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Existence of standing waves for the nonlinear Schrödinger equation with double power nonlinearity and harmonic potential

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Abstract.

In this paper we prove the existence of standing waves for the nonlinear Schrödinger equation with double power nonlinearity and harmonic potential. The nonlinearity of our problem does not satisfy the *global Ambrosetti-Rabinowitz condition.* Therefore, in general, it seems difficult to obtain a boundedness of Palais-Smale sequence for the associated functional. We overcome this by the compactness argument.

§1. Introduction and main theorem

In this paper we consider the existence of a solution to the following semilinear elliptic equation:

$$-\Delta u + (|x|^2 + \omega)u + |u|^{p-1}u - |u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $N \geq 1, 1 < q < p < 2^* - 1$ and $\omega \in \mathbb{R}$. Here, we set $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$. A motivation to study the equation (1) stems from the nonlinear Schrödinger equation:

$$i\partial_t \psi = -\Delta\psi + |x|^2\psi + |\psi|^{p-1}\psi - |\psi|^{q-1}\psi \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{2}$$

The model equation (2) describes the Bose-Einstein condensate with attractive inter-particle interactions under the magnetic trap. Recently many experiments on this phenomenon were done (see [17], [18]). We are interested in standing waves for the equation (2), that is, solutions of the form $\psi(t,x) = e^{i\omega t}u(x)$. It is observed that the function $\psi(t,x)$ of this form satisfies the equation (2) if and only if u is a solution to the equation (1).

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Many authors have studied the problem concerning the existence of standing waves ([1], [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [19]). We recall several known results. We consider the following semilinear elliptic equation:

$$(3) \qquad -\Delta u + V(x)u + f(u) = 0 \quad \text{in } \mathbb{R}^N,$$

where $N \geq 1, f \in C(\mathbb{R}, \mathbb{R})$ and $V \in C(\mathbb{R}^N, \mathbb{R})$. In the case where the potential $V(x) \equiv m > 0$ (constant), that is, in the autonomous case, Berestycki and Lions [4] ($N \geq 3$, $N = 1$) and Berestycki, Gallouët and Kavian [5] ($N = 2$) prove an existence results for a wide class of nonlinearities by the constrained minimization method. From their results, we know that if $f(s) = |s|^{p-1}s - |s|^{q-1}s$ $(1 < q < p < 2^* - 1)$ the equation (3) has a radially symmetric solution in $H^1(\mathbb{R}^N)$ under the assumption $m < m_0$ for some $m_0 > 0$. Berestycki and Lions [4] also show that if $N \geq 3$ the equation (3) does not have a nontrivial solution for $m \geq m_0$ from the Pohozaev identity. Wei and Winter [19] show the uniqueness of the positive radial solution to the equation (3).

In the case where the potential V is not constant, that is, in the nonautonomous case, Berestycki, Lions and Peletier [6] show the existence of a nontrivial solution to the equation (3) for a wide class of nonlinearities including our nonlinearity $f(s) = |s|^{p-1}s - |s|^{q-1}s$ $(1 < q < p < 2^* - 1)$ by the shooting method. However, they require the boundeness of the potential V.

In this paper we use the mountain pass theorem ([2]) to show the existence of standing waves. In order to use the mountain pass theorem, we need the following Palais-Smale condition.

Definition. Let E be a Banach space and assume that $J \in C^1(E, \mathbb{R})$.

(i) We say that a sequence $\{u_n\}$ is a Palais-Smale sequence (PS sequence, for short) associated with the functional J if and only if there exists a constant $M > 0$ such that

$$|J(u_n)| \leq M, \qquad J'(u_n) \to 0 \text{ in } E^* \ (n \to \infty).$$

Here, $J'(\cdot)$ is the Fréchet derivative of $J(\cdot)$ and E^* is the dual of E.

(ii) We say that the functional J satisfies the Palais-Smale condition (PS condition, for short) if and only if any PS sequence has a convergent subsequence.

When we show the existence of a solution to the equation (3), the following condition is often assumed: there exists a constant $\mu > 2$ such that

$$0 < \mu F(s) \leq f(s)s \quad \text{for all } s \in \mathbb{R},$$

where $F(s) = \int_0^s f(t)dt$. This condition is called the *global Ambrosetti-Rabinowitz condition.* It ensures the boundedness of the PS sequence for the functional associated with the equation (3). We explain why this condition is useful. For simplicity, we suppose that the potential $V \equiv m > 0$ (constant). We define $J : H^1 \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} m|u|^2 dx - \int_{\mathbb{R}^N} F(u)dx.$$

The functional J is a C^1 functional on H^1 and u is a solution to the equation (3) if and only if u is a critical point of the functional J. If $\{u_n\}$ is a PS sequence for the functional J, then we have

$$\begin{aligned} M + \|u_n\|_{H^1} &\geq \mu J(u_n) - \langle J'(u_n), u_n\rangle \\ &= (\frac{\mu}{2} - 1)\|u_n\|_{H^1}^2 + \int_{\mathbb{R}^N} (f(u_n)u_n - \mu F(u_n))\, dx. \end{aligned}$$

If the nonlinearity f satisfies the global Ambrosetti-Rabinowitz condition, we have

$$M + \|u_n\|_{H^1} \geq (\frac{\mu}{2} - 1)\|u_n\|_{H^1}^2.$$

This implies that the sequence $\{u_n\}$ is bounded in H^1. When the potential V is not constant, we can also obtain the boundedness of the PS sequence similarly. However, our nonlinearity $f(s) = |s|^{p-1}s - |s|^{q-1}s$ does not satisfy the global Ambrosetti-Rabinowitz condition. Note that if $p < q$, then $f(s)$ satisfies the global Ambrosetti-Rabinowitz condition. It seems difficult to show that the associated functional satisfies the PS condition without the global Ambrosetti-Rabinowitz condition. Recently there are several existence results without the global Ambrosetti-Rabinowitz condition. Jeanjean [8] obtains the existence of a positive solution to the Landesman-Lazer type problem. Jeanjean and Tanaka [10] prove the existence of semiclassical states to the nonlinear elliptic equation with potentials. Zou [22] shows the existence of infinitely many solutions to the equation (3) by the fountain theorem. These results are based on Struwe's method (see e.g. [15], [16]). However, they need the following additional condition on the nonlinearity f: there is $K \geq 1$ such that

(4) $$\hat{F}(s) \leq K\hat{F}(t) \quad \text{for all } 0 \leq s \leq t,$$

where $\hat{F} = \frac{1}{2}f(\xi)\xi - F(\xi)$. Our nonlinearity $f(s) = |s|^{p-1}s - |s|^{q-1}s$ satisfies neither global Ambrosetti-Rabinowitz condition nor the above condition (4). Recently Jeanjean and Tanaka [11] prove the existence of the solution to the equation (3) for a wide class of nonlinearities. However, they require that the nonlinearity f has a superlinear growth at infinity and that there exists a function $\phi \in L^1(\mathbb{R}^N)$ such that $|x \cdot \nabla V(x)| \leq \phi(x)$ for all $x \in \mathbb{R}^N$.

To state our theorem, we give some notation. We define the function space X by

$$X = \left\{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |x|^2 |u|^2 dx < \infty\right\}$$

equipped with the inner product

$$(u,v)_X = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + |x|^2 uv + uv) dx.$$

We use $\|\cdot\|_X$ to denote the norm of the function space X. We denote the dual of X by X^*. Note that X is continuously embedded in $H^1(\mathbb{R}^N)$. Furthermore, the embedding $X \hookrightarrow L^r$ is compact, where $2 \leq r < 2^*$ (see e.g. [3], [21]). we define two constants $m_0 > 0$ and $\theta > 0$ such that

$$m_0 = \sup\{m > 0 \mid \frac{m}{2}s^2 + \frac{1}{p+1}s^{p+1} - \frac{1}{q+1}s^{q+1} < 0 \text{ for some } s > 0\},$$
$$\theta = \pi^{-\frac{N}{2}} \left(\frac{2}{q+1}\right)^{\frac{N}{2}} \left(\frac{p+1}{q+1}\right)^{\frac{N(q-1)}{2(p-1)}}.$$

Our result is the following.

Theorem 1.1. *Assume that $1 < q < p < 2^* - 1$. Let λ_1 be the first eigenvalue of $-\Delta + |x|^2$. If $-\lambda_1 < \omega < \theta m_0 - \lambda_1$, there exists a solution to the equation (1) in X.*

Remark. We can show that the equation (1) has a family of solutions $(u(\varepsilon), \lambda(\varepsilon))$ bifurcating from $(0, \lambda_1)$. Indeed, for some ε_0, the solution can be expressed as

$$u(\varepsilon) = \varepsilon\Phi + \varepsilon z(\varepsilon) \quad \text{for } 0 < \varepsilon < \varepsilon_0,$$

where Φ is an eigenfunction corresponding to λ_1 and $z \in X$ is a continuous function of ε such that $z(0) = 0$ and $(z, \Phi)_X = 0$ (see [7]). However, we do not determine the size of ε_0. Our theorem gives a range of ω for which the equation (1) has a solution.

We prove this theorem by the variational method. We define a C^1 functional I: $X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} (|x|^2 + \omega) u^2 dx + \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} - \frac{1}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} dx.$$

Then we find that u is a critical point of the functional I if and only if u is a solution to the equation (1). We briefly explain the outline of the proof. We use the mountain pass theorem. We first prove that the functional I satisfies the mountain pass geometry in Lemma 2.2. Next, we show that the functional I satisfies the PS condition. Main difficulty is to obtain a boundedness of a PS sequence. In Lemma 2.3, we prove this by the following way. Let $\{u_j\}$ be the PS sequence of the functional I and we suppose that $\|u_j\|_X \to \infty$ as $j \to \infty$ and set $w_j = u_j\|u_j\|_X^{-1}$. Since the sequence $\{w_j\}$ is bounded in X, there exists a subsequence $\{w_j\}$ (we still denote by $\{w_j\}$) and a function $w \in X$ such that $w_j \to w$ weakly in X. We derive a contradiction in both the cases $w = 0$ and $w \neq 0$ in X. Finally, we show that any PS sequence has a convergent subsequence in Lemma 2.4.

§2. Proof of the main theorem

We recall the mountain pass theorem to prove Theorem 1.1.

Theorem 2.1 ([2]). *Let E be a Banach space equipped with the norm $\|\cdot\|$. Suppose that a functional $J \in C^1(E, \mathbb{R})$ satisfies the PS condition and*

(i) $J(0) = 0$,
(ii) *there exist constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ and $\|u\| = \rho$,*
(iii) *there exists a function $e \in E$ such that $J(e) \leq 0$ and $\|e\| > \rho$.*

Then J possesses a critical value $c \geq \alpha$. Moreover the critical value c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ \gamma(1) = e\}$.

We first show that the functional I satisfies the mountain pass geometry, that is, we check that the functional I satisfies assumptions (i), (ii) and (iii) of Theorem 2.1. We find that $I(0) = 0$. Next lemma shows that the functional I satisfies the assumptions (ii) and (iii).

Lemma 2.2. *Assume that* $1 < q < p < 2^* - 1$ *and* $-\lambda_1 < \omega < \theta m_0 - \lambda_1$. *Then we find that*

(1) *there exist constants* $\rho, \alpha > 0$ *such that* $I(u) \geq \alpha$ *for all* $u \in X$ *and* $\|u\|_X = \rho$.

(2) *there exists a function* $v \in X \setminus \{0\}$ *such that* $I(v) \leq 0$,

Proof. (1) Let $e_1 = \pi^{-\frac{N}{4}} \exp(-\frac{|x|^2}{2})$ be the eigenfunction corresponding to λ_1 for $-\Delta + |x|^2$ with $\|e_1\|_2 = 1$. For $h > 0$, we have

$$\begin{aligned} I(he_1) &= \frac{h^2}{2}\{\int_{\mathbb{R}^N} \left(|\nabla e_1|^2 + |x|^2 e_1^2 - \lambda_1 e_1^2\right) dx\} \\ &\quad + \frac{(\omega + \lambda_1)h^2}{2} \int_{\mathbb{R}^N} e_1^2 dx + \frac{h^{p+1}}{p+1} \int_{\mathbb{R}^N} e_1^{p+1} dx - \frac{h^{q+1}}{q+1} \int_{\mathbb{R}^N} e_1^{q+1} dx \\ &= \frac{(\omega + \lambda_1)h^2}{2} \int_{\mathbb{R}^N} e_1^2 dx + \frac{h^{p+1}}{p+1} \int_{\mathbb{R}^N} e_1^{p+1} dx - \frac{h^{q+1}}{q+1} \int_{\mathbb{R}^N} e_1^{q+1} dx. \end{aligned}$$

We put $L_1 = \|e_1\|_{p+1}^{p+1}$ and $L_2 = \|e_1\|_{q+1}^{q+1}$. Then we have

$$I(he_1) = \frac{(\omega + \lambda_1)h^2}{2} + \frac{L_1 h^{p+1}}{p+1} - \frac{L_2 h^{q+1}}{q+1}.$$

A simple calculation yields that for $\omega + \lambda_1 < L_1^{\frac{p-1}{p-q}} L_2^{\frac{q-1}{p-q}} m_0$ there exists a positive number h_1 such that

$$\frac{(\omega + \lambda_1)h_1^2}{2} + \frac{L_1 h_1^{p+1}}{p+1} - \frac{L_2 h_1^{q+1}}{q+1} < 0.$$

Therefore, if we set $\theta = L_1^{\frac{p-1}{p-q}} L_2^{\frac{q-1}{p-q}}$ and $e = h_1 e_1$ then we deduce that $I(e) < 0$ for all $\omega < \theta m_0 - \lambda_1$.

(2) Since $\omega > -\lambda_1$, there exists $a \in (0, 1)$ satisfying $\omega > -a\lambda_1$. Then we obtain

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (|x|^2 + \omega) u^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx \\ &\quad - \frac{1}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} dx \\ &\geq \frac{1-a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1-a}{2} \int_{\mathbb{R}^N} |x|^2 u^2 dx + \frac{a\lambda_1 + \omega}{2} \int_{\mathbb{R}^N} u^2 dx \\ &\quad + \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx - \frac{1}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} dx. \end{aligned}$$

Since $a\lambda_1 + \omega > 0$, we have

$$I(u) \geq c\|u\|_X^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \geq c\|u\|_X^2 - c\|u\|_X^{q+1}.$$

If $\|u\| = \rho$ is sufficiently small, then there exists a constant $\alpha > 0$ such that $I(u) \geq \alpha$. Q.E.D.

Next we check that the functional I satisfies the PS condition. We show this by two steps. First, we obtain the boundedness of the PS sequence. Then we find that there exists a sequence $\{u_j\}$ (we still denote by $\{u_j\}$) and a function $u \in X$ such that $u_j \to u$ weakly in X. Second, we show that $u_j \to u$ strongly in X.

Lemma 2.3. *Every PS sequence of the functional I is bounded in X.*

Proof. We shall prove this lemma by contradiction. Let $\{u_j\}$ be a PS sequence of the functional I and suppose that $\|u_j\|_X \to \infty$ as $j \to \infty$. We set $w_j = u_j\|u_j\|_X^{-1}$. There exist subsequence $\{w_j\}$ (we still denote by $\{w_j\}$) and a function w such that $w_j \to w$ weakly in X. Since X is compactly embedded in L^r for $2 \leq r < 2^*$, we deduce that $w_j \to w$ strongly in L^r for $2 \leq r < 2^*$ as $j \to \infty$. Furthermore, we find that $w_j(x) \to w(x)$ for a.a. $x \in \mathbb{R}^N$ as $j \to \infty$. We derive a contradiction in both the cases $w \neq 0$ and $w = 0$ in X.

First, we consider the case $w \neq 0$ in X. We define the subspace $\Omega \subset \mathbb{R}^N$ by $\Omega = \{x \in \mathbb{R}^N | w(x) \neq 0\}$. Since $w \neq 0$ in X, we deduce that $\Omega \neq \emptyset$ and $|u_j(x)| \to \infty$ as $j \to \infty$ for $x \in \Omega$. We have

$$\begin{aligned}(p+1)M + \|u_j\|_X &\geq |(p+1)I(u_j) - \langle I'(u_j), u_j\rangle| \\ &\geq \frac{p-q}{q+1}\int_{\mathbb{R}^N}|u_j|^{q+1}dx - \frac{p-1}{2}c\|u_j\|_X^2.\end{aligned}$$

Dividing the above inequality by $\|u_j\|_X^2$ yields that

$$\begin{aligned}\frac{(p+1)M}{\|u_j\|_X^2} + \frac{1}{\|u_j\|_X} &\geq \frac{p-q}{q+1}\int_{\mathbb{R}^N}|u_j|^{(q-1)}|w_j|^2dx - \frac{p-1}{2}c \\ &\geq \frac{p-q}{q+1}\int_{\Omega}|u_j|^{(q-1)}|w_j|^2dx - \frac{p-1}{2}c.\end{aligned}$$

By Fatou's lemma, we deduce that $\liminf_{j\to\infty}\int_\Omega|u_j|^{(q-1)}|w_j|^2dx = \infty$. However, we find that $(p+1)M\|u_j\|_X^{-2} + \|u_j\|_X^{-1} \to 0$. This contradicts the above inequality.

Second, we consider the case $w = 0$ in X. Therefore, we find that $w_j \to 0$ in L^r if $2 \le r < 2^*$. By the Hölder inequality, we have

$$\|u_j\|_{q+1} \le \|u_j\|_{p+1}^{\theta} \|u_j\|_2^{(1-\theta)},$$

where $\theta = (p+1)(q-1)(p-1)^{-1}(q+1)^{-1}$. Dividing the above inequality by $\|u_j\|_X^2$ yields that

$$\frac{\|u_j\|_{q+1}^{q+1}}{\|u_j\|_X^2} \le \|w_j\|_2^{2\frac{p-q}{p-1}} \left(\frac{\|u_j\|_{p+1}^{p+1}}{\|u_j\|_X^2} \right)^{\frac{q-1}{p-1}}.$$

Furthermore, we find that

$$\begin{aligned}(q+1)M + \|u_j\|_X &\ge |(q+1)I(u_j) - \langle I'(u_j), u_j\rangle| \\ &\ge \frac{p-q}{p+1}\|u_j\|_{p+1}^{p+1} - \frac{q-1}{2}c\|u_j\|_X^2.\end{aligned}$$

It follows that there exists a positive constant c such that $\|u_j\|_{p+1}^{p+1}\|u_j\|_X^{-2} \le c$ for sufficiently large j. Since $\|w_j\|_2^2 \to 0$ and $\|u_j\|_{p+1}^{p+1}\|u_j\|_X^{-2} \le c$, we deduce that $\|u_j\|_{q+1}^{q+1}\|u_j\|_X^{-2} \to 0$ as $j \to \infty$. On the other hand, we have

$$\begin{aligned}(p+1)M + \|u_j\|_X &\ge |(p+1)I(u_j) - \langle I'(u_j), u_j\rangle| \\ &\ge \frac{p-1}{2}c\|u_j\|_X^2 - \frac{p-q}{q+1}\|u_j\|_{q+1}^{q+1}.\end{aligned}$$

Dividing the above inequality by $\|u_j\|_X^2$ yields that

$$\frac{(p+1)M}{\|u_j\|_X^2} + \frac{1}{\|u_j\|_X} \ge \frac{p-1}{2}c - \frac{(p-q)\|u_j\|_{q+1}^{q+1}}{(q+1)\|u_j\|_X^2}.$$

Since $\|u_j\|_{q+1}^{q+1}\|u_j\|_X^{-2} \to 0$ as $j \to \infty$, we deduce that for any $\varepsilon > 0$, there exists sufficiently large j such that $\varepsilon > \frac{p-1}{2}c$. This is a contradiction. Q.E.D.

Lemma 2.4. *Assume that $1 < q < p < 2^* - 1$. The functional I satisfies the PS condition.*

Proof. Let $\{u_j\}$ be the PS sequence of the functional I. From Lemma 2.3, we deduce that $\{u_j\}$ is bounded in X. Then there exists a subsequence $\{u_j\}$ (we still denote by $\{u_j\}$) and function $u \in X$ such that $u_j \to u$ weakly in X. Since $X \hookrightarrow L^r$ is compact in L^r for $2 \le r < 2^*$, we deduce that $u_j \to u$ strongly in L^r for $2 \le r < 2^*$. Since $I'(u_j) \to 0$

in X^* as $j \to \infty$, we have $\langle I'(u_j), h\rangle \le \varepsilon_j \|h\|_X$ for all $h \in X$, where $\varepsilon_j = \|I'(u_j)\|$. We note that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$(5) \quad \left| \int_{\mathbb{R}^N} \left(\nabla u_j \cdot \nabla h + (|x|^2 + \omega) u_j h + |u_j|^{p-1} u_j h - |u_j|^{q-1} u_j h \right) dx \right| \le \varepsilon_j \|h\|_X.$$

If we put $h = u_j$ in (5) and let $j \to \infty$, then we have

$$(6) \quad \begin{aligned} &\lim_{j\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_j|^2 + (|x|^2 + \omega) u_j^2 \right) dx \\ &= \lim_{j\to\infty} \left(- \int_{\mathbb{R}^N} |u_j|^{p+1} dx + \int_{\mathbb{R}^N} |u_j|^{q+1} dx \right) \\ &= - \int_{\mathbb{R}^N} |u|^{p+1} dx + \int_{\mathbb{R}^N} |u|^{q+1} dx. \end{aligned}$$

If we put $h = u$ in (5), then we obtain

$$\left| \int_{\mathbb{R}^N} \left(\nabla u_j \cdot \nabla u + (|x|^2 + \omega) u_j u + |u_j|^{p-1} u_j u - |u_j|^{q-1} u_j u \right) dx \right| \le \varepsilon_j \|u\|_X.$$

Since $u_j \to u$ weakly in X and the embedding $L^{\frac{p+1}{p}}, L^{\frac{q+1}{q}} \hookrightarrow X^*$ is compact, letting $j \to \infty$, we have

$$(7) \quad \begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |x|^2 u^2 dx + \omega \int_{\mathbb{R}^N} u^2 dx \\ &= - \int_{\mathbb{R}^N} |u|^{p+1} dx + \int_{\mathbb{R}^N} |u|^{q+1} dx. \end{aligned}$$

From (6) and (7), we have $\lim_{j\to\infty} \|u_j\|_X^2 = \|u\|_X^2$. Therefore, we deduce that

$$\begin{aligned} \lim_{j\to\infty} \|u_j - u\|_X^2 &= \lim_{j\to\infty} (u_j - u, u_j - u)_X \\ &= \|u\|_X^2 + \lim_{j\to\infty} \|u_j\|_X^2 - 2 \lim_{j\to\infty} (u_j, u)_X \\ &= \|u\|_X^2 + \|u\|_X^2 - 2\|u\|_X^2 = 0. \end{aligned}$$

This completes the proof. Q.E.D.

From Lemmas 2.2 and 2.4, we can use the mountain pass theorem and deduce that I possesses a critical value c which is characterized as $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$.

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