective equivalence class of sprays, or equivalently whose sphere bundle SM [4] is equipped with a line-element field L; and by the latter we mean roughly speaking the $(2n-2)$-dimensional quotient SM/L, whose points are the paths of the path geometry (we have to say 'roughly speaking' because SM/L won't necessarily have a manifold structure). In the case of reversible sprays, which includes that of systems of second-order ordinary differential equations under point transformations, we can replace the sphere bundle by the projectivized tangent bundle PTM.

Another aim of our paper is to show that the structures described in [15] and [17] are in effect particular cases of a quite general one, namely the almost Grassmann structure associated with a path geometry.

As a Cartan geometry, an almost Grassmann structure falls into the general class of parabolic geometries studied by Čap and others: see [5] for a recent account of such geometries which explains how almost Grassmann structures fit into the picture. However, Čap's approach is too general for our purposes, since we are concerned with the detailed geometry of certain almost Grassmann structures rather than the position of almost Grassmann structures in the general family of parabolic geometries. Almost Grassmann structures have been studied in detail and per se by Akivis and Goldberg [1] and Dhooghe [11]. Both of these are useful references, Dhooghe especially for the structural aspects, Akivis and Goldberg for the theory of the associated Cartan connection. In particular, the general construction of a normal Cartan connection is discussed in [1]. This connection is not necessarily torsionless: Dhooghe is less helpful in the matter of the connection theory because he deals only with the torsionless case. We need to construct the normal Cartan connection explicitly in the case of interest, namely the almost Grassmann structure associated with a path geometry, and identify the components of its curvature with known projective invariants of sprays. The simplest way of doing so is to work locally with a frame adapted to the projective structure, or from the point of view of the Cartan geometry to work in a gauge (just as Cartan himself would have done). Since there is no easily accessible account of such an approach to the determination of the normal Cartan connection of an almost Grassmann structure we have thought it advisable to provide one, in an appendix.

The term 'torsion' in the title to Grossman's paper means something different from the torsion of the normal Cartan connection of an almost Grassmann structure; nevertheless, the vanishing of the torsion in the two cases has identical consequences. As we mentioned above, from the point of view of the projective geometry of sprays the condition for the torsion of the Cartan connection to vanish is that the sprays be isotropic. Of course the isotropic sprays form a distinct and important subclass of sprays; for example in Finsler geometry the geodesic sprays of a Finsler function are isotropic if and only if the space is of scalar curvature [21]. One useful by-product of our approach is therefore to identify Grossman's 'torsion-free path geometries' as projective classes of isotropic sprays.

So Grossman's paper discusses a particular case of our construction, namely that in which the torsion of the normal Cartan connection vanishes. The paper of Nurowski and Sparling also in effect discusses a particular case, that is, the case in which the base manifold is 2-dimensional. As Akivis and Goldberg point out, almost Grassmann structures of type $(2,2)$ can be identified with 4-dimensional conformal structures of neutral signature. This is a special feature of the 2-dimensional case, though it does lead to the idea that almost Grassmann structures can be thought of as generalized conformalstuctures. We won't go further into the conformal geometry here, since we have described it in detail elsewhere [10].

Our paper is organized as follows. We briefly review the facts we need from the projective differential geometry of sprays and the theory of almost Grassmann structures in Section 2. In Section 3 we define the almost Grassmann structure associated with a projective class of sprays, we derive its normal Cartan connection, and we show that vanishing of torsion means the sprays are isotropic; in Section 4 we describe some conditions on an almost Grassmann structure which ensure that it is derived in this way from a projective class of sprays. We consider the case of isotropic sprays in Section 5, and in Section 6 we discuss the geodesic sprays of a Finsler space, and conversely consider the Finsler inverse problem in the light of all this structure. Finally in Section 7 we consider the isotropic case in relation to Finsler geometry. In the first appendix we give the derivation of the normal Cartan connection of an almost Grassmann structure in a gauge, while in the second we discuss aspects of the flat case, when the almost Grassmann structure is a Grassmannian.

§2. Preliminaries

2.1. The projective geometry of sprays

A spray S on a manifold M is a second-order differential equation field on $T^\circ M$, the tangent bundle of M with its zero section deleted, say

$$S = v^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial v^i},$$

such that the coefficients Γ^i are positively homogeneous of degree 2 in the fibre coordinates v^i ((x^i) are coordinates on M, (x^i, v^i) the induced coordinates on TM). The base integral curves of the spray S are called its geodesics: they are curves in M whose lifts to $T^\circ M$ are the integral curves, in the usual sense, of S.

Two sprays $\hat{S}$, S are said to be projectively equivalent if $\hat{\Gamma}^i = \Gamma^i + \alpha v^i$, where the function α is positively homogeneous of degree 1 in the v^i.

This is the condition for the geodesics, with any given initial point and direction, of the two sprays to differ only by an orientation-preserving reparametrization. A projective equivalence class of sprays determines and is determined by a collection of unparametrized but oriented curves in M — paths — with the property that there is a unique path through each point in each direction. We may therefore describe such a collection of paths as a path geometry.

We shall be interested in a restricted class of sprays, those whose integral curves are reversible, and a corresponding restricted notion of projective equivalence. Reversible sprays (that is, sprays whose integral curves have the property that the integral curve through x with initial tangent vector $-v$ is just the integral curve through x with initial tangent vector v traversed in the opposite sense) are such that the coefficients Γ^i are homogeneous of degree 2 without qualification, that is, satisfy $\Gamma^i(x^j, \lambda v^j) = \lambda^2 \Gamma^i(x^j, v^j)$ for all non-zero λ. In order for a projective change to respect this additional condition the function α must also be homogeneous of degree 1 without qualification. The corresponding path geometry has the property that given a point $x \in M$ and a line in T_xM there is a unique path (now an unparametrized and unoriented curve) through x whose tangent line at x is the given line. The set of lines in T_xM is just $\mathrm{P}T_xM$, the projective tangent space at x; thus a path geometry in this sense determines and is determined by a congruence of paths on $\mathrm{P}TM$, the projective tangent bundle of M (one and only one path of the congruence passes through each point of $\mathrm{P}TM$); the corresponding projective equivalence class of sprays determines and is determined by a line element field on $\mathrm{P}TM$, the tangent line element field of the congruence of paths.

We may parametrize suitable geodesic paths of a projective class of sprays with one of the coordinates, say x^1; with such a parametrization the differential equations of geodesics take the form

$$\frac{d^2x^\alpha}{d(x^1)^2} = f^\alpha\left(x^1, x^\beta, \frac{dx^\gamma}{dx^1}\right), \qquad \alpha, \beta, \gamma = 2, 3, \ldots.$$

Conversely, given a system of second-order differential equations in the dependent variables x^α, with independent variable x^1, we can locally recover a spray by setting

$$\Gamma^1 = 0, \quad \Gamma^\alpha(x^i, v^i) = -\tfrac{1}{2}(v^1)^2 f^\alpha(x^i, v^\beta/v^1).$$

Under a coordinate transformation involving all of the coordinates x^i (a so-called point transformation) the spray corresponding to the new system of differential equations will be projectively equivalent to that corresponding to the original one. The invariants of the system of second-order ordinary differential equations under point transformations will be the projective invariants of the corresponding projective equivalence class of sprays.

Any spray determines a horizontal distribution on $T^\circ M$, which is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial v^j}, \qquad \Gamma^j_i = \frac{\partial \Gamma^j}{\partial v^i}.$$

A spray has two curvatures. Its Berwald curvature is

$$B^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial v^k}, \qquad \Gamma^i_{jk} = \frac{\partial \Gamma^i_j}{\partial v^k} = \frac{\partial \Gamma^i_k}{\partial v^j}.$$

The Riemann curvature R^l_{kij} of the spray is

$$R^l_{kij} = H_i\left(\Gamma^l_{jk}\right) - H_j\left(\Gamma^l_{ik}\right) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

The bracket of a pair of horizontal vector fields is given by

$$[H_i, H_j] = -R^l_{kij} v^k \frac{\partial}{\partial v^l} = -R^l_{ij} \frac{\partial}{\partial v^l}.$$

The Riemann curvature can be reconstructed from R^l_{ij}: we have

$$R^l_{ijk} = \frac{\partial R^l_{jk}}{\partial v^i},$$

since $v^m \partial R^l_{mjk}/\partial v^i = v^m \partial R^l_{ijk}/\partial v^m = 0$, due in the first place to a Bianchi identity and secondly to the fact that R^l_{ijk} is homogeneous of degree zero.

A spray whose Riemann curvature vanishes is said to be R-flat. An R-flat spray need not of course be flat, that is, rectifiable; for a spray to be flat both curvatures must vanish.

Each of the curvatures gives rise to a projectively invariant tensor. The projectively invariant Douglas tensor D^l_{ijk} is constructed from the Berwald curvature by

$$D^l_{ijk} = B^l_{ijk} - \frac{1}{n+1}\left(B_{jk}\delta^l_i + B_{ki}\delta^l_j + B_{ij}\delta^l_k + v^l \frac{\partial B_{ij}}{\partial v^k}\right),$$

where $B_{ij} = B^k_{ijk}$ and $n = \dim M$. From the Riemann curvature we obtain a projectively invariant tensor P^l_{kij} by

$$P^l_{kij} = \tilde{R}^l_{kij} - \frac{1}{n^2-1}\left(Q_{jk}\delta^l_i - Q_{ik}\delta^l_j - (Q_{ij} - Q_{ji})\delta^l_k\right),$$

where

$$\tilde{R}^l_{kij} = R^l_{kij} - \frac{1}{n+1}v^l\frac{\partial R^m_{mij}}{\partial v^k}, \qquad Q_{ij} = \tilde{R}^k_{ikj} + n\tilde{R}^k_{jki}.$$

This is the counterpart of the projective curvature tensor of the affine theory, to which it reduces in the affine case; we call it the generalized projective curvature tensor. Just as the Riemann curvature is determined by R^l_{jk}, the generalized projective curvature tensor can be derived by differentiation from $P^l_{mjk}v^m$:

$$P^l_{ijk} = \frac{\partial}{\partial v^i}\left(P^l_{mjk}v^m\right);$$

in particular, if $P^l_{mjk}v^m$ vanishes so does P^l_{ijk}.

The transformation rule for horizontal vector fields under a projective transformation of sprays is

$$\hat{H}_i = H_i - \alpha\frac{\partial}{\partial v^i} - \frac{\partial \alpha}{\partial v^i}v^j\frac{\partial}{\partial v^j}.$$

It follows easily that the quantity

$$\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1}\left(\Gamma_j\delta^k_i + \Gamma_i\delta^k_j + B_{ij}v^k\right), \qquad \Gamma_i = \Gamma^j_{ij} = \Gamma^j_{ji},$$

is projectively invariant. Douglas calls this the fundamental projective invariant; it is important to bear it in mind however that the Π^k_{ij} are not the components of a tensor, nor even of a connection. Nevertheless it is possible, and often advantageous, to express the projective invariants described above in terms of the fundamental invariants. In this regard it is worth noting that $\Pi^j_{ij} = \Pi^j_{ji} = 0$. Furthermore, if $\mathfrak{R}^l_{ijk}$ is constructed from Π^i_{jk} just as R^l_{ijk} is from Γ^i_{jk} (where we use gothic type to record the fact that $\mathfrak{R}^l_{ijk}$ is not a tensor), then $\mathfrak{R}_{ij} = \mathfrak{R}^k_{ikj}$ is symmetric. Then

$$D^l_{ijk} = \frac{\partial \Pi^l_{ij}}{\partial v^k}, \qquad P^l_{kij} = \mathfrak{R}^l_{jik} - \frac{1}{n-1}\left(\mathfrak{R}_{jk}\delta^l_i - \mathfrak{R}_{ik}\delta^l_j\right).$$

There is an important special type of spray, called isotropic. The definition is usually given as follows: a spray is isotropic if and only if there is a function ρ and covector τ_i such that

$$R^i_j = \rho\delta^i_j + \tau_j v^i$$

where $R^i_j = R^i_{kjl}v^k v^l$ is the Jacobi endomorphism; see for example [21]. From our point of view, however, the more important fact is that the property of being isotropic is equivalent to the vanishing of the generalized projective curvature tensor; in particular, it is a projective property. It may be shown [7] that for $n > 2$, a spray is isotropic if and only if it is locally projectively equivalent to one which is R-flat.

The geodesic sprays of a Finsler function form a projective equivalence class (where we do not insist that geodesics are parametrized proportionally to Finsler arc length). The geodesic class of a Finsler function consists of isotropic sprays if and only if the Finsler space is of scalar curvature. It is easy to see [8] that any R-flat spray is metrizable, that is, admits a Finsler function of which it is a geodesic spray, so every class of isotropic sprays is in fact the geodesic class of a Finsler function of scalar curvature.

2.2. Almost Grassmann structures

An almost Grassmann structure on a manifold N of dimension pq, $p \geq 2$, $q \geq 2$, is a Cartan geometry modelled on the Grassmannian of p-dimensional subspaces of $\mathbf{R}^{p+q}$. The latter may be represented as a homogeneous space $\mathrm{PGL}(p+q)/\mathrm{H}$, where the isotropy group H is (mod $\mathbf{R}$) the subgroup of matrices of the form

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

and where (for instance) A is a $p \times p$ submatrix. So in the general case we have a principal H-bundle P over N equipped with an $\mathfrak{sl}(p+q)$-valued connection 1-form ω satisfying the conditions

(1) the map $\omega_p : T_pP \to \mathfrak{sl}(p+q)$ is an isomorphism for each $p \in P$;
(2) $R^*_h\omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in \mathrm{H}$; and
(3) $\langle Z^\dagger, \omega\rangle = Z$ for each $Z \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H and where $Z^\dagger$ is the fundamental vector field corresponding to Z.

(These conditions are modelled on those for a general Cartan connection given by Sharpe [20].) When it is necessary to draw attention to the dimensions we say that an almost Grassmann structure as just described is of type (p, q).

It will be convenient to work locally, in a gauge, by which we mean simply a local section, say κ, of $P \to N$; the connection form in that gauge is $\kappa^*\omega$, a locally-defined $\mathfrak{sl}(p+q)$-valued 1-form on N. We shall however suppress any mention of the section κ.

We use indicial notation, with $\alpha, \beta, \ldots$ ranging and summing from 1 to p and $i, j, \ldots$ ranging and summing from 1 to q; there is an apparent risk of confusion here, but in practice no confusion will arise. We can write the gauged connection form and an element of H as, respectively,

$$\begin{bmatrix} \omega^\alpha_\beta & \omega^\alpha_j \\ \omega^i_\beta & \omega^i_j \end{bmatrix}, \qquad \begin{bmatrix} A^\alpha_\beta & A^\alpha_j \\ 0 & A^i_j \end{bmatrix};$$

then under a gauge transformation $\omega^i_\alpha \mapsto \bar{A}^i_j A^\beta_\alpha \omega^j_\beta$, where the bar denotes the inverse.

In any gauge, $\{\omega^i_\alpha\}$ is a local basis of 1-forms for N; thus an almost Grassmann structure determines on N a class of local bases of 1-forms, determined up to the transformation above. That is, it determines a reduction of the frame bundle to a subgroup of $\mathrm{GL}(pq)$ which can be identified with the subgroup of $\mathrm{PGL}(p+q)$ of block-diagonal matrices, or in other words a G-structure on N with this group G. In fact G is the so-called linear isotropy subgroup of $\mathrm{PGL}(p+q)$ in the homogeneous space $\mathrm{PGL}(p+q)/\mathrm{H}$. Furthermore, as it is shown in [11], we can recover the structure, at least at the algebra level, by prolongation. So this gives an alternative definition of an almost Grassmann structure. Thus we can define an almost Grassmann structure by specifying a class of local bases of 1-forms $\{\theta^i_\alpha\}$, any two such local bases of the class being related by $\hat{\theta}^i_\alpha = B^i_j A^\beta_\alpha \theta^j_\beta$ where (A^α_β) and (B^i_j) are local matrix-valued functions, respectively $p \times p$ and $q \times q$, both non-singular. This is how we shall specify the almost Grassmann structure associated with a projective equivalence class of sprays.

Given an almost Grassmann structure, we denote the local basis of vector fields dual to a local basis of 1-forms $\{\theta^i_\alpha\}$ in the structure by $\{E^\alpha_i\}$. Then any vector $v \in T_xN$ may be written as $v^i_\alpha E^\alpha_i(x)$. Of special interest are those v for which the coefficient matrix (v^i_α) has rank 1; the set of such v forms a cone in T_xN called the Segre cone. That is to say, the Segre cone at $x \in N$ consists of those elements of T_xN that can be expressed in the form $s_\alpha t^i E^\alpha_i(x)$ with respect to one, and hence any, basis $\{E^\alpha_i\}$ defined by the structure, where $(s_\alpha) \in \mathbf{R}^p$ and $(t^i) \in \mathbf{R}^q$. For fixed non-zero (t^i), as (s_α) varies over $\mathbf{R}^p$ we obtain a p-dimensional subspace of T_xN contained in the Segre cone; following [1] we call it a p-dimensional plane generator of the Segre cone. The p-dimensional plane generators of Segre cones are parametrized by the points of the projective space P^{q-1}. Similarly, on fixing non-zero (s_α), as (t^i) varies over $\mathbf{R}^q$ we obtain a q-dimensional plane generator of the Segre cone. When $p = q = 2$ the Segre cones determine a conformal structure of neutral signature, of which they are the null cones.

An almost Grassmann structure of type (p, q) for which there are sufficiently many integrable distributions of dimension p (say) that are everywhere tangent to the p-dimensional plane generators of the Segre cones is said to be semi-integrable. By 'sufficiently many' we mean that for each $x \in N$ and each $[t^i] \in \mathrm{P}^{q-1}$ there is an integrable distribution of dimension p which is everywhere a p-dimensional plane generator of the Segre cone, and which coincides at x with $\langle t^i E_i^\alpha(x)\rangle$, the generator spanned by the vectors $t^i E_i^\alpha(x)$ there. We call a foliation by submanifolds of such an integrable distribution a Segre foliation.

In the first appendix we show that if we choose some local basis of 1-forms $\{\theta^i_\alpha\}$ of the almost Grassmann structure we can fix the gauge uniquely so that $\omega^i_\alpha = \theta^i_\alpha$ and $\omega^\alpha_\alpha = \omega^i_i = 0$. We show further that we can fix the Cartan connection uniquely by imposing conditions on the curvature 2-forms. The relevant curvature forms in this gauge are given by

$$
\begin{aligned}
\Omega^i_\alpha &= d\omega^i_\alpha + \omega^i_j \wedge \theta^j_\alpha + \theta^i_\beta \wedge \omega^\beta_\alpha \\
\Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j + \theta^i_\alpha \wedge \omega^\alpha_j = K^i_j + \theta^i_\alpha \wedge \omega^\alpha_j \\
\Omega^\alpha_\beta &= d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \omega^\alpha_i \wedge \theta^i_\beta = K^\alpha_\beta + \omega^\alpha_i \wedge \theta^i_\beta .
\end{aligned}
$$

We can express any curvature 2-form Ω^*_* in terms of the basis 1-forms θ^i_b as $\Omega^*_* = \frac{1}{2}\Omega^{*\beta\gamma}_{*jk}\theta^j_\beta \wedge \theta^k_\gamma$ with $\Omega^{*\gamma\beta}_{*kj} = -\Omega^{*\beta\gamma}_{*jk}$. The conditions that fix the connection are

$$
\Omega^{j\beta\gamma}_{\alpha jk} = \Omega^{i\beta\gamma}_{\beta jk} = 0; \qquad \Omega^{i\beta\gamma}_{jik} + \Omega^{i\gamma\beta}_{kij} - \Omega^{\beta\alpha\gamma}_{\alpha jk} - \Omega^{\gamma\alpha\beta}_{\alpha kj} = 0.
$$

We call the connection determined in this way the normal Cartan connection for the almost Grassmann structure. The curvature terms Ω^i_α comprise the torsion of the connection; we do not, and in general cannot, demand that the torsion vanishes (which is usually a requirement in fixing a normal Cartan connection); the most we can do is require that its i, i and α, α traces (in an obvious sense) vanish.

§3. The almost Grassmann structure associated with a projective equivalence class of sprays

In this section, we show that each projective class of sprays on $T^\circ M$ defines an almost Grassmann structure of type $(2, n)$ on the $2n$-dimensional bundle $\mathcal{T}^\circ M \to M$ of non-zero vector densities of weight $1/(n+1)$ on M.

We could think of the vector bundle $\mathcal{T}M$ of vector densities of weight $1/(n+1)$ (from which we obtain $\mathcal{T}^\circ M$ by removing its zero section) simply as the tensor product of the ordinary tangent bundle with the bundle of scalar densities of weight $1/(n+1)$. However, it will be useful to describe $\mathcal{T}M$ in more detail; we take an approach related to the idea of a tractor bundle, introduced in [3] and developed further in, for example, [6].

We first recall some of the basic geometrical constructions that we shall need to use. These are based upon the idea of the volume bundle of a manifold, which is a line bundle that provides, for a general manifold M of dimension n, a structure similar to, but not quite the same as, that of the line bundle $\mathbf{R}^{n+1} - \{0\} \to \mathrm{P}^n$ over projective space. The idea of using an additional coordinate in this way may be found in the early work of T. Y. Thomas [23] and J. H. C. Whitehead [24] on path geometry, and has also been used more recently in [19]. Our approach [9] to this bundle is to start with the non-zero volume elements $\theta \in \bigwedge^n T^* M$; the set of pairs $[\pm\theta]$ of such elements will then be called the volume bundle of M. The projection $\nu : \mathcal{V}M \to M$ is defined by $\nu[\pm\theta] = x$ whenever $\theta, -\theta \in \bigwedge^n T^*_x M$. If (x^i) are coordinates on M then one candidate for the fibre coordinate on the fibre of ν would be $|v|$, where v satisfies

$$\theta = v(\theta)\left(dx^1 \wedge \ldots dx^n\right)_x$$

for any $\theta \in \bigwedge^n T^* M$; we shall, however, choose instead to use $x^0 = |v|^{1/(n+1)}$ as the fibre coordinate, with the convention that the positive root is to be taken if n is odd so that $x^0 > 0$. The local trivializations defined in this way describe a principal $\mathbf{R}_+$-bundle structure on ν, where $\mathbf{R}_+$ is the multiplicative group of positive reals. We shall let $\mu : \mathcal{V}M \times \mathbf{R}_+ \to \mathcal{V}M$ denote the corresponding (right) action $[\pm\theta] \mapsto [\pm s^{n+1}\theta]$ of $\mathbf{R}_+$ on the fibres of ν, and also write $\mu_s : \mathcal{V}M \to \mathcal{V}M$ for the map defined by $\mu_s([\pm\theta]) = \mu([\pm\theta], s)$. The fundamental vector field of this bundle will be denoted by Υ; in coordinates

$$\Upsilon = x^0 \frac{\partial}{\partial x^0}.$$

Consider the bundle $T\mathcal{V}M \to \mathcal{V}M$, the tangent bundle to the volume bundle. We shall use, on the total space $T\mathcal{V}M$, the vector fields Υ^{V} and $\tilde{\Upsilon}$; here, Υ^{V} is the vertical lift, and $\tilde{\Upsilon} = \Upsilon^{\mathrm{C}} - \tilde{\Delta}$ where Υ^{C} is the complete lift and $\tilde{\Delta}$ is the dilation field on $T\mathcal{V}M$. In coordinates (x^a, v^a),

$$\Upsilon^{\mathrm{V}} = x^0 \frac{\partial}{\partial v^0}, \qquad \tilde{\Upsilon} = x^0 \frac{\partial}{\partial x^0} - v^i \frac{\partial}{\partial v^i}.$$

As $[\Upsilon^{\mathrm{V}}, \tilde{\Upsilon}] = -\Upsilon^{\mathrm{V}}$ the distribution spanned by these two vector fields is integrable, and we may take the quotient, to obtain a new manifold of dimension $2n$ which no longer projects to $\mathcal{V}M$, but does project to M and defines a vector bundle over M. The fibre coordinates (u^i) on the new bundle are defined in terms of the fibre coordinates (v^i) of TM by $u^i = x^0 v^i$; the quotient manifold is $\mathcal{T}M$.

It is worth observing that while both $T\mathcal{V}M \to TM$ and $T\mathcal{V}M \to \mathcal{T}M$ are principal bundles with 2-dimensional connected fibres, the structure group of the former is the Abelian direct product $\mathbf{R}_+ \times \mathbf{R}$ whereas that of the latter is the affine group $\mathrm{A}(1)$. It is evident from the construction that the projectivization $\mathrm{P}\mathcal{T}M$ is identical to the usual projective tangent bundle $\mathrm{P}TM$.

Observe that we denote the fibre coordinates on $\mathcal{T}M$ (over a coordinate patch in M) by (u^i), and those on TM by (v^i): provided we restrict attention to a single patch we may take $x^0 = 1$, and there is no difference; but of course they transform differently, and the notation is supposed to remind one of this fact.

We now begin the construction of the almost Grassmann structure.

Lemma *Suppose given a spray*

$$v^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial v^i}$$

on $T^\circ M$; such a spray determines a well-defined horizontal distribution on $\mathcal{T}^\circ M$, spanned by the vector fields

$$\mathcal{K}_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial u^j} + \frac{1}{n+1}\Gamma_i \Delta,$$

where

$$\Gamma^j_i = \frac{\partial \Gamma^j}{\partial v^i}, \quad \Gamma = \Gamma^k_k, \quad \Gamma_i = \frac{\partial \Gamma}{\partial v^i} = \Gamma^k_{ik}$$

and where Δ is the dilation field on $\mathcal{T}M$. If two sprays are related by a projective transformation with function α, the vector fields are modified according to the rule

$$\mathcal{K}_i \mapsto \mathcal{K}_i - \alpha \frac{\partial}{\partial u^i}.$$

Proof The horizontal distribution of the spray is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j},$$

on $T^\circ M$, and the accompanying basis of 1-forms is

$$\{dx^i, dv^i + \Gamma^i_j dx^j\}.$$

Under a coordinate transformation with Jacobian matrix J^i_j we have

$$d\hat{x}^i = J^i_j dx^j, \qquad d\hat{v}^i + \hat{\Gamma}^i_k d\hat{x}^k = J^i_j(dv^j + \Gamma^j_l dx^l),$$

from which it follows that

$$\hat{\Gamma}_i = \bar{J}^j_i \left(\Gamma_j - \frac{\partial \log |J|}{\partial x^j}\right),$$

where the bar denotes the inverse, or equivalently

$$\hat{\Gamma}_i d\hat{x}^i = \left(\Gamma_j - \frac{\partial \log |J|}{\partial x^j}\right) dx^j.$$

We now use the homogeneity properties of Γ^i_j, and the quantities derived from it, all of which are defined in a coordinate patch on $T^\circ M$, to define similarly-named quantities on the corresponding coordinate patch on $\mathcal{T}^\circ M$. The local forms on $\mathcal{T}^\circ M$ given in coordinates by $du^i + \Gamma^i_k dx^k$ do not transform in such a way as to determine a global horizontal distribution. But on the other hand

$$\begin{aligned} d\hat{u}^i &+ \left(\hat{\Gamma}^i_k - \frac{1}{n+1}\hat{u}^i\hat{\Gamma}_k\right) d\hat{x}^k \\ &= \quad |J|^{-1/(n+1)} J^i_j \left(du^j + \left(\Gamma^j_l - \frac{1}{n+1}u^j\Gamma_l\right) dx^l\right), \end{aligned}$$

so that the forms

$$du^i + \left(\Gamma^i_j - \frac{1}{n+1}u^i\Gamma_j\right) dx^j$$

do determine a well-defined horizontal distribution on $\mathcal{T}^\circ M$, associated with the given spray. This is spanned at any point by the vector fields

$$\mathcal{K}_i = \frac{\partial}{\partial x^i} - \left(\Gamma^j_i - \frac{1}{n+1}u^j\Gamma_i\right) \frac{\partial}{\partial u^j} = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial u^j} + \frac{1}{n+1}\Gamma_i\Delta$$

as required. Though the horizontal distribution is globally defined, as is apparent from the coordinate transformation rule derived above, the basis of horizontal vector fields just given is of course defined only locally.

If two sprays are related by a projective transformation with function α, we have

$$\Gamma_i^j \mapsto \Gamma_i^j + \alpha_i u^j + \alpha\delta_i^j; \quad \Gamma \mapsto \Gamma + (n+1)\alpha \qquad \text{where } \alpha_i = \frac{\partial \alpha}{\partial u^i}$$

so that, as required,

$$\left(\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial u^j} + \frac{1}{n+1}\Gamma_i \Delta\right) \mapsto \left(\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial u^j} + \frac{1}{n+1}\Gamma_i \Delta\right) - \alpha \frac{\partial}{\partial u^j}.$$

■

Corollary *A projective class of sprays defines an almost Grassmann structure on $\mathcal{T}^\circ M$ of type $(2,n)$.*

Proof Consider, in a coordinate patch with coordinates (x^i, u^i), for any spray of the class, the transformation properties of the 1-forms

$$\theta_1^i = dx^i, \quad \theta_2^i = du^i + \left(\Gamma_j^i - \frac{1}{n+1} u^i \Gamma_j\right) dx^j :$$

under a coordinate transformation $\hat{\theta}_1^i = J_j^i \theta_1^j$, $\hat{\theta}_2^i = |J|^{-1/(n+1)} J_j^i \theta_2^j$, and under a projective transformation $\hat{\theta}_1^i = \theta_1^i$, $\hat{\theta}_2^i = \theta_2^i + \alpha\theta_1^i$. The set of locally defined 1-forms $\{A_j^i A_\alpha^\beta \theta_\beta^j\}$, with $\alpha, \beta = 1, 2$, $i, j = 1, 2, \ldots, n$, and with (A_j^i), (A_α^β) arbitrary local non-singular-matrix-valued functions, of size $n \times n$ and 2×2 respectively, is therefore defined independently of the choice of coordinates and of the choice of spray within the projective equivalence class; and it determines an almost Grassmann structure of type $(2,n)$ on $\mathcal{T}^\circ M$. ■

It will be observed that the choice of $\mathcal{T}^\circ M$ rather than $T^\circ M$ as the underlying manifold is essential. The construction works only because of the small simplification in the transformation of the 1-forms θ_2^i under a projective change of spray, compared with the transformation of the $dv^i + \Gamma_j^i dx^j$ on $T^\circ M$. In fact we can write the expression for θ_2^i as

$$du^i + \left(\Gamma_j^i - \frac{1}{n+1} u^i \Gamma_j\right) dx^j = du^i + \Pi_j^i dx^j + \frac{1}{n+1} \Gamma dx^i,$$

where

$$\Pi_j^i = \Gamma_j^i - \frac{1}{n+1}(u^i \Gamma_j + \Gamma \delta_j^i);$$

that is to say, $\Pi_j^i = \Pi_{jk}^i u^k$, where Π_{jk}^i are the fundamental projective invariants. Thus we can take $\theta_2^i = du^i + \Pi_j^i dx^j$ to represent the structure on any coordinate patch without loss of generality, which illustrates in another way the significance of the bundle $\mathcal{T}^\circ M$; and this choice of course makes calculations of projective quantities easier.

We now find the normal Cartan connection of this almost Grassmann structure, using the local 1-form basis $\{\theta^i_1, \theta^i_2\}$ with $\theta^i_1 = dx^i$, $\theta^i_2 = du^i + \Pi^i_j dx^j$.

Theorem 1 *The normal Cartan connection of the almost Grassmann structure, relative to the 1-form basis $\{\theta^i_1, \theta^i_2\}$, is given by*

$$\begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_j \\ \omega^2_1 & \omega^2_2 & \omega^2_j \\ \omega^i_1 & \omega^i_2 & \omega^i_j \end{bmatrix}$$

where

$$\omega^1_2 = -\frac{1}{n-1}\mathfrak{R}_k\theta^k_1, \quad \omega^1_j = -\frac{1}{n-1}\mathfrak{R}_{jk}\theta^k_1, \quad \omega^i_j = \Pi^{\,i}_{jk}\theta^k_1$$

and the remaining components are zero; in these formulæ we have set $\mathfrak{R}^{\,i}_{jk} = \mathfrak{R}^{\,i}_{ljk}u^l$ and $\mathfrak{R}_k = \mathfrak{R}^{\,j}_{jk} = \mathfrak{R}_{jk}u^j$.

Proof We start by differentiating the expression for θ^i_2 and using the results from Section 2.1 to obtain

$$d\theta^i_2 = \tfrac{1}{2}\mathfrak{R}^{\,i}_{jk}\theta^j_1 \wedge \theta^k_1 - \Pi^{\,i}_{jk}\theta^j_1 \wedge \theta^k_2;$$

recall that $\Pi^{\,i}_{jk}$ is symmetric in j and k and trace-free. We now write the components of the connection form in terms of the basis $\{\theta^i_1, \theta^i_2\}$ as, for example, $\omega^i_j = \omega^{i1}_{jk}\theta^k_1 + \omega^{i2}_{jk}\theta^k_2$, and use the equations in Appendix 1. Equation (1), obtained from the i, i trace conditions on the torsion, together with the conditions $\omega^i_i = 0$, leads to

$$\begin{aligned}
(n-1)\omega^{11}_{1k} + \omega^{i1}_{ki} &= 0 \\
n\omega^{12}_{1k} - \omega^{21}_{1k} &= 0 \\
\omega^{12}_{1k} - n\omega^{21}_{1k} - \omega^{i2}_{ki} &= 0 \\
\omega^{22}_{1k} &= 0 \\
(n-1)\omega^{11}_{2k} &= \mathfrak{R}_k \\
n\omega^{12}_{2k} - \omega^{21}_{2k} + \omega^{i1}_{ki} &= 0 \\
\omega^{12}_{2k} - n\omega^{21}_{2k} &= 0 \\
(n-1)\omega^{22}_{2k} + \omega^{i2}_{ki} &= 0.
\end{aligned}$$

Similarly, equation (2), obtained from the α, α trace conditions, gives

$$\begin{aligned}
(\omega^{11}_{1j} + \omega^{12}_{2j})\delta^i_k + (2\omega^{i1}_{jk} - \omega^{i1}_{kj}) &= \Pi^{\,i}_{jk} \\
(\omega^{21}_{1j} + \omega^{22}_{2j})\delta^i_k - (2\omega^{i2}_{jk} - \omega^{i2}_{kj}) &= 0.
\end{aligned}$$

On solving these equations we find that

$$\begin{aligned}\omega^1_2 &= -\frac{1}{n-1}\mathfrak{R}_k\theta^k_1, \quad \omega^1_1 = \omega^2_1 = \omega^2_2 = 0;\\ \omega^i_j &= \Pi^{\,i}_{jk}\theta^k_1.\end{aligned}$$

To find the remaining components of the connection form, we use equation (3) from Appendix 1. We have $K^1_1 = K^2_1 = K^2_2 = 0$ and

$$\begin{aligned}K^1_2 &= \frac{1}{2(n-1)}\mathfrak{B}_{jk}\theta^j_1 \wedge \theta^k_1 + \frac{1}{n-1}\mathfrak{R}_{jk}\theta^j_1 \wedge \theta^k_2\\ K^i_j &= \tfrac{1}{2}\mathfrak{R}^{\,i}_{jkl}\theta^k_1 \wedge \theta^l_1 - \Pi^{\,i}_{jkl}\theta^k_1 \wedge \theta^l_2,\end{aligned}$$

where $\mathfrak{B}_{jk} = \mathfrak{R}_{j|k} - \mathfrak{R}_{k|j}$, and the solidus in the subscript indicates differentiation by a horizontal vector field $\mathcal{K}_j$ corresponding to the horizontal distribution of the spray whose connection coefficients are the fundamental projective invariants $\Pi^{\,i}_{jk}$. It follows easily that

$$\omega^1_j = -\frac{1}{n-1}\mathfrak{R}_{jk}\theta^k_1, \quad \omega^2_j = 0.$$

∎

It may be seen from the formulæ and the fact that $\theta^k_1 = dx^k$ that the components of the connection form are semi-basic.

Corollary *The non-zero curvature components of the normal Cartan connection are*

$$\begin{aligned}\Omega^1_2 &= \frac{1}{2(n-1)}\mathfrak{B}_{jk}\theta^j_1 \wedge \theta^k_1\\ \Omega^1_j &= -\frac{1}{n-1}(\mathfrak{R}_{j[k|l]}\theta^k_1 \wedge \theta^l_1 + \mathfrak{R}_{jk,l}\theta^k_1 \wedge \theta^l_2)\\ \Omega^i_2 &= \tfrac{1}{2}P^i_{jkl}u^j\theta^k_1 \wedge \theta^l_1\\ \Omega^i_j &= \tfrac{1}{2}P^i_{jkl}\theta^k_1 \wedge \theta^l_1 - D^i_{jkl}\theta^k_1 \wedge \theta^l_2,\end{aligned}$$

where P^i_{jkl} denotes the generalized projective curvature tensor, and the comma in the subscript denotes differentiation by the vertical vector field $\partial/\partial u^l$. ∎

The torsion of the connection is given by $\{\Omega^i_1, \Omega^i_2\}$, and this vanishes if and only if $P^i_{jkl}u^j = 0$. But as we pointed out earlier, $P^i_{jkl}u^j = 0$ if and only if $P^i_{jkl} = 0$, and in turn this is the necessary and sufficient condition for the projective class to consist of isotropic sprays. We therefore have the further corollary:

Corollary *The normal Cartan connection is torsionless if and only if the sprays of the projective class are isotropic.* ∎

§4. Properties of the almost Grassmann structure obtained from a projective class of sprays

Suppose given an almost Grassmann structure of type $(2, n)$ on $\mathcal{T}^\circ M$. In this section we describe four conditions that may be imposed on such an almost Grassmann structure, which are satisfied when it is derived from a path geometry on M, and we show that any almost Grassmann structure satisfying those conditions is indeed derived from a path geometry on M. We use the fact that, although there is no vertical lift of vectors from M to $\mathcal{T}M$, there is a vertical lift of lines; that is, for any $u \in \mathcal{T}M$ over $x \in M$ we can identify $\mathrm{P}T_xM$ with $\mathrm{P}V_u\mathcal{T}M$ where $V_u\mathcal{T}M$ is the vertical subspace of $T_u\mathcal{T}M$.

Condition 1 The vertical subspace at each point of $\mathcal{T}^\circ M$ is an n-dimensional plane generator of the Segre cone.

In order to explain the second condition we make some preliminary remarks. If at any point $u \in \mathcal{T}^\circ M$ we take two plane generators of the Segre cone, one from each family, they intersect in a line (a generator of the Segre cone); thus from the first assumption, each 2-dimensional plane generator Σ_u of the Segre cone intersects the vertical subspace $V_u\mathcal{T}^\circ M$ in a line. Moreover, given any line in $V_u\mathcal{T}^\circ M$ there is a 2-dimensional plane generator which meets $V_u\mathcal{T}^\circ M$ in that line. Denote by τ the projection $\mathcal{T}M \to M$. Then $\tau_{u*}\Sigma_u$ is a 1-dimensional subspace of $T_{\tau(u)}M$. We thus have a map $\mathrm{P}V_u\mathcal{T}^\circ M \to \mathrm{P}T_{\tau(u)}M$ which sends a line in $V_u\mathcal{T}^\circ M$ to the 2-dimensional plane generator Σ_u it determines and then to the line $\tau_{u*}\Sigma_u$ in $T_{\tau(u)}M$.

Condition 2 The map just defined is the inverse of the vertical lift.

If the almost Grassmann structure satisfies these two assumptions then it has a local basis of the form

$$\left\{ \frac{\partial}{\partial x^i} - P_i^j \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^i} \right\},$$

where we can assume that P_i^j is trace-free.

Condition 3 The almost Grassmann structure is homogeneous, in the sense that the dilation field Δ on $\mathcal{T}^\circ M$ is an infinitesimal symmetry of it.

This means that for any local basis of the structure $\{\mathcal{E}_i^\alpha\}$ $(\alpha, \beta = 1, 2)$ there are functions A_j^i and A_β^α such that

$$[\Delta, \mathcal{E}_i^\alpha] = A_i^j \mathcal{E}_j^\alpha + A_\beta^\alpha \mathcal{E}_i^\beta.$$

This condition is well-defined, which is to say that it does not depend on the choice of basis.

Condition 4 With respect to the local 1-form basis dual to the vector field basis specified after Condition 2, namely $\{dx^i, du^i + P^i_j dx^j\}$, the curvature of the normal Cartan connection satisfies $\Omega^i_1 = 0$; that is, half of the torsion vanishes.

Theorem 2 *An almost Grassmann structure on $\mathcal{T}^\circ M$ that satisfies the four conditions given above is derived from a path geometry on M.*

Proof As we have already noted, such an almost Grassmann structure has a local basis

$$\mathcal{E}^1_i = \frac{\partial}{\partial x^i} - P^j_i \frac{\partial}{\partial u^j}, \qquad \mathcal{E}^2_i = \frac{\partial}{\partial u^i}$$

where $P^k_k = 0$. For this basis the homogeneity condition with $\alpha = 2$ is clearly satisfied. When $\alpha = 1$ we have

$$\begin{aligned}[][\Delta, \mathcal{E}^1_i] &= (P^j_i - \Delta(P^j_i))\mathcal{E}^2_j \\ &= A^j_i \mathcal{E}^1_j + A^1_1 \mathcal{E}^1_i + A^1_2 \mathcal{E}^2_i;\end{aligned}$$

so the coefficients P^j_i must satisfy $\Delta(P^j_i) = P^j_i + \lambda \delta^j_i$ for some local function λ; but $P^k_k = 0$, so $\lambda = 0$ and P^j_i is homogeneous of degree 1 in the u^i.

When we calculated the normal Cartan connection in Theorem 1 we did so under the assumption that

$$d\theta^i_2 = \tfrac{1}{2}\mathfrak{R}^i_{jk}\theta^j_1 \wedge \theta^k_1 - \Pi^{\,i}_{jk}\theta^j_1 \wedge \theta^k_2$$

where $\Pi^{\,i}_{jk}$ is symmetric in j and k and trace-free. In the present context we have

$$d\theta^i_2 = Q^i_{jk}\theta^j_1 \wedge \theta^k_1 - \frac{\partial P^i_j}{\partial u^k}\theta^j_1 \wedge \theta^k_2,$$

and we can no longer suppose in general that $\partial P^i_j / \partial u^k$ is symmetric in j and k, though its trace on i and j vanishes. (The coefficients Q^i_{jk} will not concern us.) The computation of ω^i_j and ω^α_β proceeds much as before, provided allowance is made for the loss of symmetry, and one finds that

$$\omega^1_1 = \omega^2_1 = 0, \qquad \omega^i_j = \left(\frac{2}{3}\frac{\partial P^i_j}{\partial u^k} + \frac{1}{3}\frac{\partial P^i_k}{\partial u^j} \right) \theta^k_1.$$

Now

$$\Omega^i_1 = d\theta^i_1 + \omega^i_j \wedge \theta^j_1 + \theta^i_\alpha \wedge \omega^\alpha_1 = \omega^i_j \wedge \theta^j_1,$$

and so $\Omega^i_1 = 0$ if and only if ω^{i1}_{jk} is symmetric in j and k, or in other words

$$\frac{\partial P^i_k}{\partial u^j} = \frac{\partial P^i_j}{\partial u^k}.$$

Now that we have expressed Conditions 3 and 4 in terms of properties of the P^i_j we may proceed to consideration of the main issue.

We have $u^i\mathcal{E}^2_i = \Delta$, so the distribution $\mathcal{D}$ spanned by $u^i\mathcal{E}^\alpha_i$, $\alpha = 1, 2$, is the distribution of 2-dimensional plane generators of Segre cones which intersect each vertical subspace $V_u\mathcal{T}^\circ M$ along the line determined by u. For every non-zero t the projection into $T_{\tau(tu)}M$ of $\mathcal{D}_{tu}$ is the same, namely the line determined by u via the inverse of the vertical lift. Now

$$[\Delta, u^i\mathcal{E}^1_i] = u^i\mathcal{E}^1_i + u^i[\Delta, \mathcal{E}^1_i] = u^i\mathcal{E}^1_i,$$

taking account of the analysis of Condition 2 above; thus $\mathcal{D}$ is integrable. Each integral manifold of $\mathcal{D}$ is ruled by the lines which are the integral curves of Δ, and projects onto a 1-dimensional submanifold of M, that is to say, a path; the path determined in this way by the integral submanifold of $\mathcal{D}$ through $u \in \mathcal{T}^\circ M$ has for its tangent line at $\tau(u) \in M$ the line in $T_{\tau(u)}M$ determined by u. The almost Grassmann structure therefore defines in this way a path geometry on M. We can identify any path of the path geometry locally with a base integral curve of the local vector field

$$u^i\frac{\partial}{\partial x^i} - 2P^i\frac{\partial}{\partial u^i}, \qquad P^i = \tfrac{1}{2}P^i_j u^j.$$

Then

$$\frac{\partial P^i}{\partial u^j} = \tfrac{1}{2}\left(P^i_j + \frac{\partial P^i_k}{\partial u^j}u^k\right) = \tfrac{1}{2}\left(P^i_j + \frac{\partial P^i_j}{\partial u^k}u^k\right) = P^i_j$$

by homogeneity. Thus the almost Grassmann structure is the one defined by the path geometry. ∎

§5. The almost Grassmann structure for isotropic sprays

We show in this section that when the normal Cartan connection of the almost Grassmann structure associated with a projective class of sprays has vanishing torsion, so that the class consists of isotropic sprays, then the almost Grassmann structure has two important geometrical properties: it is semi-integrable; and it is invariant under a certain 2-dimensional distribution, which means that it passes to the quotient to define an almost Grassmann structure on the path space. (Here, as elsewhere, we understand that the path space might not have a global manifold structure, and in such circumstances our results may be interpreted only locally.)

It is known ([7]) that if a spray is isotropic then there is locally a projectively equivalent spray with vanishing Riemann tensor. For such an R-flat spray the associated horizontal distribution on $T^\circ M$ is integrable. We show first that in such a case the corresponding distribution on $\mathcal{T}^\circ M$ is also integrable.

Lemma *The distribution on $\mathcal{T}^\circ M$ spanned by the local vector fields*

$$\mathcal{K}_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial u^j} + \frac{1}{n+1}\Gamma_i \Delta$$

coming from an R-flat spray is integrable.

Proof Let

$$\mathcal{H}_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial u^j}$$

be local vector fields on $\mathcal{T}^\circ M$ corresponding to a given choice of coordinates, so that

$$\mathcal{K}_i = \mathcal{H}_i + \frac{1}{n+1}\Gamma_i \Delta.$$

Then

$$[\mathcal{K}_i, \mathcal{K}_j] = [\mathcal{H}_i, \mathcal{H}_j] + \frac{1}{n+1}(\mathcal{H}_i(\Gamma_j) - \mathcal{H}_j(\Gamma_i))\Delta.$$

Now the Riemann curvature is given by

$$R^i_{jkl} = H_k(\Gamma^i_{jl}) - H_l(\Gamma^i_{jk}) + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk} = 0$$

by assumption, so that

$$\mathcal{H}_k(\Gamma_j) = \mathcal{H}_l(\Gamma^l_{jk}) - \Gamma^l_{km}\Gamma^m_{jl} + \Gamma_m \Gamma^m_{jk},$$

whence, taking the symmetry of Γ^i_{jk} into account, $\mathcal{H}_i(\Gamma_j) = \mathcal{H}_j(\Gamma_i)$, and therefore $[\mathcal{K}_i, \mathcal{K}_j] = [\mathcal{H}_i, \mathcal{H}_j]$. But comparing the local vector fields $\mathcal{H}_i$ with

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j},$$

the coordinate horizontal vector fields on $T^\circ M$, and using our assumption that $[H_i, H_j] = 0$, we see that $[\mathcal{H}_i, \mathcal{H}_j] = 0$, so $[\mathcal{K}_i, \mathcal{K}_j] = 0$. ∎

In fact, as $\mathcal{K}_i - \mathcal{H}_i$ is a multiple of the dilation field Δ, and as $\Delta(\Gamma_i) = 0$ because Γ_i is homogeneous of degree 0, we see that $\mathcal{H}_i(\Gamma_j) = \mathcal{K}_i(\Gamma_j)$, and therefore that $\mathcal{K}_i(\Gamma_j) = \mathcal{K}_j(\Gamma_i)$ also. We shall need this fact in the proof of Theorem 3 .

Theorem 3 *The almost Grassmann structure corresponding to a projective class of isotropic sprays is semi-integrable with respect to the n-dimensional plane generators of Segre cones.*

Proof This may be shown by applying the curvature test given by Akivis and Goldberg [1]; but it may be derived directly, as follows. Let us change the basis of the almost Grassmann structure to

$$\left\{ \frac{\partial}{\partial x^i} - \Pi_i^j \frac{\partial}{\partial u^j} + \gamma \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^i} \right\}, \qquad \gamma = \frac{1}{n+1}\Gamma,$$

that is, to $\{\mathcal{K}_i, \mathcal{U}_i\}$, where as before $\{\mathcal{K}_i\}$ comes from an R-flat spray, and $\{\mathcal{U}_i\}$ is a local basis for the vertical bundle on $\mathcal{T}^\circ M$. Then the n-dimensional distributions tangent to the Segre cones are those spanned by vector fields of the form $\lambda\mathcal{K}_i + \mu\mathcal{U}_i$ for functions λ, μ on $\mathcal{T}^\circ M$ (where of course if $\lambda' = f\lambda$, $\mu' = f\mu$ for some non-vanishing function f then (λ', μ') determines the same distribution as (λ, μ)). The almost Grassmann structure is semi-integrable if and only if for any point u of $\mathcal{T}^\circ M$ and any value of the ratio $\lambda(u) : \mu(u)$ there is a distribution $\langle \lambda\mathcal{K}_i + \mu\mathcal{U}_i \rangle$ which is integrable and whose coefficients have values at u in the given ratio.

Bearing in mind that $[\mathcal{K}_i, \mathcal{K}_j] = [\mathcal{U}_i, \mathcal{U}_j] = 0$ and that $[\mathcal{U}_i, \mathcal{K}_j] - [\mathcal{U}_j, \mathcal{K}_i] = \gamma_i\mathcal{U}_j - \gamma_j\mathcal{U}_i$ we see that

$$\begin{aligned}
&[\lambda\mathcal{K}_i + \mu\mathcal{U}_i, \lambda\mathcal{K}_j + \mu\mathcal{U}_j] \\
&\qquad = (\lambda\mathcal{K}_i(\lambda) + \mu\mathcal{U}_i(\lambda))\mathcal{K}_j + (\lambda\mathcal{K}_i(\mu) + \mu\mathcal{U}_i(\mu))\mathcal{U}_j \\
&\qquad\quad - (\lambda\mathcal{K}_j(\lambda) - \mu\mathcal{U}_j(\lambda))\mathcal{K}_i + (\lambda\mathcal{K}_j(\mu) + \mu\mathcal{U}_j(\mu))\mathcal{U}_i \\
&\qquad\quad + \lambda\mu(\gamma_i\mathcal{U}_j - \gamma_j\mathcal{U}_i).
\end{aligned}$$

It follows that the distribution will be integrable if and only if

$$\mu(\lambda\mathcal{K}_i(\lambda) + \mu\mathcal{U}_i(\lambda)) = \lambda(\lambda\mathcal{K}_i(\mu) + \mu\mathcal{U}_i(\mu) + \lambda\mu\gamma_i).$$

It is easy to see that this condition is unchanged if λ and μ are multiplied by the same scalar factor, so in any region in which $\lambda \neq 0$ we may take $\lambda = 1$, when the condition reduces to

$$\mathcal{K}_i(\mu) = -\mu(\mathcal{U}_i(\mu) + \gamma_i).$$

This is a system of first-order partial differential equations for μ, which is in effect solved for the derivatives $\partial\mu/\partial x^i$; it will be solvable if and only if the obvious integrability conditions are satisfied, and if they are there will be a solution in which μ takes a specified value at a chosen point of $\mathcal{T}^\circ M$. The integrability conditions are that for all i, j,

$$\mathcal{K}_j\left(\mu(\mathcal{U}_i(\mu) + \gamma_i)\right) = \mathcal{K}_i\left(\mu(\mathcal{U}_j(\mu) + \gamma_j)\right)$$

holds as a consequence of the original equations. Recalling that $\mathcal{K}_i(\Gamma_j) = \mathcal{K}_j(\Gamma_i)$ we see that this reduces to $\mu(\mathcal{K}_j(\mathcal{U}_i(\mu)) - \mathcal{K}_i(\mathcal{U}_j(\mu))) = 0$. But

$$\begin{aligned}
&\mathcal{K}_j(\mathcal{U}_i(\mu)) - \mathcal{K}_i(\mathcal{U}_j(\mu)) \\
&\quad = [\mathcal{K}_j, \mathcal{U}_i](\mu) - [\mathcal{K}_i, \mathcal{U}_j](\mu) + \mathcal{U}_i(\mathcal{K}_j(\mu)) - \mathcal{U}_j(\mathcal{K}_i(\mu)) \\
&\quad = \gamma_j \mathcal{U}_i(\mu) - \gamma_i \mathcal{U}_j(\mu) - \mathcal{U}_i(\mu(\mathcal{U}_j(\mu) + \gamma_j)) + \mathcal{U}_j(\mu(\mathcal{U}_i(\mu) + \gamma_i)) = 0.
\end{aligned}$$

A similar argument shows that the equations with $\mu = 1$ are integrable. We can therefore find a solution of the partial differential equations with given initial value of the ratio $\lambda(u) : \mu(u)$, as required, and the almost Grassmann structure is semi-integrable as asserted. ∎

It is easy to see, by a simple modification of the proof to include appropriate homogeneity conditions, that without loss of generality we can take the Segre foliations to be invariant under dilations of $\mathcal{T}^\circ M$.

One integrable distribution of plane Segre cone generators consists of the vertical distribution on $\mathcal{T}^\circ M$; the Segre foliation is just that given by the fibres. On the other hand, any point $u \in \mathcal{T}^\circ M$ and any subspace of $T_u \mathcal{T}^\circ M$ complementary to the fibre determine a dilation-invariant Segre foliation, which corresponds to the foliation of $T^\circ M$ determined by the horizontal distribution of a spray of the projective class whose Riemann curvature vanishes.

We now turn to the second geometric property of the almost Grassmann structure associated with a projective class of isotropic sprays. Consider the 2-dimensional distribution $\mathcal{D}$ on $\mathcal{T}^\circ M$ spanned by the vector fields $\{\mathcal{W}^1, \mathcal{W}^2\}$ where $\mathcal{W}^1 = u^i \mathcal{E}_i^1$, $\mathcal{W}^2 = u^i \mathcal{E}_i^2$, and where

$$\mathcal{E}_i^1 = \frac{\partial}{\partial x^i} - \Pi_i^j \frac{\partial}{\partial u^j}, \quad \mathcal{E}_i^2 = \frac{\partial}{\partial u^i}$$

are the elements of the frame dual to the θ_α^i. The distribution $\mathcal{D}$ is integrable, by homogeneity of $u^i \mathcal{E}_i^1$, since $u^i \mathcal{E}_i^2$ is the dilation field. The quotient of $\mathcal{T}^\circ M$ by $\mathcal{D}$ is the path space of the projective class of sprays.

Theorem 4 *The almost Grassmann structure passes to the quotient under $\mathcal{D}$ if and only if the projective class consists of isotropic sprays.*

Proof Since the condition for a spray to be isotropic is projectively invariant we can use the curvature associated with the fundamental invariants to state it, so that it becomes

$$\mathfrak{R}_j^i = \lambda \delta_j^i + \mu_j u^i.$$

The condition for the almost Grassmann structure to pass to the quotient is that for any $X \in \mathcal{D}$, there are functions A_j^i and A_β^α $(\alpha, \beta = 1, 2)$, depending on X, such that

$$[X, \mathcal{E}_i^\alpha] = A_i^j \mathcal{E}_j^\alpha + A_\beta^\alpha \mathcal{E}_i^\beta \pmod{\mathcal{D}};$$

this says that X is an infinitesimal symmetry of the structure, modulo $\mathcal{D}$. It will be enough to check this condition by taking for X the elements of a (local) basis for $\mathcal{D}$, in turn. In the case in point,

$$\begin{aligned}
[\mathcal{W}^1, \mathcal{E}_i^1] &= [u^j \mathcal{E}_j^1, \mathcal{E}_i^1] = -\mathcal{E}_i^1(u^j)\mathcal{E}_j^1 - u^j[\mathcal{E}_i^1, \mathcal{E}_j^1] = \Pi_i^j \mathcal{E}_j^1 + \mathfrak{R}_i^k \mathcal{E}_k^2 \\
[\mathcal{W}^1, \mathcal{E}_i^2] &= [u^j \mathcal{E}_j^1, \mathcal{E}_i^2] = -\mathcal{E}_i^2(u^j)\mathcal{E}_j^1 - u^j[\mathcal{E}_i^2, \mathcal{E}_j^1] = -\mathcal{E}_i^1 + \Pi_i^k \mathcal{E}_k^2 \\
[\mathcal{W}^2, \mathcal{E}_i^1] &= 0 \\
[\mathcal{W}^2, \mathcal{E}_i^2] &= -\mathcal{E}_i^2,
\end{aligned}$$

the second half coming from homogeneity. The condition is satisfied for $\mathcal{W}^2$. In order for it to be satisfied for $\mathcal{W}^1$, however, $R_i^k \mathcal{E}_k^2$ must differ from a scalar multiple of $\mathcal{E}_i^2$ by an element of $\mathcal{D}$; that is to say, we must have $\mathfrak{R}_i^k = \lambda \delta_i^k + \mu_i u^k$ for some λ and μ_i. We see, therefore, that the almost Grassmann structure passes to the quotient under $\mathcal{D}$ if and only if the projective class consists of isotropic sprays. ∎

§6. The Finslerian case

One source of projective classes of sprays is Finsler geometry, where the geodesics, considered as the extremals of the Finsler function rather than of the energy, are determined only up to reparametrization, and therefore define a projective class on $T^\circ M$. We examine the almost Grassmann structure associated with Finslerian geodesics. We also consider the extent to which conditions on the almost Grassmann structure associated with a general projective class of sprays might guarantee the existence of a suitable Finsler function.

Theorem 5 *To each Finsler function F on $T^\circ M$ there is associated a closed 2-form ω_F on $\mathcal{T}^\circ M$, such that the characteristic distribution of ω_F is the 2-dimensional distribution $\mathcal{D}$ corresponding to the geodesic sprays of F, and such that the n-dimensional plane generators of the Segre cones are isotropic with respect to ω_F.*

To establish this theorem, we need a lemma telling us when forms on $T^\circ M$ may be lifted to $T^\circ(\mathcal{V}M)$ and then projected to $\mathcal{T}^\circ M$. We denote by Λ the Liouville field on TM.

Lemma *If ω is a closed form on $T^\circ M$ satisfying $\Lambda \lrcorner\, \omega = 0$ then its pull-back $\nu_*^*\omega$ by $\nu_* : T^\circ(\mathcal{V}M) \to T^\circ M$ is projectable to a form on $\mathcal{T}^\circ M$.*

Proof A form $\hat{\omega}$ on $T(\mathcal{V}M)$ is projectable to $\mathcal{T}^\circ M$ if and only if

$$X \lrcorner\, \hat{\omega} = 0, \quad \mathcal{L}_X \hat{\omega} = 0, \quad \forall X \in \langle \Upsilon^V, \tilde{\Upsilon} \rangle.$$

Now if ω is closed then the second of these conditions follows from the first, because $\mathcal{L}_X \hat{\omega} = d(X \lrcorner\, \hat{\omega})$. If furthermore $\hat{\omega} = \nu^*\omega$ for some form ω

on $T^\circ M$, and X is ν_*-related to a vector field $\hat{X}$ on $T^\circ M$, then $X \lrcorner \hat{\omega} = \nu_*^*(\hat{X} \lrcorner \omega)$. Now Υ^V is ν_*-related to the zero vector field on $T^\circ M$, while $\tilde{\Upsilon}$ is ν_*-related to $-\Lambda$ on $T^\circ M$, so a closed form ω defines a form on $\mathcal{T}^\circ M$ by this process if $\Lambda \lrcorner \omega = 0$. ∎

Proof (of Theorem 5) Suppose given a Finsler function F on $T^\circ M$. Let θ be its Hilbert 1-form, so that $\theta = (\partial F/\partial v^i)dx^i$. Since θ is of homogeneity degree 0, $\Lambda \lrcorner\, d\theta = \mathcal{L}_\Lambda \theta - d\langle \Lambda, \theta \rangle = 0$; so, by the Lemma, $d\theta$ defines a 2-form on $\mathcal{T}^\circ M$. Set

$$F_{ij} = \frac{\partial^2 F}{\partial v^i \partial v^j};$$

recall that $F_{ij}v^j = 0$ by homogeneity. On $T^\circ M$, $d\theta = F_{ij}dx^i \wedge (dv^j + \Gamma^j_k dx^k)$ where Γ^i_j are the non-linear connection coefficients of any geodesic spray of F; notice that if we add to Γ^j_k the terms $\alpha\delta^j_k + \alpha_k v^j$, corresponding to a projective change, then the right-hand side is unchanged. By the Lemma, we may construct the corresponding form ω_F on $\mathcal{T}^\circ M$, and this can be represented as

$$\omega_F = F_{ij}dx^i \wedge (du^j + \Pi^j_k dx^k).$$

Now the 2-form $d\theta$ is not symplectic: it has a 2-dimensional characteristic subspace which is spanned by Λ and any geodesic spray, that is, the characteristic distribution is effectively the projective class. Note however that $d\theta$ projects onto the path space P, which is obtained by factoring by the characteristic distribution (which is integrable because the form is closed) and defines a symplectic form on this $2(n-1)$-dimensional manifold.

A similar property holds for ω_F. Recall that the n-dimensional generators of the Segre cones on $\mathcal{T}^\circ M$ are spanned by vector fields of the form $\lambda\mathcal{E}^1_i + \mu\mathcal{E}^2_i$ where

$$\mathcal{E}^1_i = \frac{\partial}{\partial x^i} - \Pi^j_i \frac{\partial}{\partial u^j}, \quad \mathcal{E}^2_i = \frac{\partial}{\partial u^i}.$$

The characteristic distribution of ω_F is spanned by $u^i\mathcal{E}^\alpha_i$, $\alpha = 1, 2$; that is, it is the distribution $\mathcal{D}$ of the previous section. Furthermore, the n-dimensional plane generators of the Segre cones are isotropic with respect to ω_F: for any λ, μ

$$\omega_F(\lambda\mathcal{E}^1_i + \mu\mathcal{E}^2_i, \lambda\mathcal{E}^1_j + \mu\mathcal{E}^2_j) = \lambda\mu(F_{ij} - F_{ji}) = 0.$$

∎

There is also a partial converse to this result. To express it, we need to use the concept of a pseudo-Finsler function. Recall that a Finsler function F must be homogeneous in the fibre coordinates, positive, and 'strongly convex': that is, if

$$g_{ij} = F\frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial F}{\partial u^i}\frac{\partial F}{\partial u^j}$$

then the matrix (g_{ij}) must be positive definite. In the result we are about to prove we show the existence of a function F which is homogeneous, and for which the sprays of a given projective class are formally geodesic. However, we cannot ensure that F is positive or strongly convex; the best that can be expected is that the corresponding (g_{ij}) is non-singular. We call a function with these properties a pseudo-Finsler function.

Theorem 6 *If for a given projective class of sprays there is on $\mathcal{T}^\circ M$ a 2-form ω such that*

- *the n-dimensional plane generators of the Segre cones are isotropic with respect to ω;*
- *the characteristic distribution of ω is $\mathcal{D}$;*
- *ω is closed*

then the projective class is the geodesic class of a locally defined pseudo-Finsler function.

Proof Set $\pi^i = du^i + \Pi^i_k dx^k$. We can express ω as

$$\omega = \tfrac{1}{2}a_{ij}dx^i \wedge dx^j + b_{ij}dx^i \wedge \pi^j + \tfrac{1}{2}c_{ij}\pi^i \wedge \pi^j,$$

where a_{ij} and c_{ij} are skew-symmetric in their indices. Then

$$\omega(\lambda\mathcal{E}^1_i + \mu\mathcal{E}^2_i, \lambda\mathcal{E}^1_j + \mu\mathcal{E}^2_j) = \lambda^2 a_{ij} + \lambda\mu(b_{ij} - b_{ji}) + \mu^2 c_{ij},$$

and this must vanish for every choice of λ and μ; so $a_{ij} = c_{ij} = 0$, $b_{ji} = b_{ij}$. Now if $X = \xi^i\mathcal{E}^1_i + \eta^i\mathcal{E}^2_i$ then $X \lrcorner\, \omega = \xi^i b_{ij}\pi^j - \eta^i b_{ji}dx^j$, so $X \lrcorner\, \omega = 0$ if and only if $b_{ij}\xi^j = b_{ij}\eta^j = 0$. Thus if the characteristic distribution of ω is $\mathcal{D}$ then $b_{ij}\xi^j = 0$ if and only if $\xi^i \propto u^i$. The exterior derivative of ω is

$$d\omega = \mathcal{E}^1_i(b_{jk})dx^i \wedge dx^j \wedge \pi^k - \mathcal{E}^2_j(b_{ik})dx^i \wedge \pi^j \wedge \pi^k - b_{ij}dx^i \wedge d\pi^j,$$

and therefore if ω is closed then $\mathcal{E}^2_j(b_{ik}) = \mathcal{E}^2_k(b_{ij})$, or

$$\frac{\partial b_{ik}}{\partial u^j} = \frac{\partial b_{ij}}{\partial u^k}.$$

Thus, using the symmetry of b_{ij}, there is a function F^* such that

$$b_{ij} = \frac{\partial^2 F^*}{\partial u^i \partial u^j}.$$

Moreover, $b_{ij}u^j = 0$, so

$$\frac{\partial^2 F^*}{\partial u^i \partial u^j} u^j = \frac{\partial}{\partial u^i}\left(\frac{\partial F^*}{\partial u^j} u^j - F^*\right) = 0,$$

so that there is some function f independent of the u^i such that

$$\frac{\partial F^*}{\partial u^j} u^j = F^* + f.$$

But F^* is determined only up to the addition of a function affine in the u^i, so without loss of generality we may take $f = 0$. Then F^* is homogeneous of degree 1 in the fibre coordinates, and determined up to the addition of a function linear in the u^i.

Next, we consider the relationship between the function F^* and the projective class of sprays with which we started the construction. For this purpose we must examine the remaining consequences of the closure condition. Now as we noted earlier,

$$d\pi^i = d\theta^i_2 = \tfrac{1}{2}\mathfrak{R}^i_{jk} dx^j \wedge dx^k - \Pi^{\,i}_{jk} dx^j \wedge \pi^k;$$

the remaining conditions are therefore

$$\mathcal{E}^1_i(b_{jk}) + b_{il}\Pi^{\,l}_{jk} = \mathcal{E}^1_j(b_{ik}) + b_{jl}\Pi^{\,l}_{ik}$$

which by contraction with u^i gives

$$\left(u^i \frac{\partial}{\partial x^i} - 2\Pi^i \frac{\partial}{\partial u^i}\right)(b_{jk}) - \Pi^l_j b_{lk} - \Pi^l_k b_{lj} = 0;$$

and

$$b_{il}\mathfrak{R}^l_{jk} + b_{jl}\mathfrak{R}^l_{ki} + b_{kl}\mathfrak{R}^l_{ij} = 0.$$

These are projectively invariant forms of conditions for the existence of a 1-homogeneous Lagrangian for a projective class of sprays given originally by Rapcsak [18]; we use the formulation to be found in a recent paper of Szilasi and Vattamány [22]. According to the theorem of Szilasi and Vattamány, when these conditions hold there is locally a 1-form η on M such that if $F = F^* + \eta_i u^i$ then each spray of the given projective class satisfies the Euler-Lagrange equations for F. Note that F is still indeterminate: one can add to it a total derivative, that is, a further linear term in which the coefficients η_i are those of an exact 1-form. In particular, by making use of this remaining freedom one can assume that in a neighbourhood of any point F is non-zero.

Finally, we show that (g_{ij}) is non-singular at any point u of $\mathcal{T}^\circ M$ where F is non-zero. For suppose that there is a vector (w^i) such that $g_{ij}w^j = 0$ at u; then in particular $g_{ij}u^iw^j = 0$, which means that

$$F(u)\frac{\partial F}{\partial u^j}w^j = 0, \qquad \text{or} \quad \frac{\partial F}{\partial u^j}w^j = 0.$$

But then

$$F(u)\frac{\partial^2 F}{\partial u^i \partial u^j}w^j = 0,$$

so that w^i must be a multiple of u^i. But $g_{ij}u^iu^j = F(u)^2 \neq 0$, so w^i must be the zero vector. ∎

§7. The isotropic case again

We end the main body of the paper by considering what happens when the spray is isotropic, in the light of the previous two sections, thereby putting our own gloss on the results of Grossman concerning the almost Grassmann structure associated with a 'torsion-free path geometry' [15], that is, a projective equivalence class of isotropic sprays. We note that, in Grossman's terminology, the 'torsion of a path geometry' is the trace-free part of the Jacobi endomorphism (see also [14]) and, as shown in [9], this is equivalent to $P^i_{jkl}u^l$. We also remind the reader that, just as in Grossman's paper, all of our results below are local, though we do not continue to mention the fact.

We note first since every isotropic spray is projectively R-flat, and every R-flat spray is Finslerian, we are guaranteed the existence of a Finsler function F of which the projective class is the geodesic class.

Secondly, by Theorem 3 the almost Grassmann structure on $\mathcal{T}^\circ M$ passes to the path space to define an almost Grassmann structure there; this is the 'almost Segre structure' defined by Grossman.

Thirdly, since the 2-form ω_F on $\mathcal{T}^\circ M$ defined in Theorem 4 has $\mathcal{D}$ as its characteristic distribution and is closed, it passes to the quotient under $\mathcal{D}$ to define a symplectic structure on the path space. This construction of a symplectic structure on the path space is a generalization of that discussed by Álvarez Paiva in [2] in the context of Hilbert's fourth problem, that is, the search for Finsler functions whose geodesic sprays are projectively flat, the so-called projective Finsler functions.

Fourthly, the Segre foliations on $\mathcal{T}^\circ M$ for which λ doesn't vanish project onto Lagrangian foliations of the symplectic structure. Grossman points out that the almost Grassmann structure on the path space is semi-integrable; we see now that the Segre foliations are Lagrangian, and are projections of the foliations on $\mathcal{T}^\circ M$ associated with those members of the projective class that are R-flat.

§ Appendix 1: almost Grassmann structures

In this first appendix we determine the normal Cartan connection for an almostGrassmann structure.

Our first task is to fix the gauge. We have already pointed out in effect that we can partially fix the gauge by choosing a particular local representative of the preferred local coframes (in other words, a local section of the G-structure) for ω^i_α. Now if $\bar{A}^i_j A^\beta_\alpha \omega^j_\beta = \omega^i_\alpha$ then (since $\{\omega^i_\alpha\}$ is a basis) $\bar{A}^i_j A^\beta_\alpha = 0$ unless $i = j$ and $\alpha = \beta$, while $\bar{A}^i_i A^\alpha_\alpha = 1$ (no sum), or $A^\alpha_\alpha = A^i_i$ for all i and α. Thus by a choice of frame we fix A^i_j and A^α_β up to the same scalar factor; that is, we may take $(A^i_j, A^\alpha_\beta) = \pm(\delta^i_j, \delta^\alpha_\beta)$, and the remaining gauge freedom is in the choice of the A^α_i. Under a gauge transformation with $(A^i_j, A^\alpha_\beta) = \pm(\delta^i_j, \delta^\alpha_\beta)$ we have $\omega^\alpha_\beta \mapsto \omega^\alpha_\beta \pm A^\alpha_i \omega^i_\beta$. Since we are in a gauge, the ω^α_β are local 1-forms on M which can be expressed in terms of the basis $\{\omega^i_\alpha\}$ as, say, $\omega^{\alpha\gamma}_{\beta j}\omega^j_\gamma$; then if we take $A^\alpha_i = \mp\omega^{\beta\alpha}_{\beta i}$ (sum over β intended) then the transformed ω^α_β will have zero trace. That is, by choice of gauge we can take $\omega^\alpha_\alpha = \omega^i_i = 0$, and not just $\omega^\alpha_\alpha + \omega^i_i = 0$. This, together with the choice of ω^i_α, fixes the gauge.

We can now proceed to fix the Cartan connection by conditions on the curvature, in the usual way. We shall write θ^i_α for the chosen coframe on M, instead of ω^i_α, to make it clear that the purpose of the exercise is to fix the remaining components of the connection form in terms of the θ^i_α. In the chosen gauge, each component of the connection form may be written as, for instance, $\omega^i_j = \omega^{i\gamma}_{jk}\theta^k_\gamma$; the aim is to determine all the coefficients, such as, in this case, $\omega^{i\gamma}_{jk}$.

The conditions imposed on the curvature must be such as to make the process global, even though it is carried out locally. Since the conditions will all be expressed in terms of vanishing of traces of components of the curvature with respect to the θ^i_α, in view of the transformation laws for the curvature and for the θ^i_α it will be clear that this requirement is satisfied.

We may write $d\theta^i_\alpha = -\frac{1}{2}C^{i\beta\gamma}_{\alpha jk}\theta^j_\beta \wedge \theta^k_\gamma$, where the coefficients $C^{i\beta\gamma}_{\alpha jk}$ (with the obvious skew- symmetry) are to be considered as known. We have first that

$$\Omega^i_\alpha = d\omega^i_\alpha + \omega^i_j \wedge \theta^j_\alpha + \theta^i_\beta \wedge \omega^\beta_\alpha,$$

from which it follows that

$$\Omega^{i\beta\gamma}_{\alpha jk} = -C^{i\beta\gamma}_{\alpha jk} + (\omega^{i\beta}_{kj}\delta^{\gamma}_{\alpha} - \omega^{i\gamma}_{jk}\delta^{\beta}_{\alpha}) + (\omega^{\beta\gamma}_{\alpha k}\delta^{i}_{j} - \omega^{\gamma\beta}_{\alpha j}\delta^{i}_{k}).$$

We now impose the trace conditions $\Omega^{j\beta\gamma}_{\alpha jk} = \Omega^{i\beta\gamma}_{\beta jk} = 0$; recalling that $\omega^{\alpha}_{\alpha} = \omega^{i}_{i} = 0$ we find that

$$\begin{aligned} q\omega^{\beta\gamma}_{\alpha k} - \omega^{\gamma\beta}_{\alpha k} + \omega^{j\beta}_{kj}\delta^{\gamma}_{\alpha} &= C^{j\beta\gamma}_{\alpha jk} \\ p\omega^{i\gamma}_{jk} - \omega^{i\gamma}_{kj} + \omega^{\gamma\beta}_{\beta j}\delta^{i}_{k} &= -C^{i\beta\gamma}_{\beta jk}. \end{aligned}$$

By taking further traces we obtain

$$\begin{aligned} q\omega^{\beta\gamma}_{\gamma k} + p\omega^{j\beta}_{kj} &= C^{j\beta\gamma}_{\gamma jk} = -C^{j\gamma\beta}_{\gamma kj} \\ -\omega^{\gamma\beta}_{\beta k} + \omega^{j\gamma}_{kj} &= C^{j\beta\gamma}_{\beta jk} \end{aligned}$$

($C^{j\beta\gamma}_{\gamma jk} = -C^{j\gamma\beta}_{\gamma kj}$ by virtue of the skew-symmetry of $C^{i\beta\gamma}_{\alpha jk}$). From these equations we can determine $\omega^{\gamma\beta}_{\beta k}$ and $\omega^{j\gamma}_{kj}$; then the previous equations become, say,

$$\begin{aligned} q\omega^{\beta\gamma}_{\alpha k} - \omega^{\gamma\beta}_{\alpha k} &= D^{\beta\gamma}_{\alpha k} \\ p\omega^{i\gamma}_{jk} - \omega^{i\gamma}_{kj} &= D^{i\gamma}_{jk}, \end{aligned}$$

where the right-hand sides can be given explicitly in terms of $C^{i\beta\gamma}_{\alpha jk}$. From the first of these

$$\begin{aligned} q\omega^{\beta\gamma}_{\alpha k} - \omega^{\gamma\beta}_{\alpha k} &= D^{\beta\gamma}_{\alpha k} \\ q\omega^{\gamma\beta}_{\alpha k} - \omega^{\beta\gamma}_{\alpha k} &= D^{\gamma\beta}_{\alpha k}, \end{aligned}$$

whence

$$(q^2 - 1)\omega^{\beta\gamma}_{\alpha k} = qD^{\beta\gamma}_{\alpha k} + D^{\gamma\beta}_{\alpha k}; \tag{1}$$

there is an analogous formula for $\omega^{i\gamma}_{jk}$, namely

$$(p^2 - 1)\omega^{i\gamma}_{jk} = pD^{i\gamma}_{jk} + D^{i\gamma}_{kj}. \tag{2}$$

Thus the conditions on the traces of Ω^{i}_{α}, together with the trace-free conditions coming from fixing the gauge, determine the connection components ω^{α}_{β} and ω^{i}_{j} completely.

It remains to determine the connection components ω^{α}_{i}; for this we need the curvature components Ω^{α}_{β} and Ω^{i}_{j}. We may express the second of these as

$$\Omega^{i}_{j} = d\omega^{i}_{j} + \omega^{i}_{k} \wedge \omega^{k}_{j} + \theta^{i}_{\alpha} \wedge \omega^{\alpha}_{j}.$$

The first two terms on the right-hand side are known; it will be convenient to write them collectively as K^i_j, so that

$$K^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

Note from this formula that $K^i_i = 0$. The first condition to impose is that $\Omega^i_i = 0$, which gives $\theta^i_\alpha \wedge \omega^\alpha_i = 0$, whence $\omega^{\alpha\beta}_{ij} = \omega^{\beta\alpha}_{ji}$. Similarly we have

$$\Omega^\alpha_\beta = K^\alpha_\beta + \omega^\alpha_i \wedge \theta^i_\beta$$

where

$$K^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta;$$

it is automatically the case that $\Omega^\alpha_\alpha = 0$ since $\Omega^\alpha_\alpha + \Omega^i_i = 0$.

Before proceeding we note that, unlike $\Omega^{i\beta\gamma}_{jkl}$, the quantity $\Omega^{\alpha\gamma\delta}_{\beta kl}$ does not transform linearly under a gauge transformation but instead picks up terms involving $\Omega^{i\beta\gamma}_{\alpha jk}$; however its traces do transform linearly because by assumption the traces of $\Omega^{i\beta\gamma}_{\alpha jk}$ vanish. We have

$$\begin{aligned} \Omega^{i\beta\gamma}_{jik} &= K^{i\beta\gamma}_{jik} + q\omega^{\beta\gamma}_{jk} - \omega^{\gamma\beta}_{jk} \\ \Omega^{\beta\alpha\gamma}_{\alpha jk} &= K^{\beta\alpha\gamma}_{\alpha jk} - p\omega^{\beta\gamma}_{jk} + \omega^{\beta\gamma}_{kj}, \end{aligned}$$

and we now need a condition which determines $\omega^{\beta\gamma}_{jk}$ for all p and q. The condition we impose is that

$$\Omega^{i\beta\gamma}_{jik} + \Omega^{i\gamma\beta}_{kij} - \Omega^{\beta\alpha\gamma}_{\alpha jk} - \Omega^{\gamma\alpha\beta}_{\alpha kj} = 0,$$

from which it follows that

$$2(p+q)\omega^{\beta\gamma}_{jk} - 4\omega^{\gamma\beta}_{jk} = K^{\beta\alpha\gamma}_{\alpha jk} + K^{\gamma\alpha\beta}_{\alpha kj} - K^{i\beta\gamma}_{jik} - K^{i\gamma\beta}_{kij}.$$

Then $\omega^{\gamma\beta}_{jk} = \omega^{\beta\gamma}_{jk}$, whence

$$2(p+q-2)\omega^{\beta\gamma}_{jk} = K^{\beta\alpha\gamma}_{\alpha jk} + K^{\gamma\alpha\beta}_{\alpha kj} - K^{i\beta\gamma}_{jik} - K^{i\gamma\beta}_{kij}, \tag{3}$$

which is consistent with the symmetry condition derived earlier.

We conclude from these calculations that the Cartan connection is determined uniquely by the stated conditions on its curvature.

§ Appendix 2: the flat case

Given an affine hyperplane in a vector space, and the projective space of rays in the vector space, we may consider affine geometry as the sub-geometry of projective geometry obtained by 'fixing the hyperplane at infinity'. The almost Grassmann structure obtained from a projective class of sprays in the way we have described is, in that sense, a sub-geometry of the full almost Grassmann geometry; we use the flat case to demonstrate this.

A related remark has already been made by Grossman [15], who observes that the second-order differential equation $y''(x) = 0$ does not have, as a solution curve, the line $x = 0$. But that is an artefact of the coordinate system, and we may obtain every line in projective space as a geodesic of the projective class of sprays containing the one whose affine coordinate representation is given above. Our remark is rather different: it is that the path space of all the lines generates only a part of the Grassmannian geometry, fixing again a hyperplane at infinity.

The intuition behind this comes from considering path geometry as a study of incidence relationships between points and lines, as in the description in [15] of the projective tangent bundle to n-dimensional projective space as a flag manifold. Now the points correspond to rays in an $(n+1)$-dimensional vector space, whereas the lines correspond to 2-planes and hence may be defined by affine lines in the vector space not passing through the origin. From this point of view, the projective incidence relationships may be described in terms of the intersection of lines in an $(n+1)$-dimensional affine space, where one of the two lines passes through a distinguished point, and thus the whole structure may be considered as a sub-geometry of $(n+1)$-dimensional projective geometry. (This is similar to, but not quite the same as, the 'twistor space' described in [13].)

To formalise this, we therefore start with an $(n+2)$-dimensional vector space V. A 2-plane in V may be identified with an equivalence class $[x \wedge y]$ of simple 2-vectors, and hence the collection of all 2-planes in V (or lines in $\mathrm{P}V$) may be identified with a subset $Q \subset \mathrm{P}W$, where $W = \bigwedge^2 V$ is the $\frac{1}{2}(n+1)(n+2)$-dimensional vector space of all 2-vectors. The classical example of this arises when $n = 2$, and then this subset Q is the Klein quadric, a 4-dimensional hypersurface in the 5-dimensional projective space $\mathrm{P}W$. In the general case, Q is a $2n$-dimensional quadratic variety; it is the image under the projection $W \to \mathrm{P}W$ of a cone $C \subset W$.

We show first that Q is a homogeneous space. The group $\mathrm{GL}(V)$ acts on the individual vectors in V, and this induces an action on 2-vectors by $x \wedge y \mapsto A(x) \wedge A(y)$ for $A \in \mathrm{GL}(V)$, mapping simple 2-vectors to simple 2-vectors and extended by linearity. Thus $\mathrm{PGL}(V)$ acts on W, and clearly this preserves the quadratic variety Q. If we take a basis $\{v^\alpha, v_i\} = \{v^1, v^2, v_1, \ldots, v_n\}$ of V, and let H be the subgroup of

PGL(V) fixing $[v^1 \wedge v^2]$, then H contains equivalence classes of matrices of the form

$$\begin{bmatrix} a^\alpha_\beta & a^\alpha_j \\ 0 & a^i_j \end{bmatrix}$$

(modulo non-zero multiples of the identity); so we have an almost Grassmann structure of type $(2, n)$.

The Segre cones and Segre planes of an almost Grassmann structure live in the tangent spaces at each point; but in this flat case we may consider the question globally. We see that certain n-dimensional and 2-dimensional linear subvarieties of Q correspond to the Segre planes, and they capture the incidence properties of the 2-planes of V in the following way. Fix a 2-plane $L \in Q$, and choose a basis $\{v^\alpha, v_i\}$ such that $L = [v^1 \wedge v^2]$. Any other 2-plane meeting L in a common ray may be written as $[x \wedge z]$, where $x = x_\alpha v^\alpha$ is a generator of the common ray, and where $z = z_\alpha v^\alpha + z^i v_i$. The 2-vector $x \wedge z \in W$ may therefore be written as

$$(x_\alpha v^\alpha) \wedge (z_\alpha v^\alpha + z^i v_i) = (x_1 z_2 - x_2 z_1) v^1 \wedge v^2 + (x_\alpha z^i) v^\alpha \wedge v_i \in C.$$

If we choose a fixed common ray $[x_0]$ then the set $\{x_0 \wedge z : z \in W\}$ is an $(n+1)$-dimensinal linear subspace of W contained in C, and so its projection is an n-dimensional linear subvariety of Q. On the other hand, if we choose a fixed ray $[L + z_0]$ in the quotient space V/L and consider only the rays $[z]$ such that $[L + z] = [L + z_0]$ then the set $\{x \wedge z : x \in L\}$ is a 3-dimensional linear subspace of W contained in C, and so its projection is a 2-dimensional linear subvariety of Q. These are the Segre planes of the structure.

Now the geometry of 2-planes in V is just the geometry of lines in the $(n+1)$-dimensional space PV; but we are concerned with n-dimensional path geometry. So suppose that V has a distinguished element e, and also a distinguished hyperplane U defined by an element $\varepsilon \in V^*$ satisfying $\varepsilon(e) = 1$, so that $V = [e] \oplus U$. We wish to consider the geometry of lines in the n-dimensional space PU, or of 2-planes in U. Now any ray in V, apart from $[e]$ itself, corresponds to a ray in U by $[e + x] \mapsto [x]$ where $x \in U$; we obtain a fibration P$V - \{[e]\} \to$ PU where two rays $[y]$, $[\hat{y}]$ are in the same fibre when $y \wedge \hat{y} \wedge e = 0$. But the situation for lines in PV, or 2-planes in V, although similar, is more complicated: there is again a subset $Q_0 \subset Q$ of 2-planes in V which project to 2-planes in U, but now the fibres of the projection are a distinguished subfamily of the Segre 2-planes described above.

To see how this arises, and to relate it to the general case described in the main body of the paper, let $i : U \to V$ be the inclusion map: this induces an injection $i_{(2)} : \bigwedge^2 U \to \bigwedge^2 V = W$. Denote the image of $i_{(2)}$ by $W_- \subset W$; it is easy to see that this image space is characterised by the condition that contraction with ε gives zero. We may also define an injection $j : U \to W$ by setting $j(u) = e \wedge i(u)$, and we shall denote the image of j by W_+.

We now claim that $W = W_+ \oplus W_-$. Certainly the dimensions are correct, because $\dim W_+ = n+1$ and $\dim W_- = \frac{1}{2}n(n+1)$, whereas $\dim W = \frac{1}{2}(n+1)(n+2) = \dim W_+ + \dim W_-$. So suppose that $e \wedge x \in W_+ \cap W_-$; then

$$0 = \varepsilon \lrcorner (e \wedge x) = \varepsilon(e)x - \varepsilon(x)e = x - \varepsilon(x)e$$

so that $e \wedge x = \varepsilon(x) e \wedge e = 0$. We shall henceforth identify $W = W_+ \oplus W_-$ with $U \oplus \bigwedge^2 U$; a point of $\mathrm{P}W$ may thus be written as $[x, y \wedge z]$ where $x, y, z \in U$.

Now define the subsets $Q_+ = \mathrm{P}W_+$ and $Q_- = \mathrm{P}W_- \cap Q$ of Q (note that, by construction, $\mathrm{P}W_+ \subset Q$); the argument above shows that Q_+ and Q_- are disjoint. We shall let Q_0 denote the remaining part of Q, so that $Q = Q_+ \cup Q_0 \cup Q_-$.

The quadratic variety Q may be used to describe incidence relationships in $\mathrm{P}U$. Take a point and a line in $\mathrm{P}U$, the former given by a ray $[x]$ and the latter by an equivalence class of simple 2-vectors $[y \wedge z]$. Thus $x \in U$ and $y \wedge z \in \bigwedge^2 U$, defined in each case to within a non-zero scalar multiple. In some circumstances the point will lie on the line, and this will happen when (for any choice of representatives) $x \wedge y \wedge z = 0$. It turns out that the set of such pairs $(x, y \wedge z)$ is the subset of $U \oplus \bigwedge^2 U$ which corresponds, under the identification with $W = W_+ \oplus W_-$, to the cone C. To see this, suppose first that we have an element of C, a simple 2-vector in W. Write it, using the decomposition $V = \langle e \rangle \oplus U$, as $(\lambda e + y) \wedge (\mu e + z)$ (where the inclusion maps have been omitted). Multiplying out, we get $e \wedge (\lambda z - \mu y) + y \wedge z$, corresponding to the pair $(\lambda z - \mu y, y \wedge z) \in U \oplus \bigwedge^2 U$; obviously $(\lambda z - \mu y) \wedge y \wedge z = 0$. On the other hand, suppose we have a pair $(x, y \wedge z) \in U \oplus \bigwedge^2 U$ where $x \wedge y \wedge z = 0$. Then x, y, z are linearly dependent, and as $y \wedge z \neq 0$ we must have $x = \lambda y + \mu z$ for some real λ, μ. Then

$$(\mu e + y) \wedge (-\lambda e + z) = \mu(e \wedge z) + \lambda(e \wedge y) + y \wedge z = e \wedge x + y \wedge z$$

so that $e \wedge x + y \wedge z$ is indeed a simple 2-vector in W and thus lies in C.

To relate this to our spray construction, we need to introduce the tangent bundle TU_0, where $U_0 = U - \{0\}$, and we do this in terms of the canonical trivialisation $TU_0 = U_0 \times U$. This is 'almost' the tangent bundle $T\mathcal{V}(\mathrm{P}U)$ to the volume bundle of the n-dimensional projective space PU: although $\mathcal{V}(\mathrm{P}U)$ may be identified with $U_0/\pm$ rather than with U_0 itself, we may nevertheless construct the bundle of weighted tangent vectors to PU by taking an additional quotient with respect to the $\mathbf{Z}_2$ symmetry. To simplify the notation slightly, we shall refer to this bundle as $\mathcal{T}_U$ rather than $\mathcal{T}(\mathrm{P}U)$.

The volume bundle coordinates $(x^a) = (x^0, x^i)$ are not the projections of the usual Cartesian coordinates on U, and in particular x^0 is a 'radial' coordinate. In fact, if we take the x^i to be affine coordinates on PU, related to Cartesian coordinates y^a by $x^i = y^i/y^0$, then the construction of the volume bundle gives $x^0 = y^0$, so Cartesian coordinates are given in terms of the x^a by $y^0 = x^0$, $y^i = x^0 x^i$. The transformation laws for the coordinate vector fields are

$$\frac{\partial}{\partial y^0} = \frac{\partial}{\partial x^0} - \frac{x^i}{x^0}\frac{\partial}{\partial x^i}, \qquad \frac{\partial}{\partial y^i} = \frac{1}{x^0}\frac{\partial}{\partial x^i},$$

and therefore

$$\Upsilon = x^0\frac{\partial}{\partial x^0} = y^0\frac{\partial}{\partial y^0} + y^i\frac{\partial}{\partial y^i} = y^a\frac{\partial}{\partial y^a}.$$

In coordinates (y^a, w^a) on TU_0,

$$\Upsilon^{\mathrm{V}} = y^a\frac{\partial}{\partial w^a}, \qquad \tilde{\Upsilon} = \Upsilon^{\mathrm{C}} - \tilde{\Delta} = y^a\frac{\partial}{\partial y^a}.$$

We may therefore define $\mathcal{T}_U$ as follows. We specify an equivalence relation on $U_0 \times U$ by $(\hat{x}, \hat{y}) \sim (x, y)$ if $\hat{x} = \lambda x$ and $\hat{y} = y + \mu x$ for $\lambda \in \mathbf{R}_0$, $\mu \in \mathbf{R}$; then $\mathcal{T}_U = \mathcal{T}(\mathrm{P}U)$ is the quotient of $U_0 \times U$ under this relation. We shall write a typical element of $\mathcal{T}_U$ as $([x], [[y]]_x)$.

We now define a map $\phi : U_0 \times U \to U \oplus \bigwedge^2 U$ by $\phi(x, y) = (x, x \wedge y)$. This gives rise to a map $\psi : \mathcal{T}_U \to \mathrm{P}(U \oplus \bigwedge^2 U)$ by $\psi([x], [[y]]_x) = [(x, x \wedge y)]$; this is easily seen to be well-defined. Using the identification $W = U \oplus \bigwedge^2 U$ we may regard ψ as a map $\mathcal{T}_U \to \mathrm{P}W$; we shall now investigate its relationship with Q.

First, $\psi(\mathcal{T}_U) \subset Q$ because $x \wedge (x \wedge y) = 0$; and $\psi(\mathcal{T}_U) \subset Q_+ \cup Q_0$ because $x \neq 0$. The map is injective, because if $[(\hat{x}, \hat{x} \wedge \hat{y})] = [(x, x \wedge y)]$ then $\hat{x} = \lambda x$ and $\hat{x} \wedge \hat{y} = \lambda(x \wedge y)$ for some $\lambda \in \mathbf{R}_0$, so that $x \wedge \hat{y} = x \wedge y$ and therefore $\hat{y} - y = \mu x$ for some $\mu \in \mathbf{R}$. And the map is surjective to $Q_+ \cup Q_0$, because if $[(u, X)] \in Q_+ \cup Q_0$ then $u \neq 0$ so that $u \in U_0$; and $u \wedge X = 0$ so that, as X is simple, $X = u \wedge v$ for some $v \in U$. We may therefore identify the bundle $\mathcal{T}_U$ with the subset $Q_+ \cup Q_0$ of Q.

We also remark that $\mathcal{T}_U$ is a vector bundle. We have already noted this in the general case, but we may see this directly here. Observe that

Q_+ is the set $\{[x,0] : x \in U_0\}$, and so is just the projective space PU; this is the base of the bundle. The projection is $([x],[\![y]\!]_x) \mapsto [x] \in \mathrm{P}U$. The linear structure in the fibres is inherited from the linear structure on $U_0 \times U \to U_0$, so that $([x],[\![y]\!]_x) + \lambda([x],[\![\hat{y}]\!]_x) = ([x],[\![y+\lambda\hat{y}]\!]_x)$. The decomposition $Q_+ \cup Q_0$ corresponds to a decomposition of $\mathcal{T}_U$ into zero and non-zero weighted vectors; we shall write $\mathcal{T}_U^\circ$ for the slit bundle containing the non-zero weighted vectors.

As a vector bundle, the $\mathcal{T}_U$ has a corresponding projective bundle $\mathrm{P}\mathcal{T}_U$ containing elements $([x],[[\![y]\!]_x])$ where $([x],[\![y]\!]_x) \in \mathcal{T}_U^\circ$. We may identify this with the projective tangent bundle of PU as follows. An element of $\mathrm{P}T(\mathrm{P}U)$ at a point $[x] \in \mathrm{P}U$ is a line in PU passing through $[x]$; but such a line is defined by a 2-plane $[x \wedge y]$ in U, and this corresponds to the element $([x],[[\![y]\!]_x]) \in \mathrm{P}\mathcal{T}_U$. So we have the sequence of projections

$$U_0 \times U_0 \to \mathrm{P}U \times U_0 \to \mathcal{T}_U^\circ \to \mathrm{P}T(\mathrm{P}U) \to \mathrm{P}U$$

where the dimensions of these manifolds are $2n+2$, $2n+1$, $2n$, $2n-1$ and n respectively.

Finally, we relate this approach to the Segre planes in Q. Take a point $([x],[\![y]\!]_x) \in \mathcal{T}_U^\circ$; this is the 2-plane $L = [(e-y) \wedge x] \in Q_0$. There are now canonical choices of rays to use in the Segre construction. We note first that $\varepsilon \lrcorner\, (e-y) \wedge x = x$, so that $[x]$ is a canonical choice of ray in the 2-plane; it is easy to see that the n-dimensional linear subvariety of Q corresponding to this choice is just (the projective completion of) the fibre of the vector bundle $\mathcal{T}_U \to \mathrm{P}U$ projecting to $[x] \in \mathrm{P}U$.

On the other hand, $L + e \in V/L$ may be used to define a canonical 2-dimensional linear subvariety of Q as described earlier, and this turns out to be a subset of Q_0. Any point in this subvariety is a 2-plane of the form

$$[(\alpha(e-y) + \beta x) \wedge (e + \lambda x + \mu y)],$$

and we may check that this is an element of Q_0 for all values of $\alpha, \beta, \lambda, \mu$ such that α and β are not both zero. The subvarieties of this kind are the leaves of a foliation of Q_0, and it is straightforward to check that two 2-planes $[(e-y) \wedge x]$, $[(e-\hat{y}) \wedge \hat{x}]$ of this form in V are in the same leaf of this foliation precisely when $[x \wedge y] = [\hat{x} \wedge \hat{y}]$, so that each leaf projects to a well-defined 2-plane in U.

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