Advanced Studies in Pure Mathematics 47-1, 2007
Asymptotic Analysis and Singularities
pp. 273–289

A Stokes approximation of two dimensional exterior Oseen flow near the boundary

Mitsuhiro Okamura, Yoshihiro Shibata[1] and Norikazu Yamaguchi[2]

Abstract.

In this paper, we are concerned with the linearized problems of the stationary Navier-Stokes equations for viscous incompressible fluids in two dimensional exterior domain. Obtained is an error estimate between solutions of the Oseen equations and the Stokes equations near the boundary. Our main theorem implies that Stokes's linearization still works well even in a two dimensional exterior domain if we consider the neighbourhoods of the boundary. In order to prove such estimate, we mainly use classical hydrodynamic potential theory.

§1. Introduction

Let $\mathcal{B} \subset \mathbb{R}^2$ be a bounded and open set whose boundary is C^2 hypersurface and let Ω be the exterior domain to $\mathcal{B}$, namely, $\Omega \equiv \mathbb{R}^2 \setminus \overline{\mathcal{B}}$. Here and hereafter we choose an $R_0 > 0$ such that $B_{R_0}(0) \supset \overline{\mathcal{B}}$ and fix it. Here $\mathcal{B}$ is an obstacle and Ω is a region which is filled by viscous incompressible Newtonian fluids.

In this paper, we are concerned with the the following boundary value problem of the Oseen equations ([5]) in Ω:

$$(1.1) \qquad \begin{cases} -\Delta \boldsymbol{u} + 2\mu \dfrac{\partial \boldsymbol{u}}{\partial x_1} + \nabla p = 0 & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{\Phi} & \text{on} \quad \partial\Omega. \end{cases}$$

Received October 31, 2005.
Revised January 27, 2006.
[1] Partially supported by Grant-in-Aid for Scientific Research (B) 15340204, Ministry of Education, Sciences, Sports and Culture, Japan.
[2] Partially supported by Waseda University Grant for Special Research Projects (2005-A-906).

Here $\boldsymbol{u} = (u_1(x), u_2(x))$ is the velocity and $p = p(x)$ is the pressure; $\boldsymbol{\Phi} = (\Phi_1, \Phi_2) \in C(\partial\Omega)^2$ is a given boundary data. $\mu > 0$ is a parameter corresponding to the dimensionless Reynolds number; Δ is the Laplace operator in $\mathbb{R}^2$, $\nabla = (\partial_1, \partial_2)$ with $\partial_j = \partial/\partial x_j$ is the gradient and $\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \partial_1 u_1 + \partial_2 u_2$ is the divergence of $\boldsymbol{u}$.

If we put $\mu \equiv 0$ formally in (1.1), we get the boundary value problem for the Stokes equations.

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = 0 & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{\Phi} & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

In two space dimension case, the fundamental solution of the Stokes equations has logarithmic singularity at infinity (see (2.18) below), therefore the strength of viscosity (diffusion) term $\Delta \boldsymbol{u}$ is weaker than that of convective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$. So Stokes's linearization does not work well in two dimensional exterior and some unbounded domains (*the Stokes paradox*). To get around this problem, in 1910 C. W. Oseen introduced another linearization of the Navier-Stokes equations (see [5]). Oseen linearized the Navier-Stokes equations at non-zero constant solution $(\boldsymbol{u}, p) = (\boldsymbol{U}_0, P_0)$. Without loss of generality, we may set $\boldsymbol{U}_0 = |\boldsymbol{U}_0|\boldsymbol{e}_1 = 2\mu\boldsymbol{e}_1$, so we obtain our problem (1.1). The details are given in Galdi [2, Chapter VII].

The main purpose of this paper is to get some error estimate between solutions of Oseen equations (1.1) and Stokes equations (1.2) for small parameter μ. Especially, we show that the Stokes's linearization still works well even in the case of exterior problem if the neighbourhoods of the boundary is considered. We believe that this fact has big significance in terms of the numerical analysis. Our main result is connected with *the boundary element method* in theory of numerical analysis.

The following theorem is our main result of the present paper.

Theorem 1.1. *Let $\boldsymbol{u}_\mu$ be a solution to the Oseen equations* (1.1) *with parameter $\mu > 0$ and $\boldsymbol{u}_0$ be a solution to the Stokes equations* (1.2). *For any $R > R_0$, there exists an $0 < \epsilon < 1$ such that if $0 < |\mu| < \epsilon$ then the following estimate holds.*

$$\sup_{x \in \Omega \cap B_R(0)} |\boldsymbol{u}_\mu - \boldsymbol{u}_0| \leq \frac{C_R}{|\log \mu|} \sup_{x \in \partial\Omega} |\boldsymbol{\Phi}|. \tag{1.3}$$

Here $C_R > 0$ is a positive constant independent of μ.

Here, we shall state our method to prove Theorem 1.1. In 1993, Borchers and Varnhorn [1] have shown the boundedness of the Stokes semigroup which is generated by the Stokes operator in two dimensional exterior domain. They have applied classical hydrodynamic potential theory to the Stokes resolvent problem and investigated the asymptotic behavior of the resolvent near the origin (see also Varnhorn [6]). Our study here is inspired by their work. Similar argument works well for the stationary Oseen equations (1.1). So, we are also due to the classical hydrodynamic potential theory to show our main theorem.

This paper is organized as follows. In Section 2, we prepare some notations and preliminaries. In order to consider (1.1) and (1.2) by hydrodynamic potential theory, we require the singular fundamental solutions to formal Oseen and Stokes derivative operators. Especially, we mainly consider their asymptotic behavior. In Section 3, we give a sketch of the proof of Theorem 1.1 by hydrodynamic potential theory. First we introduce potential ansatz for (1.1) and (1.2) and prove unique existence theorem for corresponding boundary integral equations. After presenting the solutions, we will estimate the difference of the solutions.

§2. Preliminaries

To prove Theorem 1.1, we are due to the classical hydrodynamic potential method. In order to do, in this section, we shall introduce some notations and preliminaries.

First we shall introduce *the stress tensor* related to the flow and *formal Stokes derivative operator.*

Definition 2.1 (Stress tensor). For smooth vector valued function $\boldsymbol{u}$, let $\mathbf{D}(\boldsymbol{u}) = (\partial_j u_i + \partial_i u_j)/2$ denote the deformation tensor. Let I_2 be 2×2 identity matrix and let p be a smooth scalar function. Then we define the stress tensor corresponding to the flow by

$$\mathbf{T}(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) + pI_2. \tag{2.1}$$

Analogously, we shall define the formally adjoint stress tensor by

$$\mathbf{T}'(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) - pI_2. \tag{2.2}$$

Definition 2.2 (Formal Stokes operator). For smooth vector field $\boldsymbol{u}$ and scalar function p, define the formal Stokes operator $\mathcal{S}$ by

$$\mathcal{S} : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \mathcal{S}(\boldsymbol{u}, p) = \begin{pmatrix} -\Delta \boldsymbol{u} + \nabla p \\ \operatorname{div} \boldsymbol{u} \end{pmatrix},$$

and the adjoint operator $\mathcal{S}'$ by

$$\mathcal{S}' : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \mathcal{S}'(\boldsymbol{u}, p) = \begin{pmatrix} -\Delta \boldsymbol{u} - \nabla p \\ -\operatorname{div} \boldsymbol{u} \end{pmatrix}.$$

The following Green's first and second identities for $\mathcal{S}$ and $\mathcal{S}'$ are well known (see e.g., [1, 3, 6]).

Lemma 2.3. *Let $D \subset \mathbb{R}^2$ be a bounded open set with smooth boundary. For smooth and solenoidal vector fields $\boldsymbol{u}, \boldsymbol{v}$ and scalar functions p, q in D, the following Green's identities hold.*

$$\begin{aligned} &\int_D \mathcal{S}(\boldsymbol{u}, p) \cdot \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} dx \\ &\quad = \int_{\partial D} \mathbf{T}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{v} \, d\sigma + 2 \int_D \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v}) \, dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\int_D \left\{ \mathcal{S}(\boldsymbol{u}, p) \cdot \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} - \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \cdot \mathcal{S}'(\boldsymbol{v}, q) \right\} dx \\ &\quad = \int_{\partial D} \mathbf{T}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \mathbf{T}'(\boldsymbol{v}, q)\boldsymbol{n} \, d\sigma. \end{aligned} \tag{2.4}$$

Here $\boldsymbol{n} \in \mathbb{R}^2$ denotes the unit outer normal vector on $\partial\Omega$.

Let us extend the previous formulae to the case of the Oseen equations. First we shall define the *modified* stress tensor and the formal Oseen operator.

Definition 2.4 (Modified Stress tensor). For a smooth vector field $\boldsymbol{u}$ and a scalar function p, define the modified stress tensor by

$$\mathbf{T}_\mu : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \to \mathbf{T}_\mu(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) + \mu(\boldsymbol{u}\ 0) + pI_2, \tag{2.5}$$

and the formally adjoint modified stress tensor by

$$\mathbf{T}'_\mu : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \to \mathbf{T}'_\mu(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) - \mu(\boldsymbol{u}\ 0) - pI_2.$$

Here $(\boldsymbol{u}\ 0)$ stands for the following 2×2 matrix:

$$(\boldsymbol{u}\ 0) = \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix}.$$

Definition 2.5 (Formal Oseen operator). For $\mu > 0$, we define the formal Oseen operator

$$\mathscr{O}_\mu : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \mathscr{O}_\mu(\boldsymbol{u}, p) = \begin{pmatrix} -\Delta \boldsymbol{u} + 2\mu\partial_1 \boldsymbol{u} + \nabla p \\ \operatorname{div} \boldsymbol{u} \end{pmatrix},$$

and the adjoint operator

$$\mathscr{O}'_\mu : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \mathscr{O}'_\mu(\boldsymbol{u}, p) = \begin{pmatrix} -\Delta \boldsymbol{u} - 2\mu\partial_1 \boldsymbol{u} - \nabla p \\ -\operatorname{div} \boldsymbol{u} \end{pmatrix}.$$

From Lemma 2.3 and Gauss divergence formula, we have the following Green's identities for $\mathscr{O}_\mu$ and $\mathscr{O}'_\mu$.

Lemma 2.6. *Let $D, \boldsymbol{u}, \boldsymbol{v}, p, q$ be the same as in Lemma 2.3. Then the following formulae hold.*

$$\begin{aligned} \int_D \mathscr{O}_\mu(\boldsymbol{u}, p) \cdot \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} &= \int_{\partial D} \mathbf{T}_\mu(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{v}\, d\sigma \\ &+2 \int_D \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, dx + \mu \int_D \frac{\partial \boldsymbol{u}}{\partial x_1} \cdot \boldsymbol{v}\, dx, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\int_D \mathscr{O}_\mu(\boldsymbol{u}, p) \cdot \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} - \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \cdot \mathscr{O}'_\mu(\boldsymbol{v}, q)\, dx \\ &= \int_{\partial D} \{\mathbf{T}_\mu(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \mathbf{T}'_\mu(\boldsymbol{v}, q)\boldsymbol{n}\}\, d\sigma. \end{aligned} \tag{2.7}$$

In order to represent the solutions of (1.1) and (1.2) by layer potentials, we require the fundamental solution $E_\mu = (E^\mu_{jk})_{j,k=1,2,3}$, $\mu \geq 0$. The fundamental tensor $E_\mu = (E^\mu_{jk})_{j,k=1,2,3}$ is a 3×3 matrix which satisfies the following identity.

$$\mathscr{O}_\mu E_\mu = \delta I_3 \quad \text{in } \mathscr{S}'(\mathbb{R}^2).$$

Here $\mathscr{S}'$ denotes the class of tempered distribution, δ stands for Dirac's distribution in $\mathbb{R}^2$ and I_3 denotes the 3×3 identity matrix.

In order to get explicit representation formula of E^μ_{jk}, we are due to Fourier transform and its inverse transform. For every $k = 1, 2, 3$, we have the following vector identity

$$\begin{pmatrix} (|\xi|^2 \hat{E}^\mu_{jk} + 2\mu i \xi_1 \hat{E}^\mu_{jk} + i\xi_j \hat{E}^\mu_{3k})_{j=1,2} \\ i \sum_{j=1}^2 (\xi_j \hat{E}^\mu_{jk}) \end{pmatrix} = (\delta_{jk})_{j=1,2,3}. \tag{2.8}$$

Here δ_{jk} denotes the Kronecker symbol. From (2.8), for $j,k=1,2$, we have

$$\begin{aligned} \hat{E}^{\mu}_{jk}(\xi) &= \frac{1}{|\xi|^2+2\mu i\xi_1}\left(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\right), \\ \hat{E}^{\mu}_{3k}(\xi) &= \hat{E}^{\mu}_{k3}(\xi) = -\frac{i\xi_k}{|\xi|^2}, \\ \hat{E}^{\mu}_{33}(\xi) &= \left(1+\frac{2\mu i\xi_1}{|\xi|^2}\right). \end{aligned} \tag{2.9}$$

By using inverse Fourier transform, for $j,k=1,2$ we have

$$E^{\mu}_{jk}(x) = (-\delta_{jk}\Delta+\partial_j\partial_k)\mathscr{F}_{\xi}^{-1}\left[\frac{1}{|\xi|^2(|\xi|^2+2\mu i\xi_1)}\right](x). \tag{2.10}$$

Here $\mathscr{F}_{\xi}^{-1}[\,\cdot\,]$ denotes the inverse Fourier transform with respect to ξ-variable. Since

$$\frac{i\xi_1}{|\xi|^2(|\xi|^2+2\mu i\xi_1)} = \frac{1}{2\mu}\left(\frac{1}{|\xi|^2}-\frac{1}{|\xi|^2+2\mu i\xi_1}\right), \tag{2.11}$$

in order to get the explicit representation formula for $E^{\mu}_{jk}(x)$, $j,k=1,2$, it suffices to compute $F_{\mu}(x)\equiv\mathscr{F}_{\xi}^{-1}[(|\xi|^2+2\mu i\xi_1)^{-1}](x)$, because $\mathscr{F}_{\xi}^{-1}[|\xi|^{-2}]=-\log|x|/2\pi$. Setting $F_{\mu}(x)\equiv e^{\mu x_1}G_{\mu}(x)$, G_{μ} must satisfy the Helmholtz equation (reduced wave equation) $(\mu^2-\Delta)G_{\mu}=\delta$ in $\mathscr{S}'(\mathbb{R}^2)$. It is well known that $G_{\mu}(x)=K_0(\mu|x|)/2\pi$. Thus, we have

$$F_{\mu}(x) = e^{\mu x_1}K_0(\mu|x|)/2\pi. \tag{2.12}$$

Here and in the followings, K_n, $n\in\mathbb{N}\cup\{0\}$ denotes the modified Bessel function of order n.

From (2.9), (2.10), (2.11), (2.12) the explicit representation formula of E^{μ}_{jk} $(\mu > 0)$ is given by

$$
\begin{aligned}
(2.13)\quad E^{\mu}_{11}(x) &= -E^{\mu}_{22}(x) + F_{\mu}(x) \\
&= \frac{1}{4\mu\pi}\left[-\frac{x_1}{|x|^2} + \mu e^{\mu x_1} K_0(\mu|x|) + \mu e^{\mu x_1}\frac{x_1}{|x|} K_1(\mu|x|)\right],
\end{aligned}
$$

$$(2.14)\quad E^{\mu}_{12}(x) = E^{\mu}_{21}(x) = \frac{1}{4\mu\pi}\left[-\frac{x_2}{|x|^2} + \mu e^{\mu x_1}\frac{x_2}{|x|} K_1(\mu|x|)\right],$$

$$(2.15)\quad E^{\mu}_{22}(x) = \frac{1}{4\mu\pi}\left[\frac{x_1}{|x|^2} + \mu e^{\mu x_1} K_0(\mu|x|) - \mu e^{\mu x_1}\frac{x_1}{|x|} K_1(\mu|x|)\right].$$

$$(2.16)\quad E^{\mu}_{3k}(x) = E^{\mu}_{k3}(x) = \frac{1}{2\pi}\frac{x_k}{|x|^2}, \quad k \neq 3,$$

$$(2.17)\quad E^{\mu}_{33}(x) = \delta(x) - \frac{\mu}{\pi}\frac{x_1}{|x|^2}.$$

Here we have used the fact that $K_0'(x) = -K_1(x)$.

On the Stokes equations ($\mu = 0$ case), the explicit representation formula of the fundamental tensor E^0_{jk} are well known (see e.g., [2, 6] and cited therein).

$$
(2.18)\quad
\begin{aligned}
E^0_{jk}(x) &= \frac{1}{4\pi}\left\{-\delta_{jk}\log|x| + \frac{x_j x_k}{|x|^2}\right\}, \quad j,k \neq 3, \\
E^0_{3k}(x) &= E^0_{k3}(x) = \frac{1}{2\pi}\frac{x_k}{|x|^2}, \quad k \neq 3, \\
E^0_{33}(x) &= \delta(x).
\end{aligned}
$$

It is important to know asymptotic behavior of the fundamental solutions E^{μ}_{jk} $(\mu > 0)$ when $\mu|x| \to 0$. In order to investigate such behavior, the following lemma concerning the asymptotics of the modified Bessel function plays crucial role.

Lemma 2.7. *The modified Bessel functions of second kind have the following asymptotic behavior when* $z \to 0$,

$$
\begin{aligned}
K_0(z) &= -\log z + \log 2 - \gamma + O(z^2)\log z, \\
K_1(z) &= \frac{1}{z} + \frac{z}{2}\left(\log z - \log 2 + \gamma - \frac{1}{2}\right) + O(z^3)\log z.
\end{aligned}
$$

Here $\gamma = 0.57721\ldots$ *denotes Euler's constant.*

From (2.13), (2.14), (2.15), Taylor expansion and Lemma 2.7, we have

$$E^{\mu}_{11}(x) = \frac{1}{4\pi}\left(-\log|x| + \frac{x_1^2}{|x|^2}\right) + \frac{1}{4\pi}(-\log\mu + \log 2 - \gamma) + R_{11}(\mu, x)\mu\log\mu, \tag{2.19}$$

$$E^{\mu}_{12}(x) = E^{\mu}_{21}(x) = \frac{1}{4\pi}\frac{x_1 x_2}{|x|^2} + R_{12}(\mu, x)\mu\log\mu, \tag{2.20}$$

$$E^{\mu}_{22}(x) = \frac{1}{4\pi}\left(-\log|x| + \frac{x_2^2}{|x|^2}\right) + \frac{1}{4\pi}(-\log\mu + \log 2 - \gamma - 1) + R_{22}(\mu, x)\mu\log\mu, \tag{2.21}$$

where $R_{jk}(\mu, x)$ $(j, k = 1, 2)$ stand for continuous kernel with respect to μ and x.

Remark 2.8. Since $R_{jk}(\mu, x)$ is continuous with respect to μ and x, (2.19)–(2.21) show the fundamental tensor of the Oseen equations has the same singularity as that of the Stokes equations.

With help of the singular fundamental tensor E_{jk}, we shall define the layer potentials. For $\mu \geq 0$, the single layer potential is defined by

$$E_{\mu}\boldsymbol{\Psi}(x) = \int_{\partial\Omega} E^{(c)}_{\mu}(x - y)\boldsymbol{\Psi}(y)\, d\sigma(y), \qquad x \notin \partial\Omega, \tag{2.22}$$

and the double layer potential is defined by

$$D_{\mu}\boldsymbol{\Psi}(x) = \int_{\partial\Omega} D_{\mu}(x, y)\boldsymbol{\Psi}(y)\, d\sigma(y), \qquad x \notin \partial\Omega. \tag{2.23}$$

Here 3×2 matrix $E^{(c)}_{\mu}$ is determined from the fundamental tensor E_{μ} by eliminating the last column and the double layer tensor $D_{\mu}(x, y)$ is given by the following formula

$$\begin{aligned} D_{\mu}(x, y) &\equiv {}^{T}(-\mathbf{T}_{\mu,x}E_{\mu}(x - y)\boldsymbol{n}(y)) \\ &= \left((-\mathbf{T}_{\mu,x}E^{\mu}_{k}(x - y))_{ij}n_j(y)\right)_{ki}. \end{aligned} \tag{2.24}$$

Here $\boldsymbol{n}(y)$ denotes the unit outer normal on $\partial\Omega$.

Setting $z \equiv x - y$ and $\boldsymbol{n} = \boldsymbol{n}(y)$, here we consider the asymptotic behavior of the double layer kernel $D_{\mu}(x, y)$, $\mu > 0$ when $\mu|z| \to 0$. From Lemma 2.7, (2.13)–(2.17) and (2.24), we obtain the following asymptotic expansion when $\mu|z| \to 0$.

$$(2.25)\quad D^{\mu}_{11}(x,y) = -\frac{1}{\pi}\frac{z_1^2 z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{\mu}{8\pi}n_1(y) + \frac{\mu}{4\pi}\frac{z_1z_2}{|z|^2}n_2(y) + h.o.t.,$$

$$(2.26)\quad D^{\mu}_{12}(x,y) = -\frac{1}{\pi}\frac{z_1z_2 z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{\mu}{4\pi}\log|z|n_2(y) + \frac{\mu}{2\pi}\frac{z_1z_2}{|z|^2}n_1(y) - \frac{\mu}{4\pi}(\log\mu - \log 2 + \gamma)n_2(y) - \frac{\mu}{8\pi}(5z_1^2 - z_2^2)n_2(y) + h.o.t.,$$

$$(2.27)\quad D^{\mu}_{21}(x,y) = -\frac{1}{\pi}\frac{z_1z_2 z\cdot \boldsymbol{n}(y)}{|z|^4} + \frac{\mu}{4\pi}\log|z|n_2(y) + \frac{\mu}{4\pi}\left(\log\mu - \log 2 + \gamma - \frac{1}{2}\right)n_2(y) + \frac{\mu}{4\pi}\frac{z_2(z\cdot \boldsymbol{n}(y))}{|z|^2} - \frac{\mu}{4\pi}\frac{z_1z_2}{|z|^2}n_2(y) + h.o.t.,$$

$$(2.28)\quad D^{\mu}_{22}(x,y) = -\frac{1}{\pi}\frac{z_2^2 z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{\mu}{4\pi}\log|z|n_1(y) - \frac{\mu}{4\pi}(\log\mu - \log 2 + \gamma)n_1(y) - \frac{\mu}{8\pi}\frac{5z_1^2 + z_2^2}{|z|^2}n_1(y) - \frac{\mu}{4\pi}\log|z|n_2(y) - \frac{\mu}{4\pi}\frac{3z_1z_2 + z_2^2}{|z|^2}n_2(y) + \frac{\mu}{4\pi}(-\log\mu + \log 2 - \gamma - 1)n_2(y) + h.o.t.,$$

$$(2.29)\quad D^{\mu}_{31}(x,y) = -\frac{1}{\pi}\left(\frac{2z_1 z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{1}{|z|^2}n_1(y)\right) - \delta(z)n_1(y),$$

$$(2.30)\quad D^{\mu}_{32}(x,y) = -\frac{1}{\pi}\left(\frac{2z_2 z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{1}{|z|^2}n_2(y)\right) - \frac{\mu}{2\pi}\frac{z_2}{|z|^2}n_2(y) + \frac{\mu}{2\pi}\frac{z_1}{|z|^2}n_2(y) - \delta(z)n_2(y).$$

Here we have used the relation $z_1^2 = |z|^2 - z_2^2$ to get (2.27) and (2.28).

In the case $\mu = 0$, we have (see [1, 6])

$$(2.31)\quad D^{0}_{ki}(x,y) = -\frac{1}{\pi}\frac{z_kz_i z\cdot \boldsymbol{n}(y)}{|z|^4} \quad (k,i = 1,2),$$

$$(2.32)\quad D^{0}_{3i}(x,y) = -\frac{1}{\pi}\left(\frac{2z_i z\cdot \boldsymbol{n}(y)}{|z|^4} - \frac{n_i(y)}{|z|^2}\right) \quad (i = 1,2).$$

Remark 2.9. The double layer kernel matrix $D_\mu(x,y)$ $(\mu > 0)$ has the same singularity as $D_0(x,y)$. This property is important to analyze jump relation of the potentials (see Proposition 2.10).

Here and hereafter, let

$$(E_\mu^\bullet \boldsymbol{\Psi})(x) = \int_{\partial\Omega} E_\mu^{(r,c)}(x-y)\boldsymbol{\Psi}(y)\,d\sigma(y), \tag{2.33}$$

$$(D_\mu^\bullet \Psi)(x) = \int_{\partial\Omega} D_\mu^{(r)}(x,y)\boldsymbol{\Psi}(y)\,d\sigma(y), \tag{2.34}$$

denote the single layer and double layer potentials corresponding to the velocity part of the potentials, respectively. Here the 2×2 matrix $E_\mu^{(r,c)}$ is obtained from E_μ by eliminating the last row and last column and $D_\mu^{(r)}$ is also obtained from D_μ by eliminating the last row.

In order to prove the existence theorem of solution to (1.1) and (1.2), we need the normal stress of the single layer potential.

$$\begin{aligned}(H_\mu^\bullet \boldsymbol{\Psi})(x) &= \int_{\partial\Omega} \mathbf{T}_{\mu,x}(E_\mu^{(c)}(x-y)\boldsymbol{\Psi}(y))\boldsymbol{n}(x_0)\,d\sigma(y) \\ &\equiv \int_{\partial\Omega} H_\mu(x,y)\boldsymbol{\Psi}(y)\,d\sigma(y), \quad x \notin \partial\Omega.\end{aligned} \tag{2.35}$$

Here $x_0 \in \partial\Omega$ is a uniquely determined projection of $x \in U$ onto $\partial\Omega$, where U is a tubular neighbourhood of $\partial\Omega$. From definition of $H_\mu^\bullet$, we can easily see that $H_\mu^\bullet = (D_\mu^\bullet)^*$.

From Remarks 2.8 and 2.9, we have the following jump and continuity relations for the surface potentials corresponding to velocity parts.

Proposition 2.10. *Let* $\boldsymbol{\Psi} \in C(\partial\Omega)$ *and let* $E_\mu^\bullet$, $D_\mu^\bullet$ *and* $H_\mu^\bullet$ *be the boundary layer potentials define by* (2.33), (2.34) *and* (2.35). *Then the following relations hold.*

$$(E_\mu^\bullet \boldsymbol{\Psi})^i = (E_\mu^\bullet \boldsymbol{\Psi}) = (E_\mu^\bullet \boldsymbol{\Psi})^e, \tag{2.36}$$

$$(D_\mu^\bullet \boldsymbol{\Psi})^i - D_\mu^\bullet \boldsymbol{\Psi} = +\frac{1}{2}\boldsymbol{\Psi} = D_\mu^\bullet \boldsymbol{\Psi} - (D_\mu^\bullet \boldsymbol{\Psi})^e, \tag{2.37}$$

$$(H_\mu^\bullet \boldsymbol{\Psi})^i - H_\mu^\bullet \boldsymbol{\Psi} = -\frac{1}{2}\boldsymbol{\Psi} = H_\mu^\bullet \boldsymbol{\Psi} - (H_\mu^\bullet \boldsymbol{\Psi})^e. \tag{2.38}$$

Here $\boldsymbol{w}^i$ *and* $\boldsymbol{w}^e$ *denote the interior and exterior limits, respectively. Namely,*

$$\boldsymbol{w}^i(z) = \lim_{\Omega^c \ni x \to z} \boldsymbol{w}(x), \quad \boldsymbol{w}^e(z) = \lim_{\Omega \ni x \to z} \boldsymbol{w}(x).$$

§3. A Sketch of the proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First we shall consider the solvability of the boundary value problems for the Oseen and Stokes equations. We choose the following ansatz (see [1, p. 285]).

$$(3.1)\qquad \begin{pmatrix} \boldsymbol{u}_\mu \\ p_\mu \end{pmatrix}(x) = D_\mu \boldsymbol{\Psi} - \eta E_\mu M \boldsymbol{\Psi} + \frac{4\pi\alpha}{\log \mu} E_\mu \boldsymbol{\Psi}, \quad \mu > 0,$$

$$(3.2)\qquad \begin{pmatrix} \boldsymbol{u}_0 \\ p_0 \end{pmatrix}(x) = D_0 \boldsymbol{\Psi} - \eta E_0 M \boldsymbol{\Psi} - \alpha \int_{\partial\Omega} \begin{pmatrix} \boldsymbol{\Psi} \\ 0 \end{pmatrix} d\sigma.$$

Here $M : \boldsymbol{\Psi} \to M\boldsymbol{\Psi} \equiv \boldsymbol{\Psi} - \boldsymbol{\Psi}_M$ and

$$\boldsymbol{\Psi}_M := \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \boldsymbol{\Psi}\, d\sigma.$$

From (1.1), (1.2), (3.1), (3.2) and Proposition 2.10, we obtain the following two systems of boundary integral equations.

$$(3.3)\qquad \boldsymbol{\Phi} = K_\mu \boldsymbol{\Psi} \equiv \left(-\frac{1}{2} I_2 + D_\mu^\bullet - \eta E_\mu^\bullet M + \frac{4\pi\alpha}{\log \mu} E_\mu^\bullet \right) \boldsymbol{\Psi}, \quad \mu > 0,$$

$$(3.4)\qquad \boldsymbol{\Phi} = K_0 \boldsymbol{\Psi} \equiv \left(-\frac{1}{2} I_2 + D_0^\bullet - \eta E_0^\bullet M - \alpha |\partial\Omega| (I_2 - M) \right) \boldsymbol{\Psi}$$

For unique solvability of the above integral equations (3.3) and (3.4), we have the following theorem.

Theorem 3.1. *Let $\boldsymbol{\Phi} \in C(\partial\Omega)^2$. Then*

(i) *For any $\eta > 0$, $\alpha > 0$, there exists $\mu_0 > 0$ such that if $\mu < \mu_0$ then there exists exactly one solution $\boldsymbol{\Psi} \in C(\partial\Omega)^2$ of the system of the boundary integral equations* (3.3).

(ii) *For any $\eta > 0$, $\alpha \neq 0$ there exists exactly one solution $\boldsymbol{\Psi} \in C(\partial\Omega)^2$ of the system of the boundary integral equations* (3.4).

Proof. The second assertion (ii) was already proven by Borchers and Varnhorn [1] (see also Varnhorn [6]), so here we only show (i).

Since the boundary integral operator K_μ ($\mu > 0$) is a compact operator on $C(\partial\Omega)^2$, in view of Fredholm alternative theorem, solvability of $K_\mu \boldsymbol{\Psi} = \boldsymbol{\Phi}$ follows from the uniqueness result for the following homogeneous adjoint system with respect to usual inner product in $\mathbb{R}^2$,

$$(3.5)\qquad 0 = K_\mu^* \boldsymbol{\Psi} = \left(-\frac{1}{2} I_2 + (D_\mu^\bullet)^* - \eta M (E_\mu^\bullet)^* + \frac{4\pi\alpha}{\log \mu} (E_\mu^\bullet)^* \right) \boldsymbol{\Psi}.$$

Here we have used the fact that $(D_\mu^\bullet)^* = H_\mu^\bullet$.

Assume now $\boldsymbol{\Psi}$ is a non-trivial solution of (3.5). We shall show $\boldsymbol{\Psi} \equiv 0$. Since $(D_\mu^\bullet)^* = H_\mu^\bullet$, by virtue of (2.36) and (2.38), we have

$$0 = K_\mu^* \boldsymbol{\Psi} = \left(-\frac{1}{2} I_2 + H_\mu^\bullet - \eta M (E_\mu^\bullet)^* + \frac{4\pi\alpha}{\log\mu} (E_\mu^\bullet)^* \right) \boldsymbol{\Psi}$$
$$= \left((H_\mu^\bullet)^i - \eta M (E_\mu^\bullet)^* + \frac{4\pi\alpha}{\log\mu} (E_\mu^\bullet)^* \right) \boldsymbol{\Psi}.$$

Therefore, we obtain

$$(H_\mu^\bullet \boldsymbol{\Psi})^i = \eta M (E_\mu^\bullet)^* \boldsymbol{\Psi} - \frac{4\pi\alpha}{\log\mu} (E_\mu^\bullet)^* \boldsymbol{\Psi}. \tag{3.6}$$

Set ${}^T(\boldsymbol{v}, q) = (E_\mu)^* \boldsymbol{\Psi}$. We can easily see that the pair of functions $(\boldsymbol{v}, q)$ solves the following Oseen equations in a bounded domain $\Omega_i \equiv \mathcal{B}$:

$$\begin{cases} -\Delta \boldsymbol{v} - 2\mu \dfrac{\partial \boldsymbol{v}}{\partial x_1} - \nabla q = 0 & \text{in} \quad \Omega_i, \\ - \operatorname{div} \boldsymbol{v} = 0 & \text{in} \quad \Omega_i. \end{cases}$$

Therefore, by using Green's first identity in Ω_i and (3.6), we see that

$$\begin{aligned} 0 &= \int_{\Omega_i} (-\Delta \boldsymbol{v} - \nabla q - 2\mu \frac{\partial \boldsymbol{v}}{\partial x_1}) \cdot \boldsymbol{v} \, dx \\ &= \int_{\partial\Omega_i} \mathbf{T}'_{\mu,x}(\boldsymbol{v}, q) \boldsymbol{n} \cdot \boldsymbol{v} \, d\sigma + 2 \int_{\Omega_i} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{v}) \, dx - \mu \int_{\Omega_i} \frac{\partial \boldsymbol{v}}{\partial x_1} \cdot \boldsymbol{v} \, dx \\ &= \int_{\partial\Omega_i} (H_\mu^\bullet \boldsymbol{\Psi})^i \cdot \boldsymbol{v} \, d\sigma + 2 \int_{\Omega_i} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{v}) \, dx - \frac{\mu}{2} \int_{\partial\Omega_i} n_1 |\boldsymbol{v}|^2 \, d\sigma \\ &= \eta \|M\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 - \frac{4\pi\alpha}{\log\mu} \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + 2\|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega_i)}^2 - \frac{\mu}{2} \int_{\partial\Omega} n_1 |\boldsymbol{v}|^2 \, dx \\ &\geq \eta \|M\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + \frac{1}{2} \left(\frac{8\pi\alpha}{\log(1/\mu)} - \mu \right) \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + 2\|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega_i)}^2. \end{aligned}$$

If we choose $\mu \in (0, 1)$ sufficiently small so that

$$\frac{8\pi\alpha}{\log(1/\mu)} - \mu > 0,$$

then we see that $\boldsymbol{v} = 0$ in Ω_i. Hence we have

$$(H_\mu^\bullet \boldsymbol{\Psi})^i = (q\boldsymbol{n})^i = \eta M \boldsymbol{v} - \frac{4\pi\alpha}{\log\mu} \boldsymbol{v} = 0.$$

On the other hand $E_\mu^* \boldsymbol{\Psi}$ solves the following exterior boundary value problem in $\Omega_e \equiv \Omega$,

$$\begin{cases} -\Delta \boldsymbol{v} - 2\mu \dfrac{\partial \boldsymbol{v}}{\partial x_1} - \nabla q = 0 & \text{in} \quad \Omega_e, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in} \quad \Omega_e, \\ \boldsymbol{v} = 0 & \text{on} \quad \partial\Omega_e, \end{cases}$$

Therefore from the uniqueness result due to Galdi [2], we obtain $(\boldsymbol{v}, q) = (0,0)$ in Ω_e. This implies $(H_\mu^\bullet \boldsymbol{\Psi})^e = 0$.

Combining the above facts, by (2.38), we have $\boldsymbol{\Psi} = (H_\mu^\bullet \boldsymbol{\Psi})^e - (H_\mu^\bullet \boldsymbol{\Psi})^i = 0$. This completes the proof of the lemma. Q.E.D.

Next we shall prepare some key lemmas to prove our main results.

Lemma 3.2. *For K_μ and K_0 defined in* (3.3) *and* (3.4), *there exists $0 < \mu_0 < 1$ such that if $|\mu| < \mu_0$ the following estimate holds.*

$$\| K_\mu - K_0 \| \le \frac{C}{|\log \mu|}. \tag{3.7}$$

Here and hereafter $\|\cdot\|$ denotes the operator norm $\|\cdot\|_{\mathscr{L}(C(\partial\Omega), C(\partial\Omega))}$.

Proof. Let $\boldsymbol{\Psi} \in C(\partial\Omega)^2$. Then from (3.3) and (3.4), we see that

$$\begin{aligned} &\|K_\mu \boldsymbol{\Psi} - K_0 \boldsymbol{\Psi}\|_{C(\partial\Omega)} \\ &\le \|(D_\mu^\bullet - D_0^\bullet)\boldsymbol{\Psi}\|_{C(\partial\Omega)} + |\eta| \|(E_\mu^\bullet - E_0^\bullet) M \boldsymbol{\Psi}\|_{C(\partial\Omega)} \\ &\qquad + |\alpha| \left\| \frac{4\pi}{\log \mu} E_\mu^\bullet \boldsymbol{\Psi} + |\partial\Omega| (I_2 - M) \boldsymbol{\Psi} \right\|_{C(\partial\Omega)} \\ &\equiv J_1(\mu) + |\eta| J_2(\mu) + |\alpha| J_3(\mu) \end{aligned}$$

We shall estimate $J_k(\mu)$ $(k = 1,2,3)$ separately. First we consider $J_1(\mu)$.

$$\begin{aligned} J_1(\mu) &\equiv \max_{x \in \partial\Omega} \left| \int_{\partial\Omega} (D_\mu^{(r)}(x,y) - D_0^{(r)}(x,y)) \boldsymbol{\Psi}(y)\, d\sigma(y) \right| \\ &\le \|\boldsymbol{\Psi}\|_{C(\partial\Omega)} \max_{x \in \partial\Omega} \int_{\partial\Omega} |D_\mu^{(r)}(x,y) - D_0^{(r)}(x,y)|\, d\sigma(y). \end{aligned}$$

From (2.25), (2.26), (2.27), (2.28) and (2.31), we see that

$$J_1(\mu) \le C \|\boldsymbol{\Psi}\|_{C(\partial\Omega)} \mu |\log \mu|. \tag{3.8}$$

Next we consider $J_2(\mu)$. From (2.18), (2.19) and the fact that $\int_{\partial\Omega} M\boldsymbol{\Psi}\, d\sigma = 0$, we have

$$\begin{aligned}&\left|\int_{\partial\Omega}(E^{\mu}_{11}(x-y) - E^{0}_{11}(x-y))M\boldsymbol{\Psi}(y)\, d\sigma(y)\right| \\ &\leq \left|\int_{\partial\Omega} R_{11}(\mu, x-y)\mu\log\mu M\boldsymbol{\Psi}(y)\, d\sigma(y)\right|.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\left\|\int_{\partial\Omega}(E^{\mu}_{11}(x-y) - E^{0}_{11}(x-y))M\boldsymbol{\Psi}(y)\, d\sigma(y)\right\|_{C(\partial\Omega)} \\ &\leq C\mu|\log\mu|\|\boldsymbol{\Psi}\|_{C(\partial\Omega)}.\end{aligned}$$

By similar calculation, we get the estimates of the same type for the other components. Therefore, we obtain the estimate

$$J_2(\mu) \leq C\|\boldsymbol{\Psi}\|_{C(\partial\Omega)}\mu|\log\mu|. \tag{3.9}$$

Finally, we consider $J_3(\mu)$.

$$\begin{aligned}J_3(\mu) &= \max_{x\in\partial\Omega}\left|\frac{4\pi}{\log\mu}E^{\bullet}_{\mu}\boldsymbol{\Psi} + |\partial\Omega|(I_2 - M)\boldsymbol{\Psi}\right| \\ &\leq \|\boldsymbol{\Psi}\|_{C(\partial\Omega)}\max_{x\in\partial\Omega}\left|\int_{\partial\Omega}\left(\frac{4\pi}{\log\mu}E^{(r,c)}_{\mu}(x-y) + I_2\right)d\sigma\right|.\end{aligned}$$

Set

$$A(x,y) = a_{ij}(x,y) \equiv \frac{4\pi}{\log\mu}E^{\mu}_{jk}(x-y) + \delta_{jk}, \quad j,k = 1,2.$$

From (2.19), we have

$$\begin{aligned}a_{11}(x) &= \frac{4\pi}{\log\mu}E^{\mu}_{11}(x) + 1 \\ &= \frac{1}{\log\mu}\left(-\log|x| + \frac{x_1^2}{|x|^2} + \log 2 - \gamma\right) + \mu R_{11}(\mu, x).\end{aligned}$$

Therefore we see that

$$\begin{aligned}
&\max_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{11}(x,y)\,d\sigma\right| \\
&\le \frac{1}{|\log\mu|}\int_{\partial\Omega}(|\log|x-y||+1+\log 2+\gamma)\,d\sigma \\
&\qquad +\mu\int_{\partial\Omega}|R_{11}(\mu,x-y)|\,d\sigma \\
&\le \frac{C_1}{|\log\mu|}+C_2\mu \qquad (3.10)
\end{aligned}$$

By (2.21) and similar manner of the proof of (3.10), we get the estimate of the same type for $a_{22}(x,y)$,

$$\max_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{22}(x,y)\,d\sigma\right| \le \frac{C_1}{|\log\mu|}+C_2\mu. \tag{3.11}$$

Next we consider $a_{12}(x,y)(=a_{21}(x,y))$. From (2.20), we see that

$$\begin{aligned}
a_{12}(x,y)=a_{21}(x,y)&=\frac{4\pi}{\log\mu}E^{\mu}_{12}(x-y) \\
&=\frac{1}{\log\mu}\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2}+4\pi\mu R_{12}(\mu,x-y).
\end{aligned}$$

Therefore, we obtain

$$\max_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{12}(x,y)\,d\sigma\right| \le \frac{C_1}{|\log\mu|}+C_2\mu. \tag{3.12}$$

Summing up (3.10), (3.11) and (3.12), we have desired estimate for small $\mu>0$. Q.E.D.

Lemma 3.3. *If $\mu>0$ is sufficiently small. Then we have*

$$|\!|\!|K_\mu^{-1}|\!|\!| \le 2|\!|\!|K_0^{-1}|\!|\!|. \tag{3.13}$$

Proof. From Theorem 3.1 (ii), the boundary integral operator K_0 has bounded inverse. By virtue of (3.7) we may choose μ sufficiently small so that

$$|\!|\!|K_\mu-K_0|\!|\!| \le \frac{1}{2|\!|\!|K_0^{-1}|\!|\!|}. \tag{3.14}$$

Then the Neumann series

$$(I_2-A_\mu)^{-1}=\sum_{j=0}^{\infty}A_\mu^j,\quad A_\mu=K_0^{-1}(K_0-K_\mu)$$

absolutely converges in $\mathscr{L}(C(\partial\Omega), C(\partial\Omega))$ with

$$\begin{aligned}|||(I_2 - A_\mu)^{-1}||| &\le (1 - |||A_\mu|||^{-1}) \\ &\le (1 - |||K_0|||^{-1}|||K_0 - K_\mu|||^{-1}) \le 2.\end{aligned}$$

Since $A_\mu = I_2 - K_0^{-1}K_\mu$ is equivalent to $K_\mu^{-1} = (I_2 - A_\mu)^{-1}K_0^{-1}$, we obtain

$$|||K_\mu^{-1}||| \le 2|||K_0^{-1}|||. \tag{3.15}$$

This completes the proof. Q.E.D.

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $\boldsymbol{u}_\mu(x)$ and $\boldsymbol{u}_0(x)$ be solutions to the Oseen equations and Stokes equations, respectively,

$$\begin{aligned}\boldsymbol{u}_\mu(x) &\equiv L_\mu \boldsymbol{\Psi} = \left(D_\mu^\bullet - \eta E_\mu^\bullet M + \frac{4\pi\alpha}{\log\mu}E_\mu^\bullet\right)\boldsymbol{\Psi}, \\ \boldsymbol{u}_0(x) &\equiv L_0\boldsymbol{\Psi} = (D_0^\bullet - \eta E_0^\bullet M - \alpha|\partial\Omega|(I-M))\boldsymbol{\Psi}.\end{aligned}$$

From the above two relations, we have

$$\begin{aligned}|\boldsymbol{u}_\mu(x) - \boldsymbol{u}_0(x)| &= |L_\mu K_\mu^{-1}\boldsymbol{\Phi} - L_0 K_0^{-1}\boldsymbol{\Phi}| \\ &\le |(L_\mu - L_0)K_\mu^{-1}\boldsymbol{\Phi}| + |L_0(K_\mu^{-1} - K_0^{-1})\boldsymbol{\Phi}| \equiv G_1 + G_2.\end{aligned}$$

We shall estimate G_1 and G_2, separately. First we consider G_1. From Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}\sup_{x\in\Omega_R}|(L_\mu - L_0)K_\mu^{-1}\boldsymbol{\Phi}| &\le \frac{C}{|\log\mu|}\|K_\mu^{-1}\boldsymbol{\Phi}\|_{C(\partial\Omega)} \\ &\le \frac{C}{|\log\mu|}|||K_0^{-1}|||\,\|\boldsymbol{\Phi}\|_{C(\partial\Omega)} \\ &\le \frac{C}{|\log\mu|}\|\boldsymbol{\Phi}\|_{C(\partial\Omega)}.\end{aligned} \tag{3.16}$$

Next we consider G_2. Since $K_\mu = K_0(K_0^{-1}(K_\mu - K_0) + I)$, we see that

$$K_\mu^{-1} = (I - K_0^{-1}(K_0 - K_\mu))^{-1}K_0^{-1}.$$

Setting $A_\mu \equiv K_0^{-1}(K_0 - K_\mu)$, then $K_\mu^{-1} = (I - A_\mu)K_0^{-1}$ and we have

$$K_\mu^{-1} = K_0^{-1} + \sum_{j=1}^{\infty} A_\mu^j \cdot K_0^{-1}.$$

Hence, by virtue of Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
|||K_\mu^{-1} - K_0^{-1}||| &\le \left|\left|\left| \sum_{j=1}^{\infty} A_\mu^j \right|\right|\right| \; |||K_0^{-1}||| \\
&\le |||A_\mu||| \left|\left|\left| \sum_{j=0}^{\infty} A_\mu^j \right|\right|\right| \; |||K_0^{-1}||| \\
&\le C|||K_\mu - K_0||| \le \frac{C}{|\log \mu|}
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\sup_{x \in \Omega_R} |L_0(K_\mu^{-1} - K_0^{-1})\boldsymbol{\Phi}| &\le C_R \|(K_\mu^{-1} - K_0^{-1})\boldsymbol{\Phi}\|_{C(\partial\Omega)} \\
&\le \frac{C_R}{|\log \mu|} \|\boldsymbol{\Phi}\|_{C(\partial\Omega)}. \qquad (3.17)
\end{aligned}
$$

Combining (3.16) and (3.17), we have Theorem 1.1. Q.E.D.

Acknowledgement. The authors would like to express their thanks to the referee for valuable comments.

References

[1] W. Borchers and W. Varnhorn, On the boundedness of the Stokes semigroup in two-dimensional exterior domains, Math. Z., **213** (1993), 275–299.

[2] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I, Linearized steady problems, Springer-Verlag, New York, 1994.

[3] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach Science Publishers, New York, 1969.

[4] M. Okamura, Y. Shibata and N. Yamaguchi, A Stokes approximation of two dimensional exterior Oseen flow near the boundary, in preparation.

[5] C. W. Oseen, Neuere Methoden und Ergebniss in der Hydrodynamik, Akademische Veragsgesellschaft, 1927.

[6] W. Varnhorn, The Stokes equations, Akademie-Verlag, Berlin, 1994.

Department of Mathematical Sciences
Waseda University
3-4-1 Ōkubo, Shinjuku-ku, Tokyo 169-8555, Japan