Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 479–518

Analogies between superconductors and liquid crystals: Nucleation and critical fields

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Abstract.

In this note we summarize some of our results on surface superconductivity and nucleation of smectics, with emphasis on analogies between superconductors and liquid crystals.

1. Analogies between Superconductors and Liquid Crystals

Superconductors placed in a varying magnetic field will undergo phase transitions, which have been studied successfully in mathematics based on the Ginzburg-Landau theory of superconductivity [36]. According to this theory, superconductivity is described by a complex-valued function ψ called *order parameter*, and a real vector field $\mathcal{A}$ called *magnetic potential.* $|\psi|^2$ is proportional to the density of superconducting electron pairs. If the superconductor is in the normal state then $\psi = 0$, and if it is in the superconducting state then $\psi \not\equiv 0$. $(\psi, \mathcal{A})$ is a minimizer of the Ginzburg-Landau energy:

(1.1)

$$\mathcal{GL}[\psi, \mathcal{A}] = \int_\Omega \{|\nabla_{\kappa\mathcal{A}}\psi|^2 - \mu|\psi|^2 + \frac{\mu}{2}|\psi|^4\}dx + \frac{\kappa^4}{\mu}\int_{\mathbb{R}^3} |\mathrm{curl}\,\mathcal{A} - \mathcal{H}|^2 dx,$$

where Ω is the region occupied by the superconductor, $\mathcal{H}$ is the applied

Part of the results in this note were reported at the MSJ-IRI 2005 Conference on Asymptotic Analysis and Singularity, held at Tohoku University, Sentai, Japan, in July 2005. I would like to thank the organizers of the conference, the Institute of Mathematics of Tohoku University, and Professor I. Takagi for the invitation and for the warm hospitality. The research work presented in this note was carried out with partial support from the National Natural Science Foundation of China 10471125, the Science Foundation of the Ministry of Education of China, and Shanghai Pujiang Program 05PJ14039.

Received December 28, 2005.

Revised March 16, 2006.

magnetic field, κ is the *Ginzburg-Landau parameter* of the superconductor given by the ratio of penetration depth λ and coherence length ξ, and $\mu = \xi^{-2}$. The superconductor is type I if $\kappa < 1/\sqrt{2}$ and is type II if $\kappa > 1/\sqrt{2}$. Here we assumed that the temperature is below the critical temperature T_C, and we used the notation $\nabla_{\mathcal{A}}\psi = \nabla\psi - i\mathcal{A}\psi$. We shall also use the notation

$$\nabla^2_{\mathcal{A}}\psi = \Delta\psi - i[2\mathcal{A}\cdot\nabla\psi + \psi\,\mathrm{div}\,\mathcal{A}] - |\mathcal{A}|^2\psi.$$

For a bulk superconductor, we may take λ as the unit length, and hence set $\lambda = 1$. Then $\kappa = \sqrt{\mu}$, and (1.1) reduces to

$$\mathcal{G}L[\psi,\mathcal{A}] = \int_\Omega \{|\nabla_{\kappa\mathcal{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}dx + \kappa^2\int_{\mathbb{R}^3}|\mathrm{curl}\,\mathcal{A} - \mathcal{H}|^2 dx. \tag{1.2}$$

One may think that superconducting behavior and phase transitions were material properties, and independent of sample size and geometry. However, recent research works have shown that sample geometry plays an important role in the magnetic effects to superconductivity.[1] In sections 2.2-2.4, based on functional (1.2), we discuss mathematical formulation of upper critical field H_{c_3} for type II superconductors, nucleation of superconductivity in a decreasing magnetic field, and the transitions from the normal state to the surface superconducting state. We show that surface geometry of superconductors determines the value of H_{c_3} and governs nucleation of superconductivity. Moreover we also show that nucleation and phase transitions in superconducting cylinders of infinite height (so-called 2-dimensional superconductors) are significantly different to that on bounded bulk superconductors (so called 3-dimensional superconductors). In section 2.5 we consider type I superconductors and show that there exists a critical number $\lambda(\mathbf{h})$, where $\mathbf{h}$ is the direction of the applied magnetic field, such that, only a simple (Meissner-normal) transition occurs if $\lambda \geq \lambda(\mathbf{h})$, and a hysteresis phenomenon occurs if $0 < \lambda < \lambda(\mathbf{h})$. In section 2.6 we discuss a quasilinear system arising from a problem of nucleation of instability of the Meissner state at the superheating field H_{sh}.

On the other hand, effect of magnetic fields to liquid crystals are far from being clear mathematically. In the classical theory, the transition of stability of a nematic liquid crystal configuration under an applied magnetic field is described by a modified Oseen-Frank energy functional

[1] V. Moshchalkov et al. [64] discovered effect of sample topology on the critical fields of mesoscopic superconductors.

that is given by introducing a magnetic energy density $-\chi(\mathcal{H}\cdot\mathbf{n})^2$ into the classical Oseen-Frank energy density $F_N(\mathbf{n},\nabla\mathbf{n})$ of nematic liquid crystals. The modified energy functional may be called *the Oseen-Frank model with magnetic effect*:

$$\int_\Omega \{F_N(\mathbf{n},\nabla\mathbf{n}) - \chi(\mathcal{H}\cdot\mathbf{n})^2\}dx.$$

Here $\mathbf{n}:\bar{\Omega}\to\mathbb{S}^2$ denotes the *director field* of the nematic liquid crystal, $\mathcal{H}$ is an applied field, and χ is a positive parameter, see [31, P.287], [28], [39]. It was widely accepted that adding the term $\chi(\mathcal{H}\cdot\mathbf{n})^2$ into the Oseen-Frank energy is only a lower order perturbation from analysis point views, and simplicity of the mathematical models does not match the complexity of physical reality ([54, section 2.2]). One is naturally led to look at the analogies between the mathematical theory of superconductivity and that of liquid crystals, and seek what can be predicted about magnetic effects to liquid crystals, based on knowledge of magnetic effects to superconductors.

The analogies between superconductors and smectic liquid crystals were observed by P. G. de Gennes [30] and by W. L. McMillan. The analogies were clearly revealed by comparing the Ginzburg-Landau energy functional for superconductivity (1.2) and the Landau-de Gennes energy functional for liquid crystals (in the case of no applied magnetic field)

$$\begin{aligned}\mathcal{L}G[\Psi,\mathbf{n}] = \int_\Omega \Big\{&|\nabla_{q\mathbf{n}}\Psi|^2 - \kappa^2|\Psi|^2 + \frac{\kappa^2}{2}|\Psi|^4 + K_1|\mathrm{div}\ \mathbf{n}|^2\\ &+ K_2|\mathbf{n}\cdot\mathrm{curl}\ \mathbf{n}+\tau|^2 + K_3|\mathbf{n}\times\mathrm{curl}\,\mathbf{n}|^2\\ &+ (K_2+K_4)[\mathrm{tr}(\nabla\mathbf{n})^2 - (\mathrm{div}\,\mathbf{n})^2]\Big\}dx.\end{aligned} \tag{1.3}$$

(1.3) is in a dimensionless form, where Ω is the region occupied by the liquid crystal, Ψ is the order parameter which is zero for the nematic phase and not identically zero for the smectic phase, $\mathbf{n}$ is the director field, q is a real parameter called *wave number* which describes the intensity of layering of smectics, and τ is called chiral constant. K_j's are called elastic coefficients, among them K_1 (splay constant), K_2 (twist constant) and K_3 (bend constant) are positive. $\kappa=\sqrt{|r|/c}$, where r is a constant in the smectic energy density and c is the coupling constant ([69]), and we shall call κ the *Ginzburg-Landau parameter* of the liquid crystal.

De Gennes [30] compared the effects of magnetic fields to superconductivity with the effect of twist and bend to smectics, and predicted that, "*many effects which occur in the Ginzburg-Landau system should have their counterpart in smectic* A". Let us summarize some of the analogies discovered in [30, 31][2]:

Analogy between Superconductors and Smectics

superconductivity ψ	smectic Ψ
normal state $\psi = 0$	nematic state $\Psi = 0$
magnetic potential $\mathcal{A}$	director $\mathbf{n}$
$\lvert\nabla_{\kappa\mathcal{A}}\psi\rvert^2$	$\lvert\nabla_{q\mathbf{n}}\Psi\rvert^2$
magnetic energy	Oseen-Frank elastic energy
repel magnetic fields	repel twist and bend
2 responses to magnetic fields	2 responses to twist and bend

Several questions rise naturally, and shall be discussed in this note. First, one may expect existence of a critical field for liquid crystals that is an analogue of the upper critical field H_{c_3}. For a type II superconductor subjected to an applied magnetic field, if the applied field is stronger than H_{c_3} then the superconductor is in the normal state; and when the applied field decreases to H_{c_3} from above, superconductivity will nucleate at the surface of the sample. In section 3.4, by showing that a liquid crystal under a very strong magnetic field may not be in the pure nematic state, we make it clearer that de Gennes' theory of the analogies does not mean existence of any analogy between the magnetic effects to superconductors and the magnetic effects to liquid crystals.

Second, one may expect some quantitative comparison between the behaviors of the minimizers of the Landau-de Gennes functional (1.3) when the twist and bend go to infinity, with the behaviors of the minimizers of Ginzburg-Landau functional (1.2), and explore analogies which were predicted by de Gennes. In section 3.3 we show some confirming results, and present some mathematical questions to be answered.

Third, one may ask if there exist any more analogues in liquid crystals which resemble the magnetic effects to superconductors. We show in section 3.2 that, the *critical wave number* Q_{c_3} is a good analogue of H_{c_3}, and the effect of wave number to smectics is analogous to the

[2]S. Renn and T. Lubensky [75] studied the analogies and predicted that, the counterpart of the Abrikosov lattice in type II superconductors in response to an applied magnetic field (see [1]) should be a twisted smectic which exhibits an array of screw dislocations in response to twist. This phase, called twist grain boundary, was indeed found by J. W. Goodby et al one year after. This shows the significance of de Gennes' discovery.

magnetic effect to superconductors. A liquid crystal is in the nematic state ($\Psi = 0$) if its wave number is greater than Q_{c_3}, and is in a smectic state ($\Psi \not\equiv 0$) if its wave number is less than Q_{c_3}. Our results in [69] (for type II liquid crystals) and in [71] (for type I liquid crystals) show that nucleation of smectics from nematic background as wave number decreases from above Q_{c_3} is quite similar to nucleation of superconductivity from the normal state as applied magnetic fields decreasing from above H_{c_3}. In section 3.2 we report these results and present comparison to the known results about superconductors given in sections 2.3-2.5.

Finally, the analogy between Q_{c_3} and H_{c_3} raises a question about existence of an analogue in liquid crystals which is analogous to the surface superconducting state. We discuss this question in section 3.2.

2. Magnetic Effects to Superconductivity

§2.1. Upper Critical Field H_{c_3}

Our objective is to know how the value of the upper critical field $H_{c_3}(k)$ depends on κ and on the shape of the sample, to know where on the sample superconductivity begins to nucleate as an applied magnetic field decreases and reaches H_{c_3}, and to know how the superconductor changes its state from the normal state through the surface superconducting state to the vortex state as the applied field decreases. Recently, based on the Ginzburg-Landau theory of superconductivity, these problems have been investigated by many authors. To mention some, Chapman [25] and Bernoff-Sternberg [19] working with formal analyses, Bauman-Philips-Tang [18] investigating the case of a 2-dimensional disc, Giorgi-Philips [37], Lu-Pan [56-59], Pan [65-68], Helffer-Pan [46], Pan-Kwek [73], Helffer-Morame [42, 44, 45], Almog [3-6] for mathematical rigorous analysis in general domains. See also surveys Helffer [41], Helffer-Morame [43] and Lu-Pan [60].

Let us consider a superconductor occupying a bounded and simply-connected domain Ω in $\mathbb{R}^3$ and subjected to an applied magnetic field $\mathcal{H}$. For simplicity, let us consider the case where $\mathcal{H} = \sigma\mathbf{h}$, where $\sigma > 0$ is a constant, and $\mathbf{h}$ is a unit vector. If we set $\mathcal{A} = \sigma\mathbf{A}$, then the energy (1.2) can be written as

(2.1.1)

$$\mathcal{G}[\psi, \mathbf{A}] = \int_\Omega \{|\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\}dx + \kappa^2\sigma^2 \int_{\mathbb{R}^3} |\operatorname{curl}\mathbf{A} - \mathbf{h}|^2 dx.$$

It is easily seen that the functional $\mathcal{G}$ has a trivial critical point $(0, \mathbf{F_h})$, which describes the normal state, where $\mathbf{F_h}$ is a vector field satisfying the conditions

$$\text{curl}\,\mathbf{F_h} = \mathbf{h}, \quad \text{div}\,\mathbf{F_h} = 0. \tag{2.1.2}$$

Nucleation of superconductivity can be described by follows: As an applied field decreases and reaches a critical value, functional $\mathcal{G}$ starts to have a nontrivial minimal solution $(\psi, \mathbf{A})$ with $|\psi|$ being small. The following definition of the upper critical field was first introduced in [58, 59]:

$$H_{c_3}(\kappa, \mathbf{h}) = \inf\{\sigma > 0 : \ (0, \mathbf{F_h}) \text{ is a global minimizer of } \mathcal{G}\}. \tag{2.1.3}$$

Recent research in [56-59, 65-68, 70, 73, 46] and by many mathematicians show that, this number turned out to be very useful to describe nucleation of superconductivity.

If a superconductor occupies a cylinder of infinite height with cross section $\Omega \subset \mathbb{R}^2$, and if the applied magnetic field is parallel to the axis of the cylinder, then one may consider ψ and $\mathbf{A} = (A_1, A_2)$ to be defined on Ω, and the Ginzburg-Landau energy is reduced to an integral on Ω:

(2.1.4)
$$\mathcal{G}[\psi, \mathbf{A}] = \int_\Omega \{|\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2\sigma^2|\text{curl}\,\mathbf{A} - 1|^2\}dx,$$

where for $\mathbf{A} = (A_1, A_2)$, $\text{curl}\,\mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$. The trivial critical point is $(0, \mathbf{F})$, where $\mathbf{F}$ satisfies

$$\text{curl}\,\mathbf{F} = 1 \quad \text{and} \quad \text{div}\,\mathbf{F} = 0 \quad \text{in } \Omega, \quad \nu \cdot \mathbf{F} = 0 \quad \text{on } \partial\Omega. \tag{2.1.5}$$

In this case the critical value $H_{c_3}(\kappa)$ can be defined similarly ([58]).

§2.2. Eigenvalues of Magnetic Schrödinger Operator

Given a vector field $\mathbf{A}$, let $\mu(\mathbf{A})$ denote the lowest eigenvalue of the following problem :

$$-\nabla^2_{\mathbf{A}}\psi = \mu\psi \quad \text{in } \Omega, \qquad (\nabla_{\mathbf{A}}\psi) \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{2.2.1}$$

where ν is the unit outer normal to $\partial\Omega$.

Lemma 2.2.1 ([58]). *If $\mu(\sigma\kappa\mathbf{F}) < \kappa^2$ (resp. $\mu(\sigma\kappa\mathbf{F_h}) < \kappa^2$) then the functional $\mathcal{G}$ in (2.1.4) (resp. in (2.1.1)) has a non-trivial global minimizer.*

If $\mathcal{G}$ has a non-trivial global minimizer $(\psi, \mathbf{A})$ then $\mu(\sigma\kappa\mathbf{A}) < \kappa^2$.

§2.2.1. 2-dimensional problem. In the first part of this section Ω is a bounded, simply-connected domain in $\mathbb{R}^2$ with smooth boundary, $\mathbf{F}$ is the vector field given in (2.1.5).

Let us begin with the leading term estimate in the 2-dimensional case:[3]

Theorem 2.2.2 ([57]). *Assume* $\operatorname{curl}\mathbf{A} \in C^\alpha(\bar{\Omega})$, $0 < \alpha < 1$. *Then*

$$\lim_{b\to+\infty}\frac{\mu(b\mathbf{A})}{b} = \alpha_0(\operatorname{curl}\mathbf{A}), \tag{2.2.2}$$

where

$$\alpha_0(\operatorname{curl}\mathbf{A}) = \min\left\{\inf_{x\in\Omega}|\operatorname{curl}\mathbf{A}(x)|,\ \beta_0\inf_{x\in\partial\Omega}|\operatorname{curl}\mathbf{A}(x)|\right\},$$

where β_0 is the lowest eigenvalue of the Schrödinger operator $-\nabla^2_\omega$ with the unit magnetic field ω (namely $\operatorname{curl}\omega = 1$*) on the half plane, and* $0 < \beta_0 < 1$.

To establish this result, we have to analyze the limiting equations after blowing-up, classify all bounded eigenfunctions of $-\nabla^2_\omega$ in the entire plane and in the half plane associated with the lowest eigenvalues ([56]), and give estimates of a blow-up sequence and prove the convergence of a subsequence ([57]). The discussions in [57] actually show that, as $b \to +\infty$, the eigenfunctions concentrate at the union $\Omega_m \cup (\partial\Omega)_m$, where

$$\Omega_m = \{x \in \Omega:\ |\operatorname{curl}\mathbf{A}(x)| = \inf_{y\in\Omega}|\operatorname{curl}\mathbf{A}(y)|\},$$
$$(\partial\Omega)_m = \{x \in \partial\Omega:\ |\operatorname{curl}\mathbf{A}(x)| = \inf_{y\in\partial\Omega}|\operatorname{curl}\mathbf{A}(y)|\}.$$

In fact, the eigenfunctions concentrate in Ω_m if

$$\inf_{\Omega}|\operatorname{curl}\mathbf{A}| < \beta_0\inf_{\partial\Omega}|\operatorname{curl}\mathbf{A}|, \tag{2.2.3}$$

[3] In [57] it was required that $\mathbf{A} \in C^2(\bar{\Omega}, \mathbb{R}^3)$. However, to obtain the asymptotic estimate of eigenvalues in Theorem 2.2.2, it is sufficient to assume that $\operatorname{curl}\mathbf{A}$ is Hölder continuous, see the proofs in [57]. A similar remark applies to Theorem 2.2.4: In [73] it was assumed that $\operatorname{curl}\mathbf{A}$ is smooth, however, the assumption $\operatorname{curl}\mathbf{A} \in C^{1+\alpha}(\bar{\Omega})$ is sufficient to obtain the result stated in Theorem 2.2.4.

and concentrate in $(\partial\Omega)_m$ if

$$\inf_{\Omega} |\operatorname{curl} \mathbf{A}(x)| > \beta_0 \inf_{\partial\Omega} |\operatorname{curl} \mathbf{A}|. \tag{2.2.4}$$

Let us look at two special cases.

Case 1. $\operatorname{curl} \mathbf{A}$ is constant. This is the most interesting case and is the most important in applications. As a direct consequence of (2.2.2) we get the leading term estimate of $\mu(b\mathbf{F})$ where $\mathbf{F}$ is the vector field given in (2.1.5):

Theorem 2.2.3 ([57]). *Assume* $\operatorname{curl} \mathbf{F} = 1$. *Then*

$$\lim_{b\to+\infty} \frac{\mu(b\mathbf{F})}{b} = \beta_0, \tag{2.2.5}$$

and the eigenfunctions concentrate at the boundary $\partial\Omega$ *as* $b \to +\infty$.

The equality catching the second term in the expansion of $\mu(b\mathbf{F})$ was proved by B. Helffer and A. Morame [42], while the upper bound estimate was already obtained in [58]. See the surveys [60] and [41] for details and for references of related work by other authors.

Case 2. $\operatorname{curl} \mathbf{A}$ has non-degenerate zeros. As in [73], write

$$\mathscr{Z}(\operatorname{curl} \mathbf{A}) = \{x \in \bar{\Omega} : \operatorname{curl} \mathbf{A}(x) = 0\}.$$

We say that $\operatorname{curl} \mathbf{A}$ vanishes non-degenerately if $\mathscr{Z}(\operatorname{curl} \mathbf{A})$ is the union of a finite number of smooth curves and $\nabla(\operatorname{curl} \mathbf{A}) \neq 0$ on $\mathscr{Z}(\operatorname{curl} \mathbf{A})$. The limiting equation of a blow-up sequence is an eigenvalue problem for the operator $-\nabla^2_{\mathbf{A}_\vartheta}$, where

$$\mathbf{A}_\vartheta = -\frac{|x|^2}{2}(\cos\vartheta, \sin\vartheta),$$

and $\vartheta \in (-\pi, \pi)$ is a parameter. Corresponding to an interior blowing-up sequence, we have a limiting equation in the entire plane $\mathbb{R}^2$:

$$-\nabla^2_{\mathbf{A}_\vartheta}\phi = \lambda\phi \quad \text{in } \mathbb{R}^2. \tag{2.2.6}$$

After gauge transforms we can reduce (2.2.6) to an eigenvalue variation problem for an ordinary differential operator $-\frac{d^2}{dt^2} + \frac{1}{4}(t^2+2\tau)^2$ for $t \in \mathbb{R}$, with τ being a real parameter. Let $\lambda(\tau)$ denote the lowest eigenvalue of this operator and let

$$\lambda_0 = \inf_{\tau\in\mathbb{R}} \lambda(\tau).$$

We showed in [73] that there is a unique τ_0 such that

$$\lambda(\tau_0) = \min_{\tau} \lambda(\tau) = \lambda_0,$$

which answers a question left open in Montgomery [62]. On the other hand, corresponding to a boundary blowing-up sequence we have a limiting equation in the half plane $\mathbb{R}^2_+ = \{(x_1, x_2) : \ x_2 > 0\}$,

$$-\nabla^2_{\mathbf{A}_\vartheta} \phi = \lambda \phi \quad \text{in } \mathbb{R}^2_+, \qquad (\nabla_{\mathbf{A}_\vartheta} \phi) \cdot \nu = 0 \quad \text{on } \partial \mathbb{R}^2_+. \tag{2.2.7}$$

Let $\lambda(\mathbb{R}^2_+, \vartheta)$ denote the lowest eigenvalue of (2.2.7).

Theorem 2.2.4 ([73]). *Assume* $\operatorname{curl} A \in C^{1+\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$ *and* $\operatorname{curl} \mathbf{A}$ *vanishes non-degenerately on* $\bar{\Omega}$. *Let* ν *be the unit outer normal of* $\partial\Omega$ *and* τ *be the unit tangential vector such that the orientation of* $\{\nu, \tau\}$ *is the same as that of* $x_1 x_2$ *coordinates. For any* $x \in \partial\Omega$, *let* $\vartheta(x)$ *denote the angle between* $\operatorname{curl}^2 \mathbf{A}(x)$ *and* τ. *Then we have*

$$\lim_{b \to +\infty} \frac{\mu(b\mathbf{A})}{b^{2/3}} = [\alpha_1(\operatorname{curl} \mathbf{A})]^{2/3},$$

where

$$\alpha_1(\operatorname{curl} \mathbf{A}) = \min\{\lambda_0^{3/2} \inf_{x \in \Omega \cap \mathcal{Z}(\operatorname{curl} \mathbf{A})} |\nabla \operatorname{curl} \mathbf{A}(x)|,$$
$$\inf_{x \in \partial\Omega \cap \mathcal{Z}(\operatorname{curl} \mathbf{A})} \lambda(\mathbb{R}^2_+, \vartheta(x))^{3/2} |\nabla \operatorname{curl} \mathbf{A}(x)|\}.$$

In order to prove the above results, one may use blow-up technique and then classify the solutions of limiting equations. Let us consider the case where $\inf_{x \in \bar{\Omega}} |\operatorname{curl}(x)| > 0$. If blowing up at an interior point and making gauge transforms and changing variables, one will get a limiting equation

$$-\nabla^2_{\mathbf{E}} \psi = \alpha \psi \quad \text{in } \mathbb{R}^2, \tag{2.2.8}$$

here $\mathbf{E} = (-x_2, 0)$. If blowing up at a boundary point, the limiting equation will be

$$\begin{cases} \Delta\phi + 2ix_2 \partial_1 \phi - |x_2|^2 \phi + \beta \phi = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial \phi}{\partial x_2} = 0 \quad \text{on } \partial \mathbb{R}^2_+. \end{cases} \tag{2.2.9}$$

If $\psi \not\equiv 0$ is a bounded solution of (2.2.9), then it is called an eigenfunction of (2.2.9). It was shown in [56] that the lowest eigenvalue of (2.2.8) is

$\alpha = 1$ ([56, Theorem 1])[4], and the lowest eigenvalue of (2.2.9) is $\beta = \beta_0$ ([56, Theorem 3]), where

$$\beta_0 = \min_{z \in \mathbb{R}} \beta(z) = \beta(z_0);$$

where $\beta(z)$ is the lowest eigenvalue of the following ODE

$$-\frac{d^2u}{dx_2^2} + (x_2 + z)^2 u = \beta(z) u \quad \text{for } x_2 > 0, \qquad \frac{du}{dx_2}(0) = 0, \tag{2.2.10}$$

and z_0 is the unique minimum point of $\beta(z)$. Moreover, the eigenfunctions of (2.2.9) are given by

$$\phi = c e^{i z_0 x_1} u(x_2), \tag{2.2.11}$$

where $u(x_2)$ is an eigenfunction of (2.2.10) for $z = z_0$. To prove the last fact, for a bounded eigenfunction ϕ of (2.2.9), we make a Fourier transform in x_1 in the sense of distribution. Then we show that, as a distribution with parameter x_2, $\tilde{\phi}(\cdot, x_2) = \mathcal{F}_{x_1}[\phi]$ must be supported at a single point z_0 (the unique minimum point of $\beta(z)$). Hence for each x_2,

$$\tilde{\phi}(z, x_2) = \sum_{k=0}^{N(x_2)} c_k(x_2) \frac{d^k}{dz^k} \delta(z - z_0),$$

and

$$\phi(z, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N(x_2)} c_k(x_2)(-i x_1)^k \exp(i z_0 x_1).$$

Since ϕ is bounded, we must have $c_k(x_2) = 0$ for all $k > 0$, and hence

$$\phi(x_1, x_2) = v(x_2) \exp(i z_0 x_1),$$

where $v(x_2) = \frac{c_0(x_2)}{\sqrt{2\pi}}$. Then we show that v must satisfy (2.2.11) for $z = z_0$. See [56, section 5].

For (2.2.10), the uniqueness of the minimum point z_0 of $\beta(z)$ was first proved in [29]. Without knowing [29], we gave a proof of the uniqueness in [56, section 7], and the method in [56] is based on analysis on the behavior of solutions of (2.2.10), which is different to [29]. In the case

[4]There exist two obvious typos in the proof of Theorem 1 in [56] on page 1251: Line 12, "$W^{1,2}(B_n)$" should read "$W_0^{1,2}(B_n)$"; and Line 20, "$u \in C^1[0, n]$" should read "$u \in C_0^1[0, n]$".

where $|\text{curl}\,\mathbf{A}|$ has non-degenerate zeros, the associated Sturm-Liouville problem is

$$-\frac{d^2u}{dx_2^2} + \frac{1}{4}(x_2^2 + 2z)^2 u = \lambda(z)u \quad \text{for } x_2 > 0, \quad \frac{du}{dx_2}(0) = 0,$$

and the uniqueness of the minimum point of the lowest eigenvalue $\lambda(z)$ can not be proved by using the method in [29]. In [72], to prove Theorem 2.2.4, we used the method of [56] combined with analysis on a Riccati type equation. This method has been generalized by Aramaki [10, 11] to study the case of magnetic fields with higher order zeros.

§**2.2.2. 3-dimensional problem.** In the following we assume Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with smooth boundary. It was shown in [59] that, for any $\mathbf{A} \in C^{1+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ we have[5]

$$\begin{aligned}(2.2.12)\qquad &\lim_{b\to+\infty} \frac{\mu(b\mathbf{A})}{b} \\ &= \min\left\{\inf_{x\in\Omega}\left|\text{curl }\mathbf{A}(x)\right|, \inf_{x\in\partial\Omega} B(\theta(x))\left|\text{curl }\mathbf{A}(x)\right|\right\},\end{aligned}$$

where $\theta(x)$ is the angle between curl $\mathbf{A}(x)$ and the outer-normal vector ν on $\partial\Omega$, $B(\theta)$ is a positive function, decreasing on $(0, \frac{\pi}{2})$, $B(0) = 1$, $B(\frac{\pi}{2}) = \beta_0 < 1$, and $B(\pi - \theta) = B(\theta)$. Note that $B(\theta)$ attends the minimum when $\theta = \pi/2$, which corresponds to a point on $\partial\Omega$ at which $\text{curl}\,\mathbf{A}(x)$ is tangential to the surface.

In the case where $\mathbf{A} = \mathbf{F}_\mathbf{h}$, (2.2.12) yields

$$(2.2.13)\qquad \lim_{b\to+\infty} \frac{\mu(b\mathbf{F}_\mathbf{h})}{b} = \beta_0,$$

(see (2.2.5) for the 2-dimensional case). Moreover, the eigenfunctions concentrate at the tangential set

$$(\partial\Omega)_\mathbf{h} = \{x \in \partial\Omega : \ \mathbf{h}\cdot\nu(x) = 0\},$$

which is a subset of the surface where $\mathbf{h}$ is tangential to $\partial\Omega$. We expect that

$$(2.2.14)\qquad \mu(\varepsilon^{-2}\mathbf{F}_\mathbf{h}) = \frac{1}{\varepsilon^2}\{\beta_0 + C_2 P_{\min}\varepsilon^{2/3} + O(\varepsilon^{7/9})\},$$

[5]The remark in footnote 3 applies also.

where C_2 is a positive constant, see (2.4.3) in section 2.4. In [59, Appendix] we found an upper bound with correct order in the second term. In [70] we proved the upper bound and conjectured the formula (2.2.14) independently and simultaneously with B. Helffer and A. Morame [44, 45]. Helffer and Morame [44] also proved that their formula is equivalent to (2.2.14) up to an error term $O(\varepsilon^{2/3+\delta})$ for some $\delta > 0$. It is surprising that the result in Theorem 2.2.4 played an important role in [70] and [44, 45].

Now we estimate $\mu(b\mathbf{F_h})$ for small b. Let $w_{\mathbf{h}} \in W^{1,2}(\Omega)$ be the solution of

$$\Delta w_{\mathbf{h}} = 0 \text{ in } \Omega, \qquad \frac{\partial w_{\mathbf{h}}}{\partial \nu} = \mathbf{F_h} \cdot \nu \text{ on } \partial\Omega, \qquad \int_\Omega w_{\mathbf{h}}\, dx = 0. \tag{2.2.15}$$

Define

$$\omega(\mathbf{F_h}) = \inf_{\phi \in W^{1,2}(\Omega)} \fint_\Omega |\nabla\phi - \mathbf{F_h}|^2 dx = \fint_\Omega |\nabla w_{\mathbf{h}} - \mathbf{F_h}|^2 dx. \tag{2.2.16}$$

Theorem 2.2.5 ([68]). *We have, as $b \to 0$,*

$$\mu(b\mathbf{F_h}) = \omega(\mathbf{F_h})b^2 + O(b^3).$$

Problem 2.2.6. *(Location of concentration, and multiplicity) In Theorem 2.2.2, if (2.2.3) holds then the eigenfunctions concentrate in Ω_m. If Ω_m consists of more than one point, should the eigenfunctions concentrate at only one point, or should they concentrate over all Ω_m? One may ask a similar question if (2.2.4) holds.*

In Theorem 2.2.3, the eigenfunctions concentrate in $\mathcal{N}(\partial\Omega)$, the set of the maximum points of the boundary curvature ([42]). If there are more than one maximum point, should the eigenfunctions concentrate at only one point, or should they concentrate over all $\mathcal{N}(\partial\Omega)$?

Answers to these questions will help to find the multiplicity of the lowest eigenvalue (number of linearly independent eigenfunctions module gauge equivalence).

Problem 2.2.7. *Find an asymptotic estimate for $\mu(b\mathbf{A})$ when* curl $\mathbf{A}$ *has higher order zeros.*

For recent progress on this problem see Aramaki [10, 11].

Problem 2.2.8. *In the 2-dimensional case where $\mathbf{A} = \mathbf{F}$, and in the 3-dimensional case where $\mathbf{A} = \mathbf{F_h}$, examine the second eigenvalue*

$\lambda_2(b)$, or all other eigenvalues $\lambda_j(b)$ which satisfy asymptotically $\lambda_j(b) \leq (1+o(1))b$ as $b \to \infty$.

Analysis on these eigenvalues $\lambda_j(b)$ may help to understand the behavior of minimizers of the Ginzburg-Landau functional when the applied magnetic field is close to H_{c_2}. We expect that

(i) the eigenvalues that are asymptotically equal to b are complicated; and
(ii) those that are asymptotically strictly less than b are relatively simple to understand.

A result of Morame-Truc [63] seems to support (ii), where they considered the spectrum of $-\nabla^2_{\mathbf{F}_h}$ in the half-space, and showed that the spectrum in the interval $[\beta_0, 1]$ is a finite set of eigenvalues.

Problem 2.2.9. *Find an asymptotic estimate for the eigenvalue $\mu(b\mathbf{A})$ for large b where $\operatorname{curl}\mathbf{A}$ is in $W^{1,2}$ or in L^2 but is not smooth.*

This problem rises in the study of nucleation of smectics and is needed for the estimate of critical wave number Q_{c_3}, see section 3.3.

§2.3. Surface Superconductivity in 2-Dimensional Superconductors

Throughout this section we assume that Ω is a bounded and simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let us begin with an estimate of H_{c_3} proved in [58]:

$$\lim_{\kappa\to+\infty} \frac{H_{c_3}(\kappa)}{\kappa} = \frac{1}{\beta_0}, \tag{2.3.1}$$

where β_0 was given in (2.2.5). This result was improved later:

Theorem 2.3.1 ([46]). *For large κ we have*

$$H_{c_3}(\kappa) = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} + O(\kappa^{-1/3}), \tag{2.3.2}$$

where C_1 is a positive constant, and κ_{max} is the maximum value of the curvature of $\partial\Omega$.

We mention that, the results in [58] cover more general cases where the applied field can vary in space. In [58, Proposition 1.2], a lower bound estimate for H_{c_3} involving κ_{max} was obtained.

The behavior of minimizers of $\mathcal{G}$ in (2.1.4) when the applied magnetic field H is reduced from H_{c_3} was examined in [58], [46] and [65], and the results obtained there provide a complete description of nucleation process for 2-dimensional superconductors with large κ. The results are summarized below. We use $o(1)$ to denote a function of κ which tends to zero as $\kappa \to +\infty$. For positive functions a and b of κ, $a \ll b$ means that $\frac{a}{b} \to 0$ as $\kappa \to +\infty$.

(i) As the applied field decreases from H_{c_3}, superconductivity nucleates first at the maximum points of the boundary curvature. More precisely, if

$$\frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}}\kappa_{max} - o(1) \leq H < H_{c3}(\kappa),$$

the order parameters concentrate in a small neighborhood of the maximum points of the boundary curvature.

(ii) As the applied field is reduced again but is still close to H_{c_3}, the superconducting region expands gradually, and then a *thin* superconducting sheath forms on the entire boundary of the sample. More precisely, if

$$\frac{\kappa}{\beta_0} - o(\kappa) < H \ll \frac{\kappa}{\beta_0} - L\kappa^{1/3}$$

for some constant $L > 0$, the order parameters ψ concentrate uniformly along the entire boundary, and

$$|\psi(x)|^2 \sim c_1\kappa^{-1}(\frac{\kappa}{\beta_0} - H) \quad \text{for } x \in \partial\Omega,$$

$$\mathcal{G}[\psi, \mathbf{A}] \sim -c_2|\partial\Omega|\kappa^{-1}(\frac{\kappa}{\beta_0} - H)^2.$$

(iii) As the applied field is further reduced but is still kept away above H_{c_2}, the superconducting sheath becomes strong and a boundary layer gradually raises, while the interior of the sample remains close to a normal state. More precisely:

(iiia) If $H = (b + o(1))\kappa$ for some constant $1 < b < \frac{1}{\beta_0}$, then for any $0 < \alpha < 2\sqrt{b-1}$, there exist positive constants $\kappa(\alpha)$ and $C(\alpha)$ such that, for all $\kappa > \kappa(\alpha)$ we have

$$\int_\Omega \{|\psi|^2 + \frac{1}{\kappa^2}|\nabla_{\kappa H\mathbf{A}}\psi|^2\} \exp\big(\alpha\kappa \text{dist}(x, \partial\Omega)\big)dx \leq \frac{C(\alpha)}{\kappa}.$$

(iiib) If $1 \ll H - \kappa = o(\kappa)$, then there exist positive constants α_1, C and κ_1 such that, for all $\kappa > \kappa_1$ we have

$$\int_\Omega \{|\psi|^2 + \frac{1}{\kappa(H-\kappa)}|\nabla_{\kappa H\mathbf{A}}\psi|^2\} \exp\big(\alpha_1\sqrt{\kappa(H-\kappa)}\mathrm{dist}(x,\partial\Omega)\big)\,dx \le \frac{C}{\sqrt{\kappa(H-\kappa)}}.$$

(iiic) In both cases, there exists a positive number E_b ($b = 1$ in the case (iiib)), and for any closed subdomain D of $\overline{\Omega}$, there exists $\kappa_D > 0$ such that

$$\mathcal{G}[\psi, \mathbf{A}, D] = -\sqrt{\kappa H}E_b|D \cap \partial\Omega| + o(\kappa), \quad \text{for all } \kappa > \kappa_D, \tag{2.3.3}$$

where $\mathcal{G}[\psi, \mathbf{A}, D]$ is the energy on D.

(iv) The sample will remain in a surface superconducting state until the applied field reaches H_{c_2}.

Remark 2.3.2. Conclusion (i) was proved in [46]. Conclusion (ii) was proved in [58]. Recently S. Fournails and B. Helffer [34] obtained some more precise results on the energy estimates and on the asymptotic behavior of the minimizers. Conclusion (iii) was proved in [65]. It suggests that the equality

$$H_{c_2}(\kappa) \sim \kappa$$

is asymptotically correct.

One may wish to know more about the behavior of the order parameters for (κ, H) satisfying

$$\begin{aligned} &H = (b + o(1))\kappa \quad \text{as } \kappa \to \infty, \quad \text{where} \quad 1 \le b < \frac{1}{\beta_0}; \\ &\text{when } b = 1 \text{ we further require that } 1 \ll H - \kappa = o(\kappa). \end{aligned} \tag{2.3.4}$$

Conjecture 2.3.3 ([65]). *Assume $\kappa_n \to \infty$ and (κ_n, H_n) satisfies (2.3.4). Let ψ_n be the order parameter corresponding to $\kappa = \kappa_n$ and $H = H_n$. Then*

$$\lim_{n\to\infty} |\psi_n(x)| = \begin{cases} 0 & \text{if } x \in \Omega, \\ c_b & \text{if } x \in \partial\Omega, \end{cases}$$

where c_b is a positive constant.

Recently Y. Almog and B. Helffer proved in [7] that if $1 < b < 1/\beta_0$ then the order parameters indeed converge to a constant but in a rather weak sense.

We would like to mention that, to prove the uniform estimate of the energy (2.3.3), one has to investigate the limiting equation

$$(2.3.5)\qquad -\nabla^2_{\mathbf{E}}\psi = \lambda(1-|\psi|^2)\psi \quad \text{in } \mathbb{R}^2_+, \qquad \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on } \partial\mathbb{R}^2_+,$$

where $\mathbf{E} = (-x_2, 0)$. The case where $1 < b < 1/\beta_0$ in (2.3.4) corresponds to (2.3.5) with $\beta_0 < \lambda < 1$. In [65] we discussed a particular solution of (2.3.5) of the form

$$(2.3.6)\qquad \psi = e^{izx_1} f_z(x_2),$$

where $z \in (z_1(\lambda), z_2(\lambda))$ and $f_z(t)$ is a solution of

$$-f'' + (t+z)^2 f = \lambda(1-f^2)f, \quad 0 < t < \infty, \qquad f'(0) = 0.$$

Conjecture 2.3.4 ([65]). *If $\beta_0 < \lambda < 1$, the bounded solutions of (2.3.5) are in the form given in (2.3.6).*

Solution of this conjecture is useful for understanding the boundary layer behavior of the order parameters for H lying between H_{c_2} and H_{c_3}, the effect of domain boundary to the distribution of vortices, and the bifurcation behavior as H approaches H_{c_2}. See [5, 6] for some discussions.

A result similar to (iii) was established independently by Y. Almog [3-4] by considering large domain limit. If the applied field H is below H_{c_2}, then the boundary layer solution is no longer stable ([5]). As the applied field decreases cross κ then complicated bifurcations may occur ([6]). The behavior of the minimizers with the applied field lying in between H_{c_1} and H_{c_2} was further investigated by E. Sandier and S. Serfaty [79].

§2.4. Surface Superconductivity in 3-Dimensional Superconductors

In this section Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with smooth boundary. Consider an applied magnetic field of the form $\mathcal{H} = \sigma\mathbf{h}$, where $\mathbf{h}$ is a unit vector and $\sigma > 0$. It was proved in [59] that

$$(2.4.1)\qquad \frac{\kappa}{\beta_0} - C\kappa^{1/3} \le H_{c_3}(\mathbf{h}, \kappa) = \frac{\kappa}{\beta_0} + o(\kappa);$$

and as the applied field decreases from $H_{c_3}(\mathbf{h}, \kappa)$, a superconducting sheath nucleates at the tangential set $(\partial\Omega)_{\mathbf{h}}$ where the applied field is tangential to the surface (see section 2.2)[6]. An improved version of (2.4.1) was established in [70], which was used to locate the nucleation set more precisely. To explain the result we need the following notation. For $x \in \partial\Omega$, let $\kappa_1(x)$ and $\kappa_2(x)$ be the principal curvatures of $\partial\Omega$ at x, and let $\theta_{\mathbf{h}} = \theta_{\mathbf{h}}(x)$ be the angle between $\mathbf{h}$ and the principal direction corresponding to $\kappa_1(x)$. We define a function $P(x)$ on $\partial\Omega$ by

$$P(x) = [\kappa_1^2 \cos^2\theta_{\mathbf{h}} + \kappa_2^2 \sin^2\theta_{\mathbf{h}} - \alpha_0(\kappa_1 - \kappa_2)^2 \cos^2\theta_{\mathbf{h}} \sin^2\theta_{\mathbf{h}}]^{1/3},$$

where the constant α_0 was defined in [70, (2.4)] and $0 < \alpha_0 < 1$. Let

$$(2.4.2) \qquad P_{\min} = \min_{x\in(\partial\Omega)_{\mathbf{h}}} P(x), \qquad \mathcal{P} = \{x \in (\partial\Omega)_{\mathbf{h}} : P(x) = P_{\min}\}.$$

Let λ_0 be the eigenvalue given in section 2.2, and let

$$(2.4.3) \qquad C_2 = \lambda_0(1-\alpha_0)^{1/3}.$$

Theorem 2.4.1 ([70]). *For large κ we have*

$$(2.4.4) \qquad \frac{\kappa}{\beta_0} - \frac{C_2}{\beta_0^{5/3}} P_{\min}\, \kappa^{1/3} + O(\kappa^{2/9}) \le H_{c_3}(\mathbf{h}, \kappa) \le \frac{\kappa}{\beta_0} + M\kappa^{1/2},$$

where M is a constant independent of $\mathbf{h}$ *and κ.*

(2.4.4) was used in [70] to investigate the behavior of the order parameters for a superconductor in an applied field below H_{c_3}. The results show that there exist significant differences between 2-dimensional superconductors and 3-dimensional ones:

(i) For 2-dimensional superconductors, when the applied field is homogeneous and decreases from H_{c_3}, a superconducting sheath nucleates on a subset of the lateral surface where the curvature of $\partial\Omega$ is maximal ([56-58], [46]). If the applied field lies in between and keeps away from H_{c_2} and H_{c_3}, the sample is in the surface superconducting state and superconductivity persists uniformly at a thin sheath surrounding the entire lateral surface of the sample, while the interior of the sample remains close to the normal state ([65]).

(ii) For bounded 3-dimensional superconductors, when the applied field decreases from $H_{c_3}(\kappa, \mathbf{h})$, superconductivity nucleates at a subset

[6] Different phenomena happen in superconductors with edges and corners. Non-smoothness of domain surface raises greatly the value of H_{c_3}, and localizes the location of nucleation ([66, 67]). Also see [47] and [8, 15, 22].

of the surface where the applied field is tangential to the surface. The superconducting sheath grows as the applied field decreases. However the superconducting sheath will not cover the whole surface before the applied field reaches H_{c_2}([78], [59, 70]).

The right side estimate in (2.4.4) should be improved:

Conjecture 2.4.2 ([70]). *For large κ it holds that*

$$H_{c_3}(\mathbf{h},\kappa)=\frac{\kappa}{\beta_0}-\frac{C_2}{\beta_0^{5/3}}P_{\min}\ \kappa^{1/3}+O(\kappa^{2/9}). \tag{2.4.5}$$

As the applied magnetic field decreases from H_{c_3}, superconductivity nucleates first at the subset $\mathcal{P}$ of $\partial\Omega$, where the applied field is tangential to $\partial\Omega$, and the function $P(x)$ is minimal among all points in the tangential set $(\partial\Omega)_{\mathbf{h}}$.

§2.5. Type I Superconductors and Hysteresis

Analysis based on (1.2) may persuade us to accept that *type I behavior is simpler*, in the sense that, *superconductors undergo simple transitions from the Meissner to the normal states.* However, for superconductors with small size, A. Geim et al. [35] discovered numerous phase transitions whose character changes rapidly with size and temperature. In particular, a sample can be either type I or type II depending on its size. Recall that (1.2) describes superconductors of size comparable with the penetration depth λ. In this section we examine (1.1) for small κ and small positive μ, which describes type I superconductors of small size or in temperature slightly below T_c. Let the sample be subjected to an applied magnetic field $\mathcal{H}$ and assume $\mathcal{H}=\frac{\sigma}{\kappa}\mathbf{h}$, where $\mathbf{h}$ is a unit vector. We let $\mathcal{A}=\frac{\sigma}{\kappa}\mathbf{A}$ and re-write the functional (1.1) as

$$G[\psi,\mathbf{A}]=\int_\Omega\{|\nabla_{\sigma\mathbf{A}}\psi|^2-\mu|\psi|^2+\frac{\mu}{2}|\psi|^4\}dx+\frac{\kappa^2\sigma^2}{\mu}\int_{\mathbb{R}^3}|\mathrm{curl}\,\mathbf{A}-\mathbf{h}|^2dx, \tag{2.5.1}$$

where Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with smooth boundary. The normal state is gauge equivalent to $(0,\mathbf{F_h})$, here we choose $\mathbf{F_h}$ so that

$$\mathrm{curl}\,\mathbf{F_h}=\mathbf{h},\quad \mathrm{div}\,\mathbf{F_h}=0\quad\text{in }\mathbb{R}^3,\qquad \int_\Omega\mathbf{F_h}\,dx=\mathbf{0}.$$

We look for minimizers of (2.5.1) in $\mathcal{W}_0(\Omega, \mathrm{div})$, where

$$\mathbf{D}^{1,2}(\mathbb{R}^3, \mathrm{div}) = \{\mathbf{A} \in L^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3) : \ |\nabla \mathbf{A}| \in L^2(\mathbb{R}^3), \ \mathrm{div}\,\mathbf{A} = 0 \text{ in } \mathbb{R}^3\},$$

$$\mathcal{W}_0(\Omega, \mathrm{div}) = \{\psi \in W^{1,2}(\Omega, \mathbb{C}), \ \mathbf{A} - \mathbf{F_h} \in \mathbf{D}^{1,2}(\mathbb{R}^3, \mathrm{div}), \ \int_\Omega \mathbf{A} dx = \mathbf{0}\}.$$

Given a unit vector $\mathbf{h}$, define the critical field H_c ([68]):

$$H_c(\mathbf{h}, \mu, \kappa) = \inf\{\sigma > 0 : (0, \mathbf{F_h}) \text{ is a global minimizer of (2.5.1)}\}. \tag{2.5.2}$$

Let $w_\mathbf{h} \in W^{1,2}(\Omega)$ be the solution of (2.2.15) and let $\mathbf{U_h} \in \mathbf{D}^{1,2}(\mathbb{R}^3, \mathrm{div})$ be the solution of

$$\mathrm{curl}^{\,2}\mathbf{U_h} = (\nabla w_\mathbf{h} - \mathbf{F_h})\chi_\Omega \quad \text{in } \mathbb{R}^3, \qquad \int_\Omega \mathbf{U_h} dx = \mathbf{0}, \tag{2.5.3}$$

where χ_Ω is the characteristic function of Ω. Recall the number $\omega(\mathbf{F_h})$ defined in (2.2.16). We now define $a(\mathbf{h})$ and $\lambda(\mathbf{h})$ by

$$a(\mathbf{h}) = \frac{1}{\sqrt{\omega(\mathbf{F_h})}}, \qquad \lambda(\mathbf{h}) = \frac{\sqrt{2}a(\mathbf{h})}{\sqrt{|\Omega|}}\|\mathrm{curl}\,\mathbf{U_h}\|_{L^2(\mathbb{R}^3)}. \tag{2.5.4}$$

Given a unit vector $\mathbf{h}$ and positive numbers λ and ρ, the system

$$\begin{cases} \Delta w = 0 \quad \text{in } \Omega, \qquad \lambda^2 \mathrm{curl}^{\,2}\mathbf{A} = \rho(\nabla w - \mathbf{A})\chi_\Omega \quad \text{in } \mathbb{R}^3, \\ \dfrac{\partial w}{\partial \nu} = \mathbf{A}\cdot\nu \quad \text{on } \partial\Omega, \quad w \in W^{1,2}(\Omega), \quad \mathbf{A} - \mathbf{F_h} \in \mathbf{D}^{1,2}(\mathbb{R}^3, \mathrm{div}\,). \end{cases} \tag{2.5.5}$$

has a unique solution $(w^\rho, \mathbf{A}^\rho)$. Moreover, $(w^0, \mathbf{A}^0) = (w_\mathbf{h}, \mathbf{F_h})$.

Lemma 2.5.1 ([68]). *Given $a > 0$, consider the non-negative solution ρ of the equation*

$$a^2 \fint_\Omega |\nabla w^\rho - \mathbf{A}^\rho|^2 dx + \rho = 1. \tag{2.5.6}$$

(i) *If $\lambda \geq \lambda(\mathbf{h})$, (2.5.6) has a unique solution $\rho(a) > 0$ if $0 < a < a(\mathbf{h})$, a unique solution $\rho = 0$ if $a = a(\mathbf{h})$, and no solution if $a > a(\mathbf{h})$.*

(ii) *If $0 < \lambda < \lambda(\mathbf{h})$, there exists a number $b_\lambda(\mathbf{h}) > a(\mathbf{h})$ such that, (2.5.6) has a unique solution $\rho(a) > 0$ if $0 < a < a(\mathbf{h})$, two solutions $\rho_*(a) = 0$ and $\rho(a) > 0$ if $a = a(\mathbf{h})$, two positive solutions $\rho_*(a) < \rho(a)$ if $a(\mathbf{h}) < a < b_\lambda(\mathbf{h})$, a unique solution $\rho_0 > 0$ if $a = b_\lambda(\mathbf{h})$, and no solution if $a > b_\lambda(\mathbf{h})$.*

In both cases, $\rho(a)$ is strictly decreasing, and $\lim_{a\to 0}\rho(a)=1$.

Theorem 2.5.2 ([68]). *Let $\lambda = \frac{\kappa}{\sqrt{\mu}}$. If $\lambda \ge \lambda(\mathbf{h})$, for small μ we have*

$$H_c(\mathbf{h},\mu,\kappa) = a(\mathbf{h})\sqrt{\mu} + o(\sqrt{\mu}).$$

Let $\sigma = a\sqrt{\mu}$ and $\mu \to 0$.

(i) *If $0 < a < a(\mathbf{h})$, the global minimizers are non-trivial, and as $\mu \to 0$, all critical points $(\psi_\mu, \mathbf{A}_\mu)$ satisfy*

$$\text{(A)} \quad \psi_\mu \sim c_\mu\big[1 + ia\sqrt{\mu}w^{\rho(a)}\big], \qquad \mathbf{A}_\mu \to \mathbf{A}^{\rho(a)}, \qquad |c_\mu| \to \sqrt{\rho(a)}.$$

(ii) *If $a > a(\mathbf{h})$, $(0, \mathbf{F_h})$ is the only critical point in $\mathcal{W}_0(\Omega, \mathrm{div})$ for all small μ.*

Theorem 2.5.3 ([33]). *For $0 < \lambda < \lambda(\mathbf{h})$, there exist three critical numbers*

$$a(\mathbf{h}) < a_\lambda(\mathbf{h}) < b_\lambda(\mathbf{h}),$$

such that, for small μ we have

$$H_c(\mathbf{h},\mu,\kappa) = a_\lambda(\mathbf{h})\sqrt{\mu} + o(\sqrt{\mu}).$$

Let $\sigma = a\sqrt{\mu}$ and $\mu \to 0$.

(i) *If $0 < a < a(\mathbf{h})$, then the global minimizers are non-trivial, and as $\mu \to 0$, all critical points satisfy (A).*

(ii) *If $a(\mathbf{h}) < a < a_\lambda(\mathbf{h})$, then the global minimizers satisfy (A). Other critical points, subjected to a subsequence, satisfy either (A) or (B):*

$$\text{(B)} \quad \psi_\mu \sim c_\mu\big[1 + ia\sqrt{\mu}w^{\rho_*(a)}\big], \qquad \mathbf{A}_\mu \to \mathbf{A}^{\rho_*(a)}, \qquad |c_\mu| \to \sqrt{\rho_*(a)}.$$

(iii) *If $a_\lambda(\mathbf{h}) < a < b_\lambda(\mathbf{h})$, then $(0, \mathbf{F_h})$ is a global minimizer. Other critical points, subjected to a subsequence, satisfy either (A) or (B).*

(iv) *If $a > b_\lambda(\mathbf{h})$, then $(0, \mathbf{F_h})$ is the only critical point in $\mathcal{W}_0(\Omega, \mathrm{div})$ for all small μ.*

Physical explanation. Consider type I superconductors described by (2.5.1) with small μ, and $\kappa = \lambda\sqrt{\mu}$.

(i) If $\lambda \ge \lambda(\mathbf{h})$, we have simple transitions. More precisely, there exists a critical field

$$\mathcal{H}_c \sim \frac{a(\mathbf{h})}{\lambda}$$

such that:

(ia) As the applied field increases, the Meissner state is a global minimizer for $\mathcal{H}$ below $\mathcal{H}_c$, and disappears for $\mathcal{H}$ above $\mathcal{H}_c$;

(ib) As the applied field decreases, the normal state is a global minimizer for $\mathcal{H}$ above $\mathcal{H}_c$, and is unstable for $\mathcal{H}$ below $\mathcal{H}_c$.

(ii) If $0 < \lambda < \lambda(\mathbf{h})$, we have hysteresis. More precisely, besides the critical field $\mathcal{H}_c$, there exist two more critical fields, the subcooling field $\mathcal{H}_{sc}$ and the superheating field $\mathcal{H}_{sh}$ with

$$\mathcal{H}_{sc} < \mathcal{H}_c < \mathcal{H}_{sh}, \quad \mathcal{H}_{sc} \sim \frac{a(\mathbf{h})}{\lambda}, \quad \mathcal{H}_c \sim \frac{a_\lambda(\mathbf{h})}{\lambda}, \quad \mathcal{H}_{sh} \sim \frac{b_\lambda(\mathbf{h})}{\lambda}$$

such that:

(iia) As the applied field increases, the Meissner state is a global minimizer for $\mathcal{H}$ below $\mathcal{H}_c$, is stable (local minimizer) for $\mathcal{H}$ between $\mathcal{H}_c$ and $\mathcal{H}_{sh}$, and disappears for $\mathcal{H}$ above $\mathcal{H}_{sh}$;

(iib) As the applied field decreases, the normal state is a global minimizer for $\mathcal{H}$ above $\mathcal{H}_c$, is stable (local minimizer) for $\mathcal{H}$ between $\mathcal{H}_{sc}$ and $\mathcal{H}_c$, and is unstable for $\mathcal{H}$ below $\mathcal{H}_{sc}$.

In [68] we also found that type II superconductors of size much smaller than the penetration depth may behave as type I superconductors:

(i) When temperature increases to T_c, the applied field penetrates the sample almost completely, but superconductivity may persists.

(ii) Type II superconductors may exhibit type I behavior when T is close to T_c.

Remark 2.5.4. Lin-Du [53] studied hysteresis for type II superconductors for applied magnetic fields near H_{c_1}. Richardson-Rubinstein [76] found that a material can exhibit both types of behavior depending on its geometry. Aftalion-Du [2], Jimbo-Morita [48] and Jimbo-Zhai [49] discussed the Ginzburg-Landau functional on a small domain, on a thin domain, and on a perturbed domain respectively. Bolley-Helffer [20, 21] have studied superheating phenomena.

§2.6. A Quasilinear System Related to Vortex Nucleation

For a type II superconductor in an increasing applied magnetic field $\mathcal{H}$, the Meissner state is stable if $\mathcal{H} < H_{c_1}$, and is locally stable if $H_{c_1} < \mathcal{H} < H_{sh}$, where H_{sh} is the superheating field. Instability occurs when the applied field reaches H_{sh}. If the applied field is further increases, then vortices nucleate in the samples. It is important to find the value of H_{sh}, find the way how the Meissner state losses its stability, and find the location where vortices start to nucleate. Write $\mu = \kappa^2/\lambda^2$ in (1.1), and assume that for a Meissner state the order parameter can be written in the form $\psi = fe^{i\chi}$ with f real. Let $\mathcal{A} = \mathbf{A} + \frac{\lambda}{\kappa}\nabla\chi$. Fixing λ and letting $\kappa \to \infty$, we formally have $\frac{\lambda^2}{\kappa^2}\Delta f \sim 0$, hence $f^2 \sim 1 - |\mathbf{A}|^2$. Then a reduced equation is derived from the Euler-Lagrange equation of (1.1):

$$\begin{cases} -\lambda^2 \mathrm{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} & \text{in } \Omega, \\ \mathrm{curl}^2 \mathbf{A} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\ [\mathbf{A}_T] = 0, \quad [(\mathrm{curl}\, \mathbf{A})_T] = \mathbf{0} & \text{on } \partial\Omega, \\ \lambda\, \mathrm{curl}\, \mathbf{A} - \mathcal{H} \to \mathbf{0} \quad \text{as } |x| \to \infty, \end{cases} \tag{2.6.1}$$

where $\mathcal{H}$ is the applied magnetic field, $\mathbf{A}_T$ and $(\mathrm{curl}\, \mathbf{A})_T$ denote the tangential trace on $\partial\Omega$, and $[\cdot]$ is the jump across $\partial\Omega$. It was shown by Chapman [26] by using of a formal analysis that, (a) a solution $\mathbf{A}$ of (2.6.1) satisfying

$$\|\mathbf{A}\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{3}}, \tag{2.6.2}$$

is stable and corresponds to the Meissner state; (b) the solution loses stability when the maximum value $|\mathbf{A}(x)|$ reaches the critical point $1/\sqrt{3}$.

§**2.6.1. 2-dimensional case.** Consider a cylindrical superconductor with cross section $\Omega \subset \mathbb{R}^2$ and $\mathcal{H} = h\mathbf{e}_3$, where h is a constant. (2.6.1) is reduced to

$$-\lambda^2 \mathrm{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \qquad \lambda\, \mathrm{curl}\, \mathbf{A} = h \quad \text{on } \partial\Omega. \tag{2.6.3}$$

If $\mathbf{A}$ is a classical solution of (2.6.3) satisfying (2.6.2), then $H = \lambda\, \mathrm{curl}\, \mathbf{A}$ satisfies a quasilinear equation

$$\lambda^2 \mathrm{div}\, (F(\lambda^2|\nabla H|^2)\nabla H) = H \quad \text{in } \Omega, \qquad H = h \quad \text{on } \partial\Omega, \tag{2.6.4}$$

where F is determined by

$$v = F(t^2)t \iff t = (1 - v^2)v, \quad F(0) = 1, \tag{2.6.5}$$

and F is uniquely defined for $0 \le t \le \sqrt{\frac{4}{27}}$, i.e. $0 \le v \le \frac{1}{\sqrt{3}}$. The maximum points of $|\mathbf{A}(x)|$ coincide with the maximum points of $|\nabla H(x)|$. The validity of (2.6.3) was shown in [26] and [23]. It was conjectured by S. J. Chapman [26] and proved by H. Berestycki, A. Bonnet and S. J. Chapman [16] that, under condition (2.6.2), the maximum points of $|\mathbf{A}|$

locate at boundary. Chapman [27] used a formal analysis to show that the maximum points of $|\mathbf{A}|$ locate at the most negatively curved points of boundary for small λ. This conjecture was verified in [72]:

Theorem 2.6.1 ([72]). *Assume Ω is a bounded and simply-connected domain in $\mathbb{R}^2$ with C^4 boundary, and*

$$0 < h < \sqrt{\frac{5}{18}}. \tag{2.6.6}$$

Let $\mathbf{A}^\lambda$ be the solution of (2.6.3). As $\lambda \to 0$ we have:

(i) *The maximum points of $|\mathbf{A}^\lambda|$ approach the minimum points of curvature of boundary.*
(ii) *$\mathbf{A}^\lambda$ and $\operatorname{curl}\mathbf{A}^\lambda$ exhibit boundary layer behavior: they exponentially decay away from the boundary.*

Note that the number $\sqrt{\frac{5}{18}}$ coincides with the value of superheating field found in [26].

§**2.6.2. 3-dimensional case.** Assume Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with C^4 boundary and consider a simplified equation on Ω:

$$-\lambda^2\operatorname{curl}^2\mathbf{A} = (1-|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \qquad \lambda(\operatorname{curl}\mathbf{A})_T = \mathcal{H}_T \quad \text{on } \partial\Omega. \tag{2.6.7}$$

A solution $\mathbf{A}$ of (2.6.7) yields a particular solution of (2.6.1) after extending it over $\mathbb{R}^3$ by letting $\mathbf{A} = \mathbf{A}^o$ in $\Omega^c = \mathbb{R}^3\setminus\bar{\Omega}$, where $\mathbf{A}^o$ is defined in Ω^c by

$$\operatorname{curl}\mathbf{A}^o = \mathcal{H} \quad \text{in } \Omega^c, \qquad \mathbf{A}^o_T = \mathbf{A}_T \quad \text{on } \partial\Omega.$$

Let $\mathbf{H} = \lambda\operatorname{curl}\mathbf{A}$. For classical solutions, under the condition (2.6.2), the equation (2.6.7) is equivalent to the system

$$-\lambda^2\operatorname{curl}\left[F(\lambda^2|\operatorname{curl}\mathbf{H}|^2)\operatorname{curl}\mathbf{H}\right] = \mathbf{H} \quad \text{in } \Omega, \qquad \mathbf{H}_T = \mathcal{H}_T \quad \text{on } \partial\Omega, \tag{2.6.8}$$

and

$$\lambda\|\operatorname{curl}\mathbf{H}\|_{L^\infty(\Omega)} < \sqrt{\frac{4}{27}}. \tag{2.6.9}$$

Existence and uniqueness of classical solutions of (2.6.8) with $\lambda = 1$ for small boundary data were studied by R. Monneau [61] by using the implicit function theorem. However Monneau did not give a bound of boundary data for solvability, and hence his result does not guarantee any non-zero boundary data that allows (2.6.9) to have a solution for all small λ.

Theorem 2.6.2 ([14]). *Assume*

$$\mathcal{H}_T \in C^{2+\alpha}(\partial\Omega), \qquad \|\mathcal{H}_T\|_{C^0(\partial\Omega)} < \sqrt{\frac{5}{18}}, \qquad (2.6.10)$$
$$\nu \cdot \operatorname{curl} \mathcal{H}_T = 0 \text{ on } \partial\Omega.$$

For all $\lambda > 0$ small, (2.6.8) has a unique solution $\mathbf{H}^\lambda \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ which has the following properties:

(i) $\mathbf{H}^\lambda$ *satisfies (2.6.9);*
(ii) *If $\rho_\lambda \le \frac{c}{\lambda}$ and $\rho_\lambda \to \infty$ as $\lambda \to 0$, then*

$$\lim_{\lambda\to 0} \sup_{\operatorname{dist}(x,\partial\Omega)\ge\lambda\rho_\lambda} |\mathbf{H}^\lambda(x)| = 0.$$

(iii) *Let $\mu = \|\mathcal{H}_T\|_{C^0(\partial\Omega)}$. We have*

$$\lim_{\lambda\to 0} \lambda^2 \|\operatorname{curl} \mathbf{H}^\lambda\|^2_{C^0(\partial\Omega)} = M(\mu) \equiv [1 - (1 - 2\mu^2)^{1/2}](1 - 2\mu^2).$$

(iv) *If P^λ is a maximum point of $|\operatorname{curl} \mathbf{H}^\lambda(x)|$ and if $P^\lambda \to P$ for a sequence $\lambda_n \to 0$, then $P \in \partial\Omega(\mathcal{H}_T)$, where*

$$\partial\Omega(\mathcal{H}_T) = \{x \in \partial\Omega : \ |\mathcal{H}_T(x)| = \|\mathcal{H}_T\|_{C^0(\partial\Omega)}\}.$$

(v) *In particular, if $\mathcal{H} = \mathbf{h}$, a constant vector, with*

$$|\mathbf{h}| < \sqrt{\frac{5}{18}}, \qquad (2.6.11)$$

and if $P^\lambda \to P$ for a sequence $\lambda_n \to 0$, then $P \in (\partial\Omega)_{\mathbf{h}}$.

Remark 2.6.3. Under the condition (2.6.9), smooth solutions of (2.6.8) yield smooth solutions of (2.6.7). However, without the condition (2.6.2), solutions of (2.6.7) may have singularities, see [74] and [38], and the discussions in section 3.3.

Remark 2.6.4. Our discussions in sections 2.3 and 2.4 show that the ways of nucleation of superconductivity in 2-dimensional superconductors and in 3-dimensional ones are different, see the remarks following Theorem 2.4.1. Theorems 2.6.1 and 2.6.2 show that the ways of nucleation of instability in 2-dimensional superconductors and in 3-dimensional ones are also different:

(i) For 2-dimensional superconductors:

- (ia) superconductivity nucleates at boundary where curvature is the *maximal*;
- (ib) the Meissner state losses its stability at boundary where curvature is the *minimal*.

(ii) For 3-dimensional superconductors:

- (iia) superconductivity nucleates at surface tangential to the applied field, where the curvature function P is the *minimal* among points in $(\partial\Omega)_{\mathbf{h}}$;
- (iib) the Meissner state losses its stability at the points in $(\partial\Omega)_{\mathbf{h}}$. Is P the *maximal* at these points?

Remark 2.6.5. Chapman [26, p.1250] also conjectured that the instability of the Meissner state at H_{sh} leads to the formation of vortices in the sample, and leads to the transitions from the Meissner state to the mixed state. If this conjecture is true, then information of location of the maximum points of $|\mathbf{A}(x)|$ may be useful to find the location where vortices start to nucleate. Theorems 2.6.1 and 2.6.2 suggest that the ways of nucleation of vortices in 2-dimensional superconductors and in 3-dimensional ones are also different.

3. Effects of Magnetic Fields and Parameters to Liquid Crystals

§3.1. Introduction

Despite of many analogies between the Landau-de Gennes functional for liquid crystals (1.3) and the Ginzburg-Landau functional for superconductors (1.2), there exist two important differences.

(i) First, as was observed in [31], (1.2) has gauge invariance but (1.3) does not have. De Gennes and Prost wrote in [31, p.513] that, "*How severe these differences are is not fully understood yet*". This is still under our investigation and here we may point our one consequence of these differences: The Ginzburg-Landau energy is coercive for the component $\mathbf{A}$ in the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^3)$ after fixing gauge; however, Landau-de Gennes energy is not coercive for $\mathbf{n}$ in $W^{1,2}(\Omega, \mathbb{S}^2)$ except for the cases with restricted parameters. Thus one loses control on minimizing sequences at boundary, which in term permits possibility for $\mathbf{n}$ to exhibit boundary layer behavior.

(ii) Second, Landau-de Gennes model requires the director field satisfy the unit length restriction $|\mathbf{n}(x)| = 1$. It is this restriction that makes the Landau-de Gennes model very different to the Ginzburg-Landau model. For instance, the study of the Ginzburg-Landau system is more or less related to Schrödinger operator with a magnetic field, and the geometry of the underlying domain has shown to have important effects on the behavior of the solutions, as was seen in section 2; on the other hand, there is a close relation between Landau-de Gennes system and harmonic maps into sphere, and the effect of both topology and geometry of the underlying domain is important, which is still under investigation.

Throughout section 3 Ω is a bounded, simply-connected domain in $\mathbb{R}^3$ with smooth boundary. We shall consider two types of boundary conditions:

(i) Neumann boundary conditions for both $\mathbf{n}$ and Ψ,
(ii) Dirichlet boundary condition for $\mathbf{n}$ (the strong anchoring condition) and Neumann boundary condition for Ψ.

§3.2. Minimizers under Neumann Boundary Condition for Directors

In this section we consider a simplified form of Landau-de Gennes energy ([69]):

$$\mathcal{E}[\Psi,\mathbf{n}] = \int_\Omega \Big\{|\nabla_{q\mathbf{n}}\Psi|^2 - \kappa^2|\Psi|^2 + \frac{\kappa^2}{2}|\Psi|^4 + K_1|\text{div }\mathbf{n}|^2 + K_2|\text{curl }\mathbf{n} + \tau\mathbf{n}|^2\Big\}dx, \tag{3.2.1}$$

which was obtained from the original functional by letting $K_2 = K_3$, dropping the divergence term (surface energy)[7]

$$(K_2 + K_4)[\text{tr}(\nabla\mathbf{n})^2 - (\text{div }\mathbf{n})^2],$$

and rescaling. The simplified functional $\mathcal{E}$, as we wish, keeps the main features of the original Landau-de Gennes energy, namely, it is also incoercive in Sobolev spaces, and is lack of control at boundary.

We look for the global minimizers of the functional $\mathcal{E}$ in

$$W^{1,2}(\Omega,\mathbb{C}) \times V(\Omega,\mathbb{S}^2),$$

[7]If we pose Dirichlet condition to director fields $\mathbf{n} = \mathbf{n}_0$ on $\partial\Omega$, then the integral of divergence term depends only on $\mathbf{n}_0$ and can be dropped, see [39].

where

$$V(\Omega,\mathbb{S}^2)=\{\mathbf{u}:\Omega\to\mathbb{S}^2,\ \operatorname{curl}\mathbf{u}\in L^2(\Omega,\mathbb{R}^3),\ \operatorname{div}\mathbf{u}\in L^2(\Omega)\}.$$

The existence of the minimizers was proved in [69]. Note that $\mathcal{E}$ has trivial critical points $(0,\mathbf{n})$, where $\mathbf{n}$ are the critical points of the (simplified) Oseen-Frank functional

$$\mathcal{F}_N[\mathbf{n}]=\int_\Omega\{K_1|\text{div }\mathbf{n}|^2+K_2|\text{curl }\mathbf{n}+\tau\mathbf{n}|^2\}dx,$$

and thus $\mathbf{n}$ are the solutions of

$$\text{curl }\mathbf{n}+\tau\mathbf{n}=\mathbf{0},\quad |\mathbf{n}(x)|=1, \tag{3.2.2}$$

and correspond with the chiral nematic phases. The critical wave number Q_{c_3} was introduced in [69]:
(3.2.3)

$$\begin{aligned}Q_{c_3}(K_1,K_2,\kappa,\tau)&=\inf\{q>0:\ \mathcal{E}\text{ has only trivial minimizers}\},\\ Q_{c_3}(\kappa,\tau)&=\inf_{K_1,K_2>0}Q_{c_3}(K_1,K_2,\kappa,\tau).\end{aligned}$$

The results in [69, 71] show that Q_{c_3} is an analogue of H_{c_3}, and behaviors of the minimizers of the Landau-de Gennes functional for q close to Q_{c_3} are analogous to that of the minimizers of the Ginzburg-Landau functional for applied magnetic field close to H_{c_3}. We would like to mention that Q_{c_3} is also useful to identify the parameter regime where de Gennes' theory of analogies works well. To see this, let us fix κ, τ and q, and examine the behavior of the minimizers as K_1 and K_2 increase:

(i) $0<q<Q_{c_3}(\kappa,\tau)$. In this case, for any K_1 and $K_2>0$, the minimizer is non-trivial. As K_1, K_2 increase to ∞, the liquid crystal remains in a smectic state. Therefore the responses of liquid crystals to K_1, K_2 are different to the responses of superconductors to magnetic fields.

(ii) $q>Q_{c_3}(\kappa,\tau)$. In this case, the responses of liquid crystals to K_1, K_2 have some similarity with the responses of superconductors to magnetic fields, as predicted by de Gennes' theory.

§**3.2.1. Type I behavior.** Consider the minimizers with small κ. Given a unit vector field $\mathbf{n}$, let $\zeta_{\mathbf{n}}$ be the solution of

$$\Delta\zeta_{\mathbf{n}}=\text{div }\mathbf{n}\quad\text{in }\Omega,\qquad \frac{\partial\zeta_{\mathbf{n}}}{\partial\nu}=\mathbf{n}\cdot\nu\quad\text{on }\partial\Omega,\qquad \int_\Omega\zeta_{\mathbf{n}}dx=0.$$

Then we define

$$\omega(\mathbf{n}) = \fint_\Omega |\nabla\zeta_{\mathbf{n}} - \mathbf{n}|^2 dx, \qquad \omega_*(\tau) = \inf_{\mathbf{n}\in\mathcal{C}(\tau)} \omega(\mathbf{n}),$$

$$a(\tau) = \frac{1}{\sqrt{\omega_*(\tau)}},$$

$$\mathcal{C}(\tau) = \{\text{solution of (3.2.2)}\}, \quad \mathcal{C}_*(\tau) = \{\mathbf{n}\in\mathcal{C}(\tau):\ \omega(\mathbf{n}) = \omega_*(\tau)\},$$

$$\mathcal{N}(\tau) = \{(0,\mathbf{n}):\ \mathbf{n}\in\mathcal{C}(\tau)\}, \quad \mathcal{N}_*(\tau) = \{(0,\mathbf{n}):\ \mathbf{n}\in\mathcal{C}_*(\tau)\}.$$

Theorem 3.2.1 ([71]). (i) *As $\kappa\to 0$ we have*

$$Q_{c_3}(K_1,K_2,\kappa,\tau) = a(\tau)\kappa + o(\kappa). \tag{3.2.4}$$

(ii) *Let $0 < a < a(\tau)$. Let $(\psi_\kappa, \mathbf{n}_\kappa)$ be a global minimizer of $\mathcal{E}$ for $q = a\kappa$. If $\kappa\to 0$, we have, for a subsequence,*

$$\psi_\kappa = c_\kappa[1 + ia\kappa\zeta_{\mathbf{n}_0}] + o(\kappa), \qquad \mathbf{n}_\kappa = \mathbf{n}_0 + \kappa\mathbf{u}_\kappa,$$

where $\mathbf{n}_0\in\mathcal{C}_*(\tau)$, $|c_\kappa|^2\to 1 - a^2\omega_*(\tau)$, $\operatorname{div}\mathbf{u}_\kappa\to 0$ in $L^2(\Omega)$, and $\operatorname{curl}\mathbf{u}_\kappa + \tau\mathbf{u}_\kappa\to\mathbf{0}$ in $L^2(\Omega,\mathbb{R}^3)$.

Theorem 3.2.1 shows that, although every element in $\mathcal{N}(\tau)$ represents a nematic phase, smectics can only nucleate from a nematic state in $\mathcal{N}_*(\tau)$ (depending on sample geometry). Theorem 3.2.1 also shows that, a liquid crystal with small κ is in a uniform smectic state, which is analogous to the Meissner state of type I superconductors (see Theorem 2.5.2). Here we list some comparison (with κ and q small, τ, K_1 and K_2 fixed):

Sc and Sm with Small κ: Type I Behavior

superconductors	**liquid crystals**
repel magnetic until H_c	repel stress until Q_{c_3}
$\mathcal{H}_c\sim a(\mathbf{h},\lambda)$	$Q_{c_3}\sim a(\tau)\kappa$
Meissner-normal transition	uniform smectic-nematic transition
hysteresis for small λ	?

§3.2.2. Type II behavior. In [69] we gave a few estimates of Q_{c_3} for large κ. Let $\mu(b\mathbf{F}_{\mathbf{h}})$ denote the lowest eigenvalue of (2.2.1) for $\mathbf{A} = b\mathbf{F}_{\mathbf{h}}$ and set

$$\mu_*(b) = \inf_{\mathbf{h}\in\mathbb{S}^2}\mu(b\mathbf{F}_{\mathbf{h}}),$$

$$b_0(\kappa) = \min\{b>0:\ \mu_*(b) = \kappa^2\},$$

$$b_1(\kappa) = \max\{b>0:\ \mu_*(b) = \kappa^2\}.$$

Theorem 3.2.2 ([69]). (i) *If $\kappa > 0$ is fixed and $\tau \to 0^+$, then*

$$\frac{b_0(\kappa) + o(1)}{\tau} \le Q_{c_3}(\kappa, \tau) \le \frac{b_1(\kappa) + o(1)}{\tau}.$$

(ii) *If $\tau > 0$ is bounded and $\kappa \to \infty$, then*

$$Q_{c_3}(\kappa, \tau) \ge (1 + o(1)) \frac{\kappa^2}{\beta_0 \tau}, \tag{3.2.5}$$

where β_0 was given in section 2.2.

Conjecture 3.2.3. *We believe that the equality in (3.2.5) holds.*

It is natural to expect that, if the wave number of a liquid crystal is close to Q_{c_3} and if the Ginzburg-Landau parameter κ is large, the liquid crystal will be in a state that is analogous to the surface superconducting state of type II superconductors. Let us call the analogue by *surface smectic state*, or SSS for short, if it exists. One may expect that it exhibits a boundary layer of smectic surrounding the surface of the sample, with the bulk in a nematic phase.[8] The answer to Conjecture 3.2.3 will be important to determine the existence of a surface smectic state.

Problem 3.2.4. *Find the regime of parameters for which a surface smectic state exists. Find the boundary layer behavior of the surface smectic state.*

[8] Als-Nielsen, Christensen and Pershan [9] observed alignment of smectic-A layers at the top surface in a bulk nematic liquid crystal.

Sc and Sm with Large κ:	**Type II Behavior?**
(3d) superconductors	**liquid crystals**
$H_{c_3}(\kappa,\mathbf{h}) \sim \frac{\kappa}{\beta_0} - \frac{C_2}{\beta_0^{5/3}} P_{\min}\kappa^{1/3}$	$Q_{c_3}(\kappa,\tau) \sim \frac{\kappa^2}{\beta_0\tau} - (?)$
decrease $\mathcal{H}$:	decrease q:
nucleation of Sc at H_{c_3}	nucleation of Sm at Q_{c_3}
nucleation: $\mathcal{P}_{\min} \to (\partial\Omega)_{\mathbf{h}} \to \partial\Omega$	location of nucleation of Sm?
boundary layer of ψ	boundary layer of Ψ and $\mathbf{n}$?
Surface **S**uperconducting **S**tate	**S**urface **S**mectic **S**tate?
SSS $\to$ Abrikosov lattice	SSS $\to$ TGB?
increase $\mathcal{H}$:	increase q:
Sc state$\to$ mixed state	Sm state $\to$ *mixed state*?
H_{sh}: vortex nucleation at surface?	nematic nucleation at surface?
mixed state $\to$ Abrikosov lattice	*mixed state*$\to$ TGB?

§3.3. Minimizers under Dirichlet Boundary Condition for Directors

Given $\mathbf{u}_0 \in C^1(\partial\Omega,\mathbb{S}^2)$, we look for minimizers of the Landau-de Gennes functional for $\Psi \in W^{1,2}(\Omega,\mathbb{C})$ and $\mathbf{n} \in W^{1,2}(\Omega,\mathbb{S}^2,\mathbf{u}_0)$, where

$$W^{1,2}(\Omega,\mathbb{S}^2,\mathbf{u}_0) = \{\mathbf{u} \in W^{1,2}(\Omega,\mathbb{S}^2) : \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega\}.$$

For simplicity let us assume $\tau = 0$ and write the energy as
(3.3.1)

$$\mathcal{E}_0[\Psi,\mathbf{n}] = \int_\Omega \Big\{|\nabla_{q\mathbf{n}}\Psi|^2 - \kappa^2|\Psi|^2 + \frac{\kappa^2}{2}|\Psi|^4 + K_1|\text{div }\mathbf{n}|^2 + K_2|\text{curl }\mathbf{n}|^2\Big\}dx.$$

If $\mathbf{n}_0$ is a minimizer of the Oseen-Frank energy $\mathcal{F}_N$ (with $\tau = 0$) in $W^{1,2}(\Omega,\mathbb{S}^2,\mathbf{u}_0)$, then $(0,\mathbf{n}_0)$ is a (pure nematic) critical point of $\mathcal{E}_0$, and we call it a trivial critical point. We can also define critical wave number Q_{c_3} as in section 3.2. To estimate the value of Q_{c_3} for large κ, we need an estimate of $\mu(b\mathbf{n}_0)$ for large b and for any minimizer $\mathbf{n}_0$ of $\mathcal{F}_N$, where $\mu(b\mathbf{n}_0)$ denotes the lowest eigenvalue of (2.2.1) for $\mathbf{A} = b\mathbf{n}_0$. Let $\text{Sing}(\mathbf{n}_0)$ denote the singular set of $\mathbf{n}_0$, a subset of $\bar\Omega$ such that $\mathbf{n}_0 \in C^1(\bar\Omega \setminus \text{Sing}(\mathbf{n}_0),\mathbb{R}^3)$. From [40],

$$\mathcal{H}^{1-\delta}(\text{Sing}(\mathbf{n}_0) \cap \Omega) = 0$$

for some $0 < \delta < 1$. As in (2.2.12) we have

$$\limsup_{b \to +\infty} \frac{\mu(b\mathbf{n}_0)}{b} \le \min\Big\{ \inf_{x \in \Omega \setminus \mathrm{Sing}(\mathbf{n}_0)} \big|\mathrm{curl}\ \mathbf{n}_0(x)\big|, \inf_{x \in \partial\Omega \setminus \mathrm{Sing}(\mathbf{n}_0)} B(\theta(x))\big|\mathrm{curl}\ \mathbf{n}_0(x)\big| \Big\},$$

where $B(\theta)$ is the positive function appeared in (2.2.12), and $\theta(x)$ is the angle between curl $\mathbf{n}_0(x)$ and the outer-normal vector ν on $\partial\Omega$. Note that the equality holds when $\mathbf{n}_0 \in C^1(\bar{\Omega}, \mathbb{R}^3)$. When $\mathbf{n}_0$ has singularity, in order to obtain the lower bound we need to know the behavior of the eigenfunction near its singular set for large b.

Problem 3.3.1. *Give an asymptotic estimate of $\mu(b\mathbf{n})$ for $\mathbf{n}$ with singularity.*

In order to understand the effect of twist and bend, we examined the behavior of the minimizers for large K_1 or K_2 in [69]. Here we consider the case where K_1 is fixed and $K_2 \to \infty$, with κ and q being fixed. It is easy to see that the asymptotic behavior of the minimizers as $K_2 \to \infty$ depends on whether the following number is achieved:

(3.3.2)
$$R_h(\mathbf{u}_0) = \inf_{\mathbf{u} \in W^{1,2}(\Omega, \mathbb{S}^2, \mathbf{u}_0)} \int_\Omega |\mathrm{curl}\, \mathbf{u}|^2 dx.$$

In fact, let $k_j^{(2)} \to \infty$ and let $(\Psi_j, \mathbf{n}_j)$ be a minimizer of $\mathcal{E}_0$ for $K_2 = K_2^{(j)}$. Then

$$\int_\Omega |\mathrm{curl}\, \mathbf{n}_j|^2 dx \to R_h(\mathbf{u}_0).$$

A special case was discussed in [69, Theorem 5.4] where $R_h(\mathbf{u}_0) = 0$ and is achieved. When Ω is simply-connected, this case happens if and only if $\mathbb{G}(\Omega, \mathbf{u}_0) \neq \emptyset$, where

(3.3.3)
$$\mathbb{G}(\Omega, \mathbf{u}_0) \equiv \{\phi \in W^{2,2}(\Omega) \ : \ |\nabla\phi| = 1 \text{ a.e. in } \Omega,\ \nabla\phi = \mathbf{u}_0 \text{ on } \partial\Omega\}.$$

If $\mathbb{G}(\Omega, \mathbf{u}_0) \neq \emptyset$, then $G(\mathbf{u}_0)$ is achieved:

$$G(\mathbf{u}_0) = \inf_{\phi \in \mathbb{G}(\Omega, \mathbf{u}_0)} \int_\Omega |\Delta\phi|^2 dx.$$

Proposition 3.3.2 ([69]). *Assume $G(\Omega, \mathbf{u}_0) \neq \emptyset$. Let $K_2^{(j)} \to \infty$ and let $(\Psi_j, \mathbf{n}_j) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \mathbf{u}_0)$ be a minimizer for*

$K_2 = K_2^{(j)}$. *Then as* $j \to \infty$

$$\mathcal{E}_0[\Psi_j, \mathbf{n}_j] = K_1 G(\mathbf{u}_0) - \frac{\kappa^2}{2}|\Omega| + o(1).$$

Moreover, there exists a subsequence $\{(\Psi_{j'}, \mathbf{n}_{j'})\}$ *and a function* $\phi_0 \in \mathbb{G}(\Omega, \mathbf{u}_0)$ *which achieves* $G(\mathbf{u}_0)$*, such that*

$$(\Psi_{j'}, \mathbf{n}_{j'}) \to (e^{iq\phi_0}, \nabla\phi_0) \quad \text{in } \mathcal{W}(\Omega, \mathbf{u}_0). \tag{3.3.4}$$

We may compare $(e^{iq\phi_0}, \nabla\phi_0)$ with the Meissner state of a superconductor. (3.3.4) says that, if $\mathbb{G}(\Omega, \mathbf{u}_0) \neq \emptyset$, then for any $\kappa > 0$, the liquid crystal exhibits type I behavior: large twist and bend do not destroy the smectics but affect the director, and the liquid crystal tends to be in a uniform smectic state.

We believe that in general $R_h(\mathbf{u}_0)$ is not achieved, and we need to consider a minimization problem

$$R_l(\mathbf{u}_0) = \inf_{\mathbf{u} \in \mathcal{L}(\Omega, \text{curl}, \mathbf{u}_0)} \int_\Omega |\text{curl}\, \mathbf{u}|^2 dx, \tag{3.3.5}$$

where
(3.3.6)
$$\mathcal{L}(\Omega, \text{curl}, \mathbf{u}_0) = \{\mathbf{u} \in L^2(\Omega, \mathbb{S}^2) : \text{ curl}\, \mathbf{u} \in L^2(\Omega, \mathbb{R}^3), \ \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega\}.$$

If $\mathbf{u} \in L^2(\Omega, \mathbb{S}^2)$ and $\text{curl}\, \mathbf{u} \in L^2(\Omega, \mathbb{R}^3)$, then the tangential component of $\mathbf{u}$ at $\partial\Omega$, $\mathbf{u}_T$, is well-defined in the trace sense, and $\mathbf{u}_T \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$. However, the normal component may not be well-defined. Hence in (3.3.6) the equality $\mathbf{u} = \mathbf{u}_0$ is understood that we assume the full trace of $\mathbf{u}$ at $\partial\Omega$ is well-defined and equal $\mathbf{u}_0$ at least in $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$. When $R_l(\mathbf{u}_0) = 0$, the minimization problem (3.3.5) is related to the Aviles-Giga problem [13].

Problem 3.3.3 ([69]). *If* $R_l(\mathbf{u}_0) > 0$*, is* $R_l(\mathbf{u}_0)$ *achieved?*

It was proved in [74] that, $R_l(\mathbf{u}_0)$ is achieved if Ω is a disc in $\mathbb{R}^2$ and if $\mathbf{u}_0$ makes a constant angle with the outer normal vector of $\partial\Omega$. On the other hand, the computational results in [38] for 2-dimensional domains show that the complexity of the minimizing configurations grows rapidly when the rotation number of $\mathbf{u}_0$ increases. One may consider an approximation problem ($\lambda \ll 1$):

$$\lambda^2 \text{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \qquad \mathbf{A} = \mathbf{u}_0 \quad \text{on } \partial\Omega. \tag{3.3.7}$$

Very little is known. Please note that the sign in the left side of the equation in (3.3.7) is opposite to that in (2.6.7).

If we examine the behaviors of the minimizers of $\mathcal{E}_0$ for large K_1, with K_2 fixed, we will have similar situations with $R_{\mathbf{h}}(\mathbf{u}_0)$ replaced by $D_h(\mathbf{u}_0)$:

$$D_h(\mathbf{u}_0) = \inf_{\mathbf{u}\in W^{1,2}(\Omega,\mathbb{S}^2,\mathbf{u}_0)} \int_\Omega |\mathrm{div}\,\mathbf{u}|^2 dx. \tag{3.3.8}$$

For 2-dimensional domains minimization problem (3.3.8) is equivalent to (3.3.2) after transpose $(u_1, u_2) \to (-u_2, u_1)$. In [69] we examined the behavior of the minimizers of $\mathcal{E}_0$ for large K_1 in the case where $\mathbb{R}(\Omega, \mathbf{u}_0) \neq \emptyset$, where

$$\mathbb{R}(\Omega, \mathbf{u}_0) = \{\mathbf{u} \in W^{1,2}(\Omega, \mathbb{S}^2, \mathbf{u}_0) :\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega\}.$$

For general domains and boundary data we have to consider the minimization problem

$$D_l(\mathbf{u}_0) = \inf_{\mathbf{u}\in\mathcal{L}(\Omega,\,\mathrm{div},\,\mathbf{u}_0)} \int_\Omega |\mathrm{div}\,\mathbf{u}|^2 dx, \tag{3.3.9}$$

where

(3.3.10)

$$\mathcal{L}(\Omega, \mathrm{div}, \mathbf{u}_0) = \{\mathbf{u} \in L^2(\Omega, \mathbb{S}^2) :\ \mathrm{div}\,\mathbf{u} \in L^2(\Omega),\ \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega\}.$$

The counterpart of question 3.3.3 is the following

Problem 3.3.4. *If $D_l(\mathbf{u}_0) > 0$, is $D_l(\mathbf{u}_0)$ achieved?*

§3.4. Magnetic Effects to Liquid Crystals

In this section we show that the responses of liquid crystals in magnetic fields are indeed different to that of superconductors. We then examine the behavior of the minimizers when the applied magnetic fields vary. We consider the Dirichlet boundary condition for the director fields:

$$\mathbf{n} = \mathbf{u}_0 \quad \text{on } \partial\Omega,$$

where $\mathbf{u}_0 \in C^1(\partial\Omega, \mathbb{S}^2)$, and consider a simplified form of the Landau-de Gennes energy functional obtained by introducing an applied magnetic field $\mathbf{H}$ into the functional $\mathcal{E}_0$ in section 3.3:

$$\begin{aligned}\mathcal{E}_h[\Psi, \mathbf{n}] = \int_\Omega \{&|\nabla_{q\mathbf{n}}\Psi|^2 - \kappa^2|\Psi|^2 + \frac{\kappa^2}{2}|\Psi|^4 \\ &+ K_1|\mathrm{div}\,\mathbf{n}|^2 + K_2|\mathrm{curl}\,\mathbf{n}|^2 - \chi(\mathbf{H}\cdot\mathbf{n})^2\}dx,\end{aligned} \tag{3.4.1}$$

where χ is a positive constant. Analysis shows that the minimizers of $\mathcal{E}_h$ undergo complicated changes when the applied fields vary, indicating that magnetic fields have very important influences on phase transitions of liquid crystals.

Let us restrict ourselves to a simple situation where the applied field $\mathbf{H}$ and the boundary data $\mathbf{u}_0$ are a pair of constant vectors orthogonal to each other:

$$\mathbf{H} = \sigma\mathbf{h}, \qquad \mathbf{u}_0 = \mathbf{e}, \qquad \mathbf{h}\cdot\mathbf{e} = 0. \tag{3.4.2}$$

Write

$$\begin{aligned}
&\mathcal{F}_{\sigma\mathbf{h}}[\mathbf{n}] = \int_\Omega \{K_1|\operatorname{div}\mathbf{n}|^2 + K_2|\operatorname{curl}\mathbf{n}|^2 - \chi\sigma^2(\mathbf{h}\cdot\mathbf{n})^2\}dx,\\
&\mathcal{M}(\Omega,\sigma\mathbf{h}) = \{\text{global minimizers of } \mathcal{F}_{\sigma\mathbf{h}} \text{ in } W^{1,2}(\Omega,\mathbb{S}^2,\mathbf{e})\},\\
&\mu_*(q,\sigma) = \inf_{\mathbf{n}\in\mathcal{M}(\Omega,\sigma\mathbf{h})} \mu(q\mathbf{n}).
\end{aligned}$$

The functional $\mathcal{E}_h$ has two families of trivial critical points. One is given by

$$\psi = 0, \quad \mathbf{n} = \mathbf{n}_\sigma, \tag{3.4.3}$$

where $\mathbf{n}_\sigma \in \mathcal{M}(\Omega,\sigma\mathbf{h})$. The second family is

$$\psi = ce^{iq\mathbf{e}\cdot\mathbf{x}}, \quad \mathbf{n} = \mathbf{e}, \tag{3.4.4}$$

where c is an arbitrary complex number such that $|c| = 1$. We may compare the family (3.4.3) with the normal state of superconductors, and call them *pure nematic states*; and compare the family (3.4.4) with the Meissner state of superconductors, and call them *pure smectic states.* Superconductors under a strong magnetic field will be in the normal state. However, liquid crystals under a strong magnetic field may not be in the nematic state, as shown in the following

Theorem 3.4.1 ([55]). *Fix q, κ, $\mathbf{h}$ and $\mathbf{e}$, and $K_1 = K_2$. When σ is sufficiently large, the pure nematic states are not global minimizers of $\mathcal{E}_h$.*

Theorem 3.4.1 explores the difference between Landau-de Gennes model and Ginzburg-Landau model:

(i) For superconductivity: Strong magnetic fields penetrate and destroy superconductivity. There is a critical magnetic field H_{c_3} to distinguish the normal state and the superconducting state.

(ii) For liquid crystals: Strong magnetic fields will not completely destroy smectic structure. Liquid crystals in a sufficiently strong magnetic field will not be in a pure nematic state, and hence there is no analogue of H_{c_3} for magnetic response.

Next we consider the change of stability of the pure smectic states. Two critical magnetic fields H_{sh} and H_{s} were introduced in [55].

Theorem 3.4.2 ([55]). *Fix K_1, K_2 and χ. There exist critical fields $H_s(\kappa, q) < H_{sh}(q)$, such that:*

(i) *If $0 \le \sigma < H_{\mathrm{s}}(\kappa, q)$, the pure smectic states are the only global minimizers of the functional $\mathcal{E}_h$.*
(ii) *If $H_{\mathrm{s}}(\kappa, q) < \sigma < H_{\mathrm{sh}}(q)$, the pure smectic states are local minimizers, but not global minimizers.*
(iii) *If $\sigma > H_{\mathrm{sh}}(q)$, the pure smectic states are not local minimizers.*

We believe that the critical field H_{sh} given in [55] is an analogue to the superheating field of superconductors.

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