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Small data scattering for the Klein-Gordon equation with a cubic convolution

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Abstract.

We consider the scattering problem for the Klein-Gordon equation with cubic convolution nonlinearity. We present the method to prove the existence of the scattering operator on a neighborhood of 0 in the weighted Sobolev space $H^{s,\sigma} = (1-\Delta)^{-s/2} \langle x \rangle^{-\sigma} L_2(\mathbb{R}^n)$. The method is based on the complex interpolation method of the weighted Sobolev spaces and the Strichartz estimates for the inhomogeneous Klein-Gordon equation.

§1. Introduction

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = F_\gamma(u) \tag{1.1}$$

in space-time $\mathbb{R} \times \mathbb{R}^n$, where u is a real-valued or a complex-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. The nonlinearity $F_\gamma(u)$ is a cubic convolution term $F_\gamma(u) = -(V_\gamma * |u|^2)u$ with

$$|V_\gamma(x)| \leq C|x|^{-\gamma}.$$

Here, $0 < \gamma < n$ and $*$ denotes the convolution in the space variables. The term $F_\gamma(u)$ is an approximative expression of the nonlocal interaction of specific elementary particles. Menzala and Strauss started to study this equation in [1].

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In order to treat the scattering problem, we define the scattering operator for (1.1). First, we list some notation to give the definition. Let H^s be the usual Sobolev space $(1-\Delta)^{-s/2}L_2(\mathbb{R}^n)$ and let $H^{s,\sigma}$ be the weighted Sobolev space $(1-\Delta)^{-s/2}\langle x\rangle^{-\sigma}L_2(\mathbb{R}^n)$. A Hilbert space $X^{s,\sigma}$ is denoted by $H^{s,\sigma}\oplus H^{s-1,\sigma}$. For a positive number δ and a Banach space A, we denote the set $\{a\in A; \|a|A\|\le\delta\}$ by $B(\delta;A)$. Then the scattering operator is defined as the mapping $S: B(\delta;X^{s,\sigma})\ni(f_-,g_-)\mapsto(f_+,g_+)\in X^{s,0}$ if the following condition holds for some $\delta>0$:

For any $(f_-,g_-)\in B(\delta;X^{s,\sigma})$, there uniquely exist a time-global solution $u\in C(\mathbb{R};H^s)$ of (1.1), and data $(f_+,g_+)\in X^{s,0}$ such that $u(t)$ approaches $u_\pm(t)$ in H^s as t tends to $\pm\infty$, where $u_\pm(t)$ are solutions of linear Klein-Gordon equations whose initial data are $(f_\pm,g_\pm)$, respectively.

We call that "$(S,X^{s,\sigma})$ is well-defined" if we can define the scattering operator $S: B(\delta;X^{s,\sigma})\to X^{s,0}$ for some $\delta>0$.

By Mochizuki [2], it is shown that if $n\ge 3$, $s\ge 1$, $\gamma<n$ and $2\le\gamma\le 2s+2$, then $(S,X^{s,0})$ is well-defined. By using the methods of Mochizuki and Motai [3] and Strauss [7], we see that if $n\ge 2$, $s\ge 1$, $4/3<\gamma<2$ and $\sigma>1/3$, then $(S,X^{s,\sigma})$ is well-defined. In view of the condition of σ, there is a gap between the two cases $\gamma\ge 2$ and $\gamma<2$ if we use only the methods of [2, 3, 7].

In [6], it is proved that $(S,X^{s,\sigma})$ is well-defined if $4/3<\gamma<2$ and $\sigma>(2-\gamma)/2$. The proof is based on the Strichartz estimate for pre-admissible pair, and the complex interpolation method for the weighted Sobolev spaces. Accordingly, we can fill the gap in some sense.

In this paper, we shall introduce the method of [6]. For this purpose, we first give notation.

For $s\in\mathbb{R}$ and $(1/p,1/q)\in[0,1]\times[0,1]$, let H^s_p be the Sobolev space $(1-\Delta)^{-s/2}L_p(\mathbb{R}^n)$. For $s\in\mathbb{R}$, we set $E^s[u](t)=\|(u(t),\partial_t u(t))|X^{s,0}\|$. For $s_0\in\mathbb{R}$ and $Q=(1/q,1/r)\in[0,1]\times[0,1]$, $L(s_0,Q)$ denotes $L_q(\mathbb{R};H^{s_0}_r(\mathbb{R}^n))$. Put $\omega=\sqrt{1-\Delta}$ and $U(t)=\exp(\pm it\omega)$. For a Banach space A, $\mathcal{B}^0(\mathbb{R};A)$ is the set of all A-valued, continuous and bounded functions on $\mathbb{R}$. Moreover, if f in $\mathcal{B}^0(\mathbb{R};A)$ has its derivative, and if $\partial_t f\in\mathcal{B}^0(\mathbb{R};A)$, then we write $f\in\mathcal{B}^1(\mathbb{R};A)$. For $s\in\mathbb{R}$, $\mathcal{H}^s$ denotes $\mathcal{B}^0(\mathbb{R};H^s)\cap\mathcal{B}^1(\mathbb{R};H^{s-1})$ with the norm $\|u|\mathcal{H}^s\|=$

$\|u|L(s,(0,1/2))\| + \|\partial_t u|L(s-1,(0,1/2))\|$. Furthermore, we set

$$\underline{\mathcal{H}^s} = \{u \in \mathcal{H}^s;\ \text{there exist } f, g \in \mathcal{S}'(\mathbb{R}^n) \text{ such that} \\ u(t) = \cos t\omega f + \omega^{-1}\sin t\omega g, \omega^{-1}\partial_t u(t) \in \mathcal{H}^s\}.$$

We call $u = u(t,x)$ a free solution if $u \in \underline{\mathcal{H}^s}$ for some $s \in \mathbb{R}$. For a free solution u_0, $u \in \acute{\mathcal{S}}(\mathbb{R}^n)$ is said to be a u_0-solution if

$$u(t) = u_0(t) + \int_0^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau.$$

For $s, s_0 \in \mathbb{R}$ and $Q = (1/q, 1/r) \in [0,1] \times [0,1]$, we denote $L(s_0, Q) \cap \mathcal{H}^s$ and $L(s_0, Q) \cap \underline{\mathcal{H}^s}$ by $Z(s_0, s, Q)$ and $\underline{Z}(s_0, s, Q)$, respectively. Define $1/q_\varepsilon = 1/3 - \varepsilon$ and $1/r_\theta = 1/2 - (1+\theta)/3n$. Assume that $4/3 < \gamma < 2$. Then there exist sufficiently small $\varepsilon(\gamma) > 0$ and $\theta(\gamma) \in (0,1)$ such that

$$\frac{1}{6} < \frac{n}{2}\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big) < \frac{1}{q_{\varepsilon(\gamma)}} < n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big), \\ \gamma = 2 - 2\Big\{\frac{2}{q_{\varepsilon(\gamma)}} - n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big)\Big\}.$$

For $Q_\gamma = (1/q_{\varepsilon(\gamma)}, 1/r_{\theta(\gamma)})$, we set

$$s(Q_\gamma) = \max\Big\{\frac{n+2}{n}\Big(1 - \frac{3}{q_{\varepsilon(\gamma)}}\Big), \frac{2-\gamma}{4}\Big\}.$$

We are now ready to state the results in [6].

Theorem 1.1. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $s \geq 1$, and put $s_\gamma = s(Q_\gamma)$, $Z = Z(s_\gamma + s - 1, s, Q_\gamma)$, $\underline{Z} = \underline{Z}(s_\gamma + s - 1, s, Q_\gamma)$. Then there exist some positive numbers δ_0, δ_+, δ_- satisfying the following properties:*

(i) If $u_0 \in B(\delta_0; \underline{Z})$, then there uniquely exist $u \in Z$ and $u_+, u_- \in \underline{Z}$ such that u is a u_0-solution and we have

$$\lim_{t \to \pm\infty} E^s[u - u_\pm](t) = 0. \tag{1.2}$$

Moreover, the operators $\widetilde{V_\pm} : B(\delta_0; \underline{Z}) \ni u_0 \mapsto u_\pm \in \underline{Z}$ are well defined, injective and continuous.

(ii) If $u_\pm \in B(\delta_\pm; \underline{Z})$, then there uniquely exist $u \in Z$ and $u_0 \in \underline{Z}$ such that u is a u_0 − solution and (1.2) holds.
Moreover, the operators $W_\pm : B(\delta_\pm; \underline{Z}) \ni u_\pm \mapsto u_0 \in \underline{Z}$ are well defined, injective and continuous.

(iii) *The numbers $\delta_\pm$ satisfy $B(\delta_-;\underline{Z}) \subset B(\delta_0;\underline{Z})$, $W_-(B(\delta_-;\underline{Z})) \subset B(\delta_0;\underline{Z})$ and $B(\delta_+;\underline{Z}) \subset \widetilde{V_+} \circ W_-(B(\delta_-;\underline{Z}))$. In particular, the operator $\widetilde{S} = \widetilde{V_+} \circ W_- : B(\delta_-;\underline{Z}) \to \underline{Z}$ is well defined, injective and continuous.*

The following result follows from Theorem 1.1.

Theorem 1.2. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $\sigma > (2-\gamma)/2$, $s \geq 1$ and put $s_\gamma = s(Q_\gamma)$, $Z = Z(s_\gamma + s - 1, s, Q_\gamma)$, $u_\star(t) = \cos t\omega f_\star + \omega^{-1}\sin t\omega f_\star$, where $\star$ denotes either $0, +$ or $-$. Then there exist some positive numbers η_0 and η_- satisfying the following properties:*

(i) *If $(f_0, g_0) \in B(\eta_0; X^{s,\sigma})$, then there uniquely exist $u \in Z$ and $(f_+, g_+), (f_-, g_-) \in X^{s,0}$ such that u is a u_0 – solution and (1.2) holds.*
Moreover, the operators $V_\pm : B(\eta_0; X^{s,\sigma}) \ni (f_0, g_0) \mapsto (f_\pm, g_\pm) \in X^{s,0}$ are well defined, injective and continuous.

(ii) *If $(f_-, g_-) \in B(\eta_-; X^{s,\sigma})$, there uniquely exist $u \in Z$ and $(f_+, g_+) \in X^{s,0}$ such that u satisfies*

$$u(t) = u_-(t) + \int_t^\infty \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau$$

and (1.2) holds.
Moreover, the scattering operator $S : B(\eta_-; X^{s,\sigma}) \ni (f_-, g_-) \mapsto (f_+, g_+) \in X^{s,0}$ is well defined, injective and continuous.

§2. Outline of the proof

Let us give a sketch of the proof of the theorems. We need the Strichartz estimates proved by Nakamura and Ozawa [5] (see also [4]).

Proposition 2.1. *Put $J = (0,t)$ or $(-\infty, t)$ or (t, ∞).*

(i) *If $2/q_j = n(1/2 - 1/r_j)$, $2\rho_j = (n+2)(1/2 - 1/r_j)$, $2 \leq q_j, r_j \leq \infty$, $(q_j, r_j) \neq (2, \infty)$, $j = 1, 2$, then we have*

$$\| \int_J U(t-\tau)h(\tau)d\tau | L_{q_1} H_{r_1}^{-\rho_1} \| \lesssim \| h | L_{q_2'} H_{r_2'}^{\rho_2} \|.$$

(ii) *If*

$$1/r_4' + 2/nq_4' = 1/r_3 + 2/nq_3 + 2/n, \tag{2.1}$$

$\max(0, 1/2 - 1/n) < 1/r_j < 1/2$, $0 < 1/q_j < n(1/2 - 1/r_j)$, $1/q_3 < 1/q_4'$, $\rho_3 + \rho_4 = (n+2)(1/r_4' - 1/r_3)/2$, *then we have*

$$\| \int_J U(t-\tau)h(\tau)d\tau | L_{q_3} H_{r_3}^{-\rho_1} \| \lesssim \| h | L_{q_4'} H_{r_4'}^{\rho_4} \|.$$

A pair (q,r) satisfying $2/q = n(1/2 - 1/r)$ is called an admissible pair. We immediately see that admissible pairs (q_3, r_3) and (q_4, r_4) satisfy (2.1).

We next state the estimate of the nonlinearity.

Lemma 2.2. *Assume that* $0 \le \rho \le \tilde{\rho}$, $0 < 1/\tilde{r} < 1/2 \le 1/r < 1$, $0 < \gamma < n$. *If there exist some* $\theta_j \in [0,1]$, $j = 1, 2$, *satisfying*

$$1 + \frac{1}{r} = \frac{\gamma}{n} + (\frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}) + 2(\frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n}),$$
$$\frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}, \frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n} > 0,$$

then we have

$$\|F(u)|H_r^{\rho}\| \lesssim \|u|H_{\tilde{r}}^{\tilde{\rho}}\|^3. \tag{2.2}$$

Proof. By the Hölder inequality, the Hardy-Littlewood-Sobolev inequality and the Sobolev embedding theorem, we obtain (2.2). Q.E.D.

We state the lemma which is useful to prove Theorem 1.1.

Lemma 2.3. *Assume that* $n \ge 2$, $4/3 < \gamma < 2$, $s \ge 1$ *and put* $L = L(s_\gamma + s - 1, Q_\gamma)$. *Then there exists some* $\delta > 0$ *satisfying as follows: If* $u_0 \in B(\delta; L)$, *then there uniquely exists* $u \in L$ *such that we have*

$$u(t) = u_0(t) + \int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau,$$
$$\|u|L\| \le \frac{4}{3}\|u_0|L\|,$$

$$\|\int_J U(t-\tau)F(u(\tau))d\tau|L(s-1,(0,1/2))\| \le \frac{1}{3}\|u_0|L\|. \tag{2.3}$$

Proof. (Step I.) In order to show the existence of a time-global solution, we define the contraction mapping on the suitable complete metric space. Put $Y = B(\frac{4}{3}\|u_0|L\|; L)$ and $d(u,v) = \|u - v|L\|$. Then (Y,d) is a nonempty complete metric space. We define a mapping Φ by

$$\Phi : u \longmapsto u_0 + \int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau.$$

By Proposition 2.1,(ii), we have

$$\|\int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau |L\| \lesssim \|F(u)|L(s_\gamma + s - 2 + \tilde{\rho}, (\frac{3}{q_\varepsilon}, \frac{1}{\tilde{r}_\theta}))\|.$$

Here (ε,θ) denotes $(\varepsilon(\gamma),\theta(\gamma))$ and $(\tilde{r_\theta},\tilde{\rho})$ satisfies

$$\frac{1}{\tilde{r_\theta}}+\frac{6}{nq_\varepsilon}=\frac{1}{r_\theta}+\frac{2}{nq_\varepsilon}+\frac{2}{n},$$
$$\tilde{\rho}=\frac{n+2}{2}\Big(\frac{1}{\tilde{r_\theta}}-\frac{1}{r_\theta}\Big)=\frac{n+2}{2}\Big(\frac{2}{n}-\frac{4}{nq_\varepsilon}\Big)=\frac{n+2}{3n}+2\frac{n+2}{n}\varepsilon.$$

Since $\tilde{\rho}\le 1$, we have

$$\|F(u)|L(s_\gamma+s-2+\tilde{\rho},(\frac{3}{q_\varepsilon},\frac{1}{\tilde{r_\theta}}))\|\lesssim\|F(u)|L(s_\gamma+s-1,(\frac{3}{q_\varepsilon},\frac{1}{\tilde{r_\theta}}))\|.$$

It follows from Lemma 2.2 that

$$\|F(u)|L(s_\gamma+s-1,(\frac{3}{q_\varepsilon},\frac{1}{\tilde{r_\theta}}))\|\lesssim\|u|L\|^3$$

since $1+1/\tilde{r_\theta}=\gamma/n+3/r_\theta$.

(Step II.) We estimate the left hand side of (2.3). By Proposition 2.1,(2),

$$\begin{aligned}&\|\int_J U(t-\tau)F(u(\tau))d\tau|L(s-1,(0,1/2))\|\\ &\qquad\lesssim\|F(u)|L(s-1+\dot{\rho},(\frac{3}{q_\varepsilon},\frac{1}{\dot{r_\theta}}))\|,\end{aligned}$$

where

$$\frac{1}{\dot{r_\theta}}=\frac{1}{2}+\frac{2}{n}(1-\frac{3}{q_\varepsilon}),\ \dot{\rho}=\frac{n+2}{n}(1-\frac{3}{q_\varepsilon}).$$

Put

$$\vartheta=\frac{1}{s-1+s_\gamma}\Big\{\frac{1}{q_\varepsilon}-\frac{n}{2}\Big(\frac{1}{2}-\frac{1}{r_\theta}\Big)\Big\}.$$

Then we have $0\le\dot{\rho}\le s_\gamma$, $0\le\vartheta\le 1$,

$$1+\frac{1}{\dot{r_\theta}}=\frac{\gamma}{n}+\frac{1}{r_\theta}+2\Big(\frac{1}{r_\theta}-\vartheta\frac{s-1+s_\gamma}{n}\Big),$$
$$\frac{1}{r_\theta}-\vartheta\frac{s-1+s_\gamma}{n}>0.$$

Thus, by Lemma 2.2, we have

$$\|F(u)|L(s-1+\dot{\rho},(\frac{3}{q_\varepsilon},\frac{1}{\dot{r_\theta}}))\|\lesssim\|u|L\|^3.$$

(Step III.) If $\|u_0|L\|$ is sufficient small, we see from Steps I and II that Φ is a contraction mapping from (Y,d) into itself, and that (2.3) holds.

Q.E.D.

Using Lemma 2.3, we easily show Theorem 1.1.

For the Theorem 1.2, we state the following lemma which is shown by the complex interpolation method for the weighted Sobolev spaces (see, e.g., [8]), and the Strichartz estimate for the free Klein-Gordon equation.

Lemma 2.4. *Assume that* $n \geq 2$, $\max(0, 1/2 - 1/n) < 1/r < 1/2$ *and* $(n/2 - n/r)/2 < 1/q < (n/2 - n/r)$. *Then we have*

$$\|U(\cdot)f|L_q L_r\| \lesssim \|f|H^{s,\sigma}\| \tag{2.4}$$

if

$$s > \frac{n+2}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{n+2}{2n}\left\{\frac{2}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right)\right\}$$

and

$$\sigma > \frac{2}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right).$$

By substituting $q = q_{\varepsilon(\gamma)}$ and $r = r_{\theta(\gamma)}$ for (2.4), we obtain Theorem 1.2.

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