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Diffusion phenomenon for abstract wave equations with decaying dissipation

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Abstract.

We consider the initial value problem of the abstract wave equation with dissipation whose coefficient tends to 0 as $t \to \infty$. In the case that the coefficient of the dissipation is a positive constant, Ikehata–Nishihara and Chill–Haraux obtained the decay estimate of the difference between the solution of this equation and the solution of the corresponding abstract heat equation. In the case that $H = L^2(\mathbb{R}^n)$ and A is the Laplace operator and $b(t)$ is a positive valued monotone C^2 function satisfying the sufficient assumption, Wirth obtained the decay estimate of the difference between the solution of the dissipative wave equation and the solution of the corresponding heat equation. The purpose of this paper is to show the decay estimate of the difference between the solution of the abstract wave equation with decaying dissipative term and the solution of the corresponding abstract parabolic equation.

§1. Introduction

Let H be a separable Hilbert space with norm $\|\cdot\|$. Let A be a non-negative self-adjoint operator with domain $\mathcal{D}(A)$. Then, for a non-negative number γ, the space $\mathcal{D}(A^\gamma)$ becomes a Hilbert space with the graph-norm of A^γ denoted by $\|\cdot\|_\gamma$.

For a positive valued C^1 function $b(t)$ on $[0, \infty)$, we consider the difference of the solution of the initial value problem of the abstract dissipative wave equation

$$u'' + b(t)u' + Au = 0, \quad u(0) = u_0, \quad u'(0) = u_1, \tag{1.1}$$

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and the solution of the corresponding heat equation (1.2):

$$b(t)v' + Av = 0, \quad v(0) = v_0, \tag{1.2}$$

where

$$\begin{aligned} v_0 &= u_0 + u_1 \int_0^\infty \exp(-\int_0^s b(\sigma)d\sigma)ds, \\ &= u_0 + \frac{u_1}{b(0)} - u_1 \int_0^\infty \frac{b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma)ds. \end{aligned} \tag{1.3}$$

Let Ω be an exterior domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, or $\Omega = \mathbb{R}^n$. Let $H = L^2(\Omega)$ and let $A_D = A_R = -\Delta$ with domain

$$\mathcal{D}(A_D) = \{u \in H^2(\Omega);\ u(t,x) = 0 \text{ on } \partial\Omega\},$$

$$\mathcal{D}(A_R) = \{u \in H^2(\Omega);\ \frac{\partial u}{\partial \nu}(t,x) + \sigma(x)u(t,x) = 0 \text{ on } \partial\Omega\},$$

where $\sigma(x)$ is a non-negative smooth function on $\partial\Omega$. Then A_D and A_R become non-negative self-adjoint operators (see Mizohata [8, Chapter 3, section 16] for A_R), and the abstract dissipative wave equation (1.1) becomes the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b(t)\frac{\partial u}{\partial t} = 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{1.4}$$

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x) \qquad \text{in } \Omega, \tag{1.5}$$

with the Dirichlet boundary condition

$$u(t,x) = 0 \qquad \text{on } [0,\infty) \times \partial\Omega \tag{1.6}$$

in the case $A = A_D$, or with the Robin boundary value condition

$$\frac{\partial u}{\partial \nu}(t,x) + \sigma(x)u(t,x) = 0 \qquad \text{on } [0,\infty) \times \partial\Omega. \tag{1.7}$$

The abstract heat equation (1.2)–(1.3) becomes the following initial boundary value problem:

$$b(t)\frac{\partial v}{\partial t} - \Delta v = 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{1.8}$$

$$v(0,x) = v_0(x), \qquad \text{in } \Omega, \tag{1.9}$$

where

$$v_0(x) = u_0(x) + u_1(x) \int_0^\infty \exp(-\int_0^s b(\sigma)d\sigma)ds, \tag{1.10}$$

with the Dirichlet boundary condition

$$v(t,x) = 0 \qquad \text{on } [0,\infty) \times \partial\Omega \tag{1.11}$$

in the case $A = A_D$, or with the Robin boundary value condition

$$\frac{\partial v}{\partial \nu}(t,x) + \sigma(x)v(t,x) = 0 \qquad \text{on } [0,\infty) \times \partial\Omega \tag{1.12}$$

in the case $A = A_R$. Here we note that if we take $\sigma \equiv 0$, then (1.7) and (1.12) are the Neumann boundary conditions.

In the case that $b(t)$ is a positive constant and that A is the Dirichlet-Laplace operator on $L^2(\Omega)$ in exterior domain, Ikehata [6] showed that the L^2 norm of the difference between the solution u of (1.4)–(1.6) with $b(t) \equiv 1$ and the solution v of the corresponding heat equation (1.8)–(1.11) decays faster than each of the solution does (diffusion phenomenon). Ikehata–Nishihara [7] showed the diffusion phenomenon for the abstract dissipative wave equation (1.1), by obtaining the decay estimate of the norm of the difference between the solution u of (1.1) with $b(t) \equiv 1$ and the solution v of the corresponding abstract parabolic equation (1.2)–(1.3). This estimate is an improvement of the previous one in [6]. Chill–Haraux [3] improved their estimate to the following one:

$$\|u(t) - v(t)\|_{1/2} \le Ct^{-1}(\|u_0\|_{1/2} + \|u_1\|) \quad \text{for every } t \ge 1. \tag{1.13}$$

[3] also showed that this estimate is optimal in the sense that the inequality

$$\limsup_{t\to\infty} \sup_{\|u_0\|_{1/2}, \|u_1\| \le 1} t\|u(t) - v(t)\|_{1/2} > 0$$

holds, when 0 belongs to the essential spectrum of A.

Recently, in the case that A is the Laplace operator on $L^2(\mathbb{R}^n)$, Wirth [10] considered Cauchy problem of (1.4)–(1.5) in $\mathbb{R}^n$, where $b(t)$ is a positive valued C^2 function satisfying the following assumption:

$$b(t) \text{ is a monotone function,} \tag{1.14}$$

$$\lim_{t\to\infty} \frac{|b'(t)|}{b(t)^2} = 0, \tag{1.15}$$

$$\left| \frac{d^k}{dt^k} b(t) \right| \le \frac{b(t)}{(t+1)^k} \quad \text{for} \quad k = 1, 2, \tag{1.16}$$

$$\int_0^\infty \frac{ds}{b(s)^3} < \infty. \tag{1.17}$$

$$\int_0^\infty \frac{ds}{b(s)(1+s)^2} < \infty. \tag{1.18}$$

Then, by using the WKB-representation of the solution, he obtained the estimate

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} \leq C\frac{1+t}{b(t)^3}\left(\int_0^t \frac{ds}{b(s)}\right)^{-2}, \tag{1.19}$$

in the case $b(t)$ is monotone decreasing, where u is the solution of the Cauchy problem of (1.4)–(1.5) in $\mathbb{R}^n$ and v is the solution of the corresponding parabolic equation (1.8)–(1.10) in $\mathbb{R}^n$. For the estimate in the case $b(t)$ is monotone increasing, see [10]. Wirth noted in [10] that for the WKB representation of the solution, the monotonicity of $b(t)$ is weakened to the assumption that $b(t)$ is a small perturbation of a monotone function γ; there exists $\gamma \in C^1([0,\infty);(0,\infty))$ such that γ is monotone and $\lim_{t\to\infty} t\gamma(t) = \infty$, and that

$$|b(t) - 2\gamma(t)| \leq c\frac{\gamma(t)}{1+t}. \tag{1.20}$$

The purpose of this paper is to show the decay estimates of the difference between the solution $u(t)$ of the abstract hyperbolic equation (1.1) and the solution $v(t)$ of the corresponding abstract parabolic equation (1.2)–(1.3) under the following assumption on $b(t)$:

Assumption. $b(t)$ is a positive valued C^1 function on $[0,\infty)$ satisfying the following: There exist positive constants b_0, b_1, b_2 and θ and strictly monotone decreasing continuous function $f(t)$ on $[t_0,\infty)$ and monotone decreasing continuous function $g(t)$ on $[t_0,\infty)$ for some $t_0 \geq 0$ satisfying

$$\lim_{t\to\infty} g(t) = 0 \tag{1.21}$$

such that the following inequalities for every $t \geq t_0$:

$$b_0 f(t) \leq b(t) \leq b_1 f(t), \tag{1.22}$$

$$b_2 f(t) \leq f(2t), \tag{1.23}$$

$$f(t) \geq b_3(t+1)^{\theta-1}, \tag{1.24}$$

$$\frac{|b'(t)|}{b(t)^2} \leq g(t), \tag{1.25}$$

$$\sup_{t \geq t_0} b(t)^2 \int_0^t \frac{g(s)^2}{b(s)} ds < \infty. \tag{1.26}$$

We easily see that the following functions satisfy the above assumption on $b(t)$.

Example 1. $b(t) = (t+1)^{-\alpha} \quad (0 \leq \alpha < 1)$.

Example 2. $b(t) = (t+1)^{-\alpha} \log(t+2) \quad (0 \leq \alpha < 1)$.

Example 3. $b(t) = (t+1)^{-\alpha}(\sin(t^\beta) + 2) \quad (\alpha, \beta > 0, \alpha + 2\beta < 1)$

Example 4. $b(t) = (t+1)^{-\alpha}(\sin(\log(t+1)) + 2) \quad (0 \leq \alpha < 1)$.

Here we note that $b(t)$ in Examples 1 and 2 also satisfy Wirth's assumption. However $b(t)$ in Example 3 does not satisfy (1.14) (nor (1.20)) nor (1.16), and $b(t)$ in Example 4 does not satisfy (1.14) (nor (1.20)).

Chill-Haraux [3] obtained the estimate (1.13) as follows: They showed that the restriction of each of the solutions of (1.1) and (1.2) with $b(t) \equiv 1$ in high frequency region decays exponentially, by using the energy method. In low frequency region, they estimated the difference between solutions of (1.1) and (1.2)–(1.3) by using an explicit formula of the solution of the dissipative abstract wave equation (1.1) with $b(t) \equiv 1$. Wirth [10] obtained the decay estimate (1.19) by using the WKB-representation of the solutions. Here, we use a different method. We show that the restriction of the each solution of (1.1) and (1.2) in high frequency region decays exponentially by using the energy method similar to [3], with the separating points and the energy depending on t. We cannot use the method of [3] for the estimate in low frequency region, since we do not have the direct representation formula as in [3], of the solution of (1.1) for general function $b(t)$. We transform the equation (1.1) into a system of integral equations, and from integral inequalities we give the decay estimate of the difference of the solutions in low frequency region.

As applications, we obtain the estimate of the difference between the solution of the initial boundary value problem (1.4)–(1.5) with boundary condition (1.6) or (1.7) and the solution of the corresponding parabolic equation (1.8)–(1.10) with (1.11) or (1.12), respectively.

Throughout this paper, we assume that $b(t)$ satisfies (1.21)–(1.26), and consider the solution of (1.1) as the mild solutions, the unique existence of which is well-known. As is well-known, if the initial data (u_0, u_1) belongs to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, then the solution becomes the strong solution.

Last we note that, independently of Wirth [10], we gave in [11] the detailed proof of corresponding theorems under the following assumption on $b(t)$: $b(t)$ is a C^1 function on $[0,\infty)$ satisfying the following.

$$b_0(t+1)^{-\alpha} \le b(t) \le b_1(t+1)^{-\alpha}, \tag{1.27}$$

$$|b'(t)| \le b_2(t+1)^{-\alpha-1}, \tag{1.28}$$

where $0 < \alpha < 1$, $b_0, b_1, b_2 > 0$ are constants.

It is easy to see that the assumption (1.27) and (1.28) is stronger than the assumption (1.21)–(1.26).

§2. Results

2.1. Abstract theorems

Our main result in this paper is the following, which gives the estimate of the difference between the solution $u(t)$ of the abstract wave equation (1.1) and the solution $v(t)$ of the corresponding abstract parabolic equation (1.2)–(1.3):

Theorem 1. *Let β and γ be arbitrary non-negative numbers. Let T_0 be an arbitrary positive number. Then there exists a positive constant C such that the following estimates hold for every $(u_0, u_1) \in (\mathcal{D}(A^{\beta+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^\beta) \cap \mathcal{R}(A^\gamma))$:*

$$\begin{aligned} &\left\|A^\beta(u(t)-v(t))\right\|_{1/2} \\ &\le C\frac{1}{tb(t)}\left(\frac{b(t)}{t}\right)^{\beta+\gamma}\Big(\|u_0\|_{\beta+1/2}+\|\tilde{u_0}\|+\|u_1\|_\beta+\|\tilde{u}_1\|\Big), \end{aligned} \tag{2.1}$$

for every $t \ge T_0$, where u and v are the solution of the equation (1.1) and (1.2)–(1.3) respectively, and $\tilde{u_0}$ and $\tilde{u}_1$ are elements of H such that $u_0 = A^\gamma \tilde{u_0}$ and that $v_0 = A^\gamma \tilde{v_0}$ respectively. Here $\mathcal{R}(A^\gamma)$ denotes the range of A^γ.

Remark 1. If 0 is an eigenvalue of A and $\gamma > 0$, then the elements $\tilde{u_0}$ and $\tilde{v_0}$ are not determined uniquely.

Remark 2. By the same argument as in the proof of Theorem 1.3 by Chill-Haraux [3], we have the individual estimate by using the estimate (2.1),

$$\lim_{t\to\infty} tb(t)\left(\frac{t}{b(t)}\right)^{\beta+\gamma}\left\|A^\beta(u(t)-v(t))\right\|_{1/2} = 0,$$

for each fixed initial data $(u_0, u_1) \in (\mathcal{D}(A^{\beta+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^\beta) \cap \mathcal{R}(A^\gamma))$. Similar individual estimates hold in Theorem 2 and corollaries.

Remark 3. Taking $\beta = \gamma = 0$ in Theorem 1, we obtain

$$\|u(t) - v(t)\|_{1/2} \leq C\frac{1}{b(t)t}\left(\|u_0\|_{1/2} + \|u_1\|\right) \tag{2.2}$$

for every $t \geq T_0$. By the assumption (1.22) and (1.24), we have $tb(t) \geq b_0 b_3 (t+1)^\theta$ $(\theta > 0)$ for $t \geq t_0$. On the other hand, when 0 is an eigenvalue of A or belongs to the essential spectrum of A, we easily see that

$$\limsup_{t\to\infty} \sup_{\|v(0)\|\leq 1} \|v(t)\| = 1.$$

Hence the solution of (1.2) and therefore the solution of (1.1) itself are not bounded by $C\left(\|u_0\|_{1/2} + \|u_1\|\right)/tb(t)$. Thus, (2.2) implies the diffusion phenomenon of (1.1).

Next we obtain that if the initial data belongs to $\mathcal{R}(A^\gamma)$, the solution itself decays faster accordingly to γ:

Theorem 2. *Let $\tilde{\beta}$ and γ be arbitrary non-negative numbers. Then there exists a positive constant C depending only on α, $\tilde{\beta}$, γ, b_0, b_1,b_2 and the operator A, such that the following estimates hold for every $(u_0, u_1) \in (\mathcal{D}(A^{\tilde{\beta}+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^{\tilde{\beta}}) \cap \mathcal{R}(A^\gamma))$:*

$$\begin{aligned} &\|A^\beta u(t)\| \\ &\leq C\left(1 + \frac{t}{b(t)}\right)^{-\beta} \left(\|u_0\|_\beta + \|\tilde{u}_0\| + \|u_1\|_{\max\{\beta-\frac{1}{2},0\}} + \|\tilde{u}_1\|\right) \\ &\qquad \textit{for every } \beta \textit{ such that } 0 \leq \beta \leq \tilde{\beta} + \frac{1}{2}, \end{aligned} \tag{2.3}$$

for every $t \geq 0$, where u is the solution of equation (1.1) *and $\tilde{u}_0$ and $\tilde{u}_1$ are an element of H such that $u_0 = A^\gamma \tilde{u}_0$ and $u_1 = A^\gamma \tilde{u}_1$ respectively.*

2.2. Application to dissipative wave equations in exterior domains

Throughout this subsection, let Ω be an exterior domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

We can apply our abstract theorems to the problems for wave equations with dissipative term in exterior domains (or in the whole space) (1.4)–(1.5) with Dirichlet boundary condition (1.6) or with Robin boundary condition (1.7).

Before stating our results, we introduce some notations. For $s \geq 0$, let

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n) \right\},$$

with the norm $\|f\|_{H^s} = \|\langle \xi \rangle^s \hat{f}\|_{L^2}$. For $s < \frac{n}{2}$, let

$$\dot{H}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, |\xi|^s \hat{f} \in L^2(\mathbb{R}^n) \right\},$$

with the norm $\|f\|_{\dot{H}^s} = \||\xi|^s \hat{f}\|_{L^2}$ (see Bergh–Löfström [1, Chapter 6] and Bourdaud [2]). Let

$$H^s(\Omega) = \{ f \mid \exists g \in H^s(\mathbb{R}^n) \text{ such that } g|_\Omega = f \},$$

with the norm $\|f\|_{H^s(\Omega)} = \inf \left\{ \|g\|_{H^s(\mathbb{R}^n)} \,\middle|\, g|_\Omega = f \right\}$.

Remark 4. If $0 \leq \gamma < n/2$, Hardy's inequality implies the following continuous embedding:

$$L^p_\mu(\mathbb{R}^n) \subset \dot{H}^{-\gamma}(\mathbb{R}^n) \qquad (\, \gamma = \mu + n(\frac{1}{p} - \frac{1}{2}) \,). \tag{2.4}$$

for $p \in (1, 2]$ and $\mu \geq 0$, where

$$L^p_\mu(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n); \|u\|_{L^p_\mu} = \|(1 + |x|)^\mu u(x)\|_{L^p} < \infty.$$

Let A_D and A_R be operators defined in the introduction of this paper. The characterization of the fractional powers of A_D and A_R given by Fujiwara [4] and Grisvard [5] yields the following:

(i) (The Dirichlet boundary condition)

$$\begin{aligned} \mathcal{D}(A_D^\beta) =& \{ u \in H^{2\beta}(\Omega);\ (-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ & \qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \frac{1}{4} \} \\ & \qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \frac{1}{4} \notin \mathbb{N} \cup \{0\}); \end{aligned}$$

$$\begin{aligned} \mathcal{D}(A_D^\beta) =& \{ u \in H^{2\beta}(\Omega);\ (-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ & \qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \frac{1}{4}, \\ & \qquad \text{and } \int_\Omega \frac{1}{\zeta(x)} |((-\Delta)^{\beta - \frac{1}{4}} u(x)|^2 dx < \infty \} \\ & \qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \frac{1}{4} \in \mathbb{N} \cup \{0\}). \end{aligned}$$

(ii) (The Robin boundary condition)

$$\mathcal{D}(A_R^\beta) =\{u \in H^{2\beta}(\Omega);\ (\frac{\partial}{\partial\nu}u(x)+\sigma(x))(-\Delta)^k u(x)=0 \text{ on } \partial\Omega$$
$$\text{for every non-negative integer } k \text{ such that } k<\beta-\frac{3}{4}\}$$
$$(\text{for } \beta\ge 0 \text{ such that } \beta-\frac{3}{4}\notin \mathbb{N}\cup\{0\});$$

$$\mathcal{D}(A_R^\beta) =\{u \in H^{2\beta}(\Omega);\ (\frac{\partial}{\partial\nu}u(x)+\sigma(x))(-\Delta)^k u(x)=0 \text{ on } \partial\Omega$$
$$\text{for every non-negative integer } k \text{ such that } k<\beta-\frac{3}{4},$$
$$\text{and } \int_\Omega \frac{1}{\zeta(x)}|(\frac{\partial u}{\partial\zeta}+\sigma(x))(-\Delta)^{\beta-\frac{3}{4}}u(x)|^2dx<\infty\}$$
$$(\text{for } \beta\ge 0 \text{ such that } \beta-\frac{3}{4}\in \mathbb{N}\cup\{0\}).$$

Taking $A = A_D$ or A_R, Theorem 1 with $\gamma = 0$ yields the decay estimate of the difference between the solutions of the dissipative wave equation and the solution of the corresponding parabolic equation in exterior domains.

Corollary 1. *Let $A = A_D$ or $A = A_R$. Let β be an arbitrary non-negative number. Let T_0 be an arbitrary positive number. Then there exists a positive constant C such that the following estimates hold for every $(u_0, u_1) \in \mathcal{D}(A^{\max\{\beta+1/2,1\}}) \times \mathcal{D}(A^{\max\{\beta,1/2\}})$:*

$$\left\|(-\Delta)^\beta(u(t)-v(t))\right\|_{H^1} \le C\frac{1}{tb(t)}\left(\frac{b(t)}{t}\right)^\beta (\|u_0\|_{H^{2\beta+1}}+\|u_1\|_{H^{2\beta}}),$$

for every $t \ge T_0$, where $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{\max\{2\beta+1,2\}-i}(\Omega))$ is the solution of the equation (1.4)–(1.5) and $v \in C^1((0,\infty); H^2(\Omega)) \cap C([0,\infty); L^2(\Omega))$ is the solution of the parabolic equation (1.8)–(1.10), with the boundary conditions (1.6) and (1.11) in the case $A = A_D$, and with the boundary conditions (1.7) and (1.12) in the case $A = A_R$ respectively.

Next we consider the problem in the whole space. Let

(2.5) $$H = L^2(\mathbb{R}^n), \quad A = -\Delta \quad \text{with domain} \quad \mathcal{D}(A) = H^2(\mathbb{R}^n)$$

Then $D(A^\gamma) = H^{2\gamma}(\mathbb{R}^n)$ and $\mathcal{R}(A^\gamma) = \dot{H}^{-2\gamma}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $\gamma \ge 0$. Hence, Theorem 1 implies the decay estimate of the difference between the solutions of the dissipative wave equation and the solution of the corresponding parabolic equation in the whole space.

Corollary 2. *Let β and γ be arbitrary non-negative numbers. Let T_0 be an arbitrary positive number. Then there exists a positive constant C such that the following estimates hold for every $(u_0, u_1) \in (H^{2\beta+1}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n)) \times (H^{2\beta}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n))$:*

(2.6)
$$\begin{aligned}&\left\|(-\Delta)^{\beta}(u(t)-v(t))\right\|_{H^1}\\ &\leq C\left(\frac{b(t)}{t}\right)^{\beta+\gamma}\left(\|u_0\|_{H^{2\beta+1}}+\|u_0\|_{\dot{H}^{-2\gamma}}+\|u_1\|_{H^{2\beta}}+\|u_1\|_{\dot{H}^{-2\gamma}}\right)\end{aligned}$$

for every $t \geq T_0$, where $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\beta+1-i}(\mathbb{R}^n))$ is the solution of the equation (1.4)–(1.5) with $\Omega = \mathbb{R}^n$, and $v \in C^1((0,\infty); H^2(\mathbb{R}^n)) \cap C([0,\infty); L^2(\mathbb{R}^n))$ is the solution of the parabolic equation (1.8)–(1.10) with $\Omega = \mathbb{R}^n$.

Remark 5. Under the assumption (1.22)–(1.23) for monotone decreasing function f, the decay order of (2.6) with $\beta = \gamma = 0$ is same as that of (1.19) by Wirth.

By taking $A = A_D$ or A_R, Theorem 2 with $\gamma = 0$ implies the following estimate of the solution of the dissipative wave equation.

Corollary 3. *Let $A = A_D$ or $A = A_R$. Let $\tilde{\beta}$ be an arbitrary number such that $\tilde{\beta} \geq \frac{1}{2}$. Then there exists a positive constant C such that the following estimates hold for every $(u_0, u_1) \in \mathcal{D}(A^{\tilde{\beta}+1/2}) \times \mathcal{D}(A^{\tilde{\beta}})$:*

$$\left\|(-\Delta)^{\beta}u(t)\right\|_{L^2} \leq C\left(1+\frac{t}{b(t)}\right)^{-\beta}\left(\|u_0\|_{H^{2\beta}}+\|u_1\|_{H^{\max\{2\beta-1,0\}}}\right)$$
$$\text{for every } \ t \geq 0,$$

for every β such that $0 \leq \beta \leq \tilde{\beta} + \frac{1}{2}$, where $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\tilde{\beta}+1-i}(\Omega))$ is the solution of the equation (1.4)–(1.5), with the boundary condition (1.5) in the case $A = A_D$, and with the boundary condition (1.7) in the case $A = A_R$ respectively.

In the same way as in Corollary 2, Theorem 2 yields the following decay estimate of the solution of (1.4)–(1.5) with $\Omega = \mathbb{R}^n$.

Corollary 4. *Let $\tilde{\beta}$ be an arbitrary number such that $\tilde{\beta} \geq \frac{1}{2}$. Let γ be arbitrary non-negative numbers. Then there exists a positive constant C such that the following estimates hold for every $(u_0, u_1) \in$*

$(H^{2\tilde{\beta}+1}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n)) \times (H^{2\tilde{\beta}}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n))$:

$$\begin{aligned} \left\|(-\Delta)^\beta u(t)\right\|_{L^2} \leq C\left(1+\frac{t}{b(t)}\right)^{-\beta-\gamma} & (\|u_0\|_{H^{2\beta}} + \|u_0\|_{\dot{H}^{-2\gamma}} \\ & + \|u_1\|_{H^{\max\{2\beta-1,0\}}} + \|u_1\|_{\dot{H}^{-2\gamma}}) \\ & \text{for every } t \geq 0, \end{aligned} \tag{2.7}$$

for every β such that $0 \leq \beta \leq \tilde{\beta} + \frac{1}{2}$, where $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\beta+1-i}(\mathbb{R}^n))$ is the solution of the equation (1.4)–(1.5) with $\Omega = \mathbb{R}^n$.

Remark 6. If we take $\beta = 0$ in (2.7), we have

$$\begin{aligned} & \|u(t)\|_{L^2} \\ & \leq C\left(1+\frac{t}{b(t)}\right)^{-\gamma} (\|u_0\|_{L^2} + \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{L^2} + \|u_1\|_{\dot{H}^{-2\gamma}}) \end{aligned} \tag{2.8}$$

for every $t \geq 0$, where u is the solution of the equation (1.4)–(1.5) with $\Omega = \mathbb{R}^n$. Hence by (2.4), if u_0 and u_1 belong to $L^p_\mu(\mathbb{R}^n)$ of (2.4) with γ replaced by 2γ for $0 < 2\gamma < n/2$, then $\|u(t,\cdot)\|_{L^2}$ decays as $(1+t/b(t))^{-\gamma}$. Wirth [10] obtained similar estimate

(2.9)

$$\begin{aligned} & \|u(t)\|_{L^2} \\ & \leq C\left(1+\int_0^t \frac{d\tau}{b(\tau)}\right)^{-\gamma} (\|u_0\|_{L^2} + \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{L^2} + \|u_1\|_{\dot{H}^{-2\gamma}}) \end{aligned}$$

for every $t \geq 0$, under the assumption (1.14)–(1.17) and $\int_0^\infty \frac{1}{b(s)} ds = \infty$. Estimates (2.8) and (2.9) are equivalent, under the assumption (1.22)–(1.23) if f is monotone decreasing.

§3. Sketch of the proof of Theorem 1

3.1. Reduction of the equations to ordinary differential equations

Chill–Haraux [3] derived an ordinary differential equation from (1.1) by using the spectral theorem for self-adjoint operators. We also use the spectral theorem as in [3]. The self-adjoint operator A is unitarily equivalent to a multiplication operator on some L^2 space, by the spectral theorem (see Reed–Simon [9, Theorem VIII.4, p. 260]). Namely, we can identify H as $H = L^2(E, d\mu)$ on a measure space (E, μ) and A as the multiplication operator as follows:

$$(Au)(\xi) = a(\xi)u(\xi) \qquad (\xi \in E, \quad u \in \mathcal{D}(A)),$$

where a is a nonnegative μ-measurable function, and

$$D(A^\gamma) = L^2(E; (1 + a^{2\gamma})d\mu) \quad \text{for } \gamma \geq 0.$$

Then the equation (1.1) is equivalent to

$$\text{(3.1)} \qquad \begin{cases} u''(t,\xi) + b(t)u'(t,\xi) + a(\xi)u(t,\xi) = 0, \\ u(0,\xi) = u_0(\xi), \quad u'(0,\xi) = u_1(\xi), \end{cases}$$

for every fixed $\xi \in E$, where $'$ means the derivative by t. Also, the equation (1.2)–(1.3) is equivalent to

$$\text{(3.2)} \qquad \begin{cases} b(t)v'(t,\xi) + a(\xi)v(t,\xi) = 0, \\ v(0,\xi) = v_0(\xi) = u_0(\xi) + u_1(\xi) \int_0^\infty \exp(-\int_0^s b(\sigma)d\sigma)ds, \end{cases}$$

for every fixed $\xi \in E$.

Chill–Haraux [3] showed that the restriction of the solutions of (3.1) and (3.2) with $b(t) \equiv 1$ to the region $\{a(\xi) \geq 1/16\}$ decay exponentially for $t \geq 1$ by proving that some energy decays exponentially, and showed the estimate of the difference between solutions of (3.1) and (3.2) with $b(t) \equiv 1$ restricted to the region $\{a(\xi) < 1/16\}$. Here, since $b(t)$ decays, the separating point of the spectrum depends on t.

Since $b \in C^1([0,t_0]; (0,\infty))$, we have $\inf_{t\in[0,t_0]} b(t) > 0$, $\sup_{t\in[0,t_0]} b(t) < \infty$ and $\sup_{t\in[0,t_0]} |b'(t)| < \infty$. From these facts and that $f(t_0) > 0$, we can extending the function f and g on $[0,\infty)$, such that (1.21)–(1.26) hold on $[0,\infty)$ by changing positive constants b_0 and b_1. Thus, we can assume that (1.21)–(1.26) hold on $[0,\infty)$. We define the function Λ on $[0,\infty)$ by

$$\text{(3.3)} \qquad \Lambda(t) = \frac{3}{16} b_0^2 b_2^2 f(t/2)^2 \quad \text{for} \quad 0 \leq t < \infty$$

Put

$$\text{(3.4)} \qquad \lambda_0 = \frac{3}{64} b_0^2 b_2 f(0)^2$$

Then $\Lambda : [0,\infty) \to (0,\lambda_0]$ has the inverse function $T : (0,\lambda_0] \to [0,\infty)$ defined by

$$T(\lambda) = 2f^{-1}\left(\frac{4\sqrt{\lambda}}{\sqrt{3}b_0 b_2}\right) \quad \text{for} \quad 0 < \lambda \leq \lambda_0,$$

We define the sets $\mathcal{G}_+$, $\mathcal{G}_-$ and $\mathcal{G}_0$ as follows:

$$\begin{aligned}\mathcal{G}_- :&= \{(t,\xi) \in [0,\infty) \times E; a(\xi) < \Lambda(t)\}\\ &= \{(t,\xi) \in [0,\infty) \times E; a(\xi) \le \lambda_0, 0 \le t < T(a(\xi))\}\\ \mathcal{G}_+ :&= \{(t,\xi) \in [0,\infty) \times E; \Lambda(t) \le a(\xi)\}\end{aligned} \tag{3.5}$$

For each $t \ge 0$, put

$$\begin{aligned}\mathcal{G}_-(t) &:= \{\xi \in E; 0 \le a(\xi) < \Lambda(t)\}\\ \mathcal{G}_+(t) &:= \{\xi \in E; a(\xi) \ge \Lambda(t)\}.\end{aligned} \tag{3.6}$$

3.2. Estimate for low frequency

In this subsection, we estimate the difference between the solutions of (3.1) and (3.2) for $(t,\xi) \in \mathcal{G}_-$.

By the assumptions (1.22) and (1.23) and the definition of $\Lambda(t)$, we have

$$\frac{3}{4}b(t)^2 \ge \frac{3}{4}b_0^2 b_2^2 f(t/2)^2 > 4a(\xi)$$

for $(t,\xi) \in \mathcal{G}_-$. Thus, we have

$$\sqrt{b(t)^2 - 4a(\xi)} \ge \frac{1}{2}b(t)$$

for $(t,\xi) \in \mathcal{G}_-$. Put

$$\begin{aligned}w_+(t,\xi) :&= u'(t,\xi) + \frac{1}{2}(b(t) + \sqrt{b(t)^2 - 4a(\xi)})u(t,\xi),\\ w_-(t,\xi) :&= u'(t,\xi) + \frac{1}{2}(b(t) - \sqrt{b(t)^2 - 4a(\xi)})u(t,\xi)\\ &= u'(t,\xi) + \frac{2a(\xi)u(t,\xi)}{b(t) + \sqrt{b(t)^2 - 4a(\xi)}}.\end{aligned}$$

Here we note that

$$\begin{aligned}u(t,\xi) &= \frac{w_+(t,\xi) - w_-(t,\xi)}{\sqrt{b(t)^2 - 4a(\xi)}},\\ u'(t,\xi) &= \left(\frac{b(t)}{2\sqrt{b(t)^2 - 4a(\xi)}} + \frac{1}{2}\right) w_-(t,\xi)\\ &\quad - \frac{2a(\xi)}{\sqrt{b(t)^2 - 4a(\xi)}(b(t) + \sqrt{b(t)^2 - 4a(\xi)})} w_+(t,\xi).\end{aligned}$$

We see that (3.1) is equivalent to

(3.7)
$$\begin{cases} w_-'(t,\xi) + b_+(t,\xi)w_-(t,\xi) = \phi_-(t,\xi)(w_+(t,\xi) - w_-(t,\xi)) & t > 0, \\ w_+'(t,\xi) + b_-(t,\xi)w_+(t,\xi) = \phi_+(t,\xi)(w_+(t,\xi) - w_-(t,\xi)) & t > 0, \\ w_-(0,\xi) = u_1(\xi) + \frac{1}{2}\left(b(0) - \sqrt{b^2(0) - 4a(\xi)}\right) u_0(\xi), \\ w_+(0,\xi) = u_1(\xi) + \frac{1}{2}\left(b(0) + \sqrt{b^2(0) - 4a(\xi)}\right) u_0(\xi), \end{cases}$$

where

$$\begin{aligned} b_+(t,\xi) &:= \frac{1}{2}\left(b(t) + \sqrt{b(t)^2 - 4a(\xi)}\right), \\ b_-(t,\xi) &:= \frac{1}{2}\left(b(t) - \sqrt{b(t)^2 - 4a(\xi)}\right) = \frac{2a(\xi)}{b(t) + \sqrt{b(t)^2 - 4a(\xi)}}, \\ \phi_\pm(t,\xi) &:= \frac{b_\pm'(t)}{\sqrt{b(t)^2 - 4a(\xi)}} \\ &= \frac{1}{2}\left(1 \pm \frac{b(t)}{\sqrt{b(t)^2 - 4a(\xi)}}\right) \frac{b'(t)}{\sqrt{b(t)^2 - 4a(\xi)}}. \end{aligned}$$

Put

$$\begin{aligned} B_\pm(t) &: = \int_0^t b_\pm(s)ds, \\ \Phi_\pm(t) &: = \int_0^t \phi_\pm(s)ds \\ &= \frac{1}{2}\left[\log(\tau + \sqrt{\tau^2 - 4a(\xi)}) \pm \log\sqrt{\tau^2 - 4a(\xi)}\right]_{b(0)}^{b(t)}, \\ \Phi(t) &: = \Phi_+ + \Phi_- = \left[\log(\tau + \sqrt{\tau^2 - 4a(\xi)})\right]_{b(0)}^{b(t)}, \end{aligned}$$

and put

$$\begin{aligned} W_+(t,\xi) &:= \exp(B_-(t,\xi) - \Phi_+(t,\xi))w_+(t,\xi), \\ W_-(t,\xi) &:= \exp(B_+(t,\xi) + \Phi_-(t,\xi))w_-(t,\xi). \end{aligned}$$

Equation (3.7) is equivalent to

(3.8)
$$\begin{cases} W_-'(t,\xi) = \phi_-(t,\xi)\exp((B_+ - B_- + \Phi)(t,\xi))W_+(t,\xi) & t \geq 0, \\ W_+'(t,\xi) = -\phi_+(t,\xi)\exp((B_- - B_+ - \Phi)(t,\xi))W_-(t,\xi) & t \geq 0, \\ W_-(0,\xi) = w_-(0,\xi) = u_1(\xi) + b_-(0,\xi)u_0(\xi), \\ W_+(0,\xi) = w_+(0,\xi) = u_1(\xi) + b_+(0,\xi)u_0(\xi). \end{cases}$$

From (3.8), it follows that

$$(3.9)\quad W_-(t,\xi) = w_-(0,\xi) + \int_0^t \phi_-(s,\sigma)\exp((B_+ - B_- + \Phi)(s,\xi))W_+(s,\xi)ds,$$

$$(3.10)\quad W_+(t,\xi) = G(t,\xi) + F(t,\xi),$$

where

$$(3.11)\quad G(t,\xi) := w_+(0,\xi) - \int_0^t \phi_+(s,\sigma)\exp((B_- - B_+ - \Phi)(s,\xi))ds w_-(0,\xi),$$

$$(3.12)\quad F(t,\xi) := -\int_0^t \int_0^s \phi_+(s,\sigma)\phi_-(\sigma,\xi)\exp((B_- - B_+ - \Phi)(s,\xi)) \times \exp((B_+ - B_- + \Phi)(\sigma,\xi))W_+(\sigma,\xi)d\sigma ds.$$

Then we can show that

$$(3.13)\quad |F(t,\xi)| \le C\frac{a(\xi)}{b(t)^2}(|u_0(\xi)| + |u_1(\xi)|)$$

for $(t,\xi) \in \mathcal{G}_-$. By the definition, it can be written as

(3.14)

$$\frac{w_+(t,\xi)}{\sqrt{b(t)^2 - 4a(\xi)}} = \exp\left(-\int_0^t \frac{2a(\xi)}{b(s) + \sqrt{b^2(s) - 4a(\xi)}}ds\right)\frac{h(t,a(\xi))}{h(0,a(\xi))}\frac{G(t,\xi)}{\sqrt{b(t)^2 - 4a(\xi)}} + \exp\left(-\int_0^t \frac{2a(\xi)}{b(s) + \sqrt{b^2(s) - 4a(\xi)}}ds\right)\frac{h(t,a(\xi))}{h(0,a(\xi))}\frac{F(t,\xi)}{\sqrt{b(t)^2 - 4a(\xi)}},$$

where

$$h(t,\lambda) = \left(b(t)\sqrt{b(t)^2 - 4\lambda} + b(t)^2 - 4\lambda\right)^{1/2}.$$

We estimate the difference between the first term of the right-hand side of (3.14) and

$$v(t,\xi) = \exp\left(-\int_0^t \frac{a(\xi)}{b(s)}ds\right)v_0(\xi).$$

By using (3.13), we estimate the second term of the right-hand side of (3.14). By using the integrals (3.9), we estimate $w_-(t,\xi)$ itself. Then, we obtain the following estimates on $\mathcal{G}_-$:

Lemma 1. *There exists a positive constant C such that*

$$\left|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}} - \exp\left(-\int_0^t \frac{a(\xi)}{b(s)}ds\right) v_0(\xi)\right|$$
$$\leq C\Big(\frac{a(\xi)}{b(t)^2} + a(\xi)^2\int_0^t \frac{1}{b(s)^3}ds\Big)\exp\left(-a(\xi)\int_0^t \frac{1}{b(s)}ds\right)$$
$$\times (|u_0(\xi)| + |u_1(\xi)|)$$
$$+ C\int_t^\infty \exp\left(-\int_0^s b(\sigma)d\sigma\right) ds|u_1(\xi)|.$$
$$|w_-(t,\xi)|$$
$$\leq C\left(\exp\left(-\frac{1}{2}\int_0^t b(\tau)d\tau\right) + a(\xi)\exp\left(-a(\xi)\int_0^t \frac{1}{b(s)}ds\right)\right)$$
$$\times (|u_0(\xi)| + |u_1(\xi)|)$$

for every $(t,\xi) \in \mathcal{G}_-$.

By assumptions (1.22) and (1.23) and the monotone decreasingness of f, there are positive constants c_0 and C such that

$$c_0\frac{t}{b(t)} \leq \int_0^t \frac{1}{b(s)}ds \leq C\frac{t}{b(t)}.$$
$$c_0\frac{t}{b(t)^3} \leq \int_0^t \frac{1}{b(s)^3}ds \leq C\frac{t}{b(t)^3}.$$

By using these inequality and the fact that $\sup_{s\geq 0} s^\gamma \exp(-s) < \infty$ for every fixed $\gamma \geq 0$, the next corollary follows from Lemma 1.

Corollary 5. *Let* β *be an arbitrary non-negative constant. There exists a positive constant C such that*

$$a(\xi)^\beta\left|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}} - \exp\left(-\int_0^t \frac{a(\xi)}{b(s)}ds\right) v_0(\xi)\right| \tag{3.15}$$
$$\leq Cb(t)^{\beta-1}t^{-\beta-1}(|u_0(\xi)| + |u_1(\xi)|)$$

and

$$a(\xi)^\beta\left|\frac{w_-(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}}\right| \leq Cb(t)^\beta t^{-\beta-1}(|u_0(\xi)| + |u_1(\xi)|) \tag{3.16}$$

for every $(t,\xi) \in \mathcal{G}_-$.

The sum of (3.15) and (3.16) implies the decay estimate of the difference between the solution of the wave equation and the solution of the corresponding parabolic equation for low frequency:

Corollary 6. *Let β be an arbitrary non-negative constant. There exists a positive constant C such that*

$$a(\xi)^{\beta}|u(t,\xi)-v(t,\xi)| \le Cb(t)^{\beta-1}t^{-\beta-1}(|u_0(\xi)|+|u_1(\xi)|) \tag{3.17}$$

for every $(t,\xi)\in\mathcal{G}_-$, where $u(t,\xi)$ is the solution of (3.1) *and $v(t,\xi)$ is the solution of* (3.2).

3.3. Estimate for high frequency

Let (t,ξ) be an arbitrary fixed number of $\mathcal{G}_+$. Define the energy function

$$e(\tau,\xi) := \frac{1}{2}|u'(\tau,\xi)|^2 + \frac{1}{2}a(\xi)|u(\tau,\xi)|^2 + b_4 b(t)u(\tau,\xi)\overline{u'(\tau,\xi)},$$

for $\tau \ge t/2$, for some sufficiently small positive constant b_4. Then by estimating the energy inequality for $e(\tau,\xi)$, we can prove that each solution of (3.1) itself has exponential decay estimate on $\mathcal{G}_+$.

Lemma 2. *There exists a positive constant C and c_1 such that*

$$\begin{aligned}&(1+a(\xi))|u(t,\xi)|^2 + |u'(t,\xi)|^2\\ &\le C\exp\left(-c_1(t+1)^{\theta}\right)\left(a(\xi)|u_0(\xi)|^2+|u_1(\xi)|^2\right)\end{aligned} \tag{3.18}$$

for every $(t,\xi)\in\mathcal{G}_+$, where $u(t,\xi)$ is the solution of (3.1).

We easily see that the solution of (3.2) decays exponentially.

Lemma 3. *Let T_0 be an arbitrary positive number. Let β be an arbitrary non-negative constant. There exists a positive constant C and c_2 such that*

$$\begin{aligned}&a(\xi)^{\beta}(1+a(\xi))^{1/2}|v(t,\xi)|\\ &\le C\exp\left(-c_2(t+1)^{\theta}\right)\left(|u_0(\xi)|^2+|u_1(\xi)|^2\right)^{1/2},\end{aligned} \tag{3.19}$$

for every $(t,\xi)\in\{(t,\xi);(t,a(\xi))\in\mathcal{G}_+, t\ge T_0\}$, where $v(t,\xi)$ is the solution of (3.2).

3.4. Proof of Theorem 1

Let $t \geq T_0$ be an arbitrary fixed number.
We first prove the inequality (2.1) for $\gamma = 0$.
Integrating the square of (3.17) on $\mathcal{G}_-(t)$, we obtain

$$\begin{aligned}(3.20)\qquad &\left(\int_{\xi\in\mathcal{G}_-(t)} a(\xi)^{2\beta}\left|u(t,\xi)-v(t,\xi)\right|^2 d\mu_\xi\right)^{1/2}\\ &\leq Cb(t)^{\beta-1}t^{-\beta-1}\left(\|u_0\|+\|u_1\|\right).\end{aligned}$$

Integrating (3.18) multiplied by $a(\xi)^{2\beta}$ on $\mathcal{G}_+(t)$, we obtain

$$\begin{aligned}(3.21)\qquad &\left(\int_{\xi\in\mathcal{G}_+(t)} a(\xi)^{2\beta}\left((1+a(\xi))|u(t,\xi)|^2+|u'(t,\xi)|^2\right)d\mu_\xi\right)^{1/2}\\ &\leq C\exp\left(-\frac{c_1}{2}(t+1)^\theta\right)\left(\|u_0\|_{\beta+\frac{1}{2}}+\|u_1\|_\beta\right).\end{aligned}$$

Integrating the square of (3.19) on $\mathcal{G}_+(t)$, we obtain

$$\begin{aligned}(3.22)\qquad &\left(\int_{\xi\in\mathcal{G}_+(t)} a(\xi)^{2\beta}(1+a(\xi))|v(t,\xi)|^2 d\mu_\xi\right)^{1/2}\\ &\leq C\exp\left(-c_2(t+1)^\theta\right)\left(\|u_0\|+\|u_1\|\right).\end{aligned}$$

Summing (3.20), (3.21) and (3.22) up, we obtain (2.1) with $\gamma = 0$.

Next, we consider the case $\gamma > 0$. We apply the result (2.1) to initial data $\tilde{u}_0$ and $\tilde{u}_1$. Let $\tilde{u}(t)$ and $\tilde{v}(t)$ be solutions of (1.1) and (1.2)–(1.3) with u_0 and u_1 replaced by $\tilde{u}_0$ and $\tilde{u}_1$, respectively. Inequality (2.1) with β and γ replaced by $\beta+\gamma$ and 0 yields

$$\begin{aligned}(3.23)\qquad &\left\|A^{\beta+\gamma}(\tilde{u(t)}-\tilde{v(t)})\right\|_{1/2}\\ &\leq C\frac{1}{tb(t)}\left(\frac{b(t)}{t}\right)^{\beta+\gamma}\left(\|u_0\|_{\beta+1/2}+\|\tilde{u}_0\|+\|u_1\|_\beta+\|\tilde{u}_1\|\right).\end{aligned}$$

Since $A^\gamma\tilde{u}(t) = u(t)$ and $A^\gamma\tilde{v}(t) = v(t)$, the inequality (3.23) implies (2.1).

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