Advanced Studies in Pure Mathematics 47-1, 2007
Asymptotic Analysis and Singularities
pp. 329–339

On asymptotic behavior of solutions to the fourth order cubic nonlinear Schrödinger type equation

Jun-ichi Segata

Abstract.

We present some results concerning the asymptotic behavior of solutions to the fourth order nonlinear Schrödinger type equation. The nonlinear interaction is the power type one with degree three. We prove the existence of the modified wave operators for this equation.

§1. Introduction

In this note, we review the paper [16] on the asymptotic behavior of solution to the fourth order cubic nonlinear Schrödinger type equation:

$$i\partial_t u - \frac{1}{4}\partial_x^4 u = \lambda |u|^2 u, \qquad t, x \in \mathbb{R}, \tag{1.1}$$

where $u = u(t, x)$ is a complex valued unknown function, ∂_α denotes the partial derivative with respect to α variable and $\lambda \in \mathbb{R}$. We shall state the result on the global existence of the equation (1.1) which behaves as $|t| \to \infty$ like a given asymptotic profiles. We also give the outline of the proof for this result.

Equation (1.1) is a simplified equation of the Fukumoto-Moffatt model [4] which describes the motion of a vortex filament.

We briefly review the scattering theory for the nonlinear dispersive equations. Let us consider the following equation:

$$i\partial_t v + \mathcal{P}(i^{-1}\partial_x) v = \mu |v|^{p-1} v, \qquad t > 0, x \in \mathbb{R}^n, \tag{1.2}$$

Received October 29, 2005.
Revised January 2, 2006.
2000 *Mathematics Subject Classification.* 35B40, 35Q55.
Key words and phrases. higher order dispersive equation, scattering problem, modified wave operators.
Supported by JSPS Research Fellowships for Young Scientists.

where $v = v(t,x)$ is a complex valued unknown function, $p > 1$, $\mu \in \mathbb{R}$ and $\mathcal{P}(i^{-1}\partial_x)$ is the Fourier multiplier operator with the real symbol $\mathcal{P}(\xi)$.

When the solution v to the nonlinear equation (1.2) behaves as $t \to \infty$ like the solution w to the linear equation:

$$i\partial_t w + \mathcal{P}(i^{-1}\partial_x)w = 0, \qquad t > 0, x \in \mathbb{R}^n,$$

we call the nonlinearity in (1.2) the short range interaction. When the influence of the nonlinear term is not negligible for large time, we call the nonlinear interaction in (1.2) the long range one.

According to the method of Cook-Kuroda, roughly speaking, the nonlinear interaction in (1.2) is the short range one if the nonlinear term belongs to $L^1_t([1,\infty), L^2_x(\mathbb{R}^n))$. If the estimate

$$\|e^{it\mathcal{P}(i^{-1}\partial_x)}\phi\|_{L^\infty_x} \le Ct^{-m}\|\phi\|_{L^1_x} \tag{1.3}$$

holds, then the nonlinear term $|v|^{p-1}v = \mathcal{O}(t^{-m(p-1)})$ in $L^2_x(\mathbb{R})$. Since $\int_1^\infty t^{-m(p-1)}dt < \infty$ if and only if $p > 1 + \frac{1}{m}$, the nonlinearity $|v|^{p-1}v$ will be the short range interaction if $p > 1+\frac{1}{m}$. Therefore we expect that the power of the borderline between short and long range interactions will be equal to or less than $1 + \frac{1}{m}$.

For the n dimensional (second order) nonlinear Schrödinger equation ((1.2) with $\mathcal{P}(i^{-1}\partial_x) = \frac{1}{2}\Delta = \frac{1}{2}\sum_{j=1}^n \partial^2/\partial x_j^2$), $p = 1+\frac{2}{n}$ is known to the critical exponent by Barab [1] and Tsutsumi-Yajima [20] (we note that the free Schrödinger group satisfies (1.3) with $m = \frac{n}{2}$). Therefore for the case $p \le 1 + \frac{2}{n}$ the solution to the nonlinear Schrödinger equation have the asymptotic profiles different from the solution to free Schrödinger equation.

In this regard, for the one dimensional critical case $p = 3(= 1 + \frac{2}{n}|_{n=1})$, Ozawa [15] showed the existence of solution to the nonlinear Schrödinger equation which behaves like

$$v(t,x) \sim \sqrt{2\pi}E(t,x)\exp\left(\mp i\mu\left|\hat{\phi}_\pm\left(\frac{x}{t}\right)\right|^2 \log t\right)\hat{\phi}_\pm\left(\frac{x}{t}\right),$$

as $t \to \infty$, for given *final* data $\phi_\pm$, where $E(t,x) = \frac{1}{\sqrt{2\pi i t}}\exp\left(\frac{ix^2}{2t}\right)$ is the fundamental solution to the free Schrödinger equation. Later on Hayashi-Naumkin [8] obtained the asymptotic behavior of solution to the nonlinear Schrödinger equation for given *initial* data $v(0,x)$.

For the scattering theory or asymptotic behavior of solutions to nonlinear Schrödinger equations, we refer to Cazenave [3, Chapter 7], Ginibre [5] Ginibre–Ozawa [6], Ginibre–Velo [7], Hayashi–Naumkin–Shimomura–Tonegawa [12], Shimomura–Tonegawa [17] and Y. Tsutsumi [19] for instance. As far as we know there is no result concerning the asymptotic behavior of solution to subcritical nonlinear Schrödinger equation ((1.2) with $\mathcal{L}(i^{-1}\partial_x) = \frac{1}{2}\Delta$ and $p < 1 + \frac{2}{n}$) is still an open problem.

In [9]-[11], Hayashi-Naumkin studied the large time behavior of solution to the generalized Korteweg-de Vries equation:

$$(1.4) \qquad \partial_t v + \frac{1}{3}\partial_x^3 v + \frac{1}{p}\partial_x(|v|^{p-1}v) = 0, \qquad t > 0, x \in \mathbb{R},$$

where $v(t,x)$ is a real valued unknown function and $p > 1$. We note that the free evolution group $e^{-t\partial_x^3}$ satisfies (1.3) with $m = \frac{1}{3}$. Combining this with the method of Cook-Kuroda, we see that (1.4) is short range for $p > 4$. By making use of the decay estimate of one derivative of $e^{-t\partial_x^3}v(0,x)$, Hayashi-Naumkin proved that the critical exponent between short range scattering and long range one is three.

We return to the fourth order nonlinear Schrödinger equation (1.1). It is known that the free evolution group $e^{-it\partial_x^4/4}$ satisfies (1.3) with $m = \frac{1}{4}$ (see Ben Artzi-Koch- Saut [2]). Therefore we expect from above observation that the critical power will be equal to or less than five for the fourth order case. In [16] we proved that (1.1) fall into the *critical* case for some final data (for the precise statement, see Theorem 1.1).

Before we state our main result precisely, we introduce several notations and function spaces. We denote $\{W(t)\}_{t\in\mathbb{R}}$ the free evolution group generated by the linear oprator $-i\partial_x^4/4$:

$$W(t)\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi - \frac{i}{4}t\xi^4}\hat{\phi}(\xi)d\xi = \int_{\mathbb{R}} F(t,x-y)\phi(y)dy,$$

where the function $F(t,x)$ is the fundamental solution to linearized equation of (1.1):

$$(1.5) \qquad F(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi - \frac{i}{4}t\xi^4}d\xi.$$

Let $H^{s,\alpha}$ be the weighted Sobolev space defined by

$$H^{s,\alpha} = \{\phi \in \mathcal{S}'; \|\phi\|_{H^{s,\alpha}} = \|(1+|x|^2)^{\alpha/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\}, \qquad s,\alpha \in \mathbb{R},$$

and $\dot{H}^s$ be the homogeneous Sobolev space

$$\dot{H}^s = \{\phi \in \mathcal{S}'; \|\phi\|_{\dot{H}^s} = \|(-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\}, \quad s \in \mathbb{R}.$$

For $1 \le p, q \le \infty$ and an interval $I \subset \mathbb{R}$, we define

$$L_t^p(I, L_x^q) = \left\{ f;\ \|f\|_{L_t^p(I,L_x^q)} = \left(\int_I \|f(t)\|_{L_x^q}^p dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Let us define the class of final data $\phi_\pm$.

$$\begin{aligned} (1.6)\quad \mathcal{D} &\equiv \{\phi \in \mathcal{S}'; \phi \in H^{0,4} \ \textit{and}\ x^k\phi \in \dot{H}^{k-12},\ k = 0,1,2,3,4\}. \\ \|\phi\|_{\mathcal{D}} &\equiv \|\phi\|_{H^{0,4}} + \sum_{k=0}^{4} \|x^k \phi\|_{\dot{H}^{k-12}}. \end{aligned}$$

For a given final data $\phi_\pm$, we introduce the following asymptotic profile

$$(1.7)\qquad v_\pm(t,x) = \sqrt{2\pi} F(t,x) \exp\left(iS^\pm(t,\chi)\right) \hat{\phi}_\pm(\chi),$$

$t, x \neq 0$, where $\hat{\phi}_\pm$ are the Fourier transforms of $\phi_\pm$ with respect to space variable,

$$S^\pm(t,x) = \mp \frac{\lambda}{3} |\hat{\phi}_\pm(x)|^2 |x|^{-2} \log|t|, \qquad |t| \ge e,$$

and

$$\chi = \chi(t,x) = \left| \frac{x}{t} \right|^{-\frac{2}{3}} \frac{x}{t}.$$

We note that χ is the stationary point of the integral (1.5).

The main result is as follows.

Theorem 1.1. *(i) Let $\phi_+ \in \mathcal{D}$ and $\|\phi_+\|_{\mathcal{D}}$ be sufficiently small, where $\mathcal{D}$ is the set defined by* (1.6). *Then the equation* (1.1) *has a unique solution $u \in C([0,\infty); L^2(\mathbb{R})) \cap L^8_{loc}((0,\infty); L^\infty(\mathbb{R}))$ satisfying*

$$(1.8)\qquad \sup_{t \ge e} (t^\alpha \|u(t) - v_+(t)\|_{L_x^2}) < \infty,$$

$$(1.9)\qquad \sup_{t \ge e} \left\{ t^\alpha \left(\int_t^\infty \|u(\tau) - v_+(\tau)\|_{L_x^\infty}^8 d\tau \right)^{\frac{1}{8}} \right\} < \infty,$$

where $3/8 < \alpha < 1$ and $v_+(t,x)$ is the modified free dynamics given by (1.7).

(ii) Let $\phi_- \in \mathcal{D}$ and $\|\phi_-\|_{\mathcal{D}}$ be sufficiently small. Then the equation (1.1) *has a unique solution $u \in C((-\infty,0]; L^2(\mathbb{R})) \cap L^8_{loc}((-\infty,0); L^\infty(\mathbb{R}))$*

satisfying

$$\sup_{t\le -e}\left(|t|^{\alpha}\|u(t)-v_-(t)\|_{L^2_x}\right) < \infty,$$

$$\sup_{t\le -e}\left\{|t|^{\alpha}\left(\int_{-\infty}^{t}\|u(\tau)-v_-(\tau)\|^8_{L^\infty_x}\,d\tau\right)^{\frac{1}{8}}\right\} < \infty,$$

where $3/8<\alpha<1$ *and* $v_-(t,x)$ *is the modified free dynamics given by* (1.7).

REMARK. We give several remarks.

(a): The modified wave operator $\Omega_+ : \phi_+ \mapsto u(0)$ for the positive time to the equation (1.1) is well-defined on a suitable small ball of $\mathcal{D}$, where u is the solution obtained in the first half of Theorem 1.1. Similarly, the existence of a modified wave operator for the negative time to the equation (1.1) follows from the second half of Theorem 1.1.

(b): If $\phi\in\mathcal{D}$, then $\hat{\phi}(\xi)$ is almost flat near $\xi=0$, more precisely, $\hat{\psi}(\xi)$ behaves like $|\xi|^{\alpha}$, $\alpha>\frac{23}{2}$ near $\xi=0$.

(c): The function $\sqrt{2\pi}F(t,x)\hat{\phi}_\pm(\chi)$ is the leading term of the function $W(t)\phi_\pm$. Therefore roughly speaking, the first half of Theorem 1.1 says that

$$u(t)\sim\exp\left(-i\frac{\lambda}{3}\left|\hat{\phi}_+(\chi)\right|^2|\chi|^{-2}\log|t|\right)W(t)\phi_+,$$

as $t\to\infty$.

We briefly give a point of proof for Theorem 1.1. Given data $\phi_\pm\in L^2$, we assume that the solution u to the nonlinear equation (1.1) approach $v_\pm$ in the suitable sense. Then we presume that u decays like $F(t,x)\hat{\phi}_\pm(\chi)$. By using the method of stationary phase, we have

$$F(t,x)\hat{\phi}_\pm(\chi)\sim\frac{1}{\sqrt{6i\pi t}}\frac{1}{|\chi|}\exp\left(\frac{3}{4}it|\chi|^4\right)\hat{\phi}_\pm(\chi), \tag{1.10}$$

as $t\to\pm\infty$. From this formula, if the function $\phi_\pm\in L^2$ satisfies $|\hat{\phi}(\xi)|\le|\xi|^{\alpha}$ for some $\alpha>0$, then the function $F(t,x)\hat{\phi}(\chi)$ decays like $t^{-\frac{1}{2}}$ which is the decay order same as the solution to free Schrödinger equation. Therefore under the condition that $|\hat{\phi}(\xi)|\le|\xi|^{\alpha}$ for some $\alpha>0$ near $\xi=0$, we are able to construct the modified wave operator for (1.1). This point is our conception and is different from the idea of Shimomura-Tonegawa [17]. It seems that our method also applies to other higher order nonlinear dispersive equations.

It seems that for general final data, the equation

$$i\partial_t u - \frac{1}{4}\partial_x^4 u = \lambda |u|^4 u, \qquad t, x \in \mathbb{R}, \tag{1.11}$$

will be the critical between the short range scattering and long range one. To obtain the asymptotic behavior of solution to (1.11) with general data, we will need the further information about the fundamental solution (1.5). We don't treat this problem here.

In the next section, we give the outline of the proof for Theorem 1.1.

§2. Outline of proof

In this section we give the outline of proof for Theorem 1.1. Hereafter we only treat the case $t > 0$ since other case being similar. Let $u(t, x)$ be the solution to (1.1). If $u(t, x)$ behaves like $v_+(t, x)$ as $t \to \infty$, $u(t, x)$ satisfies the integral equation of Yang-Ferdman type:

$$\begin{aligned} \text{(2.1)}\ u(t) &= \tilde{v}_+(t) \\ &+ i\int_t^{+\infty} W(t-\tau)\left\{\lambda|u|^2u(\tau) - \left(i\partial_\tau - \frac{1}{4}\partial_x^4\right)\tilde{v}_+(\tau)\right\}d\tau, \end{aligned}$$

where $\tilde{v}_+(t, x) = \psi(t)v_+(t, x)$[1] and $\psi \in C^\infty([0, \infty))$ with $\psi(t) = 1$ if $t \geq 2e$ and $\psi(t) = 0$ if $0 \leq t \leq e$. To show the existence of u which satisfies (1.8) and (1.9) in Theorem 1.1, we use the Banach fixed point theorem. More precisely, we show that

$$\begin{aligned} \text{(2.2)}\ \Phi u(t) &\equiv \tilde{v}_+(t) \\ &+ i\int_t^{\infty} W(t-\tau)\left\{\lambda|u|^2u(\tau) - \left(i\partial_\tau - \frac{1}{4}\partial_x^4\right)\tilde{v}_+(\tau)\right\}d\tau \end{aligned}$$

is a contraction on the space

$$\begin{aligned} X^\rho = \Bigg\{ & u \in C([0,\infty), L_x^2(\mathbb{R})); \\ & \sup_{t\geq 0}(t+1)^\alpha \left\{\|u(t) - \tilde{v}_+(t)\|_{L_x^2} + \left(\int_t^\infty \|u(\tau) - \tilde{v}_+(\tau)\|_{L_x^\infty}^8 d\tau\right)^{1/8}\right\} \leq \rho \Bigg\} \end{aligned}$$

[1]In order to avoid a singularity of the function v_+, we multiply v_+ by a cut-off function $\psi(t)$.

with the metric

$$\begin{aligned}\|u_1 - u_2\|_X \\ = \sup_{t\geq 0}(t+1)^{\alpha}\left\{\|u_1(t)-u_2(t)\|_{L^2_x} + \left(\int_t^{\infty}\|u_1(\tau)-u_2(\tau)\|^8_{L^\infty_x}\,d\tau\right)^{1/8}\right\},\end{aligned}$$

if $\|\phi_+\|_{\mathcal{D}}$ and ρ are sufficiently small. To guarantee that Φ is a contraction on X^ρ, we split the right hand side of (2.2) into two parts:

$$\begin{aligned}(2.3)\ \Phi u(t) - \tilde{v}_+(t) \\ &= i\lambda\int_t^{\infty} W(t-\tau)\{|u|^2u(\tau) - |\tilde{v}_+|^2\tilde{v}_+)(\tau)\}\,d\tau \\ &\quad -i\int_t^{\infty} W(t-\tau)\left\{\left(i\partial_t - \frac{1}{4}\partial_x^4\right)\tilde{v}_+ - \lambda|\tilde{v}_+|^2\tilde{v}_+(\tau)\right\}d\tau \\ &\equiv f_1(t,x) + f_2(t,x).\end{aligned}$$

The estimate for $f_1(t,x)$ in (2.3) follows from the Strichartz estimates for the free evolution group $\{W(t)\}_{t\in\mathbb{R}}$.

Lemma 2.1 (Kenig-Ponce-Vega [13]). *Let I be an interval of $\mathbb{R}$ (bounded or not) and $t_0 \in \overline{I}$. If (q_i, r_i) satisfy $8 \leq q_i \leq \infty$, $2 \leq r_i \leq \infty$ and $\frac{4}{q_i} + \frac{1}{r_i} = \frac{1}{2}$, $(i = 1,2)$. Then*

$$(2.4)\qquad \left\|\int_{t_0}^{t} W(t-t')f(t')dt'\right\|_{L_t^{q_1}(I;L_x^{r_1})} \leq C\|f\|_{L_t^{q_2'}(I;L_x^{r_2'})}$$

where p' is the Hölder conjugate exponent of p and C depends on q and not on I.

Indeed, we have that for $t \geq 0$,

$$\begin{aligned}(2.5)\qquad &\sup_{t\geq 0}(t+1)^{\alpha}(\|f_1(t)\|_{L^2_x} + \|f_1(\tau)\|_{L^8_\tau(t,\infty,L^\infty_x)}) \\ &\qquad \leq C(\rho^3 + \|\phi_+\|^2_{\mathcal{D}}\rho).\end{aligned}$$

The crucial point of the paper [16] is the estimation for $f_2(t,x)$. We note that the modification of the free dynamics comes from the estimate for the second term. We give how to modify the free dynamics and estimate f_2.

Concerning the choice of the modified free dynamics, we take the hint from Ozawa [15] and series of papers of Hayashi-Naumkin [9]-[11]. and set

$$v_+(t,x) = \sqrt{2\pi}F(t,x)\exp\left(iW_+(\chi)\log t\right)\hat{\phi}_+(\chi).$$

Let

$$\mathcal{R}(t,x) = \left(i\partial_t - \frac{1}{4}\partial_x^4\right)\tilde{v}_+(t,x) - \lambda|\tilde{v}_+|^2\tilde{v}_+(t,x).$$

By simple calculation we obtain

(2.6)

$$\begin{aligned}
&\mathcal{R}(t,x)\\
=\;& \sqrt{2\pi}\left(i\partial_t - \frac{1}{4}\partial_x^4\right) F(t,x)\exp\left(iW_+(\chi)\log t\right)\hat{\phi}_+(\chi)\\
&+\sqrt{2\pi}(iF(t,x)\partial_t - \partial_x^3 F(t,x)\partial_x)\exp\left(iW_+(\chi)\log t\right)\hat{\phi}_+(\chi)\\
&+\sqrt{2\pi}\exp\left(iW_+(\chi)\log t\right)(iF(t,x)\partial_t - \partial_x^3 F(t,x)\partial_x)\hat{\phi}_+(\chi)\\
&-\frac{\sqrt{2\pi}}{4}\sum_{\substack{j+k+\ell=4\\ j\neq 3,4}}\frac{4!}{j!k!\ell!}\partial_x^j F(t,x)\partial_x^k\exp\left(iW_+(\chi)\log t\right)\partial_x^\ell\hat{\phi}_+(\chi)\\
&-2\pi\sqrt{2\pi}\lambda|F|^2F(t,x)\exp\left(iW_+(\chi)\log t\right)\left|\hat{\phi}_+(\chi)\right|^2\hat{\phi}_+(\chi).
\end{aligned}$$

We note that by the definition of $F(t,x)$,

$$\left(i\partial_t - \frac{1}{4}\partial_x^4\right)F(t,x) = 0, \tag{2.7}$$

and

$$(iF(t,x)\partial_t - \partial_x^3 F(t,x)\partial_x)v(\chi) = 0. \tag{2.8}$$

Substituting the equalities (2.7) and (2.8) with $v(\chi) = W_+(\chi)$ and $v(\chi) = \hat{\phi}_+(\chi)$ into (2.6), we have

(2.9)

$$\begin{aligned}
&\mathcal{R}(t,x)\\
=\;& -\frac{\sqrt{2\pi}}{t}W_+(\chi)F(t,x)\exp\left(iW_+(\chi)\log t\right)\hat{\phi}_+(\chi)\\
&-\frac{\sqrt{2\pi}}{4}\sum_{\substack{j+k+\ell=4\\ j\neq 3,4}}\frac{4!}{j!k!\ell!}\partial_x^j F\partial_x^k\exp\left(iW_+(\chi)\log t\right)\partial_x^\ell\hat{\phi}_+(\chi)\\
&-2\pi\sqrt{2\pi}\lambda|F|^2F(t,x)\exp\left(iW_+(\chi)\log t\right)\left|\hat{\phi}_+(\chi)\right|^2\hat{\phi}_+(\chi)\\
\equiv\;& \mathcal{R}_1(t,x) + \mathcal{R}_2(t,x) + \mathcal{R}_3(t,x).
\end{aligned}$$

The worst terms are $\mathcal{R}_1$ and $\mathcal{R}_3$ which diverge in $L_t^1([1,\infty))$. As long as $\mathcal{R}$ includes $\mathcal{R}_1$ and $\mathcal{R}_3$, we are not able to apply the contraction principle. To avoid this, we choose W_+ so that $\mathcal{R}_1$ and $\mathcal{R}_3$ are canceled each other. Since the leading terms of $\mathcal{R}_1$ and $\mathcal{R}_3$ are

$$-\frac{1}{\sqrt{3i}}t^{-\frac{3}{2}}\frac{1}{|\chi|}W_+(\chi) \times \exp\left(\frac{3}{4}it|\chi|^4 + iW_+(\chi)\log t\right)\hat{\phi}_+(\chi),$$

and

$$-\frac{\lambda}{3\sqrt{3i}}t^{-\frac{3}{2}}\frac{1}{|\chi|^3} \times \exp\left(\frac{3}{4}it|\chi|^4 + iW_+(\chi)\log t\right)\left|\hat{\phi}_+(\chi)\right|^2\hat{\phi}_+(\chi),$$

we choose

$$W_+(x) = -\frac{\lambda}{3}|\hat{\phi}_+(x)|^2|x|^{-2}. \tag{2.10}$$

Therefore, we have

$$\|\mathcal{R}(t)\|_{L_x^2} \le t^{-2}(\log t)^4(1+\|\phi_+\|_{\mathcal{D}}^8)\|\phi_+\|_{\mathcal{D}}.$$

Combining this with the Strichartz estimate (Lemma 2.1 (2.4)), , we have that for $t \ge 0$,

$$\sup_{t\ge 0}(t+1)^{\alpha}(\|f_2(t)\|_{L_x^2} + \|f_2(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)}) \le C(1+\|\phi_+\|_{\mathcal{D}}^8)\|\phi_+\|_{\mathcal{D}}. \tag{2.11}$$

By (2.5) and (2.11), we guarantee that Φ is a map from X^ρ to itself if $\|\phi_+\|_{\mathcal{D}}$ and ρ are sufficiently small. Similarly we have that Φ is a contraction on X^ρ. Therefore we have Theorem 1.1.

Acknowledgment. The author would like to thank the referee for useful comments.

References

[1] J. E. Barab, Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation, J. Math. Phys., **25** (1984), 3270–3273.

[2] M. Ben-Artzi, H. Koch and J.-C. Saut, Dispersion estimates for fourth order Schrödinger equations, C. R. Acad. Sci. Paris Sér. I Math., **330** (2000), 87–92.

[3] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, **10**, Amer. Math. Soc., 2003.

[4] Y. Fukumoto and H. K. Moffatt, Motion and expansion of a viscous vortex ring. Part I. A higher-order asymptotic formula for the velocity, J. Fluid. Mech., **417** (2000), 1–45.

[5] J. Ginibre, An introduction to nonlinear Schrödinger equations, In: Nonlinear Waves, (eds. R. Agemi, Y. Giga and T. Ozawa), GAKUTO International Series, Math. Sci. Appl., **10** (1997), 85–133.

[6] J. Ginibre and T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, Comm. Math. Phys., **151** (1993), 619–645.

[7] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J. Math. Pures Appl., **64** (1985), 363–401.

[8] N. Hayashi and P. I. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, Amer. J. Math., **120** (1998), 369–389.

[9] N. Hayashi and P. I. Naumkin, Large time asymptotics of solutions to the generalized Korteweg-de Vries equation, J. Funct. Anal., **159** (1998), 110–136.

[10] N. Hayashi and P. I. Naumkin, Large time behavior of solutions for the modified Korteweg-de Vries equation, Internat. Math. Res. Notices, **1999** (1999), 395–418.

[11] N. Hayashi and P. I. Naumkin, On the modified Korteweg-de Vries equation, Math. Phys. Anal. Geom., **4** (2001), 197–227.

[12] N. Hayashi, P. I. Naumkin, A. Shimomura and S. Tonegawa, Modified wave operators for nonlinear Schrödinger equations in one and two dimensions, Electron. J. Differential Equations, **2004** (2004), 1–16.

[13] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., **40** (1991), 33–69.

[14] K. Moriyama, S. Tonegawa and Y. Tsutsumi, Wave operators for the nonlinear Schrödinger equation with a nonlinearity of low degree in one or two space dimensions, Commun. Contemp. Math., **5** (2003), 983–996.

[15] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, Comm. Math. Phys., **139** (1991), 479–493.

[16] J. Segata, Modified wave operators for the fourth-order non-linear Schrödinger-type equation with cubic non-linearity, Math. Methods Appl. Sci., **29** (2006), 1785–1800.

[17] A. Shimomura and S. Tonegawa, Long-range scattering for nonlinear Schrödinger equations in one and two space dimensions, Differential Integral Equations, **17** (2004), 127–150.

[18] W. A. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal., **41** (1981), 110–133.

[19] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré, Physique Théorique, **43** (1985), 321–347.

[20] Y. Tsutsumi and K. Yajima, The asymptotic behavior of nonlinear Schrödinger equations, Bull. Amer. Math. Soc., **11** (1984), 186–188.

Mathematical Institute, Tohoku University
Aoba, Sendai, 980-8578, Japan