Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 767–783

Concentration phenomena in the conformal Brezis-Nirenberg problem

Futoshi Takahashi

Abstract.

We study the blow up phenomena of least energy solutions to some semilinear elliptic boundary value problem $(P_{\varepsilon,a})$ below on domains of a manifold which has a metric pointwise conformal to the Euclidean metric. Typical examples of our problem are set on domains of spaces of constant positive or negative curvature. It is known that the least energy solutions concentrate at one point in the domain as a parameter involved tends to 0. We characterize the location of concentration point of the least energy solutions as the maximum point of some function, defined by the coefficient function, the conformal factor and the (Euclidean) Robin function on the domain.

§1. Introduction.

Main purpose of this note is to report some results recently obtained by the author concerning the concentration phenomena of some solutions to a semilinear elliptic boundary value problem with variable coefficient.

Let $D \subset \mathbf{R}^N (N \geq 3)$ be a smooth bounded domain and $p \in C^2(\overline{D}), p(x) > 0$ be given. On D, we consider a metric $g = p^2(x)g_0$ which is pointwise conformal to a standard metric g_0 on $\mathbf{R}^N$.

We consider the problem

$$(P_{\varepsilon,a}) \begin{cases} L_g u = u^{2^*-1} + \varepsilon a(x)u & \text{in } D, \\ u > 0 & \text{in } D, \\ u|_{\partial D} = 0 \end{cases}$$

Received October 31, 2005.
Revised January 19, 2006.
Partially supported by JSPS Grant-in-Aid for Scientific Research, No.17540186.

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $\varepsilon > 0$ and $a \in C^2(\overline{D})$ is a given function.

$$L_g := -\Delta_g + \frac{N-2}{4(N-1)} S_g$$

is called the conformal Laplacian relative to g, where $\Delta_g = p^{-N}\mathrm{div}(p^{N-2}\nabla)$ is the Laplace-Beltrami operator and S_g is the scalar curvature with respect to g.

If the conformal factor $p \equiv 1$, the problem $(P_{\varepsilon,a})$ was studied in the celebrated paper by Brezis and Nirenberg [6] more than twenty years ago, and since then a large number of studies concerning this type of equations involving the critical Sobolev exponent have been done. In this note, we call the problem $(P_{\varepsilon,a})$ *the conformal Brezis-Nirenberg problem.*

Typical examples of our problem are the followings: First is a conformal Brezis-Nirenberg problem on the spherical domain:

$$(1) \qquad \begin{cases} -\Delta_{S^N} u + \frac{N(N-2)}{4} u & = u^{2^*-1} + \varepsilon a(x) u \quad \text{in } D, \\ u > 0 & \qquad\qquad\qquad\quad \text{in } D, \\ u|_{\partial D} = 0 & \end{cases}$$

where $D \subset \mathbf{R}^N$ and (D, g_+) is a *spherical domain.* That is, let $\Pi : S^N \to \mathbf{R}^N$ be the stereographic projection from the south pole, then the pair (D, g_+) is a canonical representation of the domain $\Pi^{-1}(D) \subset S^N$. Here $\Delta_{S^N} = \Delta_{g_+}$, g_+ is a conformal metric on the sphere S^N

$$g_+ = p_+^2(x) g_0, \quad p_+(x) = \frac{2}{1+|x|^2}$$

which has the constant scalar curvature $S_+ = N(N-1)$.

Next is a conformal Brezis-Nirenberg problem on the hyperbolic domain:

$$(2) \qquad \begin{cases} -\Delta_{H^N} u - \frac{N(N-2)}{4} u & = u^{2^*-1} + \varepsilon a(x) u \quad \text{in } D, \\ u > 0 & \qquad\qquad\qquad\quad \text{in } D, \\ u|_{\partial D} = 0 & \end{cases}$$

where $D \subset B_1^N(0) \subset \mathbf{R}^N$. Here $\Delta_{H^N} = \Delta_{g_-}$, g_- is a conformal metric on the N-dimensional hyperbolic space H^N

$$g_- = p_-^2(x) g_0, \quad p_-(x) = \frac{2}{1-|x|^2}$$

which has the constant scalar curvature $S_- = -N(N-1)$. We call (D, g_-) is a *hyperbolic domain.*

One of motivations to study this type of problem comes from the famous Yamabe problem in Differential Geometry:

Let (M, g) be a compact manifold. On (M, g), can one find a metric $\tilde{g}$ which is conformal to g such that the scalar curvature of $\tilde{g}$ is a constant, say $\frac{4(N-1)}{N-2}$?

If we set a metric $\tilde{g} = u^{4/(N-2)}g$ for some positive function u on M, this problem is equivalent to solve the equation

$$\begin{cases} L_g u = u^{2^*-1} & \text{on } M, \\ u > 0 & \text{on } M. \end{cases}$$

Thus our equation can be considered as a perturbed version of the equation in Yamabe problem, when the given metric g is already a conformal metric to the standard one.

When the conformal factor $p \not\equiv 1$, the boundary value problem of the form

$$\begin{cases} -\Delta_g u = u^{2^*-1} + a(x)u & \text{in } D, \\ u > 0 & \text{in } D, \\ u|_{\partial D} = 0 \end{cases}$$

has been treated in [1]; see also [2], [3], [4] and [18] for the case of the spaces of constant curvature. Especially in these papers, the influence of the conformal factor p and the coefficient function a on the existence of positive solutions was investigated.

On the other hand when $p \equiv 1$, many authors studied the concentration phenomena of blowing up solutions to $(P_{\varepsilon,a})$ as $\varepsilon \to 0$; See [7], [10], [16], [17], [14], [19] and references therein.

Main Theorem in this note concerns the concentration phenomena of the problem $(P_{\varepsilon,a})$ when the conformal factor $p \not\equiv 1$.

If $N \geq 4$ and $a(x) > 0$ somewhere on D, solutions to $(P_{\varepsilon,a})$ can be obtained as a scalar multiple of minimizers to the constrained minimization problem

$$S_{\varepsilon,a}^{(p)} = \inf_{u \in E} \left\{ \int_D L_g u \cdot u p^N dx - \varepsilon \int_D a(x) u^2 p^N dx \right\}$$

where $E = \{u \in H_0^1(D, p^N dx) | \int_D |u|^{2^*} p^N dx = 1\}$. It is known that least energy solutions $\overline{u}_\varepsilon$ thus obtained concentrate at one point as $\varepsilon \to 0$ in the sense that after passing to a subsequence,

$$|\nabla \overline{u}_\varepsilon|^2 p^{N-2} dx \rightharpoonup S^{N/2} \delta_{x_\infty}$$

for some $x_\infty \in \overline{D}$ as $\varepsilon \to 0$ in the sense of measures, where S is the best Sobolev constant. In the following, we call x_∞ a concentration point of (the subsequence of) $\overline{u}_\varepsilon$ as $\varepsilon \to 0$.

We characterize the location of concentration point of the least energy solutions $\overline{u}_\varepsilon$ to $(P_{\varepsilon,a})$ as follows.

Theorem 1.1. *Let $N \geq 4$. Assume that $D_+ := \{x \in D : a(x) > 0\} \neq \phi$. Let x_∞ be a concentration point of least energy solutions $\overline{u}_\varepsilon$ to $(P_{\varepsilon,a})$ constructed in Proposition 2.2 below. Then:*

(1) $x_\infty \in D_+$, and

(2) x_∞ maximizes the function

$$F(x) = \frac{\{a(x)p^2(x)\}^{\frac{N-2}{2}}}{R(x)}, \quad x \in D_+,$$

here $R(x)$ is the (positive) Robin function of $-\Delta_{g_0}$ acting on $H_0^1(D)$.

By definition, the Robin function on D is

$$R(x) = H(x,x)$$

where $H(x,y)$ is the regular part of the Green's function $G(x,y)$ relative to $-\Delta_{g_0}$ under the Dirichlet boundary condition:

$$H(x,y) = \frac{1}{(N-2)\omega_N}|x-a|^{2-N} - G(x,y),$$

here ω_N is the $(N-1)$ dimensional volume of S^{N-1}.

When $p = a \equiv 1$, it is shown that any concentration point of the general one point blow up solutions must be a critical point of the Robin function; see [17] and [10]. Recently, Molle and Pistoia [14] treated the problem $(P_{\varepsilon,a})$ when $p \equiv 1$ and $a \not\equiv 1$, and they showed that the concentration points of general one point blow up solutions are critical points of the function F in Theorem 1.1. In fact, Molle and Pistoia studied the concentration phenomena of equations with more general perturbation term, which may be sub-critical or even super-critical nonlinearity. Their main result says that any concentration point must be a critical point of some function defined on D which involves the Robin function, coefficient function $a(x)$ and the exponent of perturbation term.

On the other hand, if we strengthen the assumption and when we treat only the least energy solutions, concentration points of the least energy solutions are the minimum points of the Robin function when $p = a \equiv 1$ [19]. This result can be extended to the case $a \not\equiv 1$ [20]. Our proof of Theorem 1.1 heavily depends on the method used in [19] [20] which originates from [21] [11], and the simple transformation technique using the invariance of the conformal Laplacian.

§2. Least energy solutions.

First, we define some notations. The gradient operator, Laplacian, and the volume element with respect to the conformal metric $g = p^2(x)g_0$ are $\nabla_g = p^{-1}\nabla_{g_0}$, $\Delta_g = p^{-N}\mathrm{div}(p^{N-2}\nabla_{g_0})$, and $dvol_g = p^N dx$ respectively. From now on, we will write $\Delta = \Delta_{g_0}$, $\nabla = \nabla_{g_0}$. Also we define Sobolev spaces

$$\begin{aligned} H^1(D, p^N dx) &= \{u \in L^1_{loc}(D) : \|u\|_{H^1(D,p^N)} < \infty\}, \\ \|u\|^2_{H^1(D,p^N)} &= \int_D (|\nabla_g u|^2 + |u|^2) dvol_g, \end{aligned}$$

and

$$H^1_0(D, p^N dx) = \overline{C_0^\infty(D)}^{\|\ \|_{H^1(D,p^N)}}.$$

For $u \in H^1_0(D, p^N dx)$, we define $v \in H^1_0(D)$ as

$$v = up^{\frac{N-2}{2}}. \tag{3}$$

Then by direct calculation using the formula

$$-\Delta p^{\frac{N-2}{2}} = \frac{N-2}{4(N-1)} S_g p^{\frac{N+2}{2}}, \quad N \geq 3,$$

we have

$$\Delta v = p^{\frac{N+2}{2}} \left\{ \Delta_g u - \frac{N-2}{4(N-1)} S_g u \right\} \tag{4}$$

and

$$\int_D |\nabla_g u|^2 p^N dx = \int_D |\nabla v|^2 dx - \frac{N-2}{4(N-1)} \int_D S_g v^2 p^2 dx. \tag{5}$$

First, we show some nonexistence result of the problem $(P_{\varepsilon,a})$.

Proposition 2.1. *Let $D \subset \mathbf{R}^N$ be a star-shaped domain with respect to the origin (measured by the standard metric g_0) and*

$$ap^2 + \frac{1}{2}\nabla(ap^2) \cdot x \leq 0 \quad \text{in } D,$$

then $(P_{\varepsilon,a})$ has no nontrivial solution.

Especially, if

$$a(x)\left(\frac{1-|x|^2}{1+|x|^2}\right)+\frac{1}{2}x\cdot\nabla a\leq 0 \quad in\ D\subset \mathbf{R}^N,$$

(resp.

$$a(x)\left(\frac{1+|x|^2}{1-|x|^2}\right)+\frac{1}{2}x\cdot\nabla a\leq 0 \quad in\ D\subset B_1^N(0),$$

) then the conformal Brezis-Nirenberg problem on the spherical (resp. hyperbolic) domain has no nontrivial solution for star-shaped $D\subset\mathbf{R}^N$ *(resp.* $D\subset B_1^N(0)$*).*

Proof: If $u\in H_0^1(D,p^N dx)$ is a solution of $(P_{\varepsilon,a})$, then $v\in H_0^1(D)$ defined by (3) is a solution of

$$\begin{cases} -\Delta v = v^{2^*-1}+\varepsilon a(x)p^2(x)v & \text{in } D,\\ v>0 & \text{in } D,\\ v|_{\partial D}=0. \end{cases} \tag{6}$$

Now, recall the Pohozaev identity [15] to the problem

$$\begin{cases} -\Delta v = f(x,v) & \text{in } D,\\ v|_{\partial D}=0. \end{cases}$$

$$N\int_D F(x,v)dx-\left(\frac{N-2}{2}\right)\int_D vf(x,v)dx+\int_D x\cdot\nabla_x F(x,v)dx$$
$$=\frac{1}{2}\int_{\partial D}(x\cdot\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 ds$$

where ν denotes the unit outer normal of the boundary of D and $F(x,v) = \int_0^v f(x,t)dt$. Applying this identity to (6), we have

$$\varepsilon\int_D\left(ap^2+\frac{1}{2}x\cdot\nabla(ap^2)\right)v^2dx=\frac{1}{2}\int_{\partial D}(x\cdot\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 ds.$$

Right hand side is nonnegative by the star-shapedness of D, then we obtain the conclusion.

□

On the other hand, we can obtain nontrivial solutions by using the result of Brezis-Nirenberg, at least $N\geq 4$.

Proposition 2.2. *Let* $N\geq 4$. *Assume* $a(x)>0$ *somewhere on* D. *Then there exists a nontrivial solution to* $(P_{\varepsilon,a})$ *for sufficiently small* $\varepsilon>0$.

Proof: To find a nontrivial solution to $(P_{\varepsilon,a})$, let us consider the constrained minimization problem

$$(7) \qquad S_{\varepsilon,a}^{(p)} := \inf_{u\in E}\left\{\int_D L_g u\cdot u\, dvol_g - \varepsilon\int_D a(x)u^2 dvol_g\right\}$$

where $E = \{u \in H_0^1(D, p^N dx) | \int_D |u|^{2^*} dvol_g = 1\}$.

By using (4) and (5), we can rewrite $S_{\varepsilon,a}^{(p)}$ in (7) as

$$(8) \qquad S_{\varepsilon,a}^{(p)} = \inf_{v\in E_0}\left\{\int_D |\nabla v|^2 dx - \varepsilon\int_D a(x)p^2(x)v^2 dx\right\},$$

where $E_0 = \{v \in H_0^1(D) | \int_D |v|^{2^*} dx = 1\}$.

Now, we recall the result of Brezis-Nirenberg:

Lemma 2.3.([6]) *Let $N \geq 4$. Assume $\varepsilon > 0$ small such that $-\Delta - \varepsilon a(x)p^2(x)$ is coercive. Then the following conditions*

(1) $a(x) > 0$ somewhere on D,

(2) $S_{\varepsilon,a}^{(p)} < S$,

(3) $S_{\varepsilon,a}^{(p)}$ is achieved

are equivalent.

By this Lemma, we have $v_\varepsilon^0 \in E_0$ which is an $S_{\varepsilon,a}^{(p)}$-minimizer for (8). As usual we can assume $v_\varepsilon^0 > 0$ and if we set $\overline{v}_\varepsilon = (S_{\varepsilon,a}^{(p)})^{(N-2)/4} v_\varepsilon^0$, then $v = \overline{v}_\varepsilon$ satisfies (6). Now, set $\overline{u}_\varepsilon = p^{-\frac{N-2}{2}}\overline{v}_\varepsilon$, $\quad \overline{u}_\varepsilon \in H_0^1(D, p^N dx)$. Then we easily see $\overline{u}_\varepsilon$ is a solution to the problem $(P_{\varepsilon,a})$.

□

From now on, we call $\overline{u}_\varepsilon$ in Proposition 2.2 as the *least energy solution* of $(P_{\varepsilon,a})$.

In [1], the authors extend the Concentration-Compactness Alternative of P. Lions [12] [13] to the spaces of conformal metrics ([1] Theorem 17). Using this and the fact that $S_{\varepsilon,a}^{(p)} = S + o(1)$ as $\varepsilon \to 0$, we can check that the least energy solutions to $(P_{\varepsilon,a})$ makes one point blow up phenomena: after passing to a subsequence, we have

$$|\nabla\overline{u}_\varepsilon|^2 p^{N-2} dx \rightharpoonup S^{N/2}\delta_{x_\infty}, \quad (\varepsilon \to 0)$$

for some $x_\infty \in \overline{D}$. Theorem 1.1 characterizes the location of concentration point of least energy solutions thus obtained by the method of Brezis and Nirenberg.

In the rest of this section, we make some remark on the case $N = 3$. By the recent result of Druet ([8]; see also [9]) which solves the conjecture proposed by H. Brezis in [5] affirmatively, we can state the following proposition on the minimization problem (7).

Proposition 2.4. *Let $N = 3$. Assume $\varepsilon > 0$ small such that $-\Delta - \varepsilon a(x)p^2(x)$ is coercive. Then the following conditions*

(1) $\exists x \in D$ such that $R_{a,p}(x) < 0$
(2) $S_{\varepsilon,a}^{(p)} < S$,
(3) $S_{\varepsilon,a}^{(p)}$ is achieved

are equivalent.

Here $R_{a,p}(x) = H_{a,p}(x,x)$, $H_{a,p}(x,y) = \frac{1}{4\pi}|x-y|^{-1} - G_{a,p}(x,y)$ is the regular part of the Green function $G_{a,p}$ of the operator $-\Delta - \varepsilon a(x)p^2(x)$, that is, $G = G_{a,p}$ is the distributional solution of

$$\begin{cases} -\Delta_x G(x,y) - \varepsilon a(x)p^2(x)G(x,y) = \delta_y & \text{in } D, \\ G|_{\partial D} = 0. \end{cases}$$

Thus in the case $N = 3$, the global nature of $a(x)$ affects on the existence of the least energy solutions.

§3. Sketch of proof of Theorem.

In this section, we will sketch the proof of Theorem 1.1. Let $k(x) = a(x)p^2(x)$ and consider

$$S_{\varepsilon,k} := \inf_{\substack{v \in H_0^1(D) \\ \|v\|_{L^{2^*}(D)}=1}} \left\{ \int_D |\nabla v|^2 dx - \varepsilon \int_D k(x) v^2 dx \right\}. \tag{9}$$

For a given sequence $\varepsilon_n \to 0$, let $v_{\varepsilon_n}^0$ be a minimizer for (9) and define

$$v_n = S^{\frac{N-2}{4}} v_{\varepsilon_n}^0.$$

Then we see that v_n and $\overline{u}_{\varepsilon_n}$ have the same concentration point x_∞ and $|\nabla v_n|^2 dx \rightharpoonup S^{N/2}\delta_{x_\infty}$ in the sense of measures on $\overline{D}$.

Now, by a result of Rey ([17] Proposition 2), we know there exists $(\alpha_n, \lambda_n, a_n) \in \mathbf{R}_+ \times \mathbf{R}_+ \times D$ such that

$$v_n = \alpha_n PU_{\lambda_n, a_n} + w_n \tag{10}$$

holds true for n large, where

$$\begin{array}{rcl} \alpha_n & \to & \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \\ a_n & \to & x_\infty, \\ \dfrac{\lambda_n}{d_n} & \to & 0 \quad \text{where } d_n = \operatorname{dist}(a_n, \partial D), \\ w_n & \in & E_{\lambda_n, a_n}, \\ w_n & \to & 0 \text{ in } H_0^1(D) \end{array}$$

as $n \to \infty$. Here for $\lambda > 0$ and $a \in D$, $PU_{\lambda,a}(x)$ denotes the projection of $U_{\lambda,a}$ to $H_0^1(D)$, where

$$U_{\lambda,a}(x) = \left(\frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbf{R}^N \tag{11}$$

and

$$\begin{aligned} E_{\lambda,a} = \{ w \in H_0^1(D) : \quad 0 = & \int_D \nabla w \cdot \nabla PU_{\lambda,a} dx \\ = & \int_D \nabla w \cdot \nabla (\frac{\partial}{\partial a_i} PU_{\lambda,a}) dx \quad (i = 1, \cdots, N) \\ = & \int_D \nabla w \cdot \nabla (\frac{\partial}{\partial \lambda} PU_{\lambda,a}) dx \}. \end{aligned}$$

By using this expression (10), we will have the precise asymptotics of the value $S_{\varepsilon_n,k}$ as $n \to \infty$. Detailed proof can be found in [20].

First two lemmas concern the H_0^1 and L^2 norm of the main part and are well known.

Lemma 3.1. *As $n \to \infty$, we have*

$$\begin{aligned} \int_D |\nabla PU_{\lambda_n,a_n}|^2 dx = & N(N-2)A - (N-2)^2 \omega_N^2 R(a_n) \lambda_n^{N-2} \\ & + O\left(\frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right), \end{aligned}$$

where

$$A = \int_{\mathbf{R}^N} U_{1,0}^{2^*} dx = \frac{\Gamma(N/2)}{\Gamma(N)} \pi^{N/2}.$$

Lemma 3.2. *We have*

$$\int_D k(x) PU_{\lambda_n,a_n}^2 dx = k(a_n) \omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad \text{as } n \to \infty,$$

when $N \geq 5$, where

$$C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)},$$

and

$$\int_D k(x) PU^2_{\lambda_n, a_n} dx = k(a_n)\omega_4 \lambda_n^2 |\log \lambda_n| + O\left(\frac{\lambda_n^2}{d_n} |\log \lambda_n|^{1/2}\right) + O\left(\frac{\lambda_n^2}{d_n^2}\right)$$

as $n \to \infty$ when $N = 4$.

By Lemma 3.1 and Lemma 3.2, we have

Proposition 3.3. (Asymptotic behavior of $S_{\varepsilon_n,k}$) *As $n \to \infty$, we have*

$$\begin{aligned} S_{\varepsilon_n,k} &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} \\ &- \varepsilon_n k(a_n)\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\ &+ S^{(2-N)/2}\left\{\|\nabla w_n\|^2_{L^2} - N(N+2)\int_D U^{4/(N-2)}_{\lambda_n,a_n} w_n^2 dx\right\} \\ &+ o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\varepsilon_n \lambda_n^2\right) \end{aligned}$$

when $N \geq 5$, and

$$\begin{aligned} S_{\varepsilon_n,k} &= S + \frac{S}{2}\left(\frac{\omega_4^2}{A}\right) R(a_n)\lambda_n^2 \\ &- \varepsilon_n k(a_n)\left(\frac{S\omega_4}{8A}\right)\lambda_n^2 |\log \lambda_n| \\ &+ S^{-1}\left\{\|\nabla w_n\|^2_{L^2} - 24\int_D U^2_{\lambda_n,a_n} w_n^2 dx\right\} \\ &+ o\left(\frac{\lambda_n^2}{d_n^2}\right) + o\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\varepsilon_n \lambda_n^2 |\log \lambda_n|\right) \end{aligned}$$

when $N = 4$.

To proceed further, we need the nondegeneracy result first shown by Rey ([17] Appendix D).

Lemma 3.4. *There exists a constant $C > 0$ which depends only on the dimension N such that for any $w_n \in E_{\lambda_n, a_n}$,*

$$\int_D |\nabla w_n|^2 dx - N(N+2) \int_D U_{\lambda_n, a_n}^{4/(N-2)} w_n^2 dx \geq C \int_D |\nabla w_n|^2 dx$$

holds true.

Furthermore, we need the appropriate bound of the value $S_{\varepsilon_n, k}$ from the above. The following Lemma is proved by the same argument of Lemma 2.7 in [19].

Lemma 3.5.(Upper bound of $S_{\varepsilon,k}$) *For any $y \in D$ such that $k(y) > 0$ and $\rho > 0$, there exists $\varepsilon_0 = \varepsilon_0(y, \rho)$ such that if $\varepsilon \in (0, \varepsilon_0)$, then the followings hold:*

$$S_{\varepsilon,k} \leq S - \left(\frac{N-4}{N-2}\right) \varepsilon k(y) \left\{ \frac{S \omega_N C_N}{N(N-2)A} - \rho \right\} \left[\frac{2 C_N \varepsilon k(y)}{(N-2)^3 \omega_N R(y)} \right]^{\frac{2}{N-4}}$$

when $N \geq 5$, and

$$S_{\varepsilon,k} \leq S - \frac{S \varepsilon k(y) \omega_4}{16Ae} \exp\left(- \frac{8 \omega_4 R(y) + \varepsilon k(y)/e + 2\rho}{\varepsilon k(y)} \right)$$

when $N = 4$.

Indeed, note that $k(y)$ is a positive constant for fixed $y \in D_+$. Only we have to do is to test the functional

$$J_\varepsilon(\psi) = \frac{\int_D |\nabla \psi|^2 dx - \varepsilon \int_D k(x) \psi^2 dx}{\left(\int_D |\psi|^{p+1} dx\right)^{\frac{2}{p+1}}}$$

defined for $\psi \in H_0^1(D) \setminus \{0\}$ by the function of the form

$$\psi_{\varepsilon,y} = S^{\frac{-(N-2)}{4}} \alpha_N P U_{\lambda_\varepsilon(y), y}.$$

Here for $y \in D_+$ and $\varepsilon > 0$, $\lambda_\varepsilon(y)$ is chosen to be the unique minimum point of the function

$$f(\lambda) = S + C_A \lambda^{N-2} - C_B \lambda^2 \quad \text{for } \lambda > 0$$

when $N \geq 5$, where

$$C_A = S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) R(y), \quad C_B = \varepsilon k(y) \frac{S \omega_N C_N}{N(N-2)A},$$

and it gives the minimum value

$$\min_{\lambda>0} f(\lambda) = S - \left(\frac{N-4}{N-2}\right) \varepsilon k(y) \left(\frac{S\omega_N C_N}{N(N-2)A}\right) \left(\frac{2C_N \varepsilon}{(N-2)^3 \omega_N R(y)}\right)^{\frac{2}{N-4}}.$$

The argument is similar when $N = 4$, by considering the function

$$f(\lambda) = S + C_A \lambda^2 - C_B \lambda^2 |\log \lambda| \quad (0 < \lambda < 1)$$

where

$$C_A = \frac{S}{2}\left(\frac{\omega_4^2}{A}\right) R(y), \quad C_B = \varepsilon k(y) \frac{S\omega_4}{8A}.$$

Now we sketch the arguments used in the proof of Theorem 1.1. We only treat the case $N \geq 5$, but the case $N = 4$ is similar.

Again, the following elementary fact is important in the argument: For constants $C_A, C_B > 0$, the function

$$f(\lambda) = S + C_A \lambda^{N-2} - C_B \lambda^2$$

has the unique global minimum value

$$\min_{\lambda>0} f(\lambda) = S - \left(\frac{N-4}{N-2}\right) C_B \left(\frac{2C_B}{(N-2)C_A}\right)^{\frac{2}{N-4}}. \tag{12}$$

First, we prove that $k(a_n) > 0$ for n sufficiently large. Assume the contrary that there exists a subsequence such that $k(a_n) \leq 0$. In addition if $k(a_n) \leq -C < 0$ for some $C > 0$ independent of n, Proposition 3.3 and Lemma 3.4 yield a contradiction to the fact $S > S_{\varepsilon_n,k}$ by Brezis and Nirenberg. Thus it must be hold that $k(a_n) \to 0$ for a sequence with $k(a_n) \leq 0$. On the other hand, by Proposition 3.3, Lemma 3.4 and the estimate of the Robin function

$$R(a_n) = \frac{1}{(N-2)\omega_N}\left(\frac{1}{2d_n}\right)^{N-2} + o\left(\frac{1}{d_n^{N-2}}\right) \quad \text{as } d_n \to 0 \tag{13}$$

([17]), we have $C_1 > 0$ independent of n such that

$$S > S_{\varepsilon_n,k} \geq S + C_1 \lambda_n^{N-2} - (k(a_n)K + p_n)\varepsilon_n \lambda_n^2$$

for some $p_n > 0, p_n \to 0$. Here $K = \frac{S\omega_N C_N}{N(N-2)A}$. Therefore we must have

$$C_B(n) := Kk(a_n) + p_n > 0$$

for n large. Thus by (12), we obtain

$$S_{\varepsilon_n,k} \geq S - \left(\frac{N-4}{N-2}\right) C_B(n)\varepsilon_n \left(\frac{2C_B(n)\varepsilon_n}{(N-2)C_1}\right)^{\frac{2}{N-4}}.$$

Connecting this with the upper bound

$$S_{\varepsilon_n,k} \leq S - \exists C_2 \varepsilon_n^{1+\frac{2}{N-4}}$$

for some $C_2 > 0$, which is assured by Lemma 3.5, we have a contradiction since we have seen that $C_B(n) \to 0$ as $n \to 0$. Thus we have proved $k(a_n) > 0$ for n sufficiently large.

The same argument shows that when $k(a_n) > 0$ for n sufficiently large, it cannot happen that $k(x_\infty) = \lim_{n\to\infty} k(a_n) = 0$.

Next, we will show that the blow up point x_∞ is not on the boundary ∂D. Indeed, suppose the contrary. Then $x_\infty \in \partial D$ and $d_n = d(a_n, \partial D) \to 0$ as $n \to \infty$. Then, as before, by Proposition 3.3, Lemma 3.4, the estimate (13) and the fact $k(a_n) \geq \exists C > 0$ for large n, we can find constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
S_{\varepsilon_n,k} &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) R(a_n)\lambda_n^{N-2} \\
&- \varepsilon_n k(a_n)\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\
&+ S^{(2-N)/2}\left\{\|\nabla w_n\|_{L^2}^2 - N(N+2)\int_D U_{\lambda_n,a_n}^{4/(N-2)} w_n^2 dx\right\} \\
&+ o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + o\left(\varepsilon_n \lambda_n^2\right) \\
&\geq S + C_1\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) - C_2\varepsilon_n\lambda_n^2 \\
&\geq S - \left(\frac{N-4}{N-2}\right) C_2\varepsilon_n \left\{\frac{2C_2\varepsilon_n}{(N-2)C_1(\frac{1}{d_n^{N-2}})}\right\}^{\frac{2}{N-4}} \\
&= S - C_3 \varepsilon_n^{\frac{N-2}{N-4}} d_n^{\frac{2(N-2)}{N-4}} = S + o(\varepsilon_n^{\frac{N-2}{N-4}}),
\end{aligned}$$

since we assume $d_n \to 0$. Here we used (12) in deriving the second inequality.

On the other hand, we know that $S_{\varepsilon_n,k} \leq S - C\varepsilon_n^{\frac{N-2}{N-4}} + o(\varepsilon_n^{\frac{N-2}{N-4}})$ for some $C > 0$ by Lemma 3.5. This contradicts the above estimate, so we conclude that x_∞ is in the interior of D.

Now, Proposition 3.3, Lemma 3.4 and $d_n \geq \exists C > 0$ uniformly in n indeed imply that

$$S_{\varepsilon_n,k} = S + C_A\lambda_n^{N-2} - C_B\lambda_n^2 + (\text{error term})$$

where

$$C_A = C_1 R(a_n), \quad C_B = C_2\varepsilon_n k(a_n)$$

for some constants $C_1, C_2 > 0$. Thus

$$S_{\varepsilon_n,k} \geq S - C_3 \varepsilon_n^{(N-2)/(N-4)} k(a_n) \left(\frac{k(a_n)}{R(a_n)} \right)^{\frac{2}{N-4}}$$

again by (12).

On the other hand, Lemma 3.5 gives an upper bound

$$S_{\varepsilon_n,k} \leq S - (C_3 - \rho) \varepsilon_n^{(N-2)/(N-4)} k(y) \left(\frac{k(y)}{R(y)} \right)^{\frac{2}{N-4}}$$

for any $y \in D_+$ and $\rho > 0$ sufficiently small. Therefore by connecting these, we have

$$-C_3 \varepsilon_n^{(N-2)/(N-4)} k(a_n) \left(\frac{k(a_n)}{R(a_n)} \right)^{\frac{2}{N-4}} \leq -(C_3 - \rho) \varepsilon_n^{(N-2)/(N-4)} k(y) \left(\frac{k(y)}{R(y)} \right)^{\frac{2}{N-4}}.$$

Dividing both sides by $\varepsilon_n^{\frac{N-2}{N-4}}$, letting $n \to \infty$ and $\rho \to 0$, we check that x_∞ will maximize

$$k(y) \left(\frac{k(y)}{R(y)} \right)^{\frac{2}{N-4}} = (F(y))^{\frac{2}{N-4}}.$$

This proves the Theorem. □

§4. Examples.

Here we show some examples of Theorem 1.1 in case that the domain is a ball and the coefficient function $a(x) = |x|^2$.

First, the conformal Brezis-Nirenberg problem on the spherical domain $(B_L(0), g_+)$ $(L > 0)$ with $a(x) = |x|^2$ is

$$\begin{cases} -\Delta_{S^N} u + \frac{N(N-2)}{4} u = u^{2^*-1} + \varepsilon |x|^2 u, & \text{in } B_L(0), \\ u > 0 & \text{in } B_L(0), \\ u|_{\partial B_L(0)} = 0. \end{cases}$$

Note that the explicit form of the Robin function on $B_L(0)$ is known as

$$R(x) = C_N \left(L - \frac{|x|^2}{L} \right)^{2-N}$$

where $C_N = (N-2)^{-1}\omega_N^{-1}$.

By Theorem 1.1, we know that the concentration point of least energy solutions is a maximum point of

$$F(x) = \frac{\{a(x)p_+^2(x)\}^{\frac{N-2}{2}}}{R(x)} = C_{N,L}\left(\frac{|x|(L^2-|x|^2)}{1+|x|^2}\right)^{N-2}$$

for some constant $C_{N,L} > 0$. Thus an easy calculation shows $x_\infty \in B_{r^*}(0)$, where

$$r_* = \sqrt{\frac{-(L^2+3)+\sqrt{(L^2+3)^2+4L^2}}{2}} \ (< L).$$

If the spherical domain is an upper hemisphere $(B_1(0), g_+)$, then $r_* = \sqrt{-2+\sqrt{5}} \simeq 0.486$.

The conformal Brezis-Nirenberg problem on the hyperbolic domain $(B_L(0), g_-)$ $(0 < L < 1)$ with $a(x) = |x|^2$ is

$$\begin{cases} -\Delta_{H^N} u - \frac{N(N-2)}{4}u = u^{2^*-1} + \varepsilon |x|^2 u, & \text{in } B_L(0), \\ u > 0 & \text{in } B_L(0), \\ u|_{\partial B_L(0)} = 0. \end{cases}$$

In this case the concentration point of least energy solutions is a maximum point of

$$F(x) = \frac{\{a(x)p_-^2(x)\}^{\frac{N-2}{2}}}{R(x)} = C_{N,L}\left(\frac{|x|(L^2-|x|^2)}{1-|x|^2}\right)^{N-2},$$

thus $x_\infty \in B_{r^*}(0)$ where

$$r_* = \sqrt{\frac{3-L^2-\sqrt{(3-L^2)^2-4L^2}}{2}} \ (< L).$$

References

[1] C. Bandle, A. Brillard and M. Flucher, Green's function, harmonic transplantation, and best Sobolev constant in spaces of constant curvature, Trans. Amer. Math. Soc., **350** (1998), 1103–1128.

[2] C. Bandle, Semilinear elliptic boundary value problems with critical Sobolev exponent, In: Elliptic and Parabolic Problems, (eds. J. Bemelmans et al.), World Sci. Publ., 2002, pp. 11–20.

[3] C. Bandle and L. A. Peletier, Best Sobolev constants and Emden equations for the critical exponent in S^3, Math. Ann., **313** (1999), 83–93.

[4] C. Bandle and R. Benguria, The Brezis-Nirenberg problem on S^3, J. Diff. Eqn., **178** (2002), 264–279.

[5] H. Brezis, Elliptic equations with limiting Sobolev exponents. The impact of topology, Comm. Pure and Appl. Math., **39** (1986), 17–39.

[6] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure and Appl. Math., **36** (1983), 437–477.

[7] H. Brezis, and L. A. Peletier, Asymptotics for elliptic equations involving critical growth, Partial differential equations and calculus of variations, Progress. Nonlinear Differential Equations Appl., **1**, Birkhüser Boston, Boston, MA, 1989, pp. 149–192.

[8] O. Druet, Elliptic equations with critical Sobolev exponents in dimension 3, Ann. Inst. H. Poincaré Anal. Non Linéaire, **19** (2002), 125–142.

[9] P. Esposito, On some conjecture proposed by Haim Brezis, Nonlinear Anal., **54** (2004), 751–759.

[10] Z. C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, Ann. Inst. Henri Poincaré Anal. Non Linéaire, **8** (1991), 159–174.

[11] T. Isobe, On the asymptotic analysis of H-systems, I: asymptotic behavior of large solutions, Adv. Diff. Eq., **6** (2001), 513–546.

[12] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. Part 1, Rev. Mat. Iberoamericana, **1** (1985), 145–201.

[13] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. Part 2, Rev. Mat. Iberoamericana, **1** (1985), 45–121.

[14] R. Molle and A. Pistoia, Concentration phenomena in elliptic problem with critical and supercritical growth, Advances in Diff. Eq., **8** (2003), 547–570.

[15] S. Pohozaev, Eigenfunctions of the equation $\Delta u = \lambda f(u)$, Soviet Math Dokl., **6** (1965), 1408–1411.

[16] O. Rey, Proof of two conjectures of H. Brezis and L.A. Peletier, Manuscripta Math., **65** (1989), 19–37.

[17] O. Rey, The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal., **89** (1990), 1–52.

[18] S. Stapelkamp, The Brezis-Nirenberg problem on $\mathbf{H}^N$: existence and uniqueness of solutions, In: Elliptic and Parabolic Problems, (eds. J. Bemelmans et al.), World Sci. Publ., 2002, pp. 283–290.

[19] F. Takahashi, On location of blow up points of least energy solutions to the Brezis-Nirenberg equations, Funkcial. Ekvac., **47** (2004), 145–166.

[20] F. Takahashi, Concentration points of least energy solutions to the Brezis-Nirenberg problem with variable coefficient, submitted to Banach Center Publications.

[21] J. Wei, Asymptotic behavior of least energy solutions to a semilinear Dirichlet problem near the critical exponent, J. Math. Soc. Japan, **50** (1998), 139–153.

Mathematical Institute
Tohoku University
Aoba-ku, Sendai, 980-8578
Japan

Current address:
Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka, 3-3-138
Japan