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Waves in two-phase flows

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Abstract.

In present paper an attempt has been made to discuss weak-non linear waves through a two phase mixture of gas and dust particle, when particle volume fraction appears as an additional variable. Asymptotic method is used to find solution up to second order for high frequency harmonic waves.

§1. Introduction

Equations of state with one rate dependent state variable arise in the study of gases subject to chemical dissociation or vibrational relaxation. In the former case the possible effects of diffusion are normally neglected so that the purely chemical phenomenon is treated in isolation. Comprehensive review articles on this field and its applications have been written by Lick [1]. The propagation of disturbances, governed by non-linear hyperbolic systems, may exhibit a distortion of wave profile. This was studied by Varley and Cumberbatch [2], Dunwoody [3], Parker and Seymour [4] by using theory of relatively undistorted waves as an extension of the idea of Courant and Hilbert [5] for linear-waves. Sharma et al [6] have considered non-linear wave propagation in a hot-electron plasma by using theory of relativity undistorted wave. They have used a simple asymptotic expansion method to calculate first and second order solutions.

The studies of non-linear effects on the wave propagation have been extensively carried out by Jeffery and Taniuti [7], Whitham [8] and Courant and Friedrichs [9]. If the amplitude of the disturbance is not sufficiently small, the wave form is also altered by non-linear effects during propagation. Vincenti and Kruger [10] and Chu [11] formulated the general non-linear equation for the relaxing gas flow and studied the

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effects of relaxation on acceleration-wave during propagation and their termination into a shock-wave. Parker [12] has considered the effect of non-linearity and relaxation on the propagation of a one-dimensional wave.

The study of wave-propagation in a mixture of gas and dust particle has received great attention. There are many engineering applications for flow of a suspension of powered material or liquid-droplets in a gas. Dusty-gas flows assume importance in such engineering problems as flow in rockets, nuclear-reactors, fuel-sprays, air-pollution etc. The mathematical analysis of two-phase flows is considerably more difficult than that of pure-gas flows and one of the usual simplifying assumption is that the volume occupied by the particle can be neglected.

At high gas densities (high pressure) or at high particle mass-fraction, the particle-volume-fraction may become sufficiently large so that it should be included into flow analysis. Since the particle may be considered as incompressible in comparison with the gas, the particle-volume-fraction enters into the basic flow equations as an additional variable. The interesting properties of such two-phase-flows are that even equilibrium flow can not be treated as perfect gas flows. Effect of finite particle-volume on dynamics of gas particle mixture was studied by Rudinger [13]. Different aspects of particle-volume-fraction have been studied by different authors [14, 15, 16]. Propagation of rapid pulses through a two-phase mixture of gas and dust-particles, when particle-volume-fraction is negligible is studied by Sharma et al [17]. However they have used second order solution to describe the far field behavior of weak shocks.

In present article an attempt has been made to discuss plane-wave in two-phase-flows of gas-particle mixture when particle volume fraction appears as an additional variable and gas is taken as dissociative diatomic. For case, when equilibrium is eventually established equations of motion, wave conditions, variation of wave-strength and weak-shock-waves are discussed. Asymptotic analysis is used to find a solution up to second order approximation for high frequency harmonic waves.

§2. Equation-of-Motion and Wave Condition

The equations governing one-dimensional motion of two-phase flow of a gas with internal dissipation are given by

$$u_{,t} + uu_{,x} + \frac{1}{\rho(1-\epsilon)(1+\eta)}\, p_{,x} = 0 \tag{2.1}$$

$$(\rho_{,t} + u\rho_{,x}) + \frac{\rho}{(1-\epsilon)}\, u_{,x} \;=\; 0 \tag{2.2}$$

$$(H_{,\rho}\rho_{,t}) + \left(H_{,p} - \frac{1}{\rho}\right) p_{,t} + H_{,\alpha}\alpha_{,t} + H_{,\epsilon}\epsilon_{,t} + (uH_{,\rho}\rho_{,x}) + u\left(H_{,p} - \frac{1}{\rho}\right) p_{,x}$$

$$+uH_{,\alpha}\alpha_{,x} + uH_{,\epsilon}\epsilon_{,x} \;=\; 0 \tag{2.3}$$

$$\alpha_{,t} + u\alpha_{,x} + f(p,\rho,\alpha,\epsilon) \;=\; 0 \tag{2.4}$$

$$\epsilon_{,t} + \epsilon u_{,x} + u\epsilon_{,x} \;=\; 0 \tag{2.5}$$

The above equations can be rewritten in the following matrix equation

$$A\,\frac{\partial U}{\partial t} + B\,\frac{\partial U}{\partial x} + C \;=\; 0 \tag{2.6}$$

where

$$U^T \;=\; [u, \rho, p, \alpha, \epsilon]$$

and A, B, C are matrices which can be obtained from equation (2.1) to (2.5) by inspection, U^T being a column vector.

Here u is the material velocity, p the pressure, ρ the density, α an internal state variable which may either represent the degree of dissociation in a diatomic gas or be some measure of vibrational energy, ϵ is the volume fraction of the particles. $H = H[p,\rho,\alpha,\epsilon]$ is the specific enthalpy of the system and a comma followed by an index denote partial differentiation with respect to that index. The propagation of waves into an equilibrium state, defined at a point $(p_0, V_0, \alpha_0, \epsilon_0)$ is given by

$$f(p_0, V_0, \alpha_0, \epsilon_0) \;=\; 0, \tag{2.7}$$

where $V = \dfrac{1}{\rho}$ which is said to be (locally) asymptotically stable at a constant pressure and volume.

If $\phi = \phi(x,t)$ defines a 'wavelet' and we assume that $\phi_{,t} \neq 0$, the transformation of coordinates (x,t) to (x,ϕ) transforms any function χ as

$$\chi(x, T(x,\phi)) \;=\; U(x,\phi) \tag{2.8}$$

Thus

$$\frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial t}\,\frac{\partial T}{\partial x} \;=\; \frac{\partial U}{\partial x} \tag{2.9}$$

Equation (2.6) can be written as

$$(BT_{,x} - A)U_{,t} \;=\; BU_{,x} + C, \tag{2.10}$$

where U must satisfy the compatibility condition

$$\ell(BU_{,x} + C) \;=\; 0 \tag{2.11}$$

Condition for relatively undistorted wave is defined by

$$\left\| \frac{\partial \chi}{\partial x} \right\| << \left\| \frac{\partial U}{\partial x} \right\| \tag{2.12}$$

where $\| \; \|$ denotes the Euclidean norm of a vector.

As equation (2.8) holds exactly at an acceleration wave-front propagating into an undistorted region in thermodynamic equilibrium and also on other 'wavelet' in a non-dissipative gas, $U_{,x} = 0$ and hence

$$\chi(x, T(x,\phi)) \;=\; U(\phi),$$

thus to a first approximation

$$(BT_{,x} - A)U_{,\phi} \;=\; 0. \tag{2.13}$$

From equation $\det(B - W^{-1}A) = 0$ eigenvalues are given by

$$(T_{,x})^{-1} \;=\; W^{-1} \;=\; u + a_R \tag{2.14}$$

where

$$a_R \;=\; \left\{ \mu \left(-\frac{\rho H_{,\rho}}{1-\epsilon} - \epsilon H_{,\epsilon} \right) \right\}^{1/2}$$

and

$$\mu \stackrel{\text{def}}{=} \frac{1}{\rho(1-\epsilon)(1+\eta)\left(H_{,p} - \dfrac{1}{\rho} \right)}$$

a_R being local resulting sound speed.

The left eigenvector associated with eigenvalue (2.14) is given by

$$\ell = \left[\frac{\rho}{1-\epsilon} H_{,\rho}, -\frac{\mu\rho}{a_R(1-\epsilon)} H_{,\rho}^2, \frac{\mu\rho}{a_R(1-\epsilon)} H_{,\rho}, -\frac{\mu\rho}{a_R(1-\epsilon)} H_{,\alpha} H_{,\rho}, \right.$$
$$\left. - \frac{\mu\rho}{a_R(1-\epsilon)} H_{,\rho} H_{,\epsilon} \right] \tag{2.15}$$

Substituting (2.14) in (2.13) and using relation $a_R^2 = \dfrac{\gamma' p}{\rho}$, the solution appropriate to a plane wave propagating into a region in thermodynamical equilibrium is given by

$$\left.\begin{aligned} &\alpha = \alpha_0,\ p = \rho^\Gamma,\ u = \frac{2\Gamma^{1/2}}{(\Gamma-1)(1-\epsilon)^{1/2}(1+\eta)^{1/2}}\left[\rho^{\frac{\Gamma-1}{2}} - \rho_0^{\frac{\Gamma-1}{2}}\right] \\ &a_R^2 \ = \ \gamma' \rho^{\Gamma-1},\ \text{and}\ \frac{\epsilon}{\epsilon_0} = \frac{\rho}{\rho_0} \end{aligned}\right\} \tag{2.16}$$

where $\Gamma = (1-\epsilon)(1+\eta)\gamma'$ and ρ_0, p_0 and α_0, ϵ_0 are the values of the state variables on the leading characteristic and without loss of generality the units of pressure have been chosen so that $\dfrac{p_0}{\rho_0^\Gamma} = 1$.

§3. Variation of Wave Strength

Since equation (2.16) gives the relations between u, ρ, p, α and ϵ substituting ℓ from equation (2.15) into equation (2.11) we have

$$u_{,x} - \frac{1}{2}\left\{\frac{\mu}{a_R(u+a_R)}\right\} H_{,\alpha}\, f(p,\rho,\alpha,\epsilon) \ = \ 0 \tag{3.1}$$

where

$$u + a_R \ = \ \frac{2\Gamma^{1/2}}{(\Gamma-1)(1-\epsilon)^{1/2}(1+\eta)^{1/2}}\left[\frac{\Gamma+1}{2}\rho^{\frac{\Gamma-1}{2}} - \rho_0^{\frac{\Gamma-1}{2}}\right] \tag{3.2}$$

and

$$\left(\frac{\partial T}{\partial x}\right) \ = \ (u+a_R)^{-1}. \tag{3.3}$$

Equation (3.1) can also be written as

$$u_{,x} + \frac{1}{2}\left(\frac{a_R}{u+a_R}\right)\frac{a_f^2}{(a_f^2+a_p^2)}\left\{\frac{(1-\epsilon)H_{,\alpha}}{\rho H_{,\rho}}\right\} f(p,\rho,\alpha,\epsilon) \ = \ 0, \tag{3.4}$$

where

$$a_R^2 \ = \ -\frac{\mu\rho H_{,\rho}}{1-\epsilon} - \mu\epsilon H_{,\epsilon} \ = \ a_f^2 + a_p^2$$

and

$$a_f^2 \ = \ \frac{-\mu\rho H_{,\rho}}{1-\epsilon},\ a_p^2 \ = \ -\mu\epsilon H_{,\epsilon}$$

Introducing linearization

$$\rho \ = \ \rho_0 + \psi\rho_1,\ a_R \ = \ a_{R_0} + \psi a_{R_1},\ u \ = \ \psi u_1$$

etc.

Equations (3.4) and (3.3) reduces to

$$\frac{\partial u_1}{\partial x} + \frac{1}{2}(1-\epsilon_0)\frac{u_1}{a_{R_0}}\left(\frac{H_{,\alpha}}{H_{,\rho}}\right)_0 \times \left\{\left(\frac{\partial f}{\partial p}\right)_0 a_{R_0}^2 + \left(\frac{\partial f}{\partial \rho}\right)_0 + \frac{\epsilon_0}{\rho_0}\left(\frac{\partial f}{\partial \epsilon}\right)_0\right\} = 0 \tag{3.5}$$

and

$$\frac{\partial T}{\partial x} = a_R^{-1}\left\{1 - \frac{\Gamma+1}{2a_{R_0}}\, u_1\right\} \tag{3.6}$$

respectively where it has been assumed that $f(p,\rho,\alpha,\epsilon)$ may be expanded in terms of its arguments at equilibrium.

Integrating equation (3.5), we have

$$u_1 = g(\phi)\exp[-\lambda(x-x^\star)], \ (x \geq x^\star) \tag{3.7}$$

where

$$\lambda = \frac{1}{2}(1-\epsilon_0)\frac{1}{a_{R_0}}\left(\frac{H_{,\alpha}}{H_{,\rho}}\right)_0 \left\{\left(\frac{\partial f}{\partial p}\right)_0 a_{R_0}^2 + \left(\frac{\partial f}{\partial \rho}\right)_0 + \frac{\epsilon_0}{\rho_0}\left(\frac{\partial f}{\partial \epsilon}\right)_0\right\}$$

and $\tilde{g}(\hat{\phi}) = u_1(x^\star, t)$, i.e. ϕ is the time that a "wavelet" leaves the station $x^\star$ when λ will be positive. On substituting equation (3.7) in equation (3.6) and integrating the equation of any "wavelet", ϕ =constant, we have

$$a_{R_0}(T-\phi) = (x-x^\star) + \frac{\Gamma+1}{2a_{R_0}\lambda}\, g(\phi)(\exp\{-\lambda(x-x^\star)\}-1). \tag{3.8}$$

The formation of a shock wave is characterized $\dfrac{\partial T}{\partial \phi} = 0$ so that by (3.8), if $g'(\phi) > 0$ (the prime denoting differentiation with respect to ϕ) we have

$$\frac{\partial T}{\partial \phi} = 1 + \frac{(\Gamma+1)g'(\phi)}{2a_{R_0}^2\lambda}(\exp\{-\lambda(x-x^\star)\}-1) = 0. \tag{3.9}$$

Value of x at which shock wave will occur can be obtained by putting $\phi = 0$ in equation (3.9). The acceleration on any characteristic or wavelet is obtained from (3.7) as

$$\frac{\partial u_1}{\partial t} = g'(\phi)\left(\frac{\partial T}{\partial \phi}\right)^{-1}\exp\{-\lambda(x-x^\star)\}. \tag{3.10}$$

Along the leading wave or wave front the strength of the discontinuity in acceleration is obtained through (3.9) and (3.10) by the relation

$$\left.\frac{\partial u_1}{\partial t}\right|_{\phi=0} = g'(0)\exp\{-\lambda(x-x^\star)\}$$
$$\times\left\{1+\frac{(\Gamma+1)g'(0)}{2a_{R_0}^2\lambda}\left(\exp\{-\lambda(x-x^\star)\}-1\right)\right\}^{-1}. \tag{3.11}$$

The relatively undistorted approximation is valid if

$$\left|\frac{g'(\phi)}{g(\phi)a_{R_0}\lambda}\right| >> \left(1+\frac{(\Gamma+1)}{2a_{R_0}}\,g(\phi)\,\exp\{-\lambda(x-x^\star)\}\right)\frac{\partial T}{\partial\phi}, \tag{3.12}$$

which is satisfied automatically at a wave front $\phi = 0$ or near a shock where $\frac{\partial T}{\partial \phi} = 0$. It is also satisfied in the degenerate case of $(a_{R_0}\lambda) \to 0$ in which case the results for an ideal classical non-dissipative gas are obtained.

At $x = x^\star = 0$, $\frac{\partial T}{\partial \phi} = 1$ the approximation is valid if the local frequency

$$\omega_L = \left|\frac{g'(\phi)}{g(\phi)}\right| >> \left(\frac{\partial T}{\partial\phi}\right) a_{R_0}\lambda = a_{R_0}\lambda. \tag{3.13}$$

The validity of the approximation may be extended to all values of (x,ϕ) provided

$$\left|\frac{g'(\phi)}{a_{R_0}^2\lambda}\right| < M,\ \left|\frac{g(\phi)}{a_{R_0}}\right| << 1 \tag{3.14}$$

where M is finite, and (3.13) is satisfied.

The condition (3.14) may be satisfied by small amplitude high frequency sound waves, i.e. the frequency is high in a sense relative to the natural time $(a_{R_0}\lambda)^{-1}$. The relation (3.13) suggests a parameter for an asymptotic analysis.

§4. Weak Shock Wave

It may be shown that the behavior of dissipative gas through a shock is exactly similar to that of its non-dissipative counterpart. In particular the relations

$$[\alpha] = 0 \qquad [S] > 0$$

where the [ ] brackets denote the discontinuity in a variable across the shock, must hold. For weak shock the entropy jump is third order in the density jump, while the shock speed u is that of the local resulting speed of sound to a first approximation i.e. $u \cong a_R$ S being entropy.

In the limits of weak shocks the relations (2.16) appropriately when linearized satisfy the compatibility condition which must hold across a shock, i.e. the jumps in any variable when computed from (2.16) for two values of ϕ satisfy these conditions.

Since two characteristics, say ϕ_1 and ϕ_2 coalesce at a shock it follows from (3.7) that

$$[u] = [g(\phi_1) - g(\phi_2)] \exp\{-\lambda(x - x^\star)\}. \tag{4.1}$$

The speed G of the shock surface is then given by

$$G = \frac{1}{2}\{(a_{R_1} + u_1) + (a_{R_2} + u_2)\}. \tag{4.2}$$

To a first approximation and through (3.3) the relation

$$\frac{1}{G} \cong a_{R_0}^{-1}\left(1 - \frac{\Gamma+1}{4a_{R_0}}\{g(\phi_1) + g(\phi_2)\} \exp\{-\lambda(x - x^\star)\}\right) \tag{4.3}$$

can be derived.

Through (3.8) it is implied that at the shock

$$\frac{\phi_1 - \phi_2}{g(\phi_1) - g(\phi_2)} = -\frac{\Gamma+1}{2a_{R_0}^2\lambda}(\exp\{-\lambda(x - x^\star)\} - 1). \tag{4.4}$$

where (x_1, t_1) and (x_2, t_2) are the coordinates of a point on ϕ_1 and ϕ_2 and at shock $t_1 = t_2$ and $x_1 = x_2$.

In general characteristics have the explicit form

$$t = f(x, \phi) + \phi \tag{4.5}$$

and any curve intersected by these curves may be represented in (x, ϕ) coordinates. Since the shock will be described by a curve $t = s(x)$ it follows from (4.5) and the implicit function theorem that along the shock

$$\phi = \Psi(x). \tag{4.6}$$

Therefore on the shock wave we have

$$t = f(x, \Psi(x)) + \Psi(x).$$

Considering the specific form of (4.5), which is (3.8), we derive a further relation for the shock speed

$$\frac{1}{G} = \frac{ds(x)}{dx} \cong a_{R_0}^{-1}\left\{1 - \frac{\Gamma+1}{2a_{R_0}}\, g(\phi)\exp\{-\lambda(x-x^\star)\}\right\}$$

$$+\left\{1 + \frac{\Gamma+1}{2a_{R_0}^2\,\lambda}\, g'(\phi)(\exp\{-\lambda(x-x^\star)\}-1)\right\}\frac{d\phi}{dx} \tag{4.7}$$

which hold for both the ϕ_1 and ϕ_2 sets of characteristics or wavelet. Equations (4.3) and (4.7) then imply that on the shock the relation

$$\{g(\phi_1) - g(\phi_2) - (\phi_1-\phi_2)g'(\phi_1)\}\frac{d\phi_1}{dx}$$

$$= \{g(\phi_2) - g(\phi_1) - (\phi_2-\phi_1)g'(\phi_2)\}\frac{d\phi_2}{dx} \tag{4.8}$$

must be satisfied by ϕ_1 and ϕ_2, and the shock path is then determined by (3.8), (4.4) and (4.8).

In the case of a shock propagating into an undistorted region equations (4.3) and (4.7) yield the relation

$$-\frac{\Gamma+1}{4a_{R_0}^2}\, g(\phi_2)\,\exp\{-\lambda(x-x^\star)\}$$

$$+\left\{1 + \frac{\Gamma+1}{2a_{R_0}^2\,\lambda}\, g'(\phi_2)(\exp\{-\lambda(x-x^\star)\}-1)\right\}\frac{d\phi_2}{dx} = 0 \tag{4.9}$$

as $\phi_1 = 0$, after integration we have

$$-\frac{\Gamma+1}{2a_{R_0}^2\lambda}\,(\exp\{-\lambda(x-x^\star)\}-1) \;=\; 2\int_0^{\phi_2}\frac{g(s)\,ds}{g^2(\phi_2)}. \tag{4.10}$$

This result is similar to that obtained by Whitham [18] whose result follows from (4.10) (in the $\lim \lambda \to 0$). Taking the limit of (4.10) as ϕ_2 tends to zero, we find

$$\lim\, \phi_2 \to 0\; 2\int_0^{\phi_2}\frac{g(s)\,ds}{g^2(\phi_2)} \;=\; \{g'(0)\}^{-1} \tag{4.11}$$

which confirms the result obtained from (3.9) viz that the shock first occurs when $\dfrac{\partial T}{\partial \phi} = 0$. If the compressive phase of a wave is followed by one of the rarefaction, then $g(\phi)$ has a zero and some of the wavelets in the neighborhood of this zero wavelet will not catch up the shock. It follows that the integral in (4.10) is bounded and that at large distance

$$g(\phi_2) \propto \left\{ -\frac{\Gamma+1}{2a_{R_0}^2\lambda} (\exp\{-\lambda(x-x^\star)\} - 1) \right\}^{-1/2} \tag{4.12}$$

From (3.7) and (4.12) it follows that

$$[u] \propto \left\{ \frac{\Gamma+1}{2a_{R_0}^2\lambda} (1-\exp\{-\lambda(x-x^\star)\}) \exp\{2\lambda(x-x^\star)\} \right\}^{-\frac{1}{2}} \tag{4.13}$$

Similarly the distance by which the shock is ahead of the zero "wavelet" $t = \phi_0 + \frac{x}{S_0}$ increases by an amount

$$L \propto \left\{ \frac{\Gamma+1}{2a_{R_0}^2\lambda} (1-\exp\{-\lambda(x-x^\star)\}) \right\}^{-\frac{1}{2}}. \tag{4.14}$$

In limit $\lambda \to 0$, all the above results reduce to those obtained by Whitham [18] for a nondissipative ideal gas.

§5. Asymptotic Analysis

The analysis in article three has suggested a parameter $a_{R_0}\lambda$ with which we form an asymptotic analysis. It was seen there that the "undistorted" approximation was valid provided (3.13) and (3.14) were satisfied. In this section the propagation of "high frequency" harmonic wave is considered . At $x = 0$, the initial conditions are taken to be

$$x = \omega^{-2}\sigma(1-\cos\beta), \ u = \omega^{-1}\sigma\sin\beta, \ t = \omega^{-1}\beta \tag{5.1}$$

so that a given wavelet is described by β =constant and the conditions (3.13) and (3.14) are seen to be satisfied for $\frac{\omega}{a_{R_0}\lambda} >> 1$. The constant σ is then maximum acceleration and is finite.

Again applying the transformation of article two we have

$$t = T(x, \beta, \omega). \tag{5.2}$$

Equations (5.1) and (5.2), imply the further boundary conditions

$$\frac{\partial T}{\partial \beta} = \omega^{-1} - \omega^{-2}\sigma(a_R + u)^{-1}\sin\beta \tag{5.3}$$

In terms of characteristic co-ordinates the equations (2.13) and (2.14) become

$$\left.\begin{aligned} \left(B_{ij}\frac{\partial T}{\partial x} - A_{ij}\right)\frac{\partial U_j}{\partial \beta} &= \frac{\partial T}{\partial \beta}\left(B_{ij}\frac{\partial U_j}{\partial x} + C_i\right) \\ \frac{\partial T}{\partial x} &= (a_R + u)^{-1} \end{aligned}\right\} \tag{5.4}$$

Considering asymptotic expansion of the form

$$\left.\begin{aligned} u &= \sum_{n=1}^{N} \psi^n u_n(x,\beta), & p &= p_0 + \sum_{n=1}^{N} \psi^n p_n(x,\beta) \\ \rho &= \rho_0 + \sum_{n=1}^{N} \psi^n \rho_n(x,\beta), & a_R &= a_{R_0} + \sum_{n=1}^{N} \psi^n a_{R_n}(x,\beta) \\ \alpha &= \alpha_0 + \sum_{n=1}^{N} \psi^n \alpha_n(x,\beta), & f &= \sum_{n=1}^{N} \psi^n f_n(x,\beta) \\ T &= T_0(x) + \sum_{n=1}^{N} \psi^n T_n(x,\beta), & \epsilon &= \epsilon_0 + \sum_{n=1}^{N} \psi^n \epsilon_n(x,\beta) \end{aligned}\right\} \tag{5.5}$$

as suggested by the conditions (5.1) and (5.3), the constants appearing in (5.5) are the equilibrium values of the respective variables and $\psi = \omega^{-1}$, where successive terms are such as U_n and U_{n+1} have the ratio $\dfrac{U_n}{U_{n+1}} = O(a_{R_0}\lambda)$. For each eigen value there exists a left eigenvector ℓ. Since there is an asymptotic expansion for each eigenvalue it is implied that a similar expansion exists for ℓ, viz.

$$\ell = \ell_0 + \sum_{n=1}^{N} \psi^n \ell_n. \tag{5.6}$$

Also, on any characteristic curve the relations

$$\ell\left(B\frac{\partial T}{\partial x} - A\right)\frac{\partial U}{\partial \beta} = \ell\left(B\frac{\partial U}{\partial x} + C\right) = 0 \tag{5.7}$$

must hold.

Equations (5.4), (5.5) and (5.6) from the basis of the approximating scheme.

Zeroth Approximation

On substituting (5.5) into (5.4) and equating coefficients of zero powers of ψ equal to zero in both the resulting relation and (5.7), we obtain

$$\left.\begin{aligned} \frac{\partial T_0}{\partial x} &= \frac{1}{a_{R_0}} \\ \ell_0\left\{B_0\frac{\partial T_0}{\partial x} - A_0\right\} &= 0 \end{aligned}\right\} \tag{5.8}$$

The solution of these equations, appropriate to the boundary conditions are

$$\left.\begin{aligned} T_0 &= \frac{x}{a_{R_0}} \\ \text{and} \qquad \ell_0 &= \left[\frac{\rho_0}{1-\epsilon_0}(H_{,\rho})_0, -\frac{\mu_0\rho_0}{a_{R_0}(1-\epsilon_0)}(H_{,\rho})_0^2, \frac{\mu_0\rho_0}{a_{R_0}(1-\epsilon_0)}(H_{,\rho})_0, \right. \\ &\quad \left. -\frac{\mu_0\rho_0}{a_{R_0}(1-\epsilon_0)}(H_{,\rho})_0(H_{,\alpha})_0, -\frac{\mu_0\rho_0}{a_{R_0}(1-\epsilon_0)}(H_{,\rho})_0(H_{,\alpha})_0\right] \end{aligned}\right\} \tag{5.9}$$

First Approximation

Similarly by equating coefficients of the first power of ψ equal to zero, we have

$$\left.\begin{aligned} &\frac{\partial T^{(1)}}{\partial x} = -(a_{R_1}+u_1)a_{R_0}^{-2} \\ &\left(B_{ij}^{(0)}\frac{\partial T^{(0)}}{\partial x} - A_{ij}^{(0)}\right)\frac{\partial U_j^{(1)}}{\partial \beta} = 0 \\ &\ell_i^{(0)}\left\{B_{ij}^{(0)}\frac{\partial U_j^{(1)}}{\partial x} + C_i^{(1)}\right\} = 0 \\ &\ell_i^{(0)}\left\{B_{ij}^{(0)}\frac{\partial T^{(1)}}{\partial x} + B_{ij}^{(1)}\frac{\partial T^{(0)}}{\partial x} - A_{ij}^{(1)}\right\} + \ell_i^{(1)}\left\{B_{ij}^{(0)}\frac{\partial T^{(0)}}{\partial x} - A_{ij}^{(0)}\right\} = 0 \end{aligned}\right\} \tag{5.10}$$

The first three of equations of (5.10) and the appropriate boundary conditions from (5.1) and (5.2), viz $u_1 = \sigma\sin\beta$, $T_1 = \beta$, give the solution

$$\left.\begin{aligned} u^{(1)} &= \sigma\sin\beta\exp\{-\lambda(x-x^\star)\} \\ a_R^{(0)}(T^{(1)}-\beta) &= \frac{\Gamma+1}{2a_R^{(0)}\lambda}\sigma\sin\beta(\exp\{-\lambda(x-x^\star)\}-1) \end{aligned}\right\} \tag{5.11}$$

Also we have the relation that

$$\left.\begin{aligned}
\rho^{(1)} &= \frac{\rho^{(0)}u^{(1)}}{a_R^{(0)}} \\
p^{(1)} &= \rho^{(0)}a_R^{(0)}(1-\epsilon^{(0)})(1+\eta)u^{(1)} \\
\epsilon^{(1)} &= \epsilon^{(0)}\frac{u^{(1)}}{a_R^{(0)}} \\
\alpha^{(1)} &= \text{constant}
\end{aligned}\right\} \quad (5.12)$$

$$\ell^{(1)} = [x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}] \quad (5.13)$$

where

$$x_1^{(1)} = \frac{\rho^{(0)}(H_{,\rho})^{(0)}\epsilon^{(1)} + \{\rho^{(0)}(H_{,\rho})^{(1)} + \rho^{(1)}(H_{,\rho})^{(0)}\}(1-\epsilon^{(0)})}{(1-\epsilon^{(0)})^2}$$

$$x_2^{(1)} = \frac{-\left[\begin{array}{l} 2\mu^{(0)}\rho^{(0)}(H_{,\rho})^{(0)}(H_{,\rho})^{(1)} + (\mu^{(0)}\rho^{(1)} + \mu^{(1)}\rho^{(0)})\{(H_{,\rho})^{(0)}\}^2 \\ +\mu^{(0)}\rho^{(0)}\{(H_{,\rho})^{(0)}\}^2\left\{\dfrac{\epsilon^{(1)}}{1-\epsilon^{(0)}} - \dfrac{a_R^{(1)}}{a_R^{(0)}}\right\} \end{array}\right]}{a_R^{(0)}(1-\epsilon^{(0)})}$$

$$x_3^{(1)} = \frac{\left[\begin{array}{l} \mu^{(0)}\rho^{(0)}(H_{,\rho})^{(0)}\left\{\dfrac{\epsilon^{(1)}}{1-\epsilon^{(0)}} - \dfrac{a_R^{(1)}}{a_R^{(0)}}\right\} \\ +\{\mu^{(0)}\rho^{(0)}(H_{,\rho})^{(1)} + (\mu^{(0)}\rho^{(1)} + \mu^{(1)}\rho^{(0)})(H_{,\rho})^{(0)}\} \end{array}\right]}{a_R^{(0)}(1-\epsilon^{(0)})}$$

$$x_4^{(1)} = \frac{-\left[\begin{array}{l} \mu^{(0)}\rho^{(0)}(H_{,\rho})^{(0)}\left\{\dfrac{\epsilon^{(1)}}{1-\epsilon^{(0)}} - \dfrac{a_R^{(1)}}{a_R^{(0)}}\right\} \\ +(\mu^{(0)}\rho^{(1)} + \mu^{(1)}\rho^{(0)})(H_{,\rho})^{(0)}(H_{,\alpha})^{(0)} \\ \qquad +\mu^{(0)}\rho^{(0)}\{(H_{,\rho})^{(0)}(H_{,\alpha})^{(1)} + (H_{,\rho})^{(1)}(H_{,\alpha})^{(0)}\} \end{array}\right]}{a_R^{(0)}(1-\epsilon^{(0)})}$$

$$x_5^{(1)} = \frac{-\left[\begin{array}{l} \mu^{(0)}\rho^{(0)}(H_{,\rho})^{(0)}(H_{,\epsilon})^{(0)}\left\{\dfrac{\epsilon^{(1)}}{1-\epsilon^{(0)}} - \dfrac{a_R^{(1)}}{a_R^{(0)}}\right\} \\ +(\mu^{(0)}\rho^{(1)} + \mu^{(1)}\rho^{(0)})(H_{,\rho})^{(0)}(H_{,\epsilon})^{(0)} \\ \qquad +\mu^{(0)}\rho^{(0)}\{(H_{,\rho})^{(0)}(H_{,\epsilon})^{(1)} + (H_{,\rho})^{(1)}(H_{,\epsilon})^{(0)}\} \end{array}\right]}{a_R^{(0)}(1-\epsilon^{(0)})}$$

By reference to the solution in article three, it is implied by (3.9) that shocks will occur on all wavelets for which $g'(\beta)$ is maximum, i.e. on $\beta = 2\Pi n$ $(n = 0, 1, 2, \cdots)$ and will first be located at the station

$$x = -\frac{1}{\lambda} \log\left(1 - \frac{2a_{R_0}^2 \lambda}{(\Gamma + 1)\sigma}\right). \tag{5.14}$$

Also through (4.3) it is seen that the shock formed on $\beta = 2\Pi n$ $(n = 0, 1, 2, \cdots)$ propagate with the speed of sound and so a constant distance apart. The leading shock however moves ahead of that on $\beta = 2\Pi$ as indicated by (4.14).

Any two wavelets $\beta_2 = 2\Pi n + \phi$ and $\beta_1 = 2\Pi n - \phi$ with $n \neq 0$ coalesce with the shock at the same instant and the substitution

$$g(\beta_2) = -g(\beta_1) = \omega^{-1}\sigma \sin\phi \tag{5.15}$$

satisfy (4.7). By (4.4) these two wavelets reach the shock where

$$\frac{\Gamma + 1}{2a_{R_0}^2 \lambda}(\exp[-\lambda(x - x^\star)] - 1) = -\frac{\phi}{\sigma \sin\phi}. \tag{5.16}$$

Therefore it is implied by (5.16) and (4.1) that the strength of these shocks
$\beta = 2\Pi n$ $(n = 0, 1, 2, \cdots)$ decay as

$$[u] \propto \frac{\exp\{-\lambda(x - x^\star)\}}{1 - \exp\{-\lambda(x - x^\star)\}}.$$

Those wavelets β which lie in the region

$$(2n - 2)\Pi + \phi < \beta < 2n\Pi - \phi \qquad (n = 1, 2, \cdots)$$

never coalesce with a shock and so form the expansion separating the shocks.

Second Approximation

The analysis required to obtain the full approximation in this case is algebraically complicated but as it is of interest to enquire into the behaviour of variable α, which has remained constant up to the first approximation, this will now sketched out. The second order equations are

$$\frac{\partial T^{(2)}}{\partial x} = \left\{\frac{(a_R^{(1)} + u^{(1)})^2}{a_{R_0}^{(0)}} - (a_R^{(2)} + u^{(2)})\right\} a_R^{(0)^{-2}} \tag{5.17a}$$

$$\left\{B_{ij}^{(0)}\frac{\partial T^{(0)}}{\partial x}-A_{ij}^{(0)}\right\}\frac{\partial U_j^{(2)}}{\partial\beta}$$
$$=-\left\{B_{ij}^{(0)}\frac{\partial T^{(1)}}{\partial x}+B_{ij}^{(1)}\frac{\partial T^{(0)}}{\partial x}-A_{ij}^{(1)}\right\}\frac{\partial U_j^{(1)}}{\partial\beta}$$
$$+\frac{\partial T^{(1)}}{\partial\beta}\left(B_{ij}^{(0)}\frac{\partial U_j^{(1)}}{\partial x}+C_i^{(1)}\right) \tag{5.17b}$$

$$\ell_i^{(0)}\left\{B_{ij}^{(0)}\frac{\partial U_j^{(2)}}{\partial x}+C_i^{(2)}\right\}=-\ell_i^{(0)}B_{ij}^{(1)}\frac{\partial U_j^{(1)}}{\partial x}-\ell_i^{(1)}\left\{B_{ij}^{(0)}\frac{\partial U_j^{(1)}}{\partial x}+C_i^{(1)}\right\} \tag{5.17c}$$

while the remaining equation for $\ell^{(2)}$ is not considered.

Employing (5.11) and (5.12) and integrating we get

$$-u^{(2)}+\frac{p^{(2)}}{\rho^{(0)}a_R^{(0)}(1-\epsilon^{(0)})(1+\eta)}$$
$$=\int_0^\beta\left[-a_R^{(0)}\lambda\left\{1+\frac{\Gamma+1}{2a_R^{(0)^2}\lambda}g'(r)(\exp(-\lambda s)-1)\right.\right.$$
$$\left.\left.+\frac{\Gamma+1}{2a_R^{(0)}}g'(r)[\exp(-\lambda s)]\right\}\right]g(r)[\exp(-\lambda s)]dr. \tag{5.18a}$$

$$\frac{\rho^{(0)}u^{(2)}}{a_R^{(0)}(1-\epsilon^{(0)})}-\rho^{(2)}$$
$$=\int_0^\beta\left[-\rho^{(0)}\lambda\left\{1+\frac{\Gamma+1}{2a_R^{(0)^2}\lambda}g'(r)(e^{(-\lambda s)}-1)\right\}\right.$$
$$\left.+\rho^{(0)}g'(r)(e^{(-\lambda s)}-1)\times\left\{\frac{\Gamma+1}{2}-\frac{2}{1-\epsilon^{(0)}}\right\}\right]g(r)e^{(-\lambda s)}\,dr \tag{5.18b}$$

$$-\alpha^{(2)}=\int_0^\beta\frac{\rho^{(0)}}{a_R^{(0)}}\left[1+\frac{\Gamma+1}{2a_R^{(0)^2}\lambda}g'(r)(e^{(-\lambda s)}-1)\right]f_1(s,r)\,dr \tag{5.18c}$$

$$\frac{\epsilon^{(0)}u^{(2)}}{a_R^{(0)}}-\epsilon^{(2)}=\int_0^\beta\left[-\epsilon^{(0)}\lambda\left\{1+\frac{\Gamma+1}{2a_R^{(0)^2}\lambda}g'(r)(e^{(-\lambda s)}-1)\right\}\right.$$
$$\left.+\left(\frac{\Gamma-3}{2a_R^{(0)^2}}\right)g'(r)e^{-\lambda s}\right]g(r)e^{(-\lambda s)}\,dr, \tag{5.18d}$$

where

$$g(\beta) = \sigma \sin(\beta)$$

$$f_1(s,\beta) = \left[\left(\frac{\partial f}{\partial p}\right)_0 a_R^{(0)^2} + \left(\frac{\partial f}{\partial \rho}\right)_0 + \frac{\epsilon^{(0)}}{\rho^{(0)}}\left(\frac{\partial f}{\partial \epsilon}\right)_0\right] g(\beta)e^{-\lambda s}$$

and

$$s = x - x^\star$$

From equation (5.17c), we have following equation

$$\begin{aligned}
p_x^{(2)} + &\left\{-(H_{,\rho})^{(0)} p^{(0)^2}(1+\eta)\frac{\mu^{(0)}}{a_R^{(0)}} - \frac{(H_{,\epsilon})^{(0)}}{a_R^{(0)}}\epsilon^{(0)}\rho^{(0)}(1-\epsilon^{(0)})(1+\eta)\right\} u_{,x}^{(2)} \\
&= -\frac{\mu^{(0)}}{a_R^{(0)}}\rho^{(0)}(1-\epsilon^{(0)})(1+\eta)\left[\left\{u^{(1)}\frac{a_R^{(0)}}{\mu^{(0)}}\right.\right. \\
&\quad -(H_{,\rho})^{(0)}\left\{\frac{\rho^{(0)}}{1-\epsilon^{(0)}}\left(\frac{\epsilon^{(1)}}{1-\epsilon^{(0)}} + \frac{\rho^{(1)}}{\rho^{(0)}}\right) + (H_{,\epsilon})^{(0)}\epsilon^{(1)}\right. \\
&\quad -\frac{\rho^{(0)}}{1-\epsilon^{(0)}}\{X^{(0)}(H_{,\rho})^{(1)} + X^{(1)}(H_{,\rho})^{(0)}\} \\
&\quad \left.+\{X^{(0)}(H_{,\epsilon})^{(1)} + X^{(1)}(H_{,\epsilon})^{(0)}\}\epsilon^{(0)}\right\} u_{,x}^{(1)} \\
&+\left\{\frac{a_R^{(0)}}{\mu^{(0)}\rho^{(0)}(1-\epsilon^{(0)})(1+\eta)}\left(\frac{\epsilon^{(1)}}{1-\epsilon^{(0)}} - \frac{\rho^{(1)}}{\rho^{(0)}}\right) + u^{(1)}\left(H_{,p} - \frac{1}{\rho}\right)^{(0)}\right. \\
&\quad \left. + \frac{X^{(1)}a_R^{(0)}}{\mu^{(0)}} + \frac{a_R^{(0)}}{\mu^{(0)}}X^{(0)}\left(\frac{a_R^{(1)}}{a_R^{(0)}} - \frac{\mu^{(1)}}{\mu^{(0)}}\right)\right\} p_{,x}^{(1)} \\
&\quad \left. + f_1\{X^{(0)}(H_{,\alpha})^{(1)} + X^{(1)}(H_{,\alpha})^{(0)}\} - f_2\{X^{(0)}(H_{,\alpha})^{(0)}\right]
\end{aligned} \tag{5.19}$$

Integrating equations (5.19) with respect to x, we have

$$p^{(2)} + A_1 u^{(2)} = B_1 u^{(1)} + B_2 p^{(1)} + B_3 x + B_4 \tag{5.20}$$

where A_1, B_1, B_2, B_3 are constants which can be obtained from equation (5.19) by inspection and B_4 is constant of integration. Substituting value of $u^{(1)}$ and $p^{(1)}$ from equations (5.11) and (5.12), we have a relation between $p^{(2)}$ and $u^{(2)}$. Solving resulting equations and equation (5.18a), we have $u^{(2)}$ and substituting the value of $u^{(2)}$ in equation (5.18), we can find $p^{(2)}$, $\rho^{(2)}$, $\epsilon^{(2)}$, the required second order solution, where $\ell^{(1)}$ given by

$$\ell^{(1)} = \Bigg[-X^{(1)} \frac{a_R^{(0)}}{\mu^{(0)}} - \frac{a_R^{(0)}}{\mu^{(0)}} X^{(0)} \left(\frac{a_R^{(1)}}{a_R^{(0)}} - \frac{\mu^{(1)}}{\mu^{(0)}} \right),$$

$$X^{(0)}(H_{,\rho})^{(1)} + X^{(1)}(H_{,\rho})^{(0)}, -X^{(1)},$$

$$X^{(0)}(H_{,\alpha})^{(1)} - X^{(1)}(H_{,\alpha})^{(0)},$$

$$-X^{(0)}(H_{,\epsilon})^{(1)} - X^{(1)}(H_{,\epsilon})^{(0)} \Bigg], \qquad (5.21)$$

where

$$X = -H_{,\rho} \frac{\mu \rho}{a_R(1-\epsilon)}.$$

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