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On the bifurcation structure of positive stationary solutions for a competition-diffusion system

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Abstract.

In this survey, we consider a generalized Lotka-Volterra competition model with diffusion, and discuss the bifurcation structure of positive stationary solutions for the model. To do this, the comparison principle, the bifurcation theory, and the numerical verification are employed.

§1. Introduction

To understand the mechanism of phenomena which appear in various fields, we often use the system of reaction-diffusion equations

$$\begin{cases} \mathbf{u}_t = \varepsilon\, D\, \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}), & x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial \nu}\mathbf{u} = \mathbf{0}, \quad x \in \partial\Omega, \quad t > 0 \end{cases} \tag{1.1}$$

with suitable initial condition, and discuss the existence and stability of stationary solutions for the system, where $\mathbf{u} \in \mathbf{R}^N$, $\varepsilon > 0$, D is a diagonal matrix whose elements are positive, $\mathbf{f} : \mathbf{R}^N \to \mathbf{R}^N$ is a smooth function in $\mathbf{u}$, Ω is a bounded domain in $\mathbf{R}^\ell$ with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$.

When $N = 1$, we employ the so-called *comparison principle*, and study the existence of stationary solutions of (1.1) and their stability property. Furthermore it is well-known that for suitable $f(u)$, the global attractor $\mathcal{A}$ of (1.1) can be represented as $\mathcal{A} = \bigcup_{e \in E} W^u(e)$, where E is the set of stationary solutions of (1.1), and $W^u(e)$ is an unstable manifold of (1.1) at $u = e$ (for example, see Chapter 4 in Hale [2]). This fact suggests that one important problem is to seek all stationary solutions of (1.1).

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In general, the comparison principle does not always hold for the case $N \geq 2$. This fact leads to the considerable complexity for the study of the existence and stability of stationary solutions for (1.1). As a first step to approach the problem, we treat a competition-diffusion system which describes the dynamics of the population for two competing species $\mathbf{u} = (u, v) \in \mathbf{R}^2$, where $d_u > 0$, $d_v > 0$, $D = \mathrm{diag}(d_u, d_v)$,

$$\mathbf{f}(\mathbf{u}) = (f, g)(\mathbf{u}), \quad f(\mathbf{u}) = f^0(\mathbf{u})\, u, \quad g(\mathbf{u}) = g^0(\mathbf{u})\, v,$$

and $\mathbf{f}^0(\mathbf{u}) = (f^0, g^0)(\mathbf{u})$ is a smooth function in $\mathbf{u}$. Moreover we set $\Omega = \{\, x \in \mathbf{R}^\ell \mid |x| < \pi \,\}$, and we restrict our discussion to radially symmetric positive stationary solutions of (1.1), where we denote by $\mathrm{Cl}\, A$ the closure of the set A, and we call $\mathbf{u}(x) = (u, v)(x)$ *positive* when $u(x) > 0$ and $v(x) > 0$ are satisfied for any $x \in \mathrm{Cl}\,\Omega$. At this point, we should note that such stationary solutions satisfy

$$(1.2) \qquad \begin{cases} \mathbf{0} = \varepsilon\, D\, r^{1-\ell} \left[r^{\ell-1}\, \mathbf{u}' \right]' + \mathbf{f}(\mathbf{u}), & r \in (0, \pi), \\ \mathbf{u}' = \mathbf{0}, \quad r = 0, \pi \end{cases}$$

for suitably fixed real number $\ell \in [1, +\infty)$, where $r = |x|$ and $' = \frac{d}{dr}$. There are many and various theorems on the existence of positive solutions for (1.2). Recently in case of $\ell = 1$, the author in the papers [6], [7] and [8] has established the global bifurcation structure of positive solutions for (1.2) with

$$(1.3) \qquad f^0(\mathbf{u}) = 1 - u^n - c\, v^n, \quad g^0(\mathbf{u}) = 1 - b\, u^n - v^n$$

relative to $\varepsilon > 0$, where the positive constants d_u, d_v, n, b and c are suitably fixed. The aim of this paper is to survey the result in the papers [6], [7] and [8] which correspond to the case $\ell = 1$, and to give a characterization on the set of positive solutions for (1.2) in case of $\ell > 1$.

§2. Assumptions

From the competitive interaction, we assume that

(A.1) there exists $M > 0$ such that $f(\mathbf{u}) < 0$ and $g(\mathbf{u}) < 0$ hold for any $\mathbf{u} \in \mathrm{Cl}\, \mathbf{R}^2_+$ with $|\,\mathbf{u}\,| \geq M$,

(A.2) $f^0_u(\mathbf{u}) < 0$, $f^0_v(\mathbf{u}) < 0$, $g^0_u(\mathbf{u}) < 0$ and $g^0_v(\mathbf{u}) < 0$ are satisfied for any $\mathbf{u} \in \mathrm{Cl}\, \mathbf{R}^2_+$, and

(A.3) there exists a solution $\hat{\mathbf{u}} = (\hat{u}, \hat{v}) \in \mathbf{R}^2_+$ of $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ with $\det \mathbf{f}_{\mathbf{u}}(\hat{\mathbf{u}}) < 0$,

where $\mathbf{R}_+ = (0, +\infty)$. We should remark that (A.1) means the boundedness of positive solutions for (1.2), and (A.2) implies that the comparison principle holds for (1.1) relative to the order relation $\preceq$ which is defined in the following manner:

$$(u_1, v_1) \preceq (u_2, v_2) \iff u_1 \le u_2,\ v_1 \ge v_2.$$

It is obvious that (1.3) is a typical example satisfying (A.1–3), when $n > 0$ and $\min(b, c) > 1$ are satisfied.

We define $\hat{E}$ by the set of solutions of $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ with $\mathbf{u} \in \mathrm{Cl}\,\mathbf{R}^2_+$. For the sake of simplicity, we assume that

(A.4) $\det \mathbf{f}_{\mathbf{u}}(\mathbf{e}) \ne 0$ is satisfied for any $\mathbf{e} \in \hat{E}$,

which implies that every $\mathbf{e} \in \hat{E}$ is a nondegenerate solution of $\mathbf{f}(\mathbf{u}) = \mathbf{0}$.

§3. Sets of Positive Solutions

We set

$$X = \{ \mathbf{u}(.) \in C^2([0, \pi]) \mid \mathbf{u}'(0) = \mathbf{0} = \mathbf{u}'(\pi) \}.$$

For each $\ell \in [1, +\infty)$, we denote by $E(\ell)$ the set of $(\varepsilon, \mathbf{u}(.)) \in \mathbf{R}_+ \times X$ such that $\mathbf{u}(r)$ is a positive solution of (1.2) for ε, and by $E_k^-(\ell)$ (respectively, $E_k^+(\ell)$) ($k \in \mathbf{N}$) the set of $(\varepsilon, \mathbf{u}(.)) \in E(\ell)$ such that there exists a strictly increasing sequence $\{ r_j \}_{j=0}^k$ ($\subset [0, \pi]$) such that $(-1)^j\, \mathbf{u}'(x) \succ \mathbf{0}$ (respectively, $(-1)^{j+1}\, \mathbf{u}'(x) \succ \mathbf{0}$) holds on (r_j, r_{j+1}) for any integer $0 \le j < k$, where $r_0 = 0$, $r_k = \pi$, and the relation $\prec$ is obtained from the order relation $\preceq$ by replacing $\le$ with $<$. Setting

$$E_0(\ell) = \mathbf{R}_+ \times \{ \hat{\mathbf{u}} \}, \quad E_k(\ell) = E_k^-(\ell) \cup E_k^+(\ell),$$

we clearly have $\bigcup_{k \ge 0} E_k(\ell) \subset E(\ell)$ for any $\ell \in [1, +\infty)$.

Lemma 1. *Let $\ell \in [1, +\infty)$, $k \in \mathbf{N}$, and $(\varepsilon_i, \mathbf{u}_i(.)) \in E_k(\ell)$ ($i = 1$, 2) be arbitrary. Suppose that $[\mathbf{u}_1(0)]_1 = [\mathbf{u}_2(0)]_1$ and/or $[\mathbf{u}_1(0)]_2 = [\mathbf{u}_2(0)]_2$ is satisfied, where $[\mathbf{u}]_j$ is the jth element of the vector $\mathbf{u}$. Then $\varepsilon_1 = \varepsilon_2$ and $\mathbf{u}_1(.) = \mathbf{u}_2(.)$ hold.*

Proof. Let $\ell \in [1, +\infty)$, $k \in \mathbf{N}$, and $(\varepsilon_i, \mathbf{u}_i(.)) \in E_k(\ell)$ ($i = 1$, 2) be arbitrary. Since the argument below is still valid for the case $[\mathbf{u}_1(0)]_2 = [\mathbf{u}_2(0)]_2$ by the change of the role between u and v, we only consider the case $[\mathbf{u}_1(0)]_1 = [\mathbf{u}_2(0)]_1$. For each i, setting

$$\mathbf{w}_i(\xi)\ (= (w_i, z_i)(\xi)) = \mathbf{u}_i(\sqrt{\varepsilon_i}\,\xi), \quad \Xi_i = \frac{\pi}{\sqrt{\varepsilon_i}},$$

we see that $\mathbf{w}_i(\xi)$ is a positive solution of

$$
\text{(3.1)} \qquad \begin{cases} \mathbf{0} = D\,\xi^{1-\ell}\,\frac{d}{d\xi}\left[\xi^{\ell-1}\,\frac{d}{d\xi}\mathbf{w}_i\right] + \mathbf{f}(\mathbf{w}_i), & \xi \in (0,\Xi_i), \\ \frac{d}{d\xi}\mathbf{w}_i(0) = \mathbf{0} = \frac{d}{d\xi}\mathbf{w}_i(\Xi_i), \end{cases}
$$

and satisfies $\frac{d}{d\xi}w_i(\xi)\,\frac{d}{d\xi}z_i(\xi) \le 0$ for any $\xi \in [0,\Xi_i]$. Moreover we have $a^0_{ij}(\xi) < 0$ for any i, j and ξ by virtue of (A.2), where

$$
\begin{pmatrix} a^0_{11} & a^0_{12} \\ a^0_{21} & a^0_{22} \end{pmatrix}(\xi) = \int_0^1 \mathbf{f}^0_{\mathbf{u}}(\theta\,\mathbf{w}_1(\xi) + (1-\theta)\,\mathbf{w}_2(\xi))\,d\theta.
$$

Suppose that $z_1(0) > z_2(0)$ holds. From

$$
d_u\,\ell\,\frac{d^2}{d\xi^2}\left[w_1 - w_2\right](0) = -a^0_{12}(0)\,w_1(0) > 0,
$$

it follows that there exists $\xi_1 \in (0,\hat{\Xi}]$ such that

(i) $w_1(\xi) > w_2(\xi)$ and $z_1(\xi) > z_2(\xi)$ hold for any $\xi \in (0,\xi_1)$, and

(ii) $w_1(\xi_1) = w_2(\xi_1)$ and/or $z_1(\xi_1) = z_2(\xi_1)$ is satisfied for the case $\xi_1 < \hat{\Xi}$,

where $\hat{\Xi} = \min(\Xi_1,\Xi_2)$. Setting

$$
\begin{aligned}
W(\xi) &= \tfrac{d}{d\xi}w_1(\xi)\,w_2(\xi) - w_1(\xi)\,\tfrac{d}{d\xi}w_2(\xi), \\
Z(\xi) &= \tfrac{d}{d\xi}z_1(\xi)\,z_2(\xi) - z_1(\xi)\,\tfrac{d}{d\xi}z_2(\xi),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{d_u\,\xi^{1-\ell}\,\frac{d}{d\xi}\left[\xi^{\ell-1}\,W(\xi)\right]}{w_1(\xi)\,w_2(\xi)} \\
&= -\,a^0_{11}(\xi)\,(w_1(\xi) - w_2(\xi)) - a^0_{12}(\xi)\,(z_1(\xi) - z_2(\xi)) > 0, \\
&\frac{d_v\,\xi^{1-\ell}\,\frac{d}{d\xi}\left[\xi^{\ell-1}\,Z(\xi)\right]}{z_1(\xi)\,z_2(\xi)} \\
&= -\,a^0_{21}(\xi)\,(w_1(\xi) - w_2(\xi)) - a^0_{22}(\xi)\,(z_1(\xi) - z_2(\xi)) > 0
\end{aligned}
$$

for any $\xi \in (0,\xi_1)$ due to (3.1). From $W(0) = 0$ and $Z(0) = 0$, we obtain $W(\xi_1) > 0$ and $Z(\xi_1) > 0$, which implies $\xi_1 = \hat{\Xi}$. If $\Xi_j \le \Xi_{3-j}$ holds, then we have

$$
0 < W(\Xi_j)\,Z(\Xi_j) = w_j(\Xi_j)\,z_j(\Xi_j)\,\tfrac{d}{d\xi}w_{3-j}(\Xi_j)\,\tfrac{d}{d\xi}z_{3-j}(\Xi_j) \le 0.
$$

This contradiction implies that $z_1(0) \le z_2(0)$ must be satisfied.

Since we can similarly derive a contradiction for the case $z_1(0) < z_2(0)$, we arrive at $z_1(0) = z_2(0)$. By the uniqueness of solutions for (3.1), we have $\mathbf{w}_1(\xi) = \mathbf{w}_2(\xi)$ for any ξ. By the definition of $E_k(\ell)$, we obtain $\varepsilon_1 = \varepsilon_2$. Q.E.D.

For each $k \in \mathbf{N}$ and $(\varepsilon, \mathbf{u}(.)) \in E_k(1)$, we can regard $\mathbf{u}(r)$ as a periodic function with period 2π satisfying $\mathbf{u}(r) = \mathbf{u}(-r)$ for any $r \geq 0$. Furthermore we see that $(k^2\varepsilon, \mathbf{u}(./k)) \in E_1(1)$ is equivalent to $(\varepsilon, \mathbf{u}(.)) \in E_k(1)$ for each $k \in \mathbf{N}$, because we can take r_j as satisfying $r_j = \pi j/k$ for any integer $0 \leq j \leq k$ due to the uniqueness of solutions for (1.2).

Lemma 2 (Section 2.1 in [5]). *$E(1) = \bigcup_{k\geq 0} E_k(1)$ holds.*

The above lemma says that we can understand the complete structure of $E(1)$ by using the information on the structure of $E_1(1)$. Unfortunately it is unknown whether Lemma 2 is valid for $\ell \in (1, +\infty)$.

Setting

$$\mathcal{P}_k(\ell) = \{\, [\mathbf{u}(0)]_1 \mid (\varepsilon, \mathbf{u}(.)) \in E_k(\ell) \,\},$$

we see from Lemma 1 that for each $\ell \in [1, +\infty)$ and $k \in \mathbf{N}$, there exist functions $\hat{\varepsilon}_k(p, \ell)$ and $\hat{\mathbf{u}}_k(., p, \ell)$ defined on $\mathcal{P}_k(\ell)$ such that (i) $[\hat{\mathbf{u}}_k(0, p, \ell)]_1 = p$ is satisfied for any $p \in \mathcal{P}_k(\ell)$, and (ii) $E_k(\ell)$ is represented as

$$E_k(\ell) = \{\, (\hat{\varepsilon}_k(p, \ell), \hat{\mathbf{u}}_k(., p, \ell)) \mid p \in \mathcal{P}_k(\ell) \,\}.$$

Hence it follows that $E_k(\ell)$ can be parameterized by the value of $[\mathbf{u}(0)]_1$, and that the secondary bifurcation from the positive solution on $E_k(\ell)$ is of saddle-node type even if it exists.

§4. Structure of $E_1(\ell)$

As $\mathbf{f}^0(\mathbf{u})$ is represented as

$$\begin{aligned} f^0(\mathbf{u}) &= f^0_{0,0} + f^0_{n_1,0}\, u^{n_1} + f^0_{0,n_2}\, v^{n_2} + \text{ the remainder term,} \\ g^0(\mathbf{u}) &= g^0_{0,0} + g^0_{n_3,0}\, u^{n_3} + g^0_{0,n_4}\, v^{n_4} + \text{ the remainder term} \end{aligned}$$

with suitable constants $f^0_{i,j}$, $g^0_{i,j}$ and n_j, we treat the simplest nonlinearity (1.3) in this section, in order to discuss the global bifurcation structure of positive solutions for (1.2), where n, b and c are positive constants. At this point, we should note that (1.2) has constant solutions $(0,0)$, $(0,1)$, $(1,0)$, and

$$\hat{\mathbf{u}} = \left(\left(\frac{1-c}{1-b\,c} \right)^{\frac{1}{n}}, \left(\frac{1-b}{1-b\,c} \right)^{\frac{1}{n}} \right)$$

which is positive for either $\max(b,c) < 1$ or $\min(b,c) > 1$. By the maximum principle, we can prove that (1.2) has no positive nonconstant solutions for the case $\min(b,c) < 1$. Hereafter we shall discuss the bifurcation structure of positive solutions of (1.2) for the case $\min(b,c) > 1$.

We set $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$,

$$\mathcal{X} = \{ u(.) \in C^2[0,\pi] \mid u'(0) = 0 = u'(\pi) \}, \quad \mathcal{Y} = C^0[0,\pi],$$

and we define the linear operator $\mathcal{K}(.;\ell) : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{K}(u;\ell) = -r^{1-\ell} \left[r^{\ell-1} u' \right]'$$

for $\ell \in [1,+\infty)$. Let $\{ \lambda_k(\ell) \}_{k \in \mathbf{N}_0}$ be eigenvalues of $\mathcal{K}(.;\ell)$ satisfying $\lambda_k(\ell) \leq \lambda_{k+1}(\ell)$ for any $k \in \mathbf{N}_0$, and let $\phi_k(r,\ell)$ $(k \in \mathbf{N}_0)$ be an eigenfunction of $\mathcal{K}(.;\ell)$ corresponding to the eigenvalue $\lambda_k(\ell)$. Without loss of generality, we may assume $\phi_k(0,\ell) > 0$ for each $k \in \mathbf{N}_0$. It is well-known that the following property holds for each $\ell \in [1,+\infty)$:

(i) $\lambda_0(\ell) = 0$, $\lambda_k(\ell) > 0$ for any $k \in \mathbf{N}$, and $\lim_{k\to\infty} \lambda_k(\ell) = +\infty$ are satisfied,

(ii) $\phi_k(r,\ell)$ has k zeros on $(0,\pi)$ for any $k \in \mathbf{N}_0$,

(iii) $\{ \phi_k(r,\ell) \}_{k \in \mathbf{N}_0}$ is a complete orthonormal set in $L^2(0,\pi)$ relative to the weight $r^{\ell-1}$, and

(iv) $\phi_k(r,\ell)$ is represented as

$$\phi_k(r,\ell) = \begin{cases} C \cos(k\,r) & \text{for } \ell = 1, \\ C\, r^{\frac{2-\ell}{2}}\, J_{\frac{\ell-2}{2}}\left(\sqrt{\lambda_k(\ell)}\, r\right) & \text{for } \ell > 1 \end{cases}$$

with suitable constant C, where $J_\nu(z)$ is the Bessel function of the first kind.

Setting

$$\Phi_k(\ell) = \int_0^\pi \phi_1(r,\ell)^k\, r^{\ell-1}\, dr,$$

we have $\Phi_2(\ell) > 0$ for any $\ell \in [1,+\infty)$, and $\Phi_3(1) = 0$. It is known that $\Phi_3(\ell) > 0$ is satisfied for any $\ell > 1$ (for example, we refer to [9]). By $\lambda_1(\ell) > 0$ and $\det \mathbf{f}_\mathbf{u}(\hat{\mathbf{u}}) < 0$, we obtain

$$\mathcal{D}(\ell) \equiv \left\{ \mathbf{d} = (d_u, d_v) \in \mathbf{R}^2_+ \mid \det\left(-\lambda_1(\ell)\, D + \mathbf{f}_\mathbf{u}(\hat{\mathbf{u}})\right) = 0 \right\} \neq \emptyset$$

for each $\ell \geq 1$.

Let $\ell \geq 1$ and $\mathbf{d} \in \mathcal{D}(\ell)$ be arbitrary, and let $\mathbf{v}$ (respectively, $\mathbf{v}^*$) be a nontrivial solution of

$$\left(-\lambda_1(\ell)\, D + \mathbf{f}_\mathbf{u}(\hat{\mathbf{u}})\right) \mathbf{v} = \mathbf{0}$$
$$\left(\text{respectively,} \quad \left(-\lambda_1(\ell)\, D + \mathbf{f}_\mathbf{u}(\hat{\mathbf{u}})^T\right) \mathbf{v}^* = \mathbf{0}\right),$$

where A^T is the transposed matrix of the matrix A. After simple calculations, we can check that the linearized operator of (1.2) around $\mathbf{u} = \hat{\mathbf{u}}$ has the only one eigenvalue (respectively, at least two eigenvalues) in the right half-plane for any ε with $\varepsilon > 1$ (respectively, $0 < \varepsilon < 1$), and that the linearized operator $\mathcal{L}$ of (1.2) around $\mathbf{u} = \hat{\mathbf{u}}$ for $\varepsilon = 1$ has the simple eigenvalue 0 with the corresponding eigenfunction $\phi_1(r, \ell)\, \mathbf{v}$. Moreover we see that $\phi_1(r, \ell)\, \mathbf{v}^*$ is an eigenfunction of the adjoint operator of $\mathcal{L}$ corresponding to the eigenvalue 0.

Substituting

$$\begin{aligned} \varepsilon = \tilde{\varepsilon}(\ell, \nu) &= 1 + \nu\, \tilde{\varepsilon}_1(\ell) + \nu^2\, \tilde{\varepsilon}_2(\ell) + \nu^3\, \tilde{\varepsilon}_3(\ell, \nu), \\ \mathbf{u} = \tilde{\mathbf{u}}(r, \ell, \nu) &= \hat{\mathbf{u}} + \nu\, \phi_1(r, \ell)\, \mathbf{v} + \nu^2\, \tilde{\mathbf{u}}_2(r, \ell, \nu) \end{aligned}$$

into (1.2), we have

$$\begin{aligned} 0 = &\frac{1}{\nu^2} \left\{ \varepsilon\, D\, r^{1-\ell} \left[r^{\ell-1}\, \mathbf{u}' \right]' + \mathbf{f}(\mathbf{u}) \right\} = \mathcal{L}\, \tilde{\mathbf{u}}_2(r, \ell, \nu) \\ &- \tilde{\varepsilon}_1(\ell)\, \lambda_1(\ell)\, \phi_1(r, \ell)\, D\, \mathbf{v} + \phi_1(r, \ell)^2\, \mathbf{f}_2(\mathbf{v}, \mathbf{v}) + o(1) \end{aligned}$$

as $\nu \to 0$, where $\mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2)$ is a bilinear map obtained from the second derivative of $\mathbf{f}(\mathbf{u})$. From the Fredholm Alternative Theorem, it follows that the above equation has a solution $\tilde{\mathbf{u}}_2(r, \ell, \nu)$ if and only if

$$\tilde{\varepsilon}_1(\ell) = \frac{(\mathbf{f}_2(\mathbf{v}, \mathbf{v}), \mathbf{v}^*)\, \Phi_3(\ell)}{\lambda_1(\ell)\, (D\, \mathbf{v}, \mathbf{v}^*)\, \Phi_2(\ell)}$$

is satisfied.

4.1. Case $\ell = 1$

From $\Phi_3(1) = 0$, we obtain $\tilde{\varepsilon}_1(1) = 0$, so that we need to determine the sign of $\tilde{\varepsilon}_2(1)$. To do this, we employ the numerical verification method such as the interval arithmetic built into *Mathematica*, and then we can establish the following:

Lemma 3 ([6], [7]). *If either $n = 1$ or $n \geq 2$ is satisfied, then there exist a constant $\nu_0(1) > 0$ and C^2-class functions $\tilde{\varepsilon}(1, \nu)$, $\tilde{\mathbf{u}}(., 1, \nu)$ defined on the interval $(-\nu_0(1), \nu_0(1))$ such that*

(i) $(\tilde{\varepsilon}(1, \nu), \tilde{\mathbf{u}}(., 1, \nu)) \in E_1(1)$ *holds for each* $\nu \neq 0$, *and*
(ii) $\tilde{\varepsilon}_1(1) = 0$ *and* $\tilde{\varepsilon}_2(1) < 0$ *are satisfied.*

From (A.1) and Theorem 1.3 in Rabinowitz [10], it follows that there exists a maximal continuum $\mathcal{C}$ ($\subset E_1(1) \cup \{ (1, \hat{\mathbf{u}}) \}$) such that $\mathcal{C}$ contains $(1, \hat{\mathbf{u}})$ and meets $\{ 0 \} \times X$. By (A.2) and the maximum principle, we have $\mathcal{C} = E_1(1) \cup \{ (1, \hat{\mathbf{u}}) \}$.

We define $\psi(r, \mathbf{p}, \varepsilon, \ell)$ by the solution of

$$\begin{cases} \mathbf{0} = \varepsilon\, D\, r^{1-\ell} \left[r^{\ell-1}\, \mathbf{u}' \right]' + \mathbf{f}(\mathbf{u}), & r > 0, \\ \mathbf{u}(0) = \mathbf{p}, \quad \mathbf{u}'(0) = \mathbf{0}, \end{cases}$$

where $\mathbf{p} = (p, q) \in \mathbf{R}^2$. It is well-known that $\psi(r, \mathbf{p}, \varepsilon, \ell)$ is analytic in $(r, \mathbf{p}, \varepsilon, \ell)$ (for example, see Corollary 3.4.6 in Henry [3]). Clearly we have

$$\psi(., \hat{\mathbf{u}}_1(0, p, \ell), \hat{\varepsilon}_1(p, \ell), \ell) = \hat{\mathbf{u}}_1(., p, \ell)$$

for any $\ell \in [1, +\infty)$ and $p \in \mathcal{P}_1(\ell)$. Since $\psi(r, \tilde{\mathbf{p}}, \varepsilon, \ell)$ is a solution of (1.2) if and only if $\psi'(\pi, \tilde{\mathbf{p}}, \varepsilon, \ell) = \mathbf{0}$ holds, we may seek solutions $(\mathbf{p}, \varepsilon)$ of $\psi'(\pi, \mathbf{p}, \varepsilon, \ell) = \mathbf{0}$. Setting

$$\Psi(p, \ell) = \psi'_{\mathbf{p}}(\pi, \hat{\mathbf{u}}_1(0, p, \ell), \hat{\varepsilon}_1(p, \ell), \ell),$$

we know the following:

Lemma 4 (Section 2.7 in [5]). *Every element of $\Psi(p, \ell)$ is positive for any $\ell \in [1, +\infty)$ and $p \in \mathcal{P}_1(\ell)$.*

The above lemma means that the eigenvalue 0 of $\Psi(p, \ell)$ is simple even if it exists. After lengthy arguments, we can establish the following by employing the above lemma, Lemma 1, Theorem 1.3 in Rabinowitz [10], Theorem 2.1 in [4], and Theorem 1.1 in [5]:

Theorem 1 ([6], [8]). *If either $n = 1$ or $n \geq 2$ is satisfied, then there exist continuous functions $\mathbf{u}_-(., \varepsilon)$ and $\mathbf{u}_+(., \varepsilon)$ such that*

(i) $E_1^{\pm}(1) = \{ (\varepsilon, \mathbf{u}_{\pm}(., \varepsilon)) \mid \varepsilon \in (0, 1) \}$,
(ii) $\pm \mathbf{u}'_{\pm}(r, \varepsilon) \prec \mathbf{0}$ *for any* $(r, \varepsilon) \in (0, \pi) \times (0, 1)$, *and*
(iii) $\lim_{\varepsilon \to 1} \mathbf{u}_{\pm}(., \varepsilon) = \hat{\mathbf{u}}$

hold (see Figure 1).

The above theorem says that the secondary bifurcation of positive solutions for (1.2) never occurs.

4.2. Case $\ell > 1$

Setting

$$\hat{w} = \hat{u}^n, \quad \hat{z} = \hat{v}^n, \quad \hat{\mathbf{w}} = (\hat{w}, \hat{z}), \quad \omega = 1 - \hat{w} - \hat{z}, \quad y = \frac{\lambda_1(\ell)\, \hat{z}}{n\, \omega}\, d_u,$$

we obtain

$$0 < \hat{w} < 1, \quad 0 < \hat{z} < 1, \quad 0 < \omega < 1, \quad 0 < y < 1$$

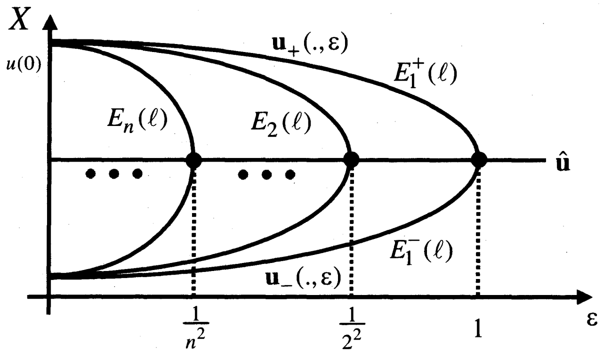


Fig. 1. Global Bifurcation Structure for $\ell = 1$

because of

$$d_v = \frac{n\,(n\,\omega - \lambda_1(\ell)\,\hat{z}\,d_u)}{\lambda_1(\ell)\,(\lambda_1(\ell)\,d_u + n\,\hat{w})}.$$

Since we can take $\mathbf{v}$ and $\mathbf{v}^*$ as satisfying

$$\mathbf{v} = \begin{pmatrix} n\,(1-\hat{w})\,\hat{u} \\ -(d_u\,\lambda_1(\ell) + n\,\hat{w})\,\hat{v} \end{pmatrix} \quad \text{and} \quad \mathbf{v}^* = \begin{pmatrix} n\,(1-\hat{z})\,\hat{v} \\ -(d_u\,\lambda_1(\ell) + n\,\hat{w})\,\hat{u} \end{pmatrix},$$

respectively, we have

$$\tilde{\varepsilon}_1(\ell) = \frac{n\,(n+1)\,r_2(y)\,\Phi_3(\ell)}{2\,\hat{z}\,r_1(y)\,\Phi_2(\ell)},$$

where $r_1(q) = \hat{w}\,\hat{z} + 2\,\omega\,q - \omega\,q^2$ and

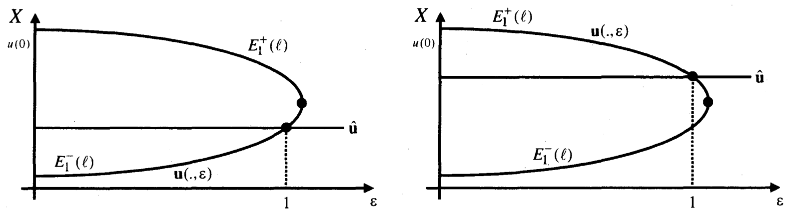
$$\begin{aligned} r_2(q) = &-\hat{w}^2\,\hat{z}^2 + \hat{z}\,(1 - 4\,\hat{w} + 3\,\hat{w}^2 - \hat{z} + 4\,\hat{w}\,\hat{z})\,q \\ &- \omega\,(1 - \hat{w} - \hat{z} - 2\,\hat{w}\,\hat{z})\,q^2 + \omega^2\,q^3. \end{aligned}$$

From

$$\begin{aligned} r_1(0) &= \hat{w}\,\hat{z} > 0, & r_1(1) &= (1-\hat{w})\,(1-\hat{z}) > 0, \\ r_2(0) &= -\hat{w}^2\,\hat{z}^2 < 0, & r_2(1) &= (1-\hat{w})^2\,(1-\hat{z})\,\hat{z} > 0, \end{aligned}$$

it follows that $r_1(y) > 0$ holds, and that there exist $0 < q_1 \le q_2 \le q_3 < 1$ such that $r_2(q_j) = 0$ for each j and

$$r_2(y) \begin{cases} < 0 & \text{if } y \in I_+ \equiv [0, q_1) \cup (q_2, q_3), \\ > 0 & \text{if } y \in I_- \equiv (q_1, q_2) \cup (q_3, 1]. \end{cases}$$



(a) $\mathbf{d} \in \mathcal{D}_-(\ell)$ (b) $\mathbf{d} \in \mathcal{D}_+(\ell)$

Fig. 2. Global Bifurcation Structure for $\ell > 1$

Setting

$$\mathcal{D}_\pm(\ell) = \left\{ \mathbf{d} \in \mathcal{D}(\ell) \;\middle|\; d_u = \frac{n\,\omega}{\lambda_1(\ell)\,\hat{z}}\,y,\; y \in I_\pm \right\},$$

we obtain $\pm\tilde{\varepsilon}_1(\ell) > 0$ for any $\mathbf{d} \in \mathcal{D}_\mp(\ell)$. By employing the similar argument with Theorem 1, we have the following:

Theorem 2. *Let $\ell > 1$ and $\sigma \in \{-,+\}$ be arbitrary. If $\mathbf{d} \in \mathcal{D}_\sigma(\ell)$ is satisfied, then there exists a continuous function $\mathbf{u}(.,\varepsilon)$ such that*

(i) $E_1^\sigma(\ell) = \{\, (\varepsilon, \mathbf{u}(.,\varepsilon)) \mid \varepsilon \in (0,1) \,\}$,
(ii) $\sigma\,\mathbf{u}'(r,\varepsilon) \prec \mathbf{0}$ *for any* $(r,\varepsilon) \in (0,\pi) \times (0,1)$, *and*
(iii) $\lim_{\varepsilon \to 1} \mathbf{u}(.,\varepsilon) = \hat{\mathbf{u}}$

hold (see Figure 2).

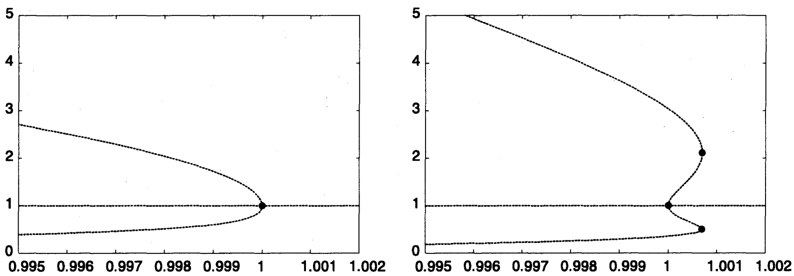
Figure 2 shows the structure of $E_1(\ell)$ which is suggested by Theorem 2, and says that the secondary bifurcation of saddle-node type appears on $E_1^+(\ell)$ (respectively, $E_1^-(\ell)$) for the case $\mathbf{d} \in \mathcal{D}_-(\ell)$ (respectively, $\mathbf{d} \in \mathcal{D}_+(\ell)$). Since Theorem 2 does not give us enough information on the structure of $E_1(\ell)$, it is open how many secondary bifurcations occur on $E_1(\ell)$.

§5. Concluding Remarks

From the result in Chafee and Infante [1], it follows that under the assumption stated in Theorem 1, the global bifurcation structure of positive solutions for (1.2) with $\ell = 1$ relative to ε is similar to that for

$$\begin{cases} 0 = \varepsilon\,u'' + u\,(1-u)\,(u-a), & r \in (0,\pi), \\ u'(0) = 0, \quad u'(\pi) = 0 \end{cases}$$

with $0 < a < 1$. Figure 3 shows the numerical bifurcation diagram for the case where $n = 1.1$ and $d_u = d_v$ are satisfied. The horizontal



(a) $b = c = 200.0$ (b) $b = c = 2000.0$

Fig. 3. Numerical Bifurcation Diagram

and vertical axes mean the value of ε and $u(0)/\hat{u}$, respectively. This figure suggests that the bifurcation structure of positive solutions for (1.2) depends on b and c, when the assumption of Theorem 1 is violated (for example, $1 < n < 2$ is satisfied).

To determine the local bifurcation structure of positive solutions for (1.2) on a neighborhood of $(\varepsilon, \mathbf{u}) = (1, \hat{\mathbf{u}})$, we employ the numerical verification method such as the interval arithmetic built into *Mathematica.* Unfortunately we have not succeeded in establishing the local bifurcation structure when $\mathbf{f}^0(\mathbf{u})$ is changed for

$$f^0(\mathbf{u}) = 1 - u^{n_1} - c\,v^{n_2}, \quad g^0(\mathbf{u}) = 1 - b\,u^{n_3} - v^{n_4}$$

with positive constants b, c and n_j, so that the global bifurcation structure for (1.2) with more general nonlinearity $\mathbf{f}^0(\mathbf{u})$ is still open.

Finally, we should remark that Theorem 1 and Theorem 2 do not give us the information on the Hopf bifurcation from positive stationary solutions of

$$\begin{cases} \mathbf{u}_t = \varepsilon D \left(\mathbf{u}_{rr} + \dfrac{\ell - 1}{r} \mathbf{u}_r \right) + \mathbf{f}(\mathbf{u}), & r \in (0, \pi), \quad t > 0, \\ \mathbf{u}_r = 0, \quad r = 0, \pi, \quad t > 0, \end{cases}$$

so that the global attractor of the above evolution equation is also still open.

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