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Perturbations of constant connection Wagner spaces

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Abstract.

This paper is dedicated to the memory of Makoto Matsumoto, Finsler teacher and friend. Prof. Matsumoto augmented work of Darboux and proved that any 2-spray is projectively a geodesic spray of a Finsler manifold, [12]. This result has a refinement for the case of constant coefficient sprays, all of which are projectively equivalent to straight lines, [5].

In the present paper, we classify 2-sprays whose coefficients are linear in x^1, x^2, the adapted coordinates, by a perturbation technique. We also study the Feynman-Kac solutions to the corresponding Finslerian diffusions. The results herein arose from applications, especially [3], [4], [7], [10], [14].

The computations in this work have been performed by the computer package Finsler [1], [13].

§1. Finsler Geometry

Our standard reference here is [1], Vol. I, Part 2. All manifolds will be C^∞ without boundary. Let M^n be an n-dimensional manifold. By *parallel transport* on M^n we mean the existence of linear (Kozul) connection. A *Finsler* connection is a linear (Kozul) connection $\mathbb{D}$ on TM^n, the tangent bundle on M^n with zero section deleted, which preserves under the action of $\mathbb{D}$ the Whitney sum decomposition $TTM = HIM \oplus VTM$ of *horizontal* and *vertical* distributions. Thus, $\mathbb{D}H = 0 = \mathbb{D}V$. The HTM subbundle of the double tangent bundle of M is often called a *nonlinear connection* on M^n. We define the *covariant derivative* (induced

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by HTM) of a vector field (C^∞) on M^n with local components X^i by

$$\nabla X^i = \partial_j X^i y^j + N^i_j X^j \tag{1}$$

where $(x^1, \dots, x^n, y^1, \dots, y^n)$ are local coordinates on TM. The summation convention on repeated upper and lower indices is used throughout. Also, $\partial_j \equiv \frac{\partial}{\partial x^j}$ throughout this paper. The n^2 functions N^i_j transform under $(x^i) \mapsto (\overline{x}^{\,i})$ non-singular just as $\gamma^i_{jk}(x)y^k$, where γ is the classical Levi-Cività connection of Riemannian geometry. The n^2-quantities are called the *local coefficients* of the non-linear connection.

Define now the *Berwald basis* for the set of all vector fields on TM^n by

$$\{\delta_i, \dot{\partial}_i\}, \quad \delta_i = \partial_i - N^j_i(x,y)\partial_j, \quad \dot{\partial}_j = \frac{\partial}{\partial y^j}\,. \tag{2}$$

Using this basis we can define the local coefficients of the Finsler connection $\mathbb{D}$ as follows:

$$\begin{aligned} \mathbb{D}_{\delta_i}\delta_j &= F^k_{ji}(x,y)\delta_k, & \mathbb{D}_{\delta_i}\dot{\partial}_j &= F^k_{ji}(x,y)\dot{\partial}_k \\ \mathbb{D}_{\dot{\partial}_i}\delta_j &= C^k_{ji}(x,y)\delta_k, & \mathbb{D}_{\dot{\partial}_i}\dot{\partial}_j &= C^k_{ji}(x,y)\dot{\partial}_k. \end{aligned} \tag{3}$$

Under non-singular coordinate change $(x^i) \mapsto (\overline{x}^{\,i})$ the $F^k_{ji}(x,y)$ transform just as a classical linear connection (like, say $\gamma^i_{jk}(x)$) while $C^k_{ji}(x,y)$ is a tensor.

Denote $\{\delta_i, \dot{\partial}_i\}$ by $\{X_a\}_{a=\overline{1,2n}}$ and by $\{\theta^a\}_{a=\overline{1,2n}}$, the *dual basis* $\{dx^i,\, \delta y^i\}$. The *connection* 1-*forms* (ω^a_b) corresponding to θ^a are defined as

$$\omega^i_j = F^i_{jk}dx^k + C^i_{jk}\delta y^k \tag{4}$$

and 1^{st}-*structure equations* for $\mathbb{D}$ are

$$\left\{ \begin{aligned} -dx^h \wedge \omega^i_j &= -\Theta^i \\ d(\delta y^i) - \delta y^h \wedge \omega^i_h &= -\widetilde{\Theta}^{\,i} \end{aligned} \right. \tag{5}$$

where the 2-*forms of torsion* $\Theta^a = \{\Theta^i, \widetilde{\Theta}^{\,i}\}$ are given as

$$\Theta^i = \frac{1}{2}T^i_{jk}dx^j \wedge dx^k + C^i_{jk}dx^j \wedge \delta y^k \tag{6}$$

$$\widetilde{\Theta}^i = \frac{1}{2}\,R^i_{jk}dx^j \wedge dx^k + P^i_{jk}dx^j \wedge \delta y^k + \frac{1}{2}S^i_{jk}\delta y^j \wedge \delta y^k.$$

The 2-*forms of curvature* for $\mathbb{D}$ are

(7) $$d\omega^i_j - \omega^h_j \wedge \omega^i_h = -\Omega^i_j,$$

where

(8) $$\Omega^i_j = \frac{1}{2} R^i_{jkh} dx^k \wedge dx^h + P^i_{jkh} dx^k \wedge \delta y^h + \frac{1}{2} S^i_{jkh} \delta y^k \wedge \delta y^h.$$

For a Finsler metric on M, we shall need to use the so-called *Cartan Finsler connection*, but there are a number of other very important connections, [1]. For Cartan we must require

(9) $$T^k_{ij} = 0, \quad S^k_{ij} = 0, \quad P^k_{ij} = 0.$$

It will follow that $C_{ijk} = g_{il} C^{\ell}_{jk} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$, where $L(x, y)$is the so-called *Finsler metric function.* If we set $\widetilde{F} = \frac{L^2}{2}$, then $g_{ij}(x, y) := \dot{\partial}_i \dot{\partial}_j \widetilde{F}$ is the so-called *fundamental metric tensor* $g = (g_{ij})$ of the *Finsler manifold* $(M^n, \widetilde{F})$. However, the Levi-Cività coefficients γ^i_{jk} depend on y^i, as well as x^i and are *not* connection coefficients in Finsler geometry proper (i.e. $C^i_{jk} \neq 0 \Longleftrightarrow g_{ij}$ depends on y^i). But, the local coefficients of Cartan satisfy

(10) $$F^i_{jk}(x, y) = \frac{1}{2} g^{ir}(x, y)\big(\delta_k g_{jr}(x, y) + \delta_j g_{ir}(x, y) - \delta_r g_{jk}(x, y)\big),$$

which is, in fact, very similar in form to the famous Levi-Cività formula which, by replacing δ_i by ∂_i, gives precisely that formula.

The last two basic properties of the Cartan connection for $(M^n, \widetilde{F})$ are *horizontal* and *vertical metricity*: if we denote this most important connection by $C\Gamma = (\Gamma^h, \Gamma^v)$ where Γ^h is given by $F^i_{jk}(x, y)$ and Γ^v by $C^i_{jk}(x, y)$, then globally,

(11) $$\begin{aligned} &h\text{-}metrical : \nabla^h g &= 0, \\ &v\text{-}metrical : \nabla^h g &= 0, \end{aligned}$$

where the first is given locally as

(12) $$\delta_k g_{ij} = F_{ijk} + F_{jik},$$

where $F_{ijk} = g_{i\ell} F^{\ell}_{jk}$, and the 2nd is given locally by

(13) $$\dot{\partial}_k g_{ij} = C_{ijk} + C_{jik} = 2C_{ijk}\,.$$

Definition. A vector field S on TM is a *semispray* if and only if $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\dot{\partial}_i$ locally on TM. G^i are the *local coefficients* and HTM is the induced nonlinear connection, $N^i_j = \dot{\partial}_j G^i = G^i_j$.

Consider the *Berwald connection* D induced by HTM. This is a Finsler connection with the local coefficients $B\Gamma = (N^i_j = \frac{\partial G^i}{\partial y^j}, F^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}, C^i_{jk} = 0)$. The connection 1-forms of the Berwald connection D are then given by

$$\omega^i_j = F^i_{jk} dx^k = \frac{\partial^2 G^i}{\partial y^j \partial y^k} dx^k = G^i_{jk} dx^k.$$

The Berwald connection has only one component of torsion, the v(h)-torsion, which also gives the 3-index curvature of the nonlinear connection:

$$vT\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = R^k_{ji}\dot{\partial}_k = \left(\frac{\delta N^k_j}{\delta x^i} - \frac{\delta N^k_i}{\delta x^j}\right)\dot{\partial}_k. \tag{14}$$

The horizontal two forms of torsion Θ^i of the Berwald connection vanish and the vertical two-forms of torsion of the Berwald connection are given by:

$$\tilde{\Theta}^i = \frac{1}{2} R^i_{jk} dx^j \wedge dx^k.$$

The two nonzero components of curvature for the Berwald connection D are:

$$\begin{aligned} R_{h\ jk}^{\ i} &= \frac{\delta F^i_{hj}}{\delta x^k} - \frac{\delta F^i_{hk}}{\delta x^j} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj}; \\ D_{h\ jk}^{\ i} &= \frac{\partial^3 G^i}{\partial y^h \partial y^j \partial y^k}. \end{aligned} \tag{15}$$

The curvature 2-forms of the Berwald connection are given by:

$$\Omega^i_j = \frac{1}{2} R_{j\ kh}^{\ i} dx^k \wedge dx^h + D_{j\ kh}^{\ i} dx^k \wedge \delta y^h.$$

The first structure equations of the Berwald connection D are given by:

$$\begin{aligned} & -dx^h \wedge \omega^i_h = 0, \\ d(\delta y^i) \quad & -\delta y^h \wedge \omega^i_h = -\tfrac{1}{2} R^i_{jk} dx^j \wedge dx^k. \end{aligned} \tag{16}$$

The second structure equations of the Berwald connection D are given by:

$$(17) \qquad d\omega_j^i - \omega_j^h \wedge \omega_h^i = -\frac{1}{2} R_{j\ kh}^{\ i} dx^k \wedge dx^h - D_{j\ kh}^{\ i} dx^k \wedge \delta y^h.$$

Theorem 0. *The Berwald connection of a semispray S has zero curvature (is flat, i.e. $R = 0$, $D = 0$ in (15)) if and only if about every point $p \in M$ there are local coordinates (x^i) in M such that with respect to the induced coordinates (x^i, y^i) on TM, the local coefficients of the semispray S have the form:*

$$(18) \qquad 2G^i(x, y) = A_j^i(x) y^j + B^i(x).$$

Proof. If there exist induced coordinates on TM such that the semispray S has the local coefficients $2G^i(x, y) = A_j^i(x) y^j + B^i(x)$, then the local coefficients of the Berwald connection D vanish, that is $F_{jk}^i = \frac{\partial^2 G^i}{\partial y^j \partial y^k} = 0$. From (15) we can see that the curvature components of D vanish so the Berwald connection is flat.

Now let us assume that the curvature two forms Ω_j^i of the Berwald connection vanish. As the horizontal torsion two forms Θ^i are zero, there are induced coordinates on TM with respect to which the local coefficients of the Berwald connection vanish: $F_{jk}^i = 0$ and $C_{jk}^i = 0$. But $F_{jk}^i = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$, so with respect to these coordinates we have that $2G^i(x, y) = A_j^i(x) y^j + B^i(x)$.

§2. Projective geometry

2.1. Local sprays

Consider a smooth connected n-manifold M^n and select a trivializing chart (U, h) on M^n for the slit tangent bundle $\tilde{T}M^n$ (i.e. with the zero section removed). A (local) *spray* in (U, h) is a system of ode's

$$(19) \qquad \frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0, \quad (i = 1, \ldots, n),$$

where the n functions G^i are C^∞ on U in $x^i, \ldots, x^n$ and in $dx^1/ds, \ldots,$ dx^n/ds (off the zero section), are otherwise continuous and are second-degree positively homogeneous in the dx^i/ds. The path parameter s is special. For a general parameter t along solutions of Eq.(19) we have

$$(20) \qquad \ddot{x}^i + 2G^i(x, \dot{x}) = \frac{s''}{s'} \dot{x}^i,$$

where $s' := ds/dt$, $\dot{x}^i := dx^i/dt$ and $\ddot{x}^i := d^2x^i/dt^2$.

Consider $\psi(x, \dot{x})$, a smooth scalar function on $\tilde{T}M^n$, which is first-degree positively homogeneous in $\dot{x}^1, \ldots, \dot{x}^n$.

The quantities

$$\frac{\ddot{x}^i + 2G^i}{\dot{x}^i} = \frac{\ddot{x}^j + 2G^j}{\dot{x}^j}, \quad \forall i, j \in \{1, \ldots, n\}$$

remain unchanged by the transformation

$$G^i \to \bar{G}^i := G^i + \psi.\dot{x}^i \tag{21}$$

which sends the spray G in (U, h) to spray $\bar{G}$ in (U, h). That is, there exists a diffeomorphism which smoothly maps solutions of G into solutions of $\bar{G}$. Such a mapping is called the *projective transformation* of G onto $\bar{G}$ in (U, h).

One obtains from the *spray parameter* s (i.e. one which makes the RHS of Eq.(20) vanish) a new spray parameter determined by ψ. Namely,

$$\bar{s} = A + B \,.\int \mathrm{e}^{2/(\mathrm{n}+1)\int_\gamma \psi(\mathrm{x}, \mathrm{dx}/\mathrm{d}\tilde{\mathrm{t}})\mathrm{d}\tilde{\mathrm{t}}}\mathrm{ds}, \tag{22}$$

where $\tilde{t}$ is any parameter along any path γ, that is a solution of G, and A, B are constants of integration.

We can see the effect of this projective change, or *time-sequencing change*, by considering the *canonical spray connection coefficients* in (U, h):

$$G^i_j := \dot{\partial}_j G^i, \quad G^i_{jk} := \dot{\partial}_k G^i_j, \tag{23}$$

where $\dot{\partial}_l$ indicates partial differentiation with respect to $\dot{x}^l$. The transformation of coordinates from (U, h) to $(\bar{U}, \bar{h})$, i.e. from $x^1, \ldots, x^n$ to $\bar{x}^1, \ldots, \bar{x}^n$, has the effect [1],

$$\frac{\partial \bar{x}^r}{\partial x^j}\frac{\partial \bar{x}^s}{\partial x^k}\bar{G}^i_{rs} = \frac{\partial \bar{x}^i}{\partial x^r}G^r_{jk} - \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}. \tag{24}$$

Because G^i are homogeneous of the second degree in $\dot{x}^l$, we have the equivalent expression for Eq.(19)[1]

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{25}$$

[1]Einstein summation convention throughout.

Upon time-sequencing change ψ of Eq.(25), we have by differentiation in (U, h)

$$\bar{G}^i_{jk} = G^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j + \dot{x}^i \dot{\partial}_k \psi_j, \tag{26}$$

where $\psi_l = \dot{\partial}_l \psi$.

Define

$$\begin{aligned} \Pi^i &:= G^i - \frac{1}{n+1} G^a_a \dot{x}^i, \\ \Pi^i_j &:= \dot{\partial}_j \Pi^i, \quad \Pi^i_{jk} := \dot{\partial}_k \Pi^i_j \end{aligned} \tag{27}$$

for a given spray G in (U, h). It is easy to see that

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} (\delta^i_j G^a_{ak} + \delta^i_k G^a_{aj} + \dot{x}^i \mathbb{D}^a_{ajk}) \tag{28}$$

and that

$$\Pi^a_{ak} = 0.$$

$\mathbb{D}^i_{jkl} := \dot{\partial}_l G^i_{jk}$, called the *(non-projective) Douglas tensor*, transforms as a *classical* fourth-rank tensor. Its importance lies in the fact that G^i_{jk} *are independent of* $\dot{x}^l$ *if and only if* $\mathbb{D}^i_{jkl} = 0$. That is, the vanishing of tensor $\mathbb{D}$ is necessary and sufficient for G to be a *quadratic spray*, as in classical affine geometry and its specialization to Riemannian geometry. If G^i_{jk} are constants in (U, h), then we say (25) is a *constant spray* and (U, h) is an *adapted coordinate* system.

Furthermore, Π^i_{jk} remains *unchanged when G is projectively mapped onto $\bar{G}$*. Π is called the *normal spray connection* in (U, h) for G. Its spray curves are solutions of

$$\frac{d^2 x^i}{d\bar{s}^2} + \Pi^i_{jk} \frac{dx^j}{d\bar{s}} \frac{dx^k}{d\bar{s}} = 0. \tag{29}$$

Remark. (1) $\bar{s}$ remains *unchanged* under coordinate transformations $(U, h) \to (\bar{U}, \bar{h})$, whose Jacobians lie in $SL(n, \mathbb{R})$, the *real unimodular group on* $\mathbb{R}^n$, and *only those* (i.e. the structural group of $\tilde{T}M^n$ is reduced from $GL^+(n, \mathbb{R})$, the nonsingular real $n \times n$ matrices with positive determinant, to $SL(n, \mathbb{R})$).

(2) Π^i_{jk} transforms as a classical connection (i.e. like G^i_{jk} above) if and only if transformations have constant Jacobian determinant.

(3) Π^i_{jk} is a tensor if and only if the structural group of $\tilde{T}M^n$ is reduced to the transformation of coordinates $(U, h) \to (\bar{U}, \bar{h})$ of the

form:

$$\bar{x}^i = \frac{a^i_j x^j + b^i}{c_k x^k + h} \quad \text{and} \quad \begin{pmatrix} & & b^1 \\ & a^i_j & \vdots \\ & & b^n \\ c_1 \dots c_n & & h \end{pmatrix}$$

$(n+1) \times (n+1)$ constant matrix. This is the *classical projective group.*

Performing path-deviation for spray Eq.(29), we obtain the analogue of the usual "geodesic" deviation equation:

$$\frac{D^2 u^i}{d\bar{s}^2} + W^i_j u^j = 0, \tag{30}$$

where

$$W^i_j = 2\partial_j \Pi^i - \partial_r \Pi^i_j \dot{x}^r + 2\Pi^i_{jr}\Pi^r - \Pi^i_r \Pi^r_j. \tag{31}$$

This occurs as follows: We are *given the local spray* Π in (U, h) and let $x^i(\bar{s};\eta)$ be a smooth 1-parameter family of solutions with initial conditions $x^i(0;\eta), \dot{x}^i(0)$. Since a spray will have a solution through any point $p \in U$ and in any direction, these are called *arbitrary smooth initial conditions.*

By Taylor's theorem,

$$x^i(\bar{s};\eta) = x^i(\bar{s}) + \eta u^i(\bar{s}) + \eta^2(\dots)$$

and substituting this into $\Pi^i_{jk}(x,\dot{x})$ passage to the limit $\eta \to 0$, yields the *variational equations*

$$\frac{d^2u^i}{d\bar{s}^2} + \partial_l \Pi^i_{jk}(x,\dot{x}) u^l \frac{dx^j}{d\bar{s}} \frac{dx^k}{d\bar{s}} + 2\Pi^i_{jk}(x,\dot{x}) \frac{dx^k}{d\bar{s}} \frac{du^j}{d\bar{s}} = 0. \tag{32}$$

Defining the *projective covariant differential operation* as, for example,

$$A^i_{/l} := \partial_l A^i + \Pi^i_{jl}(x,\dot{x}) A^j, \tag{33}$$

and

$$\frac{DA^i}{d\bar{s}} := A^i_{/l} \frac{dx^l}{d\bar{s}}, \tag{34}$$

with similar formulas holding for higher order tensors.

Using this we can rewrite Eq.(32) as

(35)

$$\frac{D}{d\bar{s}}\left(\frac{du^i}{d\bar{s}}+\Pi^i_r u^r\right)+\Pi^i_l\left(\frac{du^l}{d\bar{s}}+\Pi^l_r u^r\right)+\left(2\partial_r\Pi^i-\frac{d\Pi^i_r}{d\bar{s}}-\Pi^i_l\Pi^l_r\right)u^r=0,$$

which is precisely Eq.(30) because of Eq.(29) and the second degree homogeneity of $\Pi^i(x,\dot{x})$ in $\dot{x}$ and Eq.(27).

Now, following Berwarld's technique, define

$$W^i_{jk} := \frac{1}{3}(\dot{\partial}_k W^i_j - \dot{\partial}_j W^i_k) \tag{36}$$

and (Weyl's Projective Curvature)

$$W^i_{jkl} := \dot{\partial}_l W^i_{jk}. \tag{37}$$

This four-index quantity actually is a *tensor*. However, the *projective covariant derivative of a tensor is not necessarily a tensor.*

We are now able to state the two *main theorems of local projective differential geometry*, [11].

Theorem A. *There is a coordinate chart (U,h) on M^n, $n\geq 3$, such that $\Pi^i_{jk}=0$, if and only if, $W^i_{jkl}=0$, and $\dot{\partial}_l\Pi^i_{jk}:=\Pi^i_{jkl}=0$. The tensor, Π^i_{jkl}, is called the (projective) Douglas tensor.*

Theorem B. *There is a coordinate chart (U,h) on M^2 such that $\Pi^i_{jk}=0$, if and only if, $\Pi^i_{jkl}=0$ and $\rho_{jkl}=0$, where $\rho_{jkl}:=r_{jk/l}-r_{jl/k}$, $r_{jk}:=\mathbb{B}^h_{jkh}$, where*

$$\mathbb{B}^i_{jkl}=\partial_l\Pi^i_{jk}+\Pi^s_{jk}\Pi^i_{sl}+\Pi^i_{jls}\Pi^s_{mk}\dot{x}^m-(k/l).$$

The symbol, $-(k/l)$, means repeat all terms that come before but interchange k and l and put a minus in front of the whole expression.

Remark. The four-index tensor $\mathbb{B}$ is analogous to the usual curvature of a spray except that G^i_j, G^i_{jk} are replaced by Π^i_j, Π^i_{jk}.

The condition $\Pi^i_{jk}=0$ for all $i,j,k\in\{1,\ldots,n\}$ in some (U,h) coordinate system is the so-called *condition of projective flatness.* We now consider the $n=2$ case of a normal spray connection curves (29) associated with a given constant spray. We know from Eq.(28) that

$$\Pi^1_{11}=-\Pi^2_{21} \quad\text{and}\quad \Pi^2_{22}=-\Pi^1_{12}.$$

We can therefore set $\Pi^1_{11} = \bar{\alpha}_1$, $\Pi^2_{22} = \bar{\beta}_1$, $\Pi^1_{22} = \bar{\alpha}_2$ and $\Pi^2_{11} = \bar{\beta}_2$ in Eq.(29), which becomes

$$\begin{aligned} &\frac{d^2x^1}{d\bar{s}^2} + \bar{\alpha}_1\left(\frac{dx^1}{d\bar{s}}\right)^2 - 2\bar{\beta}_1\frac{dx^1}{d\bar{s}}\frac{dx^2}{d\bar{s}} + \bar{\alpha}_2\left(\frac{dx^2}{d\bar{s}}\right)^2 = 0, \\ &\frac{d^2x^2}{d\bar{s}^2} + \bar{\beta}_1\left(\frac{dx^2}{d\bar{s}}\right)^2 - 2\bar{\alpha}_1\frac{dx^1}{d\bar{s}}\frac{dx^2}{d\bar{s}} + \bar{\beta}_2\left(\frac{dx^1}{d\bar{s}}\right)^2 = 0. \end{aligned} \tag{38}$$

Now,

$$\rho_{121} = \Pi^1_{12}r_{11} + \Pi^2_{12}r_{21} - \Pi^1_{11}r_{12} - \Pi^2_{11}r_{22} \tag{39}$$

from Theorem B. But,

$$r_{12} = \Pi^1_{11}\Pi^2_{22} - \Pi^2_{11}\Pi^1_{22} = r_{21}. \tag{40}$$

Also,

$$\Pi^2_{12}r_{21} - \Pi^1_{11}r_{12} = -\Pi^1_{11}[2\Pi^1_{11}\Pi^2_{22} - \Pi^1_{22}\Pi^1_{11} - \Pi^1_{22}\Pi^2_{11}]. \tag{41}$$

Furthermore,

$$\begin{aligned} r_{22} &= 2[\Pi^1_{22}\Pi^1_{22} - \Pi^2_{22}\Pi^2_{22}], \\ r_{11} &= 2[-\Pi^1_{11}\Pi^1_{11} + \Pi^2_{22}\Pi^2_{11}], \end{aligned} \tag{42}$$

so that substitution of Eqs.(40) and (42) into Eq.(39), yields

$$\rho_{121} = 0, \tag{43}$$

by using Eq.(41). Similarly, one can prove that

$$\rho_{212} = 0. \tag{44}$$

It is now clear that $\rho_{jkl} = 0$. Also, $\Pi^i_{jkl} = 0$ because in this constant connection case the normal spray is quadratic since, in general,

$$\Pi^i_{jkl} = \mathbb{D}^i_{jkl} - P\left(\frac{1}{n+1}\delta^i_j\mathbb{D}^a_{akl}\right) - \frac{1}{n+1}y^i\dot{\partial}_a\mathbb{D}^a_{jkl},$$

where P means a sum of the three terms obtained by the cyclic permutation of j, k, l. Therefore, $\Pi^i_{jk} = 0$ in some coordinate chart $(\bar{U}, \bar{h})$. We have therefore, proved the following theorem.

Theorem C (Part I). *Every two-dimensional constant spray is projectively flat,* [5].

Remark. It is not true that there is a projective time-sequencing change from, say,

$$\text{(45)}\qquad \begin{aligned} \frac{d^2x^1}{ds^2} &= -2\alpha_2 \frac{dx^1}{ds}\frac{dx^2}{ds} + \alpha_1 \left[\left(\frac{dx^2}{ds}\right)^2 - \left(\frac{dx^1}{ds}\right)^2 \right], \\ \frac{d^2x^2}{ds^2} &= -2\alpha_1 \frac{dx^1}{ds}\frac{dx^2}{ds} + \alpha_2 \left[\left(\frac{dx^1}{ds}\right)^2 - \left(\frac{dx^2}{ds}\right)^2 \right] \end{aligned}$$

to $d^2x^1/d\bar{s}^2 = 0$, $d^2x^2/d\bar{s}^2 = 0$, by assuming that α_1, α_2 are not zero. The reason is that Eq.(28) implies

$$\Pi^1_{22} \neq 0, \quad \Pi^2_{11} \neq 0,$$

since $\mathbb{D}^i_{jkl} = 0$ holds for Eq.(45). Theorem C states only that *there is some coordinate system* $(\bar{U}, \bar{h})$ for which $\bar{\Pi}^i_{jk}$ in Eq.(28), vanish. This is where the tensor character of Theorems A and B play an important role.

Theorem C (Part II). *In every dimension* ≥ 3 *there exists a constant spray which is not projectively flat.*

Proof. Consider the n-dimensional conformally flat Riemannian metric $(g_{ij}) = e^{2\phi(x)}.(\delta_{ij})$, with $\phi(x) = \alpha_i x^i$, α_i constants. It is a well known fact that the Riemannian scalar curvatute $\mathbb{R}$ is never constant and vanishes if and only if $n = 2$. Yet, the (geodesic) spray of this metric has constant coefficients. But, in Riemannian geometry, projective flatness is equivalent to constant sectional curvatures. Therefore, $\mathbb{R}$ must be a constant as well, and the proof is complete (see [6]).

Remark. There exist two-dimensional projectively flat *Finsler metrics* which are *not* of constant curvature [1]. Obviously, these can not be Riemannian metrics.

In the next section we briefly describe some of the basics on Wagner theories.

§3. Semiprojective geometry

3.1. Local Finsler theory

Euler-Lagrange equations suggest studying a geometry for which the trajectories are geodesics. To find such a geometry, let M^n denote a closed, connected, C^∞-manifold and $\tilde{T}M^n$ its tangent bundle with the 0-section removed. Let

$$\text{(46)}\qquad F : \tilde{T}M^n \to \mathbb{R}$$

be a C^∞ function (positively) homogeneous of degree one in $y^i = \dot{x}^i, i = 1, 2, \ldots, n$.

If the Hessian matrix

$$g_{ij}(x,y) = \dot{\partial}_i \dot{\partial}_j \left(\frac{1}{2} F^2 \right) \tag{47}$$

of second partial derivatives with respect to y^i and y^j (or, what is the same, $dx^i/dt = \dot{x}^i$ and $dx^j/dt = \dot{x}^j$) is *nonsingular in some open conical subset of* $\tilde{T}M^n$, then the Euler-Lagrange equations are equivalent to the geodesic equations

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,y)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \quad i = 1, \ldots, n, \tag{48}$$

where

$$\gamma^i_{jk}(x,y) = \frac{1}{2} g^{ir}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}) \tag{49}$$

are the so-called Christoffel symbols of the second kind. Here t is *travel time* and

$$(dt)^2 = F^2(x, dx) = g_{ij}(x,y) dx^i dx^j \tag{50}$$

and $g^{il} g_{lk} = \delta^i_k$, that is, (g^{ij}) is the inverse of (g_{ij}), and ∂_k is the partial derivative with respect to x^k. Moreover, upon nonsingular coordinate transformation $x^i \to \bar{x}^i$, and the induced transformation $y^i \to \bar{y}^i$ by the Jacobian, $g_{ij}(x,y)$ transforms as a covariant *Finsler tensor* of rank 2, which is to say, it *transforms as in classical tensor analysis.* (This is true of all Finsler tensors regardless of type.) We remark that $F(x, dx/dt)$ *is conserved along geodesics.* It has value one and defines the *indicatrix surface* at each point x. We introduce the *unit length element of support* $l^i = y^i/F$ and the *angular metric tensor*

$$h_{ij} = g_{ij} - l_i l_j, \tag{51}$$

where $l_i = g_{ir} l^r$. Here, h_{ij} is the induced *metric tensor defined on the indicatrix surface.* It is globally defined on the *indicatrix subbundle* of the slit tangent bundle $\tilde{T}M^n$, just as g_{ij} is globally defined on $\tilde{T}M^n$. Another important Finsler object is the *Cartan "torsion tensor"*

$$C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}(x,y), \tag{52}$$

from where we get

$$V^i_{jk} = C^i_{jk} := g^{ir} C_{jrk}, \tag{53}$$

which defines the *vertical-connection coefficients*, that is, a *vertical covariant differentiation* (∇^v). For example, for any tensor $A^i_j(x, y)$,

(54) $$\nabla^v_k A^i_j := \dot{\partial}_k A^i_j + A^r_j V^i_{rk} - A^i_r V^r_{jk}.$$

Using the geodesic equations (48) in the local form

(55) $$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \quad i = 1, \ldots, n,$$

the *nonlinear Berwald connection* coefficients of (M^n, F) are

(56) $$G^i_j := \dot{\partial}_j G^i$$

where $\delta_i = \partial_i - G^r_i \dot{\partial}_r$. The *horizontal and vertical local Berwald connection coefficients* are defined by

(57) $$G^i_{jk} := \dot{\partial}_k G^i_j \quad \text{and} \quad V^i_{jk} = 0$$

in (54). From Eqs.(56) and (57), we define the *horizontal covariant derivative* ∇^h, for example,

(58) $$\nabla^h_k A^i_j := \partial_k A^i_j - (\dot{\partial}_r A^i_j) G^r_k + A^r_j G^i_{rk} - A^i_r G^r_{jk}.$$

The *Ricci identities* are given by the usual commutation relations

$$\nabla^h_k \nabla^h_s A^i_j - \nabla^h_s \nabla^h_k A^i_j = A^r_j G^i_{rsk} - A^i_r G^r_{jsk} - (\dot{\partial}_r A^i_j)\mathbb{R}^r_{sk},$$

$$\nabla^h_k \nabla^v_s A^i_j - \nabla^v_s \nabla^h_k A^i_j = A^r_j \mathbb{D}^i_{rsk} - A^i_r \mathbb{D}^r_{jsk},$$

where

$$G^i_{jsk} = \delta_k G^i_{sk} + G^r_{js} G^i_{rk} - \delta_s G^i_{jk} - G^r_{jk} G^i_{rs}$$

is the so-called (h) *h-curvature* and

(59) $$\mathbb{D}^i_{jhk} := \dot{\partial}_k G^i_{jh},$$

which detects angular dependence in the local connection coefficients G^i_{jk}, is the *Douglas tensor* (or (v) *h-curvature* of the Berwald connection), while

(60) $$\mathbb{R}^i_{hk} = \partial_k G^i_h - G^r_k G^i_{hr} - \partial_h G^i_k + G^r_h G^i_{kr}$$

agrees with (15). We remark that *geodesiscs are straight lines if and only if* $\mathbb{D} = 0 = \mathbb{R}$ *in (59) and (60).*

The *Cartan connection* $\mathbb{C}\Gamma = (\Gamma^i_{jk}, G^i_j, C^i_{jk})$ of (M^n, F) is characterized by *Matsumoto's axioms*, as follows[2]

1. $\nabla^h_k g_{ij} = 0$ (h-metrical),
2. $\nabla^v_k g_{ij} = 0$ (v-metrical),
3. $S^i_{jk} := C^i_{jk} - C^i_{kj} = 0$ (v-symmetric),
4. $T^i_{jk} := \Gamma^i_{jk} - \Gamma^i_{kj} = 0$ (h-symmetric),
5. $D^i_j = y^r \Gamma^i_{rj} - G^i_j = 0$ (deflection tensor D vanishes).

Note that axiom 3 is superfluous in our development here because we defined the vertical covariant derivative in terms of the tensor of Cartan (53). Had we used a general tensor V^i_{jk}, then axiom 3 would have been necessary to secure Γ^i_{jk} as the coefficients of the Cartan connection. Note that $\delta_i f$ is a covariant Finsler vector field, while, in general, $\partial_i f$ is not, when f is a smooth function on $\tilde{T}M^n$. Of course, if f has no y-dependence, then $\partial_i f$ is a vector.

We now have the following theorem.

Theorem 1. (Matsumoto) *The (horizontal) Cartan connection coefficients are given locally by (49) with ∂_k being replaced by δ_k.*

Using the triple notation we have $\mathbb{C}\Gamma = (\Gamma^i_{jk}, G^i_j, C^i_{jk})$ for the Cartan connection $\mathbb{C}\Gamma$ and $\mathbb{B}\Gamma = (G^i_{jk}, G^i_j, 0)$ for the Berwald connection $\mathbb{B}\Gamma$. Thus, (54) is the vertical *covariant derivative of A^i_j according to the Cartan connection* $\mathbb{C}\Gamma$, while

$$ {}_B\nabla^v_k A^i_j := \dot{\partial}_k A^i_j \tag{61}$$

gives it for the Berwald connection $\mathbb{B}\Gamma$. (The missing term, compared to (54), explains the zero in the third slot of the Berwald triple.) We remark that if there are coordinates $\bar{x}$ for which F is independent of $\bar{x}$, then (55) has $G^i \equiv 0$. Such a space is called *locally Minkowski.* However, C_{ijk} are not generally zero even in this case. In fact, vanishing of C_{ijk} implies that the geometry is Riemannian.

Both the above connections are important in the Finsler geometry. The Cartan connection $\mathbb{C}\Gamma$ is defined entirely in terms of the metric function F and its derivatives. The Berwald connection $\mathbb{B}\Gamma$ comes directly from the geodesic equations of (M^n, F). However, the Berwald connection satisfies

$$ {}_B\nabla^h_k g_{ij} = -2\,{}_B\nabla^h_l C_{ijk} y^l. \tag{62}$$

[2]Here we use the "triple" notation of Matsumoto.

This expression is generally *not zero*! If we replace ${}_B\nabla$ (Berwald) by ∇ (Cartan) in (58), the left side must be equal to zero. This is the so-called *h-Ricci lemma.* In fact, both h and *v-Ricci* lemmas hold for $\mathbb{C}\Gamma$ and both fail for $\mathbb{B}\Gamma$. For the well-known axiomatic characterization of the Berwald connection and more details on that of Cartan see [1].

We wish to consider yet another connection, called the *Wagner connection* $\mathbb{W}\Gamma = (T^i_{jk}, G^i_j, C^i_{jk})$. A Wagner connection $\mathbb{W}\Gamma$ on (M^n, F) is similar to the Cartan connection in that the above axioms are the same except for axiom 4, which is replaced by

$$\tau^i_{jk} = T^i_{jk} - \frac{1}{n+1}\delta^i_j T^a_{ak} - \frac{1}{n+1}\delta^i_k T^a_{ja} = 0,$$

where τ^i_{jk} is called the *Thomas' tensor* (J.M. Thomas). The vanishing of Thomas' tensor is equivalent to the existence of a covariant field $\sigma_i(x, y)$ such that

$$T^i_{jk} = \delta^i_j \sigma_k - \delta^i_k \sigma_j. \tag{63}$$

In the classical literature, the Wagner connection is thus said to have *semi-symmetric torsion.* Let us recall the following theorem:

Theorem 2 (Matsumoto, Hashiguchi, Ichijyo, Tamássy). *A Finsler space (M^n, F) is conformal to a locally Minkowski space if and only if there exists a Wagner connection $\mathbb{W}\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ on (M^n, F) such that F^i_{jk} depends at most on x^i, $\sigma_i(x) = \partial_i\sigma(x)$, and the h-curvature of $\mathbb{W}\Gamma$ vanishes. This means that $F(x, \dot{x})$ has the form $F = \exp[\sigma(x)]F(\dot{x})$, where $\exp[\sigma(x)]$ is the so-called conformal factor, which depends only on x,* [8], [9]. Such a space is called a *σ-Wagner space.*

Note that many spaces are conformally Minkowski but by no means all of them, even in dimension two! Also recall that every two-dimensional *Riemannian space* is conformally Euclidean. We remark that the vertical Wagner connection is identical to that of the Cartan connection.

The simplest kind of Finsler spaces beyond the locally Minkowski (whose tangent planes are curved for $n \geq 3$, in general) are *Berwald spaces.* These are characterized by

$$_C\nabla^h_l C_{ijk} = 0, \tag{64}$$

or, equivalently, by

$$_B\nabla^h_l C_{ijk} = 0. \tag{65}$$

For $n = 2$ a complete isometric classification was given by L. Berwald (see [1]). All of these two-dimensional Berwald spaces that are not locally

Minkowskian have *principal scalar I* equal to a constant (see (84) below). Of those, exactly four classes are distinguished; three are positive definite with $I^2 < 4$ $I^2 = 4$ and $I^2 < 4$. In this case $\mathbb{R} = 0$.

Wagner spaces of dimension n are by definition Finsler spaces which have a Wagner connection with its σ_i-field being a gradient, $\sigma_i(x) = \partial_i\sigma(x)$. They are generalizations of Berwald spaces in many respects. A notable example of this relationship is the

Theorem 3. (Hashiguchi) *(M^n, F) is σ-Wagner if and only if*

$$ {}_W\nabla_l^h C_{ijk} = 0, \tag{66} $$

while (M^n, F) is Berwald if and only if (65) holds (see [9]).

All Berwald spaces are trivial (i.e., $\sigma_i = 0$) examples of Wagner spaces. From Theorem 2, we can start with any locally Minkowski space $(M^n, \bar{F})$ and form a Wagner space by using $F = \exp[\sigma(x)]\bar{F}$ in M^n. This Wagner space (M^n, F) has a linear (affine) connection $F^i_{jk}(x)$ and its (usual) curvature tensor is just the horizontal Wagner curvature, which vanishes. It is notable that the geodesics of $(M^n, \bar{F})$ are never, for $\sigma_i \neq 0$, the autoparallels

$$ \frac{d^2x^i}{dt^2} + F^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \tag{67} $$

of $\mathbb{W}\Gamma$ ([1], vol. II, p.735). As an example, let us write

$$ F\left(z, \frac{\dot{x}}{\dot{z}}\right) = e^{\sigma(z)}\frac{\sqrt{(\dot{x})^2 + (\dot{z})^2}}{\phi(\dot{x}/\dot{z})} = e^{\sigma(z)}\bar{F}\left(\frac{\dot{x}}{\dot{z}}\right) \tag{68} $$

with $\sigma(z) = -\ln f(z)$. Here, $x^1 = x$, $x^2 = z$, $\dot{x}^1 = \dot{x}$, $\dot{x}^2 = \dot{z}$ and $\bar{F}$, the anisotropic part, is a Finsler function of a Minkowski space. The Wagner autoparallels are solutions of

$$ \frac{d^2x^i}{dt^2} + (\delta^i_j\sigma_k)\frac{dx^j}{dt}\frac{dx^j}{dt} = 0, \tag{69} $$

where $\sigma_1 = \partial_1\sigma = \partial_x\sigma$, $\sigma_2 = \partial_2\sigma = \partial_z\sigma(z)$. The geodesics are solutions of

$$ \frac{d^2x^i}{dt^2} + (\delta^i_j\sigma_k)\frac{dx^j}{dt}\frac{dx^k}{dt} = Q^i, \tag{70} $$

where

$$ Q^i = \bar{F}^2\bar{g}^{ij}\sigma_j - \delta^i_j\sigma_k\frac{dx^j}{dt}\frac{dx^k}{dt} \tag{71} $$

with $\bar{g}^{ij}$ being the inverse of

$$\bar{g}_{ij} = \dot{\partial}_i \dot{\partial}_j \left(\frac{1}{2} \bar{F}^2 \right). \tag{72}$$

Moreover, Q^i is orthogonal to dx^i/dt, namely,

$$\bar{g}_{ij} Q^i \frac{dx^j}{dt} = 0, \tag{73}$$

so that Q^i is the *Wagner curvature of a solution* of Eqs.(70) and (71). This means that geodesics are *curved in Wagner geometry* and Q^i measures the "curvature". Of course, geodesics are *not* curved in their usual geometry.

Let us take a specific form for $\phi(\dot{x}/\dot{z})$ in (68) above, say

$$\phi\left(\frac{\dot{x}}{\dot{z}}\right) = \frac{[(\dot{x})^2 + (\dot{z})^2]^{1/2}}{[(\dot{x})^m + (\dot{z})^m]^{1/m}}, \tag{74}$$

where m is an *even interger* ≥ 2. Furthermore, let us take the linear form

$$\sigma = -\ln f(x, z) = -\ln(a + bz), \tag{75}$$

where a and b are positive constants, to allow z dependence. We then obtain the Finsler space (M^2, F) with the metric function

$$F\left(x, z, \frac{\dot{x}}{\dot{z}}\right) = e^{-\ln f} [(\dot{x})^m + (\dot{z})^m]^{1/m}, \tag{76}$$

where

$$I = \frac{m-2}{2\sqrt{m-1}} \left(\frac{1 - (\dot{z}/\dot{x})^m}{\sqrt{(\dot{z}/\dot{x})^m}} \right).$$

The Berwald Gauss curvature scalar $\mathbb{K}$ of this space is

$$\mathbb{K} = \frac{\frac{1}{4} m b^2 (\dot{z})^{2-2m} [(m-2)(\dot{x})^m - m(\dot{z})^m]}{(m-1)^2 ((\dot{x})^m + (\dot{z})^m)^{(2-m)/m}}. \tag{77}$$

3.2. More on two-dimensional Finsler spaces

We assume $g_{ij}(x, y)$ to be positive definite on an open conical region of TM^2. *Berwald* discovered the *frame* (l^i, m^i) with

$$g_{ij} \,.\, m^i m^j = 1 = g_{ij} \,.\, l^i l^j, \tag{78}$$

$$g_{ij} \,.\, m^i l^j = 0, \tag{79}$$

(80) $$h_{ij} = m_i m_j,$$

as in (51), and

(81) $$m_i = g_{ir} m^r$$

and finally

(82) $$g_{ij} = l_i l_j + m_i m_j.$$

Using

(83) $$C_{ijk} l^k = 0,$$

we have

(84) $$F C_{ijk} = I m_i m_j m_k,$$

where $I(x, y)$ is the *principal scalar* of (M^2, F). The 3-index curvature formula (60) can now be written

(85) $$\mathbb{R}^i_{jk} = F.\mathbb{K}.m^i (l_j m_k - l_k m_j),$$

where $\mathbb{K}$ is the so-called *Berwald's Gauss curvature.*

Consider now the 3 local semisprays on (M^2, F)

(86) $$\frac{dy^i}{ds} = -G^i_{jk}\ y^j y^k + A^i_j y^j + B^i,$$

where $y^i = dx^i/ds$ and where F^2 has one of the following 3 forms:

(i) $F^2 = L^2.\ \exp\left\{2\left[-\alpha_1 x^1 + (\lambda + 1)\ \alpha_2 x^2 + \nu_3 x^1 x^2\right]\right\},$
$L = (y^2)^{1+\frac{1}{\lambda}}/(y^1)^{\frac{1}{\lambda}},\ \alpha_i > 0;$

(ii) $F^2 = (y^2)^2.\ \exp\left\{2\left[y^1/y^2 + (c_1 - c_2)\ x^1 + c_1 x^2 - \nu_3 x^1 (x^2)^2\right]\right\};$

(iii) $F^2 = \left[(y^1)^2 + (y^2)^2\right].\ \exp\left\{2\left[\frac{\alpha_1^2+\alpha_2^2}{(\alpha_1+\alpha_2)^2}\ (\alpha_1 x^1 + \alpha_2 x^2) + \frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\tan^{-1}\frac{y^1}{y^2} + \psi(x)\right]\right\}.$

Here, $F = F(x, y)$ and $\psi(x) = \frac{1}{2}[\nu_1 (x^1)^2 + \nu_2 (x^2)^2]$ (see [2] and [5]).

The geodesic equations (i.e. $A^i_j = 0$, $B^i = 0$) for these 3 geometries are, respectively,

(i)' $dy^1/ds + \lambda\left(\alpha_1 - \nu_3 x^2\right).\left(y^1\right)^2 = 0$
$dy^2/ds + \lambda\left(\alpha_2 + \frac{\nu_3}{\lambda+1}\ x^1\right).\left(y^2\right)^2 = 0;$

(ii)' $dy^1/ds + \left(c_1 - 2\nu_3 x^1 x^2\right).\left(y^2\right)^2 = 0$
$dy^2/ds + \left[\nu_3\left(x^2 - x^1\right).(-x^2) - c_2\right].\left(y^1\right)^2 + 2\left(c_1 - 2\nu_3 x^1 x^2\right).y^1 y^2 = 0;$

and

$$\text{(iii)'}\ dy^1/ds + 2\,(\alpha_2 + \nu_2 x^2)\, y^1 y^2 + (\alpha_1 + \nu_1 x^1)\left(\left(y^1\right)^2 - \left(y^2\right)^2\right) = 0$$
$$dy^2/ds + 2\,(\alpha_1 + \nu_1 x^1)\, y^1 y^2 + (\alpha_2 + \nu_2 x^2)\left(\left(y^2\right)^2 - \left(y^1\right)^2\right) = 0.$$

The Berwald-Gauss curvature scalar $\mathbb{K}$ for each case (ibid) is given as

$$\text{(i)"}\ \mathbb{K} = \tfrac{\lambda^2}{\lambda+1}\cdot \nu_3 \cdot \left(y^1/y^2\right)^{1+2/\lambda}\cdot \exp\left\{-2\left[-\alpha_1 x^1 + (\lambda+1)\,x^2 + \nu_3\, x^1 x^2\right]\right\};$$

$$\text{(ii)"}\ \mathbb{K} = 2\nu_3 x^1 \cdot \exp\left\{-2\left[\,y^1/y^2 + (c_1 - c_2)\,x^1 + c_1 x^2 - \nu_3\, x^1 (x^2)^2\right]\right\};$$

and

$$\text{(iii)"}\ \mathbb{K} = -2\,\tfrac{\alpha_1^2+\alpha_2^2}{(\alpha_1+\alpha_2)^2}\,(\nu_1+\nu_2)\exp\left\{\,2\left[\phi(x) + \tfrac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\tan^{-1}(y^1/y^2)\right]\right\},$$

where $\phi(x) = \frac{\alpha_1^2+\alpha_2^2}{(\alpha_1+\alpha_2)^2}\,(\alpha_i x^i) + \frac{1}{2}\left[\nu_1 (x^1)^2 + \nu_2 (x^2)^2\right]$.

We can see that the trajectories, i.e., geodesics in this case, are Jacobi stable in (i)" and (ii)", if and only if $\nu_3 > 0$. Likewise, in the case (iii)", trajectories are Jacobi stable if and only if $\nu_1 + \nu_2 < 0$.

Theorem 4. (Antonelli, Matsumoto) *With $\nu_3 = 0$ in (i) and (ii) and $\nu_1 = \nu_2 = 0$ in (iii), the equations (i)', (ii)' and (iii)' give the only constant coefficients Finsler geodesics in dimension 2.*

Proof. See [1] or appendix of [8].

Let us now consider (86) with $B^i = -\delta^i_j \sigma_k(x) N^j N^k$, $\sigma_k = \partial_k \sigma$ with $\sigma(x) = \sigma_k x^k$, a linear function. Thus, $B^1 = -\sigma_1\,(N^1)^2 - \sigma_2\, N^1 N^2$ and $B^2 = -\sigma_2\,(N^2)^2 - \sigma_1\, N^1 N^2$, and, with $A^i_j = \lambda \delta^i_j$, $\lambda > 0$, exactly 3 sprays emerge with Γ given by the coefficients in each of (i)', (ii)' and (iii)'. But, the result is a new connection $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \sigma_k$. Writing only the second equation for the new connection $\bar{\Gamma}$, these are as follows:

$$\text{(i)'''}\ dN^1/dt = \lambda\, N^1 - \tilde{\alpha}_1\,(N^1)^2 - \sigma_2\, N^1 N^2$$
$$dN^2/dt = \lambda\, N^2 - \tilde{\alpha}_2\,(N^2)2 - \sigma_1\, N^1 N^2$$

$\mathrm{Sgn}(\sigma_1, \sigma_2)$ is $(+,+)$ for *competition* , $(+,-)$ or $(-,+)$ for *parasitism* and $(-,-)$ for *mutualism.* All 3 cases exhibit linearly stable positive steady-states.

$$\text{(ii)'''}\ dN^1/dt = \lambda\, N^1 - \tilde{c}_1\,(N^1)^2 - \sigma_2\, N^1 N^2$$
$$dN^2/dt = \lambda\, N^2 - \sigma_2\,(N^2)^2 + \tilde{c}_2\,(N^1)^2 - (\tilde{c}_1 + c_1)\, N^1 N^2$$

An unique linearly stable positive steady-state exists.

$$\text{(iii)'''}\ dN^1/dt = \lambda\, N^1 - \tilde{\alpha}_1\,(N^1)^2 - \sigma_2\, N^1 N^2 + (\sigma_1 - \tilde{\alpha}_1)\,(N^2)^2$$
$$dN^2/dt = \lambda\, N^2 - \tilde{\alpha}_2\,(N^2)^2 - \sigma_1\, N^1 N^2 + (\sigma_2 - \tilde{\alpha}_2)\,(N^1)^2$$

An unique linearly stable positive steady-state exists.

Let us now pass to the parameter s in each of the above 3 systems. In the following, $dx^i/ds = y^i$:

$$\begin{aligned}
&\text{(A)}\ dy^1/ds + \tilde{\alpha}_1\ (y^1)^2 + \sigma_2\ y^1y^2 = 0\\
&\qquad dy^2/ds + \tilde{\alpha}_2\ (y^2)^2 + \sigma_1\ y^1y^2 = 0,\\
&\text{(B)}\ dy^1/ds + \tilde{c}_1\ (y^1)^2 + \sigma_2\ y^1y^2 = 0\\
&\qquad dy^2/ds + \sigma_2\ (y^2)^2 - \tilde{c}_2\ (y^1)^2 + (\tilde{c}_1 + c_1)\ y^1y^2 = 0,\\
&\text{(C)}\ dy^1/ds + \tilde{\alpha}_1\ (y^1)^2 + \sigma_2\ y^1y^2 + (\tilde{\alpha}_1 - \sigma_1)\ (y^2)^2 = 0\\
&\qquad dy^2/ds + \tilde{\alpha}_2\ (y^2)^2 + \sigma_1\ y^1y^2 + (\tilde{\alpha}_2 - \sigma_2)\ (y^1)^2 = 0.
\end{aligned}$$

The tildes over the coefficients indicate that the coefficients are approximations and constants up to order ε^2.

Theorem 5. *Along any solution of (A), $e^{\sigma_k x^k}$. F_i is constant, F_i being the Finsler function F in (i) above. Likewise, along any solution of (B), $e^{\sigma_k x^k}$. F_{ii} is constant, where F_{ii} is the Finsler functional in (ii) above. Similarly, $e^{\sigma_k x^k}$. F_{iii} is constant along any solution of (C), F_{iii} being the cost functional in (iii) above* [5].

Remark. The reader may verify that $d\tilde{F}/ds = 0$ along solutions of (A), (B), (C) where $\tilde{F}$ denotes the appropriate $e^{\sigma_k x^k}$. F function.

Difinition. Given a spray S on M^n, if for each pont $p \in M^n$ there are local coordinates in an open set U, $p \in U$, in which the local Berwald coefficients, G^i_{jk}, are linear, then we say S is a *locally linear Berwald spray*, or (LLB)-spray. The local coordinates are called *adapted.*

Theorem 6. *There are exactly eight 2-dimensional LLB-sprays which are autoparallels of a Wagner connection. Each is a semiprojective transformation of one of (i)', (ii)', (iii)' above. Furthermore, each conserves a functional of the form $e^{\phi(x)}.\bar{F}(y)$, with ϕ quadratic in adapted coordinates.*

§4. Stochastic Finsler geometry

Consider a solution $x(s)$, $y(s)$ of

$$\text{(87)}\qquad \begin{cases} \frac{dx^i}{ds} = y^i \\ \frac{dy^i}{ds} = -F^i_{jk}(x,y)y^iy^k. \end{cases}$$

The *hv-rolling along this solution* defines a smooth curve $\mu(s)$, $\nu(s)$ in $\mathbb{R}^{2n}$ given by

$$(88)\qquad \begin{cases} \frac{dx^i}{ds} = y^i + z^i_j \frac{d\mu^j}{ds} \\ \frac{\delta y^i}{ds} = z^i_j \frac{d\nu^j}{ds} \\ \frac{dz^i_j}{ds} = -F^i_{kl}(x,y) z^l_j \frac{dx^k}{ds} - C^i_{kl}(x,y) z^l_j \frac{\delta y^k}{ds}, \end{cases}$$

where

$$(89)\qquad \frac{\delta y^i}{ds} = \frac{dy^i}{ds} + G^i_j(x,y)\frac{dx^j}{ds}.$$

For stochastic differential equations which are (88) perturbed by noise, we have the general form

$$(90)\qquad \begin{cases} dx^i = y^i ds + z^i_j(s) \circ dv^j \\ \delta y^i = z^i_j(s) \circ dw^j \\ dz^i_j = -F^i_{kl}(x,y) z^\ell_j(s) \circ dx^k - C^i_{kl}(x,y) z^\ell_j(s) \circ \delta y^k \end{cases}$$

where

$$(91)\qquad \delta y^i = dy^i + G^i_j(x,y) \circ dx^j.$$

Here $z^i_j(s)$ is the *orthonormal frame process* and the circle notation indicates *Stratonovich stochastic theory* is employed, [1], [8].

Let us use the above to examine the Riemannian case, i.e. $C^i_{jk} = 0$ in (87). Thus,

$$(92)\qquad \begin{cases} \frac{dx^i}{ds} = y^i \\ \frac{dy^i}{ds} = -\Gamma^i_{jk}(x) y^j y^k \end{cases}$$

where $\Gamma^i_{jk}(x)$ must be the Levi-Cività connection of Riemannian geometry. Because the metric tensor $g_{ij}(x)$ is independent of y, the fibers TM^*_p of the slit tangent bundle are full n-dimensional flat Euclidean spaces (i.e. zeros included).

Next we perturb $\mu(s)$, $\nu(s)$ by adding independent white noises $v(s)$, $w(s)$ and use *hv-isometric rolling* (stochastic) to transfer the diffusion

$\mu(s)+v(s)$, $\nu(s)+w(s)$ back to $(M^n, \widetilde{F})$. Here, we assume the *stochastic Ansatz for noise addition.*

$$\begin{bmatrix} dx^i dx^j & dx^i dy^j \\ dy^i dx^j & dy^i dy^j \end{bmatrix} = ds \begin{bmatrix} g^{ij} & -g^{ik}G^i_k \\ -g^{jk}G^i_k & g^{ij} + g^{k\ell}G^i_k G^j_\ell \end{bmatrix}. \tag{93}$$

This says simply that the *distance* by which the state (x, y) will be displaced is proportional to the magnitude of $(dv^i,\ dw^i)$. A suitable metric on TM^* must be chosen. The most natural choice (for many reasons) is the *diagonal lift* of the Finsler metric tensor. This is "diagonal" only when written in terms of the Berwald basis $\{X_a\}_{a=\overline{1,2n}} = \{\delta_i, \dot{\partial}_j\}$. In natural coordinates it is not diagonal. Thus the covariance matrix (93) above has this special form: $[G_{AB}] = \begin{bmatrix} g_{ij)}(x,y) & 0 \\ 0 & g_{ij}(x,y) \end{bmatrix}$, relative to $\{\delta_i, \dot{\partial}_j\}$.

Retaining the same notation $x(s)$, $y(s)$ for the diffusion on TM^*, we obtain the system so when we apply the stochastic Ansatz for noise addition we use

$$\begin{bmatrix} dx^i dx^j & dx^i dy^j \\ dy^i dx^j & dy^i dy^j \end{bmatrix} = ds \begin{bmatrix} g^{ij}(x) & 0 \\ 0 & \delta^{ij} \end{bmatrix} \tag{94}$$

where δ^{ij} is the Kronecker delta instead of the diagonal lift. (This lift does not work since tangent spaces are *not always flat* in Finsler geometry!) This condition (94) is fulfilled by

$$\begin{cases} dx^i = y^i ds + z^i_j \circ dv^j \\ dy^i = -\Gamma^i_{jk}(x) y^j y^k ds + dw^i \\ dz^i_j = -\Gamma^i_{\ell k}(x) z^\ell_j \circ dx^k, \end{cases} \tag{95}$$

a diffusion $\big(x(s),\ y(s),\ z(s)\big)$ on the *orthonormal frame bundle.* Here, as above, $v^i(s)$ and $w^i(s)$ are independent standard Brownian motions in $\mathbb{R}^n$ and $z^i_j(s)$ is the auxiliary orthonormal frame process. The resulting Markov diffusion $x(s), y(s)$ on TM^n has generator

$$\mathbb{D} = \frac{1}{2}\, g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k) + \frac{1}{2}\, \delta^{ij}\dot{\partial}_i\dot{\partial}_j + y^i\partial_i - \Gamma^i_{jk} y^j y^k \dot{\partial}_i \tag{96}$$

the probability density $\rho(s, x, y)$ of this process satisfies the initial-boundary value problem

$$
\begin{cases}
\frac{\partial \rho}{\partial s} = \mathbb{D}^* \rho \\
\lim_{s\searrow 0}\ \rho(s,x,y) = \rho_0(x,y) \\
\rho(s,x,y)\big|_{\partial D} = 0,
\end{cases}
\tag{97}
$$

where the initial density $\rho_0(x,y)$ is supported on $B = \{(x,y)\big| y^i > 0,\ \forall_i,\ (x,y) \in TM^n\}$ and $\mathbb{D}^*$ is the formal adjoint of $\mathbb{D}$ relative to the metric

$$
[G_{AB}] = \begin{bmatrix} g_{ij} & 0 \\ 0 & \delta_{\bar{i}\bar{j}} \end{bmatrix},
\tag{98}
$$

$A, B = 1, \ldots, 2n$; $i, j = 1, \ldots, n$; $\bar{i}, \bar{j} = n+1, \ldots, 2n$, on TM^n.

§5. Results on Evolution Models with Noise

The deterministic equations are of the form (92) but with constant coefficients, thusly,

$$
\begin{array}{lll}
\Gamma^i_{ii} & = \alpha_i, \quad \Gamma^i_{jk} = 0, & i \neq j \neq k \\
\Gamma^i_{ij} & = \Gamma^i_{ji} = \alpha_j, & i \neq j \\
\Gamma^i_{jj} & = -\alpha_i, & i \neq j.
\end{array}
$$

See [1], [4] or [6], Chapter 5. Here, $\alpha_1, \ldots, \alpha_n$ are non-zero constants, so this means $x^1, \ldots, x^n$ is an *adapted* coordinate system on $(M^n, \widetilde{F}) = \big(M^n, e^{\phi(x)} \cdot F(\dot{x})\big)$ where

$$
F(\dot{x}) = F(\dot{x}^1, \ldots, \dot{x}^n) = \sqrt{(\dot{x}^1)^2 + \cdots + (\dot{x}^n)^2}
$$

is the flat Euclidean metric function.

Remark These constant coefficients characterize a model of Woese's *Ancestral Commune Theory* of proto-cell evolution, [10], [14]. One consequence is that $G^i_{jk} \equiv \Gamma^i_{jk}$ = constant, with all indices different (so $n \geq 3$) *must vanish.* This means that where $i \neq j$, $i \neq k$ and $j \neq k$ (so $n \geq 3$) there is no interaction between j^{th} and k^{th} type to influence the i^{th} type. There are *no higher-order interactions in this sense.*

The system perturbed by noise has form (96) generator

$$
\mathbb{D} = \frac{1}{2}\,\Delta_G + y^i \partial_i - \partial_j \phi (2 y^i y^j - \delta^{ij} |y|^2) \dot{\partial}_i,
\tag{99}
$$

here Δ_G being the Riemannian Laplacian on TM^n.

We shall compute the adjoint operator $\mathbb{D}^*$ for the problem (97) in terms of the curvature $\mathbb{R} := g^{ij}\mathbb{R}_{ij}$, $\mathbb{R}_{ij}$ being the Ricci tensor the contraction of $R^i_{jk\ell}$ given explicitly as

$$\mathbb{R}_{ij} = \partial_k \Gamma^k_{ii} - \Gamma^r_{ik}\Gamma^k_{rj} + \Gamma^k_{ij}\partial_k(\ell\text{n}\sqrt{g}) - \partial_i\partial_j(\ell\text{n}\sqrt{g}), \tag{100}$$

where $g = \det[g_{ij}(x)] = e^{2n\phi(x)}$. From this

$$\mathbb{R} = -(n-1)e^{-2\phi(x)}\delta^{ij}[2\partial_i\partial_j\phi + (n-2)\partial_i\phi \cdot \partial_j\phi]. \tag{101}$$

The adjoint is straightforwardly

$$\mathbb{D}^* = \frac{1}{2}\Delta_G + A^i\partial_i + B^i\dot{\partial}_i + V, \tag{102}$$

where $A^i = -y^i$, $B^j = \partial_j\phi(2y^iy^j - \delta^{ij}|y|^2)$ and the so-called Feynman-Kac potential is

$$\frac{V}{n} = \Big(\frac{\mathbb{R}}{2(n-1)} + \frac{n+2}{2}|\nabla_g\phi|^2\Big) + 2\partial_i\phi y^i. \tag{103}$$

The forward initial-boundary value problem (97) can be solved by first introducing an auxiliary diffusion $\big(X(s), Y(s)\big)$ on TM^n following by projection of the diffusion on the *orthonormal frame bundle*

$$\begin{aligned} dX^i &= A^i ds + z^i_j \circ dv^j \\ dY^i &= B^i ds + dw^i \\ dZ^i_j &= -\Gamma^i_{\ell k}(X) z^\ell_j \circ dX^k, \end{aligned} \tag{104}$$

where $Z^i_j(s)$ is an *orthonormal frame process.* To ensure the existence of solutions of (104) without explosions we can employ the C^∞ bump function technique, assuming that ϕ is compactly supported with supp ϕ large enough to contain the region of interest with $\partial_i\phi$ non-vanishing everywhere in that region for any $i = 1, \ldots, n$, (see [8]).

By the Feynman-Kac formula, the solution of (104) can now be expressed

(105)

$$\rho(s,x,y) = \mathbb{E}_{x,y}\Big\{\chi_{\{\sigma>s\}}\rho_o\big(X(s),Y(s)\big)\exp\Big[\int_0^S V\big(X(r),Y(r)\big)dr\Big]\Big\}$$

where $\sigma = \inf\{s \geq 0 \,|\, \big(X(s),Y(s)\big) \in \partial B\}$ is the first time of hitting the boundary ∂B of B and $\mathbb{E}_{x,y}$ is the conditional expectation given that $X(0) = x$ and $Y(0) = y$.

Inspection of the (negative) Riemann scalar curvature $\mathbb{R}$ shows it becomes less negative with distance from the origin and that $|\mathbb{R}|$ increases at least quadratically with n, all other things being equal. Thus, the probability density ρ is reduced because of negative $\mathbb{R}$ values and even more so as n becomes larger. We interpret this as an indication that the process is speeding up, in a relative sense, as n increases, thereby contributing to greater *stochastic chaos* for the system.

The orthonormal frame bundle exhibits (90) as a diffusion whose projection onto TM^* has generator

$$\mathbb{D} = \frac{1}{2}\, g^{ij}(\delta_i\delta_j - F^k_{ij}\delta_k) + \frac{1}{2}\, g^{ij}(\dot{\partial}_i\dot{\partial}_j - C^k_{ij}\dot{\partial}_k) + y^j\delta_j \tag{106}$$

where $\big(F^k_{ij}(x,y), G^i_j(x,y), C^i_{jk}(x,y)\big)$ are the local coefficients of the Cartan connection for the 2-dimensional Berwald type Finsler space whose metric function is

$$L = \exp\left[\phi(x)\right] \cdot \frac{(y^2)^{1+\frac{1}{\lambda}}}{(y^1)^{\frac{1}{\lambda}}} \tag{107}$$

and whose scalar curvature is (see [3])

$$\mathbb{K} = \frac{\lambda^2}{\lambda+1}\,\nu_3\left(\frac{y^1}{y^2}\right)^{1+2/\lambda} \exp\left[-2\phi\right] \tag{108}$$

where $\phi(x) = -\alpha_1 x^1 + (\lambda+1)\alpha_2 x^2 + \nu_3 x^1 x^2$ with $\lambda > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\nu_3 > 0$. The geodesics of $\widetilde{F} = \frac{1}{2}\, L^2$ are given as

$$\begin{aligned} \tfrac{d^2x^1}{ds^2} + \lambda(\alpha_1 - \nu_3 x^2)\left(\tfrac{dx^1}{ds}\right)^2 &= 0 \\ \tfrac{d^2x^2}{ds^2} + \lambda\left(\alpha_2 + \tfrac{\nu_3}{\lambda+1}\, x^1\right)(y^2)^2 &= 0 \end{aligned} \tag{109}$$

or written as real time (ecological/physiological) interactions, for *open growth*,

$$\begin{cases} \tfrac{dx^i}{dt} = k_{(i)}N^i \quad i = 1,2 \\ \tfrac{dN^1}{dt} = \lambda N^1 - \lambda(\alpha_1 - \nu_3 x^2)(N^1)^2 \\ \tfrac{dN^2}{dt} = \lambda N^2 - \lambda\left(\alpha_2 + \tfrac{\nu_3}{\lambda+1}\, x^1\right)(N^2)^2. \end{cases} \tag{110}$$

Biological Remark. The parameter ν_3 is called the *exchanges parameter* and serves to measure *information exchange* (see [3]). Type #1 are proto-mitochondrians while type #2 are ancient bacteria.

Let $G_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3, 4)$ denote the diagonal lift of $g_{ij}(x, y)$ obtained from (107) above by $g_{ij}(x, y) := \dot{\partial}_i \dot{\partial}_j \big(\frac{1}{2} L^2\big)$, as before. If $G = \det[G_{\alpha\beta}]$ then $G = g^2$ with $g = \det[g_{ij}]$. We denote by $\rho(s, x, y)$ the probability density of the process with generator (106) *stopped at time* σ to be in a region $A \subseteq TM^* \cap \{y^1 > 0, y^2 > 0\}$, relative to the measure $\sqrt{G}\, dxdy$ on TM^*, thus,

$$\operatorname{Prob}\big\{\big(x(s), y(x)\big) \in A\big\} = \int_A \rho(s, x, y) \sqrt{G}(x, y)\, dxdy$$

the function $\rho(s, x, y)$ satisfies the forward initial boundary value problem

$$\begin{cases} \mathbb{D}^* \rho = \frac{\partial \rho}{\partial s} \\ \lim_{s \searrow 0} \rho(s, x, y) = \rho_0(x, y) \\ \rho(s, x, y)\big|_{y^1 y^2 = 0} = 0. \end{cases} \tag{111}$$

The solution can be found, much as above, to be (after [8] or [1], Vol. I, Part 3)

$$\rho(s, x, y) = \mathbb{E}\Big\{\chi_{\{\sigma_{x,y} > s\}} \rho_0\big(\xi_{x,y}(s), \eta_{x,y}(x)\big) \exp \int V\big(\xi_{x,y}(r), \eta_{x,y}(r)\big)\, dr\Big\}$$

where χ is the indicator function, and

$$V = 2\mathbb{R} + \Phi(x, y)$$

where $\{m^i, \ell^i\}$ is the Berwald orthonormal frame and I is a constant principal scalar, with

$$\begin{aligned} \Phi(x, y) &= 2I m^i \ell^j \phi_{ij} - 4I m^i \ell^i \phi_i \phi_j - 2I^2 m^i m^j \phi_i \phi_j \\ &\quad + 8 g^{ij} \phi_i \phi_j + 4 y^i \phi_i - \tfrac{1}{2} I^2 / L^2. \end{aligned} \tag{112}$$

Here $\phi_i = \partial_i \phi$ and $\phi_{ij} = \partial_j \phi_i$ from (108) we see that $\rho(s, x, y)$ is increased because of the positivity of $\mathbb{R}$ so that the process is slowed down, relatively speaking. The curvature contributes to stochastic stability rather that stochastic chaos. But, $I^2 = \frac{(\lambda+2)^2}{\lambda+1}$ so large growth rate λ offsets this slow down.

Remark. Similar results to those above obtained for the cases $I^2 = 4$ and $I^2 < 4$. The stochastic treatment of non-Berwald geometry (76) has been carried out as well [1].

Remark. The computations in this work have been performed by the computer package Finsler [1], [13].

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