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On thermoelastic systems arising in shape memory alloys

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Abstract.

In this paper, we make a brief review on recent results concerning to the global existence and uniqueness of solution for thermo-elastic systems of shape memory alloys. The derivation of these systems is based on the Landau-Ginzburg theory using the shear strain (tensor) as an order parameter.

§1. Introduction

Shape memory alloys are well-known materials which, after being strained, revert back to their original shape at a certain temperature. We refer to shape memory alloys as SMA. The most effective and widely used alloys include NiTi (Nickel - Titanium), CuZnAl, and CuAlNi. Shape memory effect is due to first-order phase transitions between different equilibrium configurations of the metallic lattice, called *austenite* and *martensite.*

1.1. One-dimensional Falk model system

Falk [14] presents the Landau-Ginzburg type theory using the shear strain $\varepsilon := \partial_x u$ as an order parameter in order to explain the martensitic-austenitic phase transitions occurring in 1-D SMA. Let u denote the displacement and θ the absolute temperature. In this paper we assume that the Helmholtz free energy density F takes the following simple form:

$$\begin{aligned} F &= F(\varepsilon, \varepsilon_x, \theta) \\ &= F_0(\theta) + \widetilde{F}(\varepsilon, \theta) + \frac{\kappa}{2}\varepsilon_x^2, \end{aligned} \tag{1}$$

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where $\widetilde{F}(\varepsilon,\theta) = (\theta-\theta_c)F_1(\varepsilon) + F_2(\varepsilon)$ such that

$$F_0(\theta) = -c_v\theta\log(\theta/\theta_3) + c_v\theta + \tilde{c}, \tag{2}$$

$$F_1(\varepsilon) = \alpha_1\varepsilon^2, \tag{3}$$

$$F_2(\varepsilon) = -\alpha_2\varepsilon^4 + \alpha_3\varepsilon^6. \tag{4}$$

Here, $\tilde{c}$, α_1, α_2, α_3, and θ_3 are positive physical constants. Positive constants c_v and θ_c are carolic specific heat and critical temperature, respectively. This form accounts quite well for the experimentally observed behavior. Corresponding to this free energy density, the shear stress σ is given by

$$\sigma(\varepsilon,\theta) = \frac{\partial F}{\partial\varepsilon}(\varepsilon,\theta) = 2\alpha_1(\theta-\theta_c)\varepsilon - 4\alpha_2\varepsilon^3 + 6\alpha_3\varepsilon^5 - \kappa\varepsilon_{xx}.$$

From this we can deduce the system

$$(1F)\begin{cases} \rho u_{tt} + \kappa u_{xxxx} = (f_1(u_x)(\theta-\theta_c) + f_2(u_x))_x, \\ c_v\theta_t - k\theta_{xx} = f_1(u_x)\theta u_{xt}, \qquad (t,x)\in\mathbb{R}^+\times(0,l), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \\ u(t,0) = u(t,l) = u_{xx}(t,0) = u_{xx}(t,l) = \theta_x(t,0) = \theta_x(t,l) = 0, \end{cases}$$

where $\mathbb{R}^+ = \{t\in\mathbb{R} \mid t>0\}$, $f_1(r) = F_1'(r) = 2\alpha_1\varepsilon$ and $f_2(r) = F_2'(r) = -4\alpha_2\varepsilon^3 + 6\alpha\varepsilon^5$. Here ρ and l denote mass density and the length of a SMA rod. For more details of the Falk model system, we refer to Chapter 5 in the literature [9].

1.2. Three-dimensional Falk-Konopka model system

In 3-D case the problem is complicated. The model is based on the linearized strain tensor $\epsilon(\mathbf{u}) = (\epsilon_{ij})$ such that $\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. Here $\mathbf{u} = (u_1,u_2,u_3)$ is the displacement vector. Corresponding to (3) and (4), Falk and Konopka [15] give the form of the free energy density

$$\begin{aligned} F(\epsilon,Qu,\theta) &= F_0(\theta) + \widetilde{F}(\theta,\epsilon) + \frac{\kappa}{2}|Qu|^2 \\ &= F_0(\theta) + (\theta-\theta_c)F_1(\epsilon) + F_2(\epsilon) + \frac{\kappa}{2}|Qu|^2, \end{aligned} \tag{5}$$

where

$$F_1(\epsilon) = \sum_{i=1}^{3}\alpha_i^2 J_i^2(\epsilon) + \sum_{i=1}^{5}\alpha_i^2 J_i^4(\epsilon), \tag{6}$$

$$F_2(\epsilon) = \sum_{i=1}^{2}\alpha_i^2 J_i^6(\epsilon) \tag{7}$$

and $F_0(\theta)$ is given by (2). Here α_i^k and θ_c are constants, and J_i^k is given as follows:

$$\begin{array}{lll} J_1^2 = \epsilon_{\bar{1}}^2, & J_2^2 = 3\epsilon_{\bar{2}}^2 + \epsilon_{\bar{3}}^2, & J_3^2 = \epsilon_{\bar{4}}^2 + \epsilon_{\bar{5}}^2 + \epsilon_{\bar{6}}^2, \\ J_1^4 = (J_2^2)^2, & J_2^4 = \epsilon_{\bar{4}}^4 + \epsilon_{\bar{5}}^4 + \epsilon_{\bar{6}}^4, & J_3^4 = (J_3^2)^2, \\ J_4^4 = J_2^2 J_3^2, & \multicolumn{2}{l}{J_5^4 = \epsilon_{\bar{4}}^2(\epsilon_{\bar{2}} - \epsilon_{\bar{3}})^2 + \epsilon_{\bar{5}}^2(\epsilon_{\bar{2}} + \epsilon_{\bar{3}})^2 + 4\epsilon_{\bar{6}}^2\epsilon_{\bar{2}}^2,} \\ J_1^6 = (J_2^2)^3, & J_2^6 = \epsilon_{\bar{2}}^2(\epsilon_{\bar{2}}^2 - \epsilon_{\bar{3}}^2)^2 & \end{array}$$

with

$$\epsilon_{\bar{1}} = \operatorname{trace}\epsilon/3, \qquad \epsilon_{\bar{2}} = (2\epsilon_{33} - \epsilon_{11} - \epsilon_{22})/6,$$
$$\epsilon_{\bar{3}} = (\epsilon_{11} - \epsilon_{22})/2, \qquad \epsilon_{\bar{4}} = \epsilon_{23}, \qquad \epsilon_{\bar{5}} = \epsilon_{13}, \qquad \epsilon_{\bar{6}} = \epsilon_{12}.$$

Comparing with 1-D case, $F_1(\epsilon)$ must be taken the fourth order with respect to ϵ.

From an argument similar to 1-D case, Pawłow [26] derives 3-D thermoelasticity system of shape memory alloys:

$$(3FK)\begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} = \nabla\cdot F_{,\epsilon}(\epsilon,\theta), & \\ c_v\theta_t - k\Delta\theta = \theta F_{,\theta\epsilon}(\epsilon,\theta) : \epsilon_t & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta\cdot n = 0 & \text{on } S_T, \\ (\mathbf{u}(0,\cdot),\mathbf{u}_t(0,\cdot)) = (\mathbf{u}_0,\mathbf{u}_1), \quad \theta(0,\cdot) = \theta_0 \ge 0 & \text{in } \Omega, \end{cases}$$

where $\Omega_T = (0,T]\times\Omega$ and $S_T = [0,T]\times\partial\Omega$. We use the notations $F_{,\epsilon} = (\frac{\partial F}{\partial\epsilon_{ij}})_{i,j=1}^3$, $F_{,\theta} = (\frac{\partial F}{\partial\theta})$ and $\tilde{\epsilon}:\epsilon = \sum_{i,j=1}^3 \tilde{\epsilon}_{ij}\epsilon_{ij}$. We define the linearized elasticity operator Q by the following second order differential operator

$$Qu = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}).$$

We assume that the Lamé constants λ and μ satisfy

$$(8) \qquad \mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0,$$

which assure the strong ellipticity of Q.

In Sections 2 and 3, we introduce the results for the one-dimensional Falk model system and the results for the multi-dimensional system, respectively. In Section 4, we remark several results for the model with hysteresis represented by a subdifferential operator. We denote the i-th equation in the system (X) by $(X)_i$.

§2. One-dimensional system

In this section we introduce the results for the one-dimensional system $(1F)$ and the related topics. Sprekels and Zheng [23] prove the unique global existence of smooth solution for $(1F)$. In [10], Bubner and Sprekels establish the unique global existence of $(1F)$ for data $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$, and discussed the optimal control problem in the case

$$(A0) \qquad f_1(r) = 2\alpha_1 r \quad \text{and} \quad f_2(r) = 6\alpha_3 r^5 - 4\alpha_2 r^3.$$

Aiki [2] prove the unique global existence of solution with $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$ for more general nonlinearity, that is,

$$(A1) \qquad f_1, f_2 \in C^2(\mathbb{R})$$

and

$$(A2) \qquad F_2(r) \geq -C \text{ for } r \in \mathbb{R}.$$

We note that the condition $(A0)$ implies the conditions $(A1)$ and $(A2)$.

Although the literature [9] says that there is no interior friction from the experimental evidence, systems related to $(1F)$ have been studied for the case of viscous materials which has the stress σ containing additional viscous component of the following form:

$$\sigma = \frac{\partial F}{\partial \varepsilon} + \nu \varepsilon_t,$$

where the viscosity coefficient ν is positive constant. Correspondingly, the equations $(1F)$ are modified as follows:

$$(1FV) \begin{cases} \rho u_{tt} + \kappa u_{xxxx} - \nu u_{xxt} = (f_1(\varepsilon)\theta + f_2(\varepsilon))_x, \\ c_v \theta_t - k\theta_{xx} = f_1(\varepsilon)\theta\varepsilon_t + \nu|\varepsilon_t|^2. \end{cases}$$

The viscosity term simplifies the analysis because this term has smoothing property. In fact, K.-H. Hoffmann and Żochowski establish the unique global existence result by decomposing $(1FV)_1$ into a system of two parabolic equations in [17]. There are also some results for the system without capillarity (i.e. $\kappa = 0$ and $\nu > 0$) called *thermoviscoelasticity* (see e.g. [12]). Sprekels, Zheng and Zhu [24] study the asymptotic behavior of the solution for $(1FV)$ as $t \to \infty$. However, it seems that the asymptotic behavior of the solution for $(1F)$ has not been determined.

Here the spaces W_p^m and H^m are the standard Sobolev spaces, that is, W_p^m is equipped with the norm

$$\|f\|_{W_p^m} = \sum_{0\le k\le m} \|D_x^k f\|_{L^p},$$

and $H^m = W_2^m$.

Our result is the unique global existence for $(1F)$ in energy class. We first consider the initial value problem: (1F)'

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \qquad (t,x) \in \mathbb{R}^+ \times \Omega, \tag{9}$$

$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt}, \tag{10}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \tag{11}$$

which is closely related to $(1F)$.

Theorem 2.1 ([32], [35]). *Suppose that $\theta_0 \ge 0$.*
(i) Assume that $\Omega = \mathbb{R}$ and (A0). Let any p, $q \in [2,\infty]$ and r be fixed such that

$$\frac{2}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{12}$$

Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$, there exists a unique solution (u, θ) to the problem $(1F)'$ satisfying

$$\begin{aligned} &u \in C(\mathbb{R}^+; H^2(\mathbb{R})), && u_{xx} \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{R})),\\ &u_t \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R})), && u_t \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{R})),\\ &\theta \in C(\mathbb{R}^+; L^1(\mathbb{R})), && \theta_x \in L^r_{loc}(\mathbb{R}^+; L^{q'}(\mathbb{R})). \end{aligned}$$

(ii) Assume that $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and (A1)–(A2). Let any p, $q \in [2,4]$ and r be fixed such that

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}, \qquad r > p', \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{2q'} > 1. \tag{13}$$

Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$, there exists a unique solution (u, θ) to the problem $(1F)'$ satisfying

$$\begin{aligned} &u \in C(\mathbb{R}^+; H^2(\mathbb{T})), && u_{xx} \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{T})),\\ &u_t \in L^\infty(\mathbb{R}^+; L^2(\mathbb{T})), && u_t \in L^p_{loc}(\mathbb{R}^+; L^q(\mathbb{T})),\\ &\theta \in C(\mathbb{R}^+; L^1(\mathbb{T})), && \theta_x \in L^r_{loc}(\mathbb{R}^+; L^{q'}(\mathbb{T})). \end{aligned}$$

Our main tools of the proof of this theorem are the maximal regularity estimate and the Strichartz estimate. In general, the derivative of a solution for most of the equations is less regular than the right-hand side of the corresponding equations. However for parabolic equations such a loss of regularity does not occur, as in the case of elliptic equations. The estimate ensuring this regularity is called the maximal regularity. For this estimate, we refer to [7], [20] and [21]. The Strichartz estimate established in [31] is closely related to the restriction theory of the Fourier transform to surfaces and used often in various areas of the study of nonlinear wave and dispersive equations (see [11]).

Corresponding results in the spatially periodic setting are established by J. Bourgain [8], and more transparent version is given by Fang and Grillakis in [16]. Therefore we consider the following initial value problem with periodic boundary conditions, which is closely related to $(1F)$. From a physical point of view, the problem $(1F)'$ describes the dynamics of the ring made of shape memory alloys. Then it is a very interesting problem. Moreover, the following theorem for $(1F)$ can be proved in the same way as Theorem 2.1. Roughly speaking, extending the solutions u and θ of $(1F)$ as odd and even periodic functions respectively, we regard the initial boundary value problem as the problem with periodic boundary conditions.

Theorem 2.2 ([32], [35]). *For the initial boundary value problem $(1F)$, the same conclusion as Theorem 2.1 (ii) holds.*

Remark. (i) We note that the nonlinear term of $(1F)_2'$ and $(1F)_2$ is rewritten as the following form:

$$f_1(u_x)\theta u_{tx} = (f_1(u_x)\theta u_t)_x - f_1'(u_x)u_{xx}\theta u_t - f(u_x)\theta_x u_t,$$

which makes sense in the distribution class.
(ii) If we take $q = 2$ in Theorem 2.1, the Strichartz estimate does not necessarily needed. This is because we can take a number p greater than q in the maximal regularity although the estimate appeared only for the condition $p = q$ in [20]. In other words, we can say that the smoothing effect of the heat equation contribute the estimate of the Boussinesq type equation.

§3. Multi-dimensional system

In this section we treat the multi-dimensional thermoelastic system. Comparing (6) with 1-D case (3), in 3-D case $F_1(\epsilon)$ must be taken as the fourth order with respect to ϵ. This causes some difficulties to treat the system $(3FK)$. Moreover it also causes the difficulty that the useful embedding $H^1 \hookrightarrow L^\infty$ does not hold in multi-dimensional case. Indeed, there have been no papers on the solvability of $(3FK)$. Therefore we first consider the system with viscosity such as $(1FV)$, namely, the system with shear stress tensor σ satisfying that

$$\sigma = F_{,\epsilon}(\epsilon, \theta) - \kappa A\epsilon(\nabla \cdot A\epsilon(\mathbf{u})) + \nu A\epsilon_t.$$

Here, the fourth order tensor A represents linear isotropic Hooke's law, defining by

$$A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We note that the tensor has the following symmetry properties

$$A_{ijkl} = A_{klij}, \quad A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk},$$

and the relation $Qu = \nabla \cdot \varepsilon(\mathbf{u})A$ holds. The assumption (8) assures the strong ellipticity of the operator Q and the following inequality

$$a_*|\epsilon|^2 \leq (A\epsilon) : \epsilon \leq a^*|\epsilon|^2,$$

where $a_* = \min\{3\lambda + 2\mu, 2\mu\}$ and $a^* = \max\{3\lambda + 2\mu, 2\mu\}$.

In addition, we generalize the nonlinearity $\widetilde{F}(\epsilon, \theta) = G(\theta)F_1(\epsilon) + F_2(\epsilon)$. Then we can deduce the following quasilinear system:

$$(3FV) \begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla \cdot [G(\theta)F_{1,\epsilon}(\epsilon) + F_{2,\epsilon}(\epsilon)], & \\ \{c_v - \theta G''(\theta)H(\epsilon)\}\theta_t - k\Delta\theta = \theta G'(\theta)\partial_t F_1(\epsilon) & \\ \qquad\qquad\qquad\qquad\qquad\qquad + \nu(A\varepsilon_t) : \varepsilon_t & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta \cdot n = 0 & \text{on } S_T, \\ (\mathbf{u}(0,\cdot), \mathbf{u}_t(0,\cdot)) = (\mathbf{u}_0, \mathbf{u}_1), \quad \theta(0,\cdot) = \theta_0 \geq 0 & \text{in } \Omega. \end{cases}$$

We note that if $G(\theta) = C(\theta - \theta_c)$ then the quasilinear term $\theta G''(\theta)H(\varepsilon)\theta_t$ does not appear. In this article, we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$. We consider the following structure of the nonlinearity: $\widetilde{F}(\theta, \epsilon) = G(\theta)F_1(\epsilon) + F_2(\epsilon)$ satisfies that

(i) $G \in \mathrm{C}^4(\mathbb{R}, \mathbb{R})$ is as follows:

$$G(\theta) = \begin{cases} C_1\theta & \text{if } \theta \in [0, \theta_1], \\ \varphi(\theta) & \text{if } \theta \in [\theta_1, \theta_2], \\ C_2\theta^r & \text{if } \theta \in [\theta_2, \infty), \end{cases}$$

where $\varphi \in \mathrm{C}^4(\mathbb{R}, \mathbb{R})$, $\varphi'' \le 0$ and C_1 and C_2 are positive constants for some fixed θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < \infty$. We extend G defined on $\mathbb{R}$ as an odd function.

(ii) $F_1 \in \mathrm{C}^4(\mathbb{S}^2, \mathbb{R})$ satisfies that $F_1(\varepsilon) \ge 0$, where $\mathbb{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^3$.

(iii) $F_2 \in \mathrm{C}^4(\mathbb{S}^2, \mathbb{R})$ satisfies that $F_2(\varepsilon) \ge -C_3$, where C_3 is a real constant.

(iv) $F_1(\varepsilon)$ and $F_2(\varepsilon)$ satisfy the following growth conditions:

$$
\begin{aligned}
&|F_{1,\epsilon}(\epsilon)| \le C|\epsilon|^{K_1-1}, && |F_{2,\epsilon}(\epsilon)| \le C|\epsilon|^{K_2-1},\\
&|F_{1,\epsilon\epsilon}(\epsilon)| \le C|\epsilon|^{K_1-2}, && |F_{2,\epsilon\epsilon}(\epsilon)| \le C|\epsilon|^{K_2-2},\\
&|F_{1,\epsilon\epsilon\epsilon}(\epsilon)| \le C|\epsilon|^{K_1-3}, && |F_{2,\epsilon\epsilon\epsilon}(\epsilon)| \le C|\epsilon|^{K_2-3}
\end{aligned}
$$

for large $|\epsilon|$.

Pawłow and Żochowski [27] study the energy density F under several stronger assumptions than (6)–(7), namely, lower order powers of nonlinearity. Moreover, for the simplification of treatments they first consider the semilinearized equations:

$$
(3SLFV)\begin{cases} \rho \mathbf{u}_{tt} + \kappa Q^2 \mathbf{u} - \nu Q \mathbf{u}_t = \nabla \cdot F_{,\epsilon}(\epsilon, \theta), \\ c_v \theta_t - k\Delta\theta = \theta F_{,\theta\epsilon}(\epsilon,\theta) : \varepsilon_t + \nu(A\epsilon_t) : \epsilon_t, \end{cases}
$$

which is the model $(3FV)$ removed quasilinear term $\theta G''(\theta)\theta_t H(\varepsilon)$. They show unique global existence of the sufficiently smooth solution for $(3SLFV)$ under

$$
(14)\quad 0 \le r < \frac{1}{2}, \qquad 0 \le K_1 \le \left(\frac{1}{2} - r\right) K_2 + 1, \qquad \text{and} \qquad 0 \le K_2 \le \frac{7}{2}.
$$

In addition, due to the applied parabolic decomposition of elasticity system, they assumed the $0 < 2\sqrt{\kappa} < \nu$ between viscosity and capillarity. Such assumption, however, seems not realistic for SMA viscosity effects of which are negligible small. In [33] the unique global existence to $(3SLFV)$ of the solution in a larger class by using the contraction mapping principle. The result is proved under no conditions between κ and ν, and the class of nonlinearities is generalized to $K_2 < 6$. We remark again that if we take $r = 1$ then quasilinear term $\theta G''(\theta)H(\varepsilon)\theta_t$ of $(3FV)_2$ does not appear. The first two assumptions of (14) are appeared due to the semilinearization which causes the lack of energy conservation laws. Three-dimensional thermoviscoelasticity system corresponding to [12] is treated in [37]. For the viscoelastic system neglecting heat conduction we refer to [30]. Recently, Pawłow and Zajączkowski [28] have proved the unique global existence for the quasilinear system $(3FV)$ under

$$(15) \qquad 0 \le r < \frac{2}{3}, \qquad 15K_1 + 4r \le 15, \qquad \text{and} \qquad 0 < K_2 < \frac{7}{2}.$$

We [36] show the unique global existence of solution for $(3FV)$ under the following power of nonlinearity:

$$(3\mathrm{A}) \qquad 0 \le r < \frac{5}{6}, \qquad 0 \le K_1,\ K_2 < 6, \qquad \text{and} \qquad 6r + K_1 < 6.$$

In addition, we admit arbitrary positive coefficients of capillarity $\kappa > 0$ and viscosity $\nu > 0$. Unfortunately, our assumption $(3A)$ can not also cover the case (6)–(7).

Here we add some remarks on 2-D case. We can deduce 2-D model $(2FV)$ from obvious modifications. In [27] they also show the unique global existence for the 2-D system $(2SLFV)$ which is the semilinearized model of $(2FV)$. The unique global existence for the quasilinear system $(2FV)$ is established in [29] under the assumption:

$$(16) \qquad 0 \le r < \frac{7}{8} \qquad \text{and} \qquad 0 \le K_1, K_2 < \infty.$$

In [34] the unique global existence under $r = 1$ are proved under several other strong assumptions. Roughly speaking, the restrictions in [34] are $K_1 = 0$ and the energy of initial data $\|u_0\|_{H^2} + \|u_1\|_{L^2} + \|\theta_0\|_{L^1}$ is sufficiently small. We show that the system $(2FV)$ has a unique global solution under the assumptions:

$$(2\mathrm{A}) \qquad 0 \le r < 1 \qquad \text{and} \qquad 0 \le K_1, K_2 < \infty.$$

Comparing these assumptions with (16), we see that the restriction for r is weaker, nevertheless we cannot admit $r = 1$.

Before stating our results more precisely, we introduce some function spaces.

- $W_p^{2l,l}(\Omega_T)$ is the Sobolev space equipped with the norm

$$\|u\|_{W_p^{2l,l}(\Omega_T)} := \sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \|D_t^r D_x^\alpha u\|_{L^p(\Omega_T)},$$

where $D_t := i\frac{\partial}{\partial t}$, $D_x^\alpha = \prod_{\alpha=\alpha_1+\alpha_2+\alpha_3} D_k^{\alpha_k}$ and $D_k := i\frac{\partial}{\partial x_k}$ for multi index $\alpha = (\alpha_i)_{i=1}^d$.

- $B^s_{p,q} = B^s_{p,q}(\Omega)$ is the Besov space. Namely, $B^s_{p,q} := [L^p(\Omega), W^j_p(\Omega)]_{s/j,q}$, where $[X,Y]_{s/j,q}$ is the real interpolation space. For more details we refer to [1] and [25].

We now state our result.

Theorem 3.1 ([36]). *Let d is 2 or 3. Fix $d+2 < p \le q < \infty$. Assume that $\min_\Omega \theta_0 \ge 0$, $\nu > 0$ and (dA) holds. Then for any $T > 0$ and $(u_0, u_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times B^{2-2/q}_{q,q}$, there exists a unique solution (u, θ) to (dFV) satisfying*

$$(u,\theta) \in W^{4,2}_p(\Omega_T) \times W^{2,1}_q(\Omega_T) =: V_T(p,q).$$

Moreover, if we assume $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$\theta \ge \theta_* \exp(-\omega t),$$

where ω depends only on A, θ_* and F.

We prove the existence part in Theorem 3.1 by using the Leray-Schauder fixed point principle. Key estimates are the maximal regularity estimate for the first equation of $(3FV)$ and the classical energy estimate and the parabolic De Giorgi method for the second equation of $(3FV)$. In general, the derivative of a solution is less regular than the right-hand side of the corresponding equations. However for parabolic equations such a loss of regularity does not occur, as in the case of elliptic equations. The estimate ensuring this regularity is called the maximal regularity. For more precisely information of the maximal regularity, we refer to [7]. In particular, for more recently topics of the maximal L^p-regularity we refer to [13]. Since the maximal regularity theory is limited to linear parabolic equations, we cannot use it directly for the quasilinear equation $(3FV)_2$. To obtain the higher order a priori estimates we also use the classical energy methods and the parabolic De Giorgi method (see [20], [22]). Using these methods we can show the Hölder continuity of θ. From this regularity assertion, we arrive at the estimate for the higher Sobolev norm $V_T(p,q)$.

§4. Models including a hysteresis operator

Another interesting property of shape memory alloys is *hysteresis*. There are a lot of models and results from this point of view. Krejčí and Sprekels [18] have derived the system with the hysteresis by using the Prandtl-Ishlinskii operator, and have established the unique global existence of the weak solution for this system. For this model with internal viscosity, the unique global existence of the strong solution is showed by Krejčí and Sprekels in [19].

Recently, Aiki and Kenmochi derive the system in which the hysteresis effect is represented by using a subdifferential operator:

$$(1H)\begin{cases} \rho u_{tt} + \kappa u_{xxxx} - \nu u_{xxt} = \sigma_x, & \\ c_v\theta_t - k\theta_{xx} = \sigma\varepsilon_t + \nu\varepsilon_t^2, & \\ \sigma_t - \gamma\sigma_{xx} + \partial I(\theta,\varepsilon;\sigma) \ni c\varepsilon_t & \text{in } (0,T]\times(0,1), \\ u = u_{xx} = \theta_x = 0 & \text{on } [0,T]\times\{0,1\}, \\ u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1, & \\ \theta(0,\cdot) = \theta_0,\ \sigma(0,\cdot) = \sigma_0 & \text{on } [0,1], \end{cases}$$

where $(1H)_3$ for shear stress σ includes the subdifferential ∂ of the indicator function I of the closed interval $[f_a(\theta,\varepsilon), f_d(\theta,\varepsilon)]$ for given continuous functions f_a and f_d on $\mathbb{R}\times\mathbb{R}$ with $f_a \le f_d$, that is,

$$I(\theta,\varepsilon;\sigma) = \begin{cases} 0 & \text{if } f_a(\theta,\varepsilon) \le \sigma \le f_d(\theta,\varepsilon), \\ +\infty & \text{otherwise.} \end{cases}$$

This system is introduced and established the unique global existence of this system under $\gamma > 0$ and $\nu^2 > 4\kappa$ in [6]. The assumption $\gamma > 0$ is needed from the technical reason to obtain the smoothness of σ. In [4] Aiki proved the unique global existence for $(1H)$ under $\gamma = 0$. Moreover, in [5], the assumption $\nu^2 > 4\kappa$ also has been removed.

For 3-D case Aiki [3] consider the following system:

$$(3H)\begin{cases} \rho\mathbf{u}_{tt} + \kappa Q^2\mathbf{u} - \nu Q\mathbf{u}_t = \nabla\cdot\sigma, & \\ c_v\theta_t - k\Delta\theta = \sigma : \epsilon_t + \nu A\epsilon_t : \epsilon_t, & \\ \sigma_t - \gamma\Delta\sigma + \partial I(\theta,\epsilon;\sigma) \ni c\epsilon_t & \text{in } \Omega_T, \\ \mathbf{u} = Q\mathbf{u} = \nabla\theta\cdot n = 0 & \text{on } S_T, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{u}_t(0,\cdot) = \mathbf{u}_1, & \\ \theta(0,\cdot) = \theta_0,\ \sigma(0,\cdot) = \sigma_0 & \text{on } \Omega, \end{cases}$$

and prove the unique global existence for the system under $\gamma > 0$. Although in the paper Δ was used instead of Q and the assumption $\nu^2 > 4\kappa$

was also needed, these restriction can be removed by combining with the argument as in [33]. However, the assumption $\gamma > 0$ is still needed. The unique global existence for $(3H)$ in $\gamma = 0$ has not been shown yet. There seems not to be the research for the system without viscosity (i.e. $\nu = 0$) seems not to be also known even for 1-D case.

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