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On the dynamical system of a parabolic equation with non-local term

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Abstract.

We consider the parabolic equation and the associated stationary equation with non-local term. We prove the global existence of a solution for parabolic equation and discuss the property of ω-limit set.

§1. Introduction

In this paper we consider the following parabolic equation with non-local term:

$$(1) \qquad \begin{cases} b(x)v_t = (a(x)v_x)_x + \lambda \frac{K(x)e^v}{\int_\Omega K(x)e^v dx} & x \in \Omega, t > 0, \\ v|_{\partial\Omega} = 0 & t > 0, \\ v(x,0) = v_0(x) & x \in \Omega, \end{cases}$$

where λ is a positive constant, $\Omega = (0,1)$ and $a(x), b(x), K(x)$ are $C^2(\overline{\Omega})$ and positive in $\overline{\Omega}$. This equation is the one dimensional model for

$$(2) \qquad \begin{cases} v_t = \Delta v + \lambda \frac{e^v}{\int_\Omega e^v dx} & x \in \Omega, t > 0, \\ v|_{\partial\Omega} = 0 & t > 0, \end{cases}$$

in higher dimensional domain $\Omega \subset \mathbf{R}^n$. Note that if the domain Ω is an annulus $A_a = \{x \in \mathbf{R}^n \mid a < |x| < 1\}$ $(0 < a < 1)$ with $n \in \mathbf{N}$ and the solution $v = v(x)$ is radially symmetric, (2) is given as

$$\begin{cases} r^{n-1}v_t = (r^{n-1}v_r)_r + \frac{\lambda}{\omega_n} \frac{r^{n-1}e^v}{\int_a^1 r^{n-1}e^v dr} & r \in (a,1), t > 0, \\ v(a) = v(1) = 0 & t > 0, \end{cases}$$

where ω_n denotes the $(n-1)$ dimensional volume of the surface of the unit ball in $\mathbf{R}^n$.

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In [3], Bebernes and Lacey study such non-local problems motivated by Ohmic heating and the occurence of shear bands in metals deformed under high strain rates. Another motivation is a Keller-Segel system ([10]) for the chemotactic aggregation of cellular slime molds which is given, for instance, by

$$
\left\{
\begin{array}{lll}
\varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in} & \Omega \times (0, T), \\
\tau v_t = \Delta v + u & \text{in} & \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = v = 0 & \text{on} & \partial\Omega \times (0, T), \\
u|_{t=0} = u_0(x) \geq 0 & \text{in} & \Omega, \\
v|_{t=0} = v_0(x) & \text{in} & \Omega,
\end{array}
\right. \tag{3}
$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, τ, ε are positive constants, and ν is the outer unit normal vector. In the limit state $\varepsilon = 0$ and $\tau = 1$ of (3), it is reduced to (2)([17],[12]). In the case of $\tau = 0$ and $\varepsilon = 1$, see [15].

The stationary problem (1) or (2) is given by

$$
\left\{
\begin{array}{ll}
0 = (a(x) v_x)_x + \lambda \frac{K(x) e^v}{\int_\Omega K(x) e^v dx} & x \in \Omega, \\
v|_{\partial\Omega} = 0,
\end{array}
\right. \tag{4}
$$

or

$$
\left\{
\begin{array}{ll}
0 = \Delta v + \lambda \frac{e^v}{\int_\Omega e^v dx} & x \in \Omega, \\
v|_{\partial\Omega} = 0.
\end{array}
\right. \tag{5}
$$

In the case of $n = 2$, the equation (4) and (5) have the origin in self-dual gauge theories ([18]) and the equlibrium stste of the mean field of many point vortices of perfect fluid in Onsager's formulation ([5], [6]). In this case, the quantized blow-up mechanisms for the family of solutions of (5) happen ([16]) and equation (3) with $\tau = 0$ inherits these mechanisms ([15]). In [12], Miyasita ans Suzuki prove that if Ω is star-shaped with $n \geq 3$, then there is an upper bound of λ for the existence of the solution of (5). Moreover if Ω is the unit ball with $3 \leq n \leq 9$, we have infinitely many bendings in λ of the connected component of the solution set (λ, v). Further we can compute the Morse index for the solution (λ, v) under the same circumstances ([15]). On the other hand, since the behaviour of the solution of (3) with $\tau > 0$ and $n = 3$ is not studied so much. Now we consider the dynamics of one dimensional model in the case of $\varepsilon = 0$ and $\tau > 0$.

To describe the statement, we begin with some definitions. The mapping $T(t) : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by

$$(T(t)v_0)(x) = v(x, t),$$

forms a semiflow on $H_0^1(\Omega)$, where $v(x,t)$ is the solution of (1) with initial value. Here $v = v(x,t)$ is said to be a solution of (1) on $(0,T)$ in the sense of [8] provided that
(1) $v \in C\left([0,T) : H_0^1(\Omega)\right)$,
(2) $v(\cdot,t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\frac{dv}{dt}$ exists for $t \in (0,T)$,
(3) $t \mapsto \lambda \frac{K(x)e^{v(x,t)}}{\int_\Omega K(\xi)e^{v(\xi,t)}d\xi}$ is locally Hölder continuous for $t \in (0,T)$,
(4) $v(x,t)$ satisfies (1) for $t > 0$.
We remark that Henry imposes the locally integrable condition on the definition of solution. But in this case, this condition satisfied automatically becase of $v \in C\left([0,T) : H_0^1(\Omega)\right)$.

A family of mappings $T(t) : H_0^1(\Omega) \to H_0^1(\Omega)$ for $t \geq 0$ is said to be a dynamical system on $H_0^1(\Omega)$, provided that
(1) $T(t)$ is a continuous mapping from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ for all $t \geq 0$,
(2) $t \mapsto T(t)v$ is continuous for all $v \in H_0^1(\Omega)$,
(3) $T(0) = I_{H_0^1(\Omega)}$ holds, where $I_{H_0^1(\Omega)}$ is an identity mapping on $H_0^1(\Omega)$,
(4) $T(t)(T(s)v) = T(t+s)v$ holds for all $v \in H_0^1(\Omega)$ and $t, s \geq 0$.

Then we call $\gamma(v) = \{T(t)v \mid t \geq 0\}$ the orbit through v. For any $v \in H_0^1(\Omega)$, we define the ω-limit set of v or the orbit $\gamma(v)$ as

$$\omega(v) = \omega(\gamma(v)) = \bigcap_{s\geq 0} \overline{\bigcup_{t \geq s} T(t)v}.$$

A set $\Sigma \subset H_0^1(\Omega)$ is said to be positively invariant under $T(t)$ if $T(t)\Sigma \subset \Sigma$. A set $\Sigma \subset H_0^1(\Omega)$ is said to be negatively invariant under $T(t)$ if $T(t)\Sigma \supset \Sigma$. Σ is invariant if it satiesfies both the conditions.

Then the following theorems are the main results in this paper:

Theorem 1. *For any $v_0 \in H_0^1(\Omega)$, (1) has a unique global solution $v = v(x,t)$. For a solution $v(x,t)$ of (1), we define a mapping $T(t) : H_0^1(\Omega) \to H_0^1(\Omega)$ by $T(t)v_0 = v(x,t)$ for $t \geq 0$. Then $T(t)$ defines a dynamical system on $H_0^1(\Omega)$.*

Theorem 2. *For $v_0 \in H_0^1(\Omega)$, $\omega(v_0)$ is nonempty, compact, invariant and connected. Furthermore it holds that $\omega(v_0) \subset E$, where E denotes a set of all stationary solutions of (1).*

We apply the results in [8] to prove these two theorems. In paticular, we show the boundedness of the solution of (1) for smaller space than

$H_0^1(\Omega)$. If we treat the equation without non-local term, we cannot ensure time global solution for some λ. Thus we cannot always consider the above properties of ω-limit set.

Generally if we treat the semi-linear parabolic equation without non-local term, its linearized equation has parabolic Liouville(Matano-Angenent) theorems which implies the Morse-Smale property ([1],[4], [9]). But if we treat the equation with non-local term, we cannot apply such an argument because of the lack of comparison principle.

On the other hand, (1) is the gradient flow $v_t = \delta J_\lambda(v)$ for

$$J_\lambda(v) = \frac{1}{2}\int_\Omega a(x)v_x^2 dx - \lambda \log\left(\int_\Omega K(x)e^v dx\right)$$

and $J_\lambda(v)$ is the Lyapunov function. Hence this means that the ω-limit set of the time-global solution is contained in the set of stationary solutions. If $n = 1$, it also follows that this dynamical system is dissipative ([7]).

According to [13] and [11], equation (4) has solutions with high Morse indices provided $0 < a \ll 1$. Hence the structure of attractor to (1) can be nontrivial. Thus it is meaningful to study a dynamical system of (1). We remark that [14] constructs a global and exponential attractor for a system of two reaction-diffusion equations similar to (3) in a bounded domain $\Omega \subset \mathbf{R}^2$ when the Naumann boundary condition is imposed. In this case, we can derive an apriori estimate because of Neumann boundary condition and the sign of coefficient of linear term. This paper is composed of three sections. We prove Theorems 1 and 2 in Sections 2 and 3, respectively.

§2. The proof of Theorem 1

Proof of Theorem 1: First of all, we prove the global existence of a solution of (1). Through the following Liouville transformation in [2]

$$\begin{cases} k = \int_0^1 \sqrt{\frac{b}{a}}dx, \\ y(x) = \frac{1}{k}\int_0^x \sqrt{\frac{b}{a}}d\xi, \\ s(t) = \frac{1}{k^2}t, \end{cases}$$

Eqution (1) is reduced to

$$\begin{cases} v_s = v_{yy} + h(y)v_y + \frac{k^2}{b}u & y \in \Omega, s > 0, \\ v|_{\partial\Omega} = 0 & s > 0, \\ v(y,0) = v_0(y) & y \in \Omega, \end{cases}$$

where

$$h(y) = \frac{k}{2\sqrt{ab}}\frac{(ab)'}{b} \qquad \text{and} \qquad u(y, v(y,s)) = \lambda \frac{K(y)e^v}{\int_\Omega K(y)e^v k\sqrt{\frac{a}{b}}dy}.$$

Next substituting

$$v(y,s) = e^{-\frac{1}{2}\int_0^y h(\xi)d\xi} w(y,s), \tag{6}$$

we have

$$\begin{cases} w_s = w_{yy} - \left(\frac{1}{4}h^2 + \frac{1}{2}h_y\right) w + \frac{k^2}{b}c(y)u & y \in \Omega, s > 0, \\ w|_{\partial\Omega} = 0 & s > 0, \\ w(y,0) = w_0(y) & y \in \Omega, \end{cases} \tag{7}$$

where

$$c(y) = e^{\frac{1}{2}\int_0^y h(\xi)d\xi}, \qquad \text{and} \qquad w_0(y) = c(y)v_0(y),$$

and

$$u(y, w(y,s)) = \lambda \frac{K(y)e^{\frac{w(y,s)}{c(y)}}}{\int_\Omega K(y)e^{\frac{w(y,s)}{c(y)}} k\sqrt{\frac{a}{b}}dy}. \tag{8}$$

We denotes variables of (7) by (x,t) unless there is any confusion:

$$w_t = w_{xx} - \left(\frac{1}{4}h^2 + \frac{1}{2}h_x\right) w + \frac{k^2}{b}c(x)u.$$

Then we consider the local existence of a solution of (7) instead of (1). We rewrite (7) in the following integral equation equivalent to (7):
(9)

$$w = e^{-At}w_0 + \int_0^t e^{-A(t-s)} \left\{ -\left(\frac{1}{4}h^2 + \frac{1}{2}h_x\right) w + \frac{k^2}{b}cu(x, w(x,s)) \right\} ds.$$

Here $Aw = -\Delta w$ and we extend A to be a self-adjoint positive definite operator in $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$. We adopt norm $\|w\|_{H_0^1} = \|\nabla w\|_2$ and $\|\cdot\|_p$ denotes the standard L^p norm. Then the following lemma holds.

Lemma 1 ([8]Theorem 1.4.3). *For any $\alpha \geq 0$, there exists a constant $C_\alpha < +\infty$ such that*

$$\left\|A^\alpha e^{-At}\right\| \leq C_\alpha t^{-\alpha} e^{-\eta t} \qquad \textit{for} \;\; t > 0,$$

where η is a positive constant independent of α.

We solve this integral equation (9) by the contraction mapping theorem. We take $\delta = \|w_0\|_{H_0^1}$. We define

$$V \equiv \left\{ w \in C\left([0,\tau]; H_0^1(\Omega)\right) \;\middle|\; \|w\|_{C([0,\tau];H_0^1)} \leq 3\delta \right\},$$

where τ is a positive constant to be determined later. Here the norm is equipped with

$$\|w\|_{C([0,\tau];H_0^1)} = \sup_{0 \leq t \leq \tau} \|w(\cdot,t)\|_{H_0^1}.$$

For $w \in V$, we define a mapping F as follows:

$$F(w) = e^{-At} w_0 + \int_0^t e^{-A(t-s)} \left(-gw + \frac{k^2}{b} cu(x, w(x,s)) \right) ds,$$

where

$$g \equiv \frac{1}{4} h^2 + \frac{1}{2} h_x.$$

By Sobolev embedding and Poincaré inequality, we have

$$\begin{aligned}
&\|g\|_\infty \|w\|_2 + \left\| \frac{k^2}{b} c \right\|_\infty \|u\|_2 \\
&\leq \; m_1 \|g\|_\infty \|w\|_{H_0^1} + k^2 \left\| \frac{c}{b} \right\|_\infty \frac{\|K\|_\infty e^{\frac{\|w\|_\infty}{\min_{x\in\overline{\Omega}} c(x)}}}{k\,|\Omega| \min_{x\in\overline{\Omega}} \left(K(x) \sqrt{\frac{a(x)}{b(x)}} \right)} \\
&\leq \; 3m_1\delta \|g\|_\infty + k \left\| \frac{c}{b} \right\|_\infty \frac{\|K\|_\infty e^{\frac{m_2\|w\|_{H_0^1}}{\min_{x\in\overline{\Omega}} c(x)}}}{\min_{x\in\overline{\Omega}} \left(K(x) \sqrt{\frac{a(x)}{b(x)}} \right)} \\
&\leq \; 3m_1\delta \|g\|_\infty + k \left\| \frac{c}{b} \right\|_\infty \frac{\|K\|_\infty e^{\frac{3m_2\delta}{\min_{x\in\overline{\Omega}} c(x)}}}{\min_{x\in\overline{\Omega}} \left(K(x) \sqrt{\frac{a(x)}{b(x)}} \right)},
\end{aligned}$$

for $w \in V$ and $0 \leq t \leq \tau$, where m_i are positive constants for $i = 1, 2$. Here $|\Omega|$ denotes the measure of Ω in $\mathbf{R}$. We put

$$M = \sup_{0 \leq t \leq \tau} \left\{ \sup_{w \in V} \left(\|g\|_\infty \|w\|_2 + \left\| \frac{k^2}{b} c \right\|_\infty \|u\|_2 \right) \right\} < \infty.$$

L_2 a denotes positive constant satisfying the following inequality:

$$\begin{aligned}&\left\| -gw_1(\cdot,t) + \frac{k^2}{b}cu_1(\cdot,t) + gw_2(\cdot,t) - \frac{k^2}{b}cu_2(\cdot,t) \right\|_2 \\ &\le L_2 \left\| w_1(\cdot,t) - w_2(\cdot,t) \right\|_{H_0^1}, \quad 0 \le t \le \tau,\end{aligned}$$

for $w_1 \in V$ and $w_2 \in V$, where $u_i(\cdot,t) = u(\cdot, w_i(\cdot,t))$ for $i = 1,2$. We prove the estimate above later (see(10)). We take $\tau > 0$ sufficiently small so it holds that

$$\left\| e^{-At} w_0 \right\|_{H_0^1} \le 2\delta$$

for $0 \le t \le \tau$ and that

$$C_{\frac{1}{2}} \left(M + 2L_2\delta \right) \int_0^\tau t^{-\frac{1}{2}} e^{-\eta t} dt \le \delta,$$

where η and $C_{\frac{1}{2}}$ denote positive constants in Lemma 1. First we prove that F is a mapping from V into V. From Lemma 1, we have

$$\begin{aligned}\|F(w)\|_{H_0^1} &\le \left\| e^{-At} w_0 \right\|_{H_0^1} + \int_0^t \left\| e^{-A(t-s)} \left(gw - \frac{k^2}{b}cu \right) \right\|_{H_0^1} ds \\ &\le \; 2\delta + \int_0^t \left\| A^{\frac{1}{2}} e^{-A(t-s)} \right\| \left\| gw - \frac{k^2}{b}cu \right\|_2 ds \\ &\le \; 2\delta + C_{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\eta(t-s)} \left(\|g\|_\infty \|w\|_2 + \left\| \frac{k^2}{b}c \right\|_\infty \|u\|_2 \right) ds \\ &\le \; 2\delta + \delta = 3\delta,\end{aligned}$$

for $0 \le t \le \tau$. Thus we have

$$\|F(w)\|_{C([0,\tau];H_0^1)} \le 3\delta.$$

Next we show that F is a contraction mapping. We have by Lemma 1

$$\begin{aligned}&\|F(w_1) - F(w_2)\|_{H_0^1} \\ &\le \int_0^t \left\| A^{\frac{1}{2}} e^{-A(t-s)} \right\| \left\| g(w_1 - w_2) - \frac{k^2}{b}c(u_1 - u_2) \right\|_2 ds \\ &\le C_{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\eta(t-s)} \left\| g(w_1 - w_2) - \frac{k^2}{b}c(u_1 - u_2) \right\|_2 ds.\end{aligned}$$

Therefore, we have only to estimate the integrand on the right hand side of the inequality above.

$$\begin{aligned}
&\left\|g\left(w_1-w_2\right)-\frac{k^2}{b}c\left(u_1-u_2\right)\right\|_2 \\
\leq\ & \sup_{x\in\overline{\Omega}}|g(x)|\cdot\|w_1-w_2\|_2+\sup_{x\in\overline{\Omega}}\left|\frac{k^2}{b}c\lambda\right| \\
\times\ & \left(\frac{K\left(e^{\frac{w_1}{c}}-e^{\frac{w_2}{c}}\right)}{\int_\Omega Ke^{\frac{w_1}{c}}k\sqrt{\frac{a}{b}}dx}+\frac{\int_\Omega K\left|e^{\frac{w_1}{c}}-e^{\frac{w_2}{c}}\right|k\sqrt{\frac{a}{b}}dx}{\int_\Omega Ke^{\frac{w_1}{c}}k\sqrt{\frac{a}{b}}dy\int_\Omega Ke^{\frac{w_2}{c}}k\sqrt{\frac{a}{b}}dx}\left\|Ke^{\frac{w_2}{c}}\right\|_2\right)
\end{aligned}$$

for $w_i\in V$, $i=1,2$. By the boundedness of coefficient of (7), the mean value theorem and $H_0^1(\Omega)\subset C(\overline{\Omega})$, we have

$$\left\|-gw_1+\frac{k^2}{b}cu_1+gw_2-\frac{k^2}{b}cu_2\right\|_2 \leq L_1\|w_1-w_2\|_2,$$

for $w_i\in V$,$i=1,2$ with some constant L_1. Since the Poincaré inequality holds,

$$\left\|-gw_1+\frac{k^2}{b}cu_1+gw_2-\frac{k^2}{b}cu_2\right\|_2 \leq L_2\|w_1-w_2\|_{H_0^1}, \tag{10}$$

for $w_i\in V$ with $i=1,2$ for some constant L_2. Hence we have

$$\begin{aligned}
\|F(w_1)-F(w_2)\|_{H_0^1} &\leq C_{\frac{1}{2}}L_2\int_0^t(t-s)^{-\frac{1}{2}}e^{-\eta(t-s)}\|w_1-w_2\|_{H_0^1}ds \\
&\leq C_{\frac{1}{2}}L_2\int_0^t s^{-\frac{1}{2}}e^{-\eta s}ds\,\|w_1-w_2\|_{C([0,\tau];H_0^1)} \\
&\leq \frac{1}{2}\|w_1-w_2\|_{C([0,\tau];H_0^1)}.
\end{aligned}$$

Thus we conclude that

$$\|F(w_1)-F(w_2)\|_{C([0,\tau];H_0^1)}\leq\frac{1}{2}\|w_1-w_2\|_{C([0,\tau];H_0^1)}.$$

Hence we can apply the contraction mapping theorem. Then (7) has a unique time local solution.

Next we show that we can extend this solution globally in time. We utilize the Lyapunov function of (1)

$$J_\lambda(v)=\frac{1}{2}\int_\Omega a(x)v_x^2dx-\lambda\log\left(\int_\Omega K(x)e^v dx\right)$$

to obtain a priori estimates. In fact, since $J_\lambda(v) \le J_\lambda(v_0)$ for $v_0 \in H_0^1(\Omega)$, we have

$$\begin{aligned} \frac{1}{2}\min_{x\in\Omega} a(x)\,\|v\|^2_{H_0^1(\Omega)} &\le \frac{1}{2}\int_\Omega a(x)v_x^2 dx \\ &= J_\lambda(v) + \lambda \log \int_\Omega K(x)e^{v(x)}dx \\ &\le J_\lambda(v_0) + \lambda \log \int_\Omega K(x)e^{|v(x)|}dx, \end{aligned}$$

and moreover

$$\begin{aligned} \log \int_\Omega K(x)e^{v(x)}dx &\le \log \int_\Omega K(x)e^{|v(x)|}dx \\ &\le \log \max_{x\in\overline{\Omega}} K(x) + \max_{x\in\overline{\Omega}} |v(x)| \\ &\le \log \max_{x\in\overline{\Omega}} K(x) + C\,\|v\|_{H_0^1}, \end{aligned}$$

where C is a positive constant. Now that $H_0^1(\Omega) \subset C(\overline{\Omega})$, we have

$$\|v\|_{H_0^1} \le C', \tag{11}$$

where C' is positive and dependent on a,K,λ and v_0. Through the relation (6), $\|w\|_{H_0^1(\Omega)}$ is also uniformly bounded as long as a local solution of (7) exists. This ensures that the unique solution $w = w(x,t)$ of (7) exists for all time $t \ge 0$. Then we denote by $S(t)$ the mapping which gives a solution $w = w(\cdot,t)$ for given $w_0 \in H_0^1(\Omega)$. To show that $S(t)$ defines a dynamical system on $H_0^1(\Omega)$, we apply the continuous dependence theorem. In fact, owing to boundedness of solution of (7), we can utilize estimates similar to those used in proving the existence (for details, see the proof of Theorem 3.4.1 in [8]). Hence, Theorem 1 follows from relation (6). □

§3. The proof of Theorem 2

We consider (7) instead of (1) and put $g = \frac{1}{4}h^2 + \frac{1}{2}h_x$ in (7). $S(t)$ denotes the mapping which gives a solution $w = w(\cdot,t)$ for given $w_0 \in H_0^1(\Omega)$. In this section, we denote $L^2(\Omega)$ by X. We define X^α as the domain of A^α for $\alpha \ge 0$ with graph norm $\|w\|_{X^\alpha} = \|A^\alpha w\|_X$ for $w \in X^\alpha$. Then the inclusion $X^\beta \subset X^\alpha$ is compact for $\beta > \alpha \ge 0$ according to Theorem 1.4.8 in [8]. Let $\varepsilon \in (0, \frac{1}{2})$ be fixed. Then if for $\frac{1}{2} < \beta < \frac{3}{4} - \frac{\varepsilon}{2} <$

1 we prove $\sup_{t>1} \|S(t)w_0\|_{X^\beta} < +\infty$, it holds that $\sup_{t>1} \|T(t)v_0\|_{X^\beta} < +\infty$ through relation (6). Here noting that $X^{\frac{1}{2}} = H_0^1(\Omega)$, then $\gamma(v_0)$ is relatively compact in $H_0^1(\Omega)$. Then the properties of ω-limit set follows from Lemma 2. The rest of Theorem 2 is proven by the existence of Lyapunov function from Lemma 3.

Lemma 2 ([8]Theorem 4.3.3). *Let $\{T(t) \mid t \geq 0\}$ be a dynamical system on a complete metric space C Suppose that $x_0 \in C$ and that $\{T(t) \mid t \geq 0\}$ lies in a compact set in C. Then $\omega(x_0)$ is nonempty, compact, invariant and connected.*

Lemma 3 ([8]Theorem 4.3.4). *Let V be a Lyapunov function on C and let $E = \left\{x \in C \mid \dot{V}(x) = 0\right\}$. We define a maximal invariant subset of E by M. If $\{T(t)x_0 \mid t \geq 0\}$ lies in a compact set in C, then $T(t)x_0 \to M$ as $t \to +\infty$.*

Proof of Theorem 2: Now we show that $\sup_{t>1} \|S(t)w_0\|_{X^\beta} < +\infty$. Hence we have

$$
\begin{aligned}
&\|w\|_{X^\beta} \\
\leq\ & \left\| A^{\frac{1}{2}} A^{\beta-\frac{1}{2}} e^{-tA} w_0 \right\|_X \\
+\ & \int_0^t \left\| A^\beta e^{-A(t-s)} g w - A^{\frac{1}{4}+\frac{\varepsilon}{2}+\beta} e^{-A(t-s)} A^{-\left(\frac{1}{4}+\frac{\varepsilon}{2}\right)} e^{-A(t-s)} \frac{k^2}{b} c u \right\|_X ds \\
\leq\ & C_1 t^{-(\beta-\frac{1}{2})} e^{-C_2 t} \|w_0\|_{H_0^1} + C_3 \int_0^t (t-s)^{-\beta} e^{-C_2(t-s)} \|g w\|_X \, ds \\
+\ & C_3 \int_0^t (t-s)^{-\left(\frac{1}{4}+\frac{\varepsilon}{2}+\beta\right)} e^{-C_2(t-s)} \left\| A^{-\left(\frac{1}{4}+\frac{\varepsilon}{2}\right)} \frac{k^2}{b} c u \right\|_X ds.
\end{aligned}
$$

where C_i is a positive constant for $i = 1, 2, 3$. First by the Poincaré inequality and (11), we have

$$
\|g w\|_2 \leq C_4 \max_{x \in [0,1]} |g| \, \|w\|_{H_0^1} \leq C_5,
$$

where C_4 and C_5 are positive constants. Next by the definition of u, boundedness of a,b,K and c in $\overline{\Omega}$ and the embedding

$$
L^1(\Omega) \subset H^{-s}(\Omega),
$$

for $s > \frac{1}{2}$, we have

$$\begin{aligned}
\left\| A^{-\left(\frac{1}{4}+\frac{\varepsilon}{2}\right)} \frac{k^2}{b} cu \right\|_2 &= \left\| \frac{k^2}{b} cu \right\|_{H^{-2\left(\frac{1}{4}+\frac{\varepsilon}{2}\right)}} \\
&\le C_6 \left\| \frac{k^2}{b} cu \right\|_{L^1} \\
&\le C_6 \max_{x\in[0,1]} \left(\frac{k^2}{b} c \right) \|u\|_{L^1}
\end{aligned}$$

and

$$\begin{aligned}
\|u\|_1 &= \lambda \frac{\left\| K e^{\frac{w}{c}} k \sqrt{\frac{a}{b}} \frac{1}{k} \sqrt{\frac{b}{a}} \right\|_1}{\int_\Omega K(y) e^{\frac{w(x,t)}{c(x)}} k \sqrt{\frac{a}{b}} dx} \\
&\le \frac{\lambda}{k} \max_{x\in[0,1]} \sqrt{\frac{b}{a}} \frac{\left\| K e^{\frac{w}{c}} k \sqrt{\frac{a}{b}} \right\|_1}{\int_\Omega K(y) e^{\frac{w(x,t)}{c(x)}} k \sqrt{\frac{a}{b}} dx} \\
&\le C_7 \lambda,
\end{aligned}$$

where C_6 and C_7 are positive constants. Using these estimates, we obtain

$$\begin{aligned}
\|w\|_\beta &\le C_1 t^{-(\beta-\frac{1}{2})} e^{-C_2 t} \|w_0\|_{H_0^1} + C_3 C_5 \int_0^t (t-s)^{-\beta} e^{-C_2(t-s)} ds \\
&+ C_3 C_6 C_7 \lambda \int_0^t (t-s)^{-\left(\frac{1}{4}+\frac{\varepsilon}{2}+\beta\right)} e^{-C_2(t-s)} ds \\
&\le C_1 t^{-(\beta-\frac{1}{2})} e^{-C_2 t} \|w_0\|_{H_0^1} + C_2^{\beta-1} C_3 C_5 \Gamma(1-\beta) \\
&+ C_2^{-\frac{3}{4}+\frac{\varepsilon}{2}+\beta} C_3 C_6 C_7 \lambda \Gamma\left(\frac{3}{4} - \frac{\varepsilon}{2} - \beta\right),
\end{aligned}$$

where $\Gamma(\alpha)$ is a gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx.$$

$\|w\|_{X^\beta}$ and hence $\|v\|_{X^\beta}$ are bounded for $t > 1$, which implies that $\gamma(v_0)$ is relatively compact in $H_0^1(\Omega)$. Thus, it follows from Lemma 2 that $\omega(w_0)$ is nonempty, compact, invariant and connected. Since there exists a Lyapunov function

$$J_\lambda(v) = \frac{1}{2} \int_\Omega a(x) v_x^2 dx - \lambda \log\left(\int_\Omega K(x) e^v dx \right),$$

we conclude by Lemma 3 that

$$\omega(v_0) \subset \left\{ v \in H_0^1(\Omega) \mid \frac{d}{dt} J_\lambda(v) = 0 \right\}.$$

□

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