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On the Stokes and Navier-Stokes equations in a perturbed half-space and an aperture domain

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Abstract.

We discuss $L^p - L^q$ type estimate of the Stokes semigroup and its application to the Navier-Stokes equations in a perturbed half-space and an aperture domain. Especially, we have the L^p-L^q type estimate of the gradient of the Stokes semigroup for any p and q with $1 \leq p \leq q < \infty$, while the same estimate holds only for the exponents p and q with $1 < p \leq q \leq n$ in the exterior domain case, where n denotes the space dimension. And therefore, we can get better results concerning the asymptotic behavior of solutions to the Navier-Stokes equations compared with the exterior domain case.

Our proof of the L^p-L^q type estimate of the Stokes semigroup is based on the local energy decay estimate obtained by investigation of the asymptotic behavior of the Stokes resolvent near the origin. The order of asymptotic expansion of the Stokes resolvent near the origin is one half better compared with the exterior domain case, because we have the reflection principle on the boundary in the half-space case unlike the whole space case. And then, such better asymptotics near the boundary is also obtained in a perturbed half-space and an aperture domain by the perturbation argument. This is one of the reason why the result in our case is essentially better compared with the exterior domain case.

§1. Introduction

We study the global existence and asymptotic behavior of a strong solution to the Navier-Stokes initial value problem in a perturbed half-space and an aperture domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

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$$\text{(NS)}\qquad \begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla\pi = 0 & \text{in} \quad \Omega\times(0,\infty),\\ \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,\infty),\\ u(x,t) = 0 & \text{on} \quad \partial\Omega\times(0,\infty),\\ u(x,0) = a(x) & \text{in} \quad \Omega \end{cases}$$

for the unknown velocity field $u(x,t) = (u_1(x,t),\dots,u_n(x,t)) \in W^{2,p}(\Omega)^n$ and the unknown scalar pressure term $\nabla\pi \in L^p(\Omega)$ where $1 < p < \infty$.

The perturbed half-space is such a domain whose boundary is not flat only around the origin; to be precise, we call an open set the perturbed half-space if there is a positive number R such that $\Omega\backslash B_R = H\backslash B_R$ where $B_R = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n \mid |x| < R\}$ and $H := \{x \in \mathbb{R}^n \mid x_n > 1\}$. The aperture domain is a compact perturbation of two separated half-space $H_+ \cup H_-$ where $H_\pm = \{x \in \mathbb{R}^n \mid \pm x_n > 1\}$; to be accurate, we call an open set the aperture domain if there is a positive number R such that $\Omega\backslash B_R = (H_+ \cup H_-)\backslash B_R$. Since the aperture domain Ω is connected, we may choose a smooth $n-1$ dimensional manifold $M \subset \Omega \cup B_R$ such that $\Omega\backslash M$ consists of two disjoint "half-spaces" Ω_+ and Ω_- with $M = \partial\Omega_+ \cap \partial\Omega_-$, $\Omega = \Omega_+ \cup M \cup \Omega_-$ and $\Omega_\pm\backslash B_R = H_\pm\backslash B_R$. Let N denote the outward unit normal vector on $\partial\Omega$ or the normal vector on M directed to Ω_-.

The aperture domain is a particularly interesting class of domains with noncompact boundaries. In 1976, Heywood [18] pointed out that the solution is not uniquely determined by usual boundary conditions even for the stationary Stokes system in this domain and therefore in order to get a unique solution u we may have to prescribe either the pressure drop $[\pi]$ at infinity between the upper and lower subdomains $\Omega_\pm$ or the flux $\phi(u)$ through the aperture M as an additional boundary condition: more precisely, we must prescribe either the pressure drop $[\pi]$ which is a number defined by

$$[\pi] := \lim_{|x|\to\infty,\ x\in\Omega_+} \pi(x) - \lim_{|x|\to\infty,\ x\in\Omega_-} \pi(x),$$

or the flux $\phi(u)$ which is a number defined by

$$\phi(u) := \int_M N\cdot u\, d\sigma.$$

Here the flux $\phi(u)$ is independent of the choice of M since $\nabla\cdot u = 0$ in Ω.

We shall introduce the known results concerning the half-space, the perturbed half-space and the aperture domain. To this end, we first define the spaces $J^p(\Omega)$ and $G^p(\Omega)$ by the relations:

$$J^p(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n \mid \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)^n}},$$
$$G^p(\Omega) = \{\nabla p \in L^p(\Omega)^n \mid p \in L^p_{loc}(\Omega)\}.$$

For our domains, Farwig and Sohr [15], [16] and Miyakawa [27] proved that the Banach space $L^p(\Omega)^n$ $(1 < p < \infty)$ admits the Helmholtz decomposition :

$$\text{(HD)} \qquad L^p(\Omega)^n = J^p(\Omega) \oplus G^p(\Omega),$$

where $\oplus$ denotes the direct sum. Let P be a continuous projection from $L^p(\Omega)^n$ to $J^p(\Omega)$. The Stokes operator A is defined by $A = -P\Delta$ with domain

$$\text{(SD)} \qquad D(A) = \{u \in J^p(\Omega) \cap W^{2,p}(\Omega)^n \mid u|_{\partial\Omega} = 0\}.$$

We consider the following non-stationary Stokes equation:

$$\text{(S)} \qquad \begin{cases} \partial_t u - \Delta u + \nabla \pi = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ u(x,t)|_{\partial\Omega} = 0, \quad u(x,0) = a(x), \end{cases}$$

subject to the flux condition $\phi(u) = 0$ if Ω is an aperture domain (see Farwig and Sohr [15] and Miyakawa [27]).

By use of the Stokes operator A, the Stokes equation (S) can be formulated as an ordinary differential equation in the Banach space $J^p(\Omega)$:

$$\text{(O)} \qquad \frac{d}{dt}u(t) + Au(t) = 0, \quad u(0) = a.$$

By the resolvent estimate

$$\text{(RE)} \qquad |\lambda| \|(\lambda + A)^{-1} f\|_{L^p(\Omega)} + \|(\lambda + A)^{-1} f\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

for $|\arg\lambda| \le \pi - \delta$ with arbitrary small $\delta > 0$, we see that $-A$ generates a bounded analytic semigroup $T(t)$ on $J^p(\Omega)$ (see Farwig and Sohr [15], [16]). Through the inverse of the Laplace transform, we have the representation formula:

$$\text{(Rf)} \qquad T(t)f = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)^{-1} f d\lambda, \quad f \in J^p(\Omega),$$

where $\Gamma = \{\lambda = e^{i\theta}s \mid s \ge \varepsilon\} \cup \{\lambda = e^{-i\theta}s \mid s \ge \varepsilon\} \cup \{\lambda = \varepsilon e^{i\theta} \mid -\theta \le s \le \theta\}$ with $\theta \in (\pi/2, \pi)$ and $\varepsilon > 0$.

The purpose of this paper is to prove the $L^p - L^q$ estimate of Stokes semigroup:

$$\|T(t)a\|_{L^q(\Omega)^n} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L^p(\Omega)^n}, \tag{1}$$

$$\|\nabla T(t)a\|_{L^q(\Omega)^{n^2}} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L^p(\Omega)^n} \tag{2}$$

for $a \in J^p(\Omega)$ and $t > 0$, where $1 < p \leq q < \infty$. In particular, the gradient estimate (2) without any restriction on (p, q) is our important contribution.

The $L^p - L^q$ estimates of the Stokes semigroup have been already studied by many authors in the different cases of the domains. In fact, when $\Omega = \mathbb{R}^n$, applying the Young inequality to the concrete solution formula, we have (1) and (2) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). When Ω is a half-space H, applying the Fourier multiplier theorem to the concrete solution formula obtained by Ukai [34], we have (1) and (2) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) (cf. Borchers and Miyakawa [5] and Desch, Hieber and Prüss [12]).

When Ω is an exterior domain, (1) holds for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) but (2) holds only for $1 \leq p \leq q \leq n$ ($q \neq 1$). This result was first proved by Iwashita [20] for $1 < p \leq q < \infty$ in (1) and $1 < p \leq q \leq n$ in (2) when $n \geqq 3$. The refinement of his result was done by the following authors: Chen [8] ($n = 3$, $q = \infty$), Shibata [31] ($n = 3$, $q = \infty$), Borchers and Varnhorn [7] ($n = 2$, (1) for $p = q$), Dan and Shibata [9], [10] ($n = 2$), Dan, Kobayashi and Shibata [11] ($n = 2, 3$), and Maremonti and Solonnikov [29] ($n \geqq 2$). Especially, that Iwashita's restriction: $q \leq n$ in (2) is unavoidable was shown by Maremonti and Solonnikov [29].

When Ω is an aperture domain, Abels [2] proved (1) for $1 < p \leq q < \infty$ and (2) for $1 < p \leq q < n$ when $n \geqq 3$; and Hishida [19] proved (1) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) and (2) for $1 \leq p \leq q \leq n$ ($q \neq 1$) and $1 \leq p < n < q < \infty$ when $n \geqq 3$. We prove (1) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) and (2) for $1 \leq p \leq q < \infty$ ($q \neq 1$) when $n \geqq 2$ ([24]).

When Ω is a perturbed half-space, we prove (1) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) and (2) for $1 \leq p \leq q < \infty$ ($q \neq 1$) when $n \geqq 2$([25]).

Our proof of the L^p-L^q estimates of the Stokes semigroup is based on the local energy decay estimate obtained by investigation of the asymptotic behavior of the Stokes resolvent near the origin. The order of asymptotic expansion of the Stokes resolvent near the origin is one half better compared with the exterior domain case, because we have the reflection principle on the boundary in the half-space case unlike the whole space case.

Next we consider the application of the $L^p - L^q$ estimate to the Navier-Stokes equations. For the following domains, so far, the global existence of the Navier-Stokes flow with small L^n data has been proved: Giga and Miyakawa [17] for bounded domain, Kato [21] for whole space, Ukai [34] for half-space, Iwashita [20] for exterior domain, Abe and Shibata [1] for infinite layer and Hishida [19] for aperture domain.

We derive various decay properties of the global strong solution as $t \to \infty$ in a perturbed half-space case and in an aperture domain case. For the aperture domain case, compared with the previous results of Hishida ($n \geq 3$) [19] and Kozono and Ogawa ($n = 2$) [22], the new points in Theorem 2.5 are decay properties of $\|\nabla u(t)\|_{L^r}$ $(3 \leq n < r < \infty)$ and $\|u(t)\|_{L^\infty}$ $(n = 2)$.

To discuss our results more precisely, at first we outline at this point our notation used throughout the paper. We fix R_0 enjoying (4) if Ω is the perturbed half-space and enjoying (5) if Ω is the aperture domain. Given $R \geq R_0$, we define the cut-off function $\psi_{\pm,R}$ as follows:

$$\psi_{\pm,R} \in C^\infty(\mathbb{R}^n; [0,1]), \quad \psi_{\pm,R} = \begin{cases} 1 & \text{for } H_\pm \backslash B_{R+1}, \\ 0 & \text{for } H_\mp \cup B_R. \end{cases} \tag{3}$$

To denote the special sets, we use the following symbols:

$$\begin{aligned} &B_R = \{x \in \mathbb{R}^n \mid |x| < R\}, \ \Omega_R = \Omega \cap B_R, \ B_R^\pm = H_\pm \cap B_R, \\ &D_R = \{x \in \mathbb{R}^n \mid R < |x| < R+1\}, \ D_R^\pm = H_\pm \cap D_R, \\ &C_R = \{x \in \mathbb{R}^n \mid |x'| < R, |x_n| < R\}, \ C_R^\pm = H_\pm \cap C_R. \end{aligned} \tag{D}$$

And we set the sectorial domain Σ_ε and the ball U_r in $\mathbb{C}$ as follows:

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} \mid |\arg \lambda| < \pi - \varepsilon\}, \tag{s}$$

$$U_r = \{\lambda \in \mathbb{C} \mid |\lambda| < r\} \tag{b}$$

for $\varepsilon > 0$. We will use the standard symbol $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)}$. For $N \geq 1$ we set

$$\begin{aligned} L_R^p(\Omega)^n &= \{u \in L^p(\Omega)^n \mid u(x) = 0 \text{ for } |x| > R\}, \\ W_0^{N,p}(D) &= \{f \in W^{N,p}(D) \mid \partial_x^\alpha f|_{\partial D} = 0 \quad \text{for} \quad |\alpha| \leq N-1\}, \\ \dot{W}^{N,p}(D) &= \{f \in W_0^{N,p}(D) \mid \int_D f dx = 0\}, \\ \dot{W}^{0,p}(D) &= \{f \in L^p(D) \mid \int_D f dx = 0\}, \\ \widehat{W}^{1,p}(D) &= \{f \in L_{\mathrm{loc}}^p(D) \mid \nabla f \in L^p(D)\}. \end{aligned}$$

For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $\mathcal{L}(X) = \mathcal{L}(X, X)$. $\mathcal{B}(U; X)$ defines the set of all X-valued bounded holomorphic functions on U. And $\mathcal{A}(U_\varepsilon, X)$ denotes the set of all X-valued holomorphic function defined on U_ε which is defined by (b). To denote various constants we use the same letter C, and by $C_{A,B,\ldots}$, we denote the constant depending on the quantities $A, B, \cdots$. The constants C and $C_{A,B,\cdots}$ may change from line to line.

§2. Main results

In this section, we will state our main results concerning the Navier-Stokes system (NS) in the perturbed half-space and the aperture domain. As we already stated in the section 1, Farwig and Sohr [15], [16] and Miyakawa [27] proved the Helmholtz decomposition and the resolvent estimate (RE) in the perturbed half-space case and the aperture domain case. Therefore, we know that the Stokes operator with domain (SD) generates the analytic semigroup $\{T(t)\}_{t\geq 0}$ on $J^p(\Omega)$. We need to impose the following assumption on the domain Ω.

Assumption 2.1. *Let $n \geq 2$. B_R is defined by* (D).

(Case I) *Ω is the perturbed half-space, namely, there is $R > 0$ such that*

$$\Omega \backslash B_R = H_+ \backslash B_R. \tag{4}$$

(Case II) *Ω is the aperture domain, namely, there is $R > 0$ such that*

$$\Omega \backslash B_R = (H_+ \cup H_-) \backslash B_R. \tag{5}$$

We obtain the following theorem.

Theorem 2.1 ($L^p - L^q$ estimate of Stokes semigroup). *Let $n \geq 2$.*

(i) *For all $t > 0$, $f \in J^p(\Omega)$ and $1 \leq p \leq q \leq \infty (p \neq \infty,\ q \neq 1)$, there holds the estimate:*

$$\|T(t)f\|_{L^q(\Omega)^n} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)^n}. \tag{6}$$

(ii) *For all $t > 0$, $f \in J^p(\Omega)$ and $1 \leq p \leq q < \infty$ $(q \neq 1)$, there holds the estimate:*

$$\|\nabla T(t)f\|_{L^q(\Omega)^{n^2}} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^p(\Omega)^n}. \tag{7}$$

Main step in our proof of Theorem 2.1 is to show the following local energy decay estimate.

Theorem 2.2 (Local energy decay estimate). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ satisfy Assumption 2.1. Let $1 < p < \infty$ and let m be a nonnegative integer. R is any positive number satisfying* (4) *if Ω is a perturbed half-space or* (5) *if Ω is an aperture domain. Then there exists a positive constant $C_{p,m}$ such that*

$$\|\partial_t^m T(t)Pa\|_{W^{2,p}(\Omega_R)^n} \leq C_{p,m} t^{-\frac{n+1}{2}-m} \|a\|_{L^p(\Omega)^n} \tag{8}$$

for any $t \geq 1$ and $a \in L^p_R(\Omega)^n$.

If we consider the Stokes system in the half-space H:

$$\begin{aligned} &\partial_t v - \Delta v + \nabla\pi = 0, \ \ \text{div}\, v = 0 \ \text{ in } (0,\infty) \times H, \\ &v|_{x_n=0} = 0, \quad v|_{t=0} = b, \end{aligned} \tag{9}$$

then we know by Ukai [34] and Borchers and Miyakawa [5] that the solution v of (9) satisfies the L^p–L^q estimate:

$$\|v(t)\|_{L^q(H)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|b\|_{L^p(H)}, \tag{10}$$

$$\|\nabla v(t)\|_{L^q(H)} \leq C_{p,q} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|b\|_{L^p(H)} \tag{11}$$

for any $t > 0$ and $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). Since

$$\|v(t)\|_{L^p(C_R)} \leq C_R \|\nabla v(t)\|_{L^p(C_R)}$$

as follows from the boundary condition: $v|_{x_n=0} = 0$, using (11) and Theorem 2.2, we have

$$\|T(t)Pa\|_{W^{2,p}(\Omega_R)} \leq C_{p,R} t^{-\frac{n}{2p}-\frac{1}{2}} \|a\|_{L^p(\Omega)} \tag{12}$$

for any $a \in L^p(\Omega)$ and $t \geq 1$. Combining (10), (11) and (12) by the cut-off technique and following the argument due to Hishida [19, the proof of Theorem 2.1], we can show Theorem 2.1.

In order to prove Theorem 2.2, we need some precise information about solutions to the resolvent problem in H:

$$\begin{aligned} &(\lambda - \Delta) w + \nabla\theta = f, \ \ \text{div}\, w = 0 \ \ \text{in } H, \\ &w|_{x_n=0} = 0, \end{aligned} \tag{13}$$

which are stated in the following two theorems.

Theorem 2.3. *Let $R(\lambda)$ and $\Pi(\lambda)$ denote the solution operators of* (13) *which are defined by*

$$w = R(\lambda)f = {}^T(R_1(\lambda)f, \ldots, R_n(\lambda)f) \ \ \textit{and} \ \ \theta = \Pi(\lambda)f$$

for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. *Let* $R > 0$, $1 < p < \infty$ *and set*

$$\mathcal{B}^j_{p,R} = \mathcal{L}(L^p_R(H)^n, W^{j,p}(B_R))$$

for $j = 1, 2$. *Then there exist operators* $G^k_j(\lambda) \in \mathcal{A}(U_{1/16}, \mathcal{B}^2_{p,R})$, $k = 1, 2, 3$, $j = 1, \ldots, n$, *and* $G^k_\pi(\lambda) \in \mathcal{A}(U_{1/16}, \mathcal{B}^1_{p,R})$, $k = 1, 2, 3$ *such that*

$$R_j(\lambda)f = \lambda^{\frac{n-1}{2}} G^1_j(\lambda)f + (\lambda^{\frac{n}{2}} \log\lambda) G^2_j(\lambda)f + G^3_j(\lambda),$$
$$\Pi(\lambda)f = \lambda^{\frac{n-1}{2}} G^1_\pi(\lambda)f + (\lambda^{\frac{n}{2}} \log\lambda) G^2_\pi(\lambda)f + G^3_\pi(\lambda)$$

in B_R *when* $n \geq 2$ *and* n *is even; and*

$$R_j(\lambda)f = \lambda^{\frac{n}{2}} G^1_j(\lambda)f + (\lambda^{\frac{n-1}{2}} \log\lambda) G^2_j(\lambda)f + G^3_j(\lambda),$$
$$\Pi(\lambda)f = \lambda^{\frac{n}{2}} G^1_\pi(\lambda)f + (\lambda^{\frac{n-1}{2}} \log\lambda) G^2_\pi(\lambda)f + G^3_\pi(\lambda)$$

in B_R *when* $n \geq 3$ *and* n *is odd, provided that* $\lambda \in U_{1/16}$ *and* $f \in L^p_R(\Omega)$.

Theorem 2.4. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$, *and let* $R(\lambda)$ *and* $\Pi(\lambda)$ *be the operators given in Theorem 2.3 for* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. *Let* Σ_ε *be the set defined by* (s). *Then, there exist operators* $R(0) \in \mathcal{L}(L^p_R(H)^n, W^{2,p}_{\mathrm{loc}}(H)^n)$ *and* $\Pi(0) \in \mathcal{L}(L^p_R(H)^n, W^{1,p}_{\mathrm{loc}}(H)^n)$ *which satisfy the following three conditions:*

(i) *Given* $f \in L^p_R(H)^n$, $v = R(0)f$ *and* $\theta = \Pi(0)f$ *satisfy the equation:*

$$-\Delta v + \nabla\theta = f, \quad \mathrm{div}\ v = 0 \ \text{ in } H,$$
$$v|_{x_n=0} = 0,$$

(ii) *We have*

$$\|R(\lambda)f - R(0)f\|_{W^{1,p}(B_R)} \leq C|\lambda|^{\frac{1}{4}} \|f\|_{L^p(H)},$$
$$\|\Pi(\lambda)f - \Pi(0)f\|_{L^p(B_R)} \leq C|\lambda|^{\frac{1}{4}} \|f\|_{L^p(H)}$$

for any $f \in L^p_R(H)^n$ *and* $\lambda \in \Sigma_\varepsilon$ *with* $|\lambda| \leq 1/16$, *where* $C = C_{p,R,\varepsilon}$ *is a constant independent of* f *and* λ.

(iii) *We have*

$$|[R(0)f](x)| \leq C_{p,R}|x|^{-(n-1)} \|f\|_{L^p(H)},$$
$$|\nabla[R(0)f](x)| \leq C_{p,R}|x|^{-(n-1)} \|f\|_{L^p(H)},$$
$$|[\Pi(0)f](x)| \leq C_{p,R}|x|^{-(n-1)} \|f\|_{L^p(H)}$$

for any $f \in L^p_R(H)^n$ *and* $x \in H$ *with* $|x| \geq 2\sqrt{2}R$, *where* $C_{p,R}$ *is a constant independent of* f *and* x.

Constructing a parametrix to the resolvent problem in a perturbed half-space and an aperture domain, we can derive from Theorem 2.3 and Theorem 2.4 that the resolvent operator $(\lambda + A)^{-1}$ has the expansion formula of the same type near $\lambda = 0$ in the space $\mathcal{L}(L^p_R(\Omega)^n, W^{2,p}(\Omega_R)^n)$ as in the half-space case, which is applied to the representation formula (Rf) implies Theorem 2.2. The fundamental idea of the proofs of Theorems 2.1 and 2.2 by using Theorems 2.3 and 2.4 goes back to a paper due to Shibata [30].

Next we apply the $L^p - L^q$ estimate to the Navier-Stokes initial value problem. To this end we consider the Navier-Stokes equations (NS) in a perturbed half-space and an aperture domain : Applying the solenoidal projection P to (NS), we can rewrite (NS) as follows:

$$\text{(P-NS)} \qquad \partial_t u + Au + P(u \cdot \nabla u) = 0, \quad u(0) = a$$

where $A = -P\Delta$ is the Stokes operator.

For given $a \in J^n(\Omega)$ and $0 < T \leq \infty$ a measurable function u defined on $\Omega \times (0, T)$ is called a strong solution of (NS)(with $\phi(u) = 0$ if Ω is an aperture domain) on $(0, T)$ if u belongs to

$$u \in C([0,T); J^n(\Omega)) \cap C((0,T); D(A)) \cap C^1((0,T); J^n(\Omega))$$

together with $\lim_{t \to 0} \|u(t) - a\|_{L^n} = 0$ and satisfies (P-NS) for $0 < t < T$ in $J^n(\Omega)$.

We can show the next theorem which tells us the global existence of a strong solution to (NS) that has several decay properties with small $\|a\|_{L^n}$:

Theorem 2.5. *Let $n \geq 2$. There is a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $a \in J^n(\Omega)$ satisfies $\|a\|_{L^n} \leq \delta$, the problem* (NS) *(with $\phi(u) = 0$ if Ω is an aperture domain) admits a unique strong solution $u(t)$ on $(0, \infty)$. Moreover as $t \to \infty$,*

$$(14) \qquad \|u(t)\|_{L^r} = o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad \textit{for} \quad n \leq r \leq \infty,$$

$$(15) \qquad \|\nabla u(t)\|_{L^r} = o(t^{-1+\frac{n}{2r}}) \qquad \textit{for} \quad n \leq r < \infty.$$

If we assume that $a \in L^1 \cap J^n(\Omega)$ has small $\|a\|_{L^n}$, then we can show the better decay properties of the solutions as in the following theorem:

Theorem 2.6. *Let $n \geq 2$. There is a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $a \in L^1 \cap J^n(\Omega)$ and a satisfies $\|a\|_{L^n} \leq \eta$, then the solution $u(t)$ obtained in Theorem 2.5 enjoys*

$$(16) \qquad \|u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \qquad \textit{for} \quad 1 < r \leq \infty,$$

$$(17) \qquad \|\nabla u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \qquad \textit{for} \quad 1 < r < \infty$$

as $t \to \infty$.

Remark 2.1. *Due to Kozono and Ogawa* [22], *we can remove the smallness condition of* $\|a\|_{L^2}$ *in Theorem 2.5 and in Theorem 2.6 when* $n = 2$.

§3. Outline of the proof

In this section, we shall show the outline of the proof of Theorem 2.1 and Theorem 2.2, comparing with the exterior domain case. For an aperture domain case, we can show Theorem 2.1 and Theorem 2.2 in a similar way to a perturbed half-space case, so we shall introduce the only perturbed half-space case. (see Kubo and Shibata [25] for a perturbed half-space case and Kubo [24] for an aperture domain case, for details)

3.1. Local energy decay estimate

Constructing a parametrix to the resolvent problem in a perturbed half-space, we can derive from Theorem 2.3 and Theorem 2.4 that the resolvent operator $(\lambda + A)^{-1}$ has expansion formula of the same type near $\lambda = 0$:

Theorem 3.1. *Let* $1 < p < \infty$ *and* $R > R_0$. *Set*

$$B_\Omega = \mathcal{L}(L^p_R(\Omega); W^{2,p}(\Omega_R)^n \times W^{1,p}(\Omega_R))$$

and $\dot{U}_{\lambda_0} = U_{\lambda_0} \backslash (-\infty, 0]$. *Then there exists a* $\lambda_0 > 0$ *and* $(U(\lambda), \Theta(\lambda))$ *such that*

$$U(\lambda)f = (\lambda + A)^{-1}Pf$$

for $f \in L^p_R(\Omega)$ *and* $\lambda \in U_{\lambda_0}$, *and*

$$(U(\lambda), \Theta(\lambda)) = \begin{cases} H_1(\lambda)\lambda^{\frac{n-1}{2}} + H_2(\lambda)\lambda^{\frac{n}{2}} \log \lambda + H_3(\lambda), & \text{where } n \text{ is even,} \\ H_1(\lambda)\lambda^{\frac{n}{2}} + H_2(\lambda)\lambda^{\frac{n-1}{2}} \log \lambda + H_3(\lambda), & \text{where } n \text{ is odd} \end{cases}$$

for any $\lambda \in \dot{U}_{\lambda_0}$ *where* $H_j \in \mathcal{B}(\dot{U}_{\lambda_0}; B_\Omega)$, $j = 1, 2$ *and* $H_3 \in \mathcal{B}(U_{\lambda_0}; B_\Omega)$.

From (Rf), we know that the semigroup is described as follows :

$$T(t)Pf = \frac{1}{2\pi i} \int_{\Gamma_1} e^{t\lambda} U(\lambda) f d\lambda + \frac{1}{2\pi i} \int_{\Gamma_2} e^{t\lambda} (A + \lambda)^{-1} Pf d\lambda,$$

where

$$\Gamma_1 = \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < \varepsilon, \ \arg \lambda = \pm\delta\},$$
$$\Gamma_2 = \{\lambda \in \mathbb{C} \mid |\lambda| > \varepsilon, \ \arg \lambda = \pm\delta\}$$

for $\frac{\pi}{2} < \delta_0 < \delta < \pi$ and $0 < \varepsilon < \lambda_0$, where λ_0 is the same constant as in Theorem 3.1

In the same manner as in Iwashita [20], we can prove Theorem 2.2 by using the following Lemma:

Lemma 3.1. (i) *For $\sigma > 0$ and $t > 0$, it holds that*

$$\frac{1}{2\pi i}\int_\Gamma e^{tz} z^{\sigma-1} dz = \frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}t^{-\sigma}.$$

(ii) *For a nonnegative integer j and any $t > 0$,*

$$\frac{1}{2\pi i}\int_\Gamma e^{tz} z^{j} \log z dz = \frac{d}{d\sigma}\left[\frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}\right]\Bigg|_{\sigma=j+1} t^{-j-1}.$$

3.2. $L^p - L^q$ estimates of Stokes semigroup

In this subsection, we introduce the outline of the proof of $L^p - L^q$ estimate. At first we shall prove the following lemma.

Lemma 3.2. *Let $n \geq 2$, $1 < p < \infty$ and $R \geq R_0$. Then there exists a positive number $C = C(\Omega, n, p, R)$ such that*

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \qquad t \geq 2$$

for $f \in J^p(\Omega)$.

Proof. Fix $R \geq R_0 + 2$. For $f \in J^p(\Omega)$ we set $g = T(1)f$. Then $g \in D(A^N)$ for any $N \in \mathbb{N}$ and there holds the estimate:

$$\|A^N g\|_{L^p(\Omega)} \leq C_{N,p}\|f\|_{L^p(\Omega)}.$$

We set $u(t) = T(t)g = T(t+1)f$ for $f \in J^p(\Omega)$. Then $u(t)$ belongs to $C^1([0,\infty); J^p(\Omega)) \cap C^0([0,\infty); D(A))$ and u is the solution of the following Stokes problem with some pressure term $\pi(t)$:

$$(18) \qquad \begin{cases} \partial_t u(t) - \Delta u(t) + \nabla\pi(t) = 0 & \text{in } \Omega \times (0,\infty), \\ \nabla\cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = g. \end{cases}$$

Set $h = \psi_{+,R}g - \mathbb{B}[(\nabla\psi_{+,R})\cdot g]$, where $\mathbb{B}[\cdot]$ is the Bogovskiĭ operator on the bounded domain D_R^+ (see Bogovskiĭ [4], Borchers and Sohr [6]). And then by (3) we have $h = g$ for $|x| \geq R+1$. Moreover we can see that $h \in D(A_H)$ where A_H denotes the Stokes operator on H. In the course of the proof of this lemma, for simplicity, we abbreviate $\psi_{+,R}$ to ψ_R.

By the solvability of the Stokes equation in the half-space (cf. Ukai [34]) we know that there exists a (v, ρ) such that

$$v(t) \in C^1([0,\infty); J^p(H)) \cap C^0([0,\infty); D(A_H)),$$
$$\nabla\rho(t) \in C^0([0,\infty); L^p(H))$$

and (v, ρ) solves the following equation:

$$\begin{cases} \partial_t v(t) - \Delta v(t) + \nabla\rho(t) = 0, \ \nabla\cdot v = 0 \quad \text{in} \quad H\times(0,\infty), \\ v|_{t=0} = h, \ v|_{x_n=0} = 0, \end{cases} \tag{19}$$

where we choose ρ so that

$$\int_{D_R^+} \rho(t,x)dx = 0. \tag{20}$$

Moreover from the $L^q - L^r$ type estimate in the half-space which is proved by Ukai [34] and Borchers and Miyakawa [5] we have

$$\|\nabla^j v(t,\cdot)\|_{L^r(H)} \le C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{j}{2}}\|h\|_{W^{2,q}(H)} \tag{21}$$

for $j = 0, 1$, $t \ge 1$ and $1 \le q \le r \le \infty$ with $(q, r) \ne (1, 1)$ and

(22)

$$\|\nabla^2 v(t,\cdot)\|_{L^r(H)} + \|\partial_t v(t,\cdot)\|_{L^r(H)} \le C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-1}\|h\|_{W^{2,q}(H)}$$

for $t \ge 1$ and $1 < q \le r < \infty$.

By (19) and (22) we have

$$\|\nabla\rho(t)\|_{L^r(H)} \le C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-1}\|h\|_{W^{2,q}(H)} \tag{23}$$

for $t \ge 1$ and $1 < q \le r < \infty$. And in the cylinder C_R^+ which is defined by (D), we know the estimates:

(24)

$$\|\nabla^j v(t,\cdot)\|_{L^p(C_R^+)} \le C_R\|\nabla^j v(t,\cdot)\|_{L^\infty(H)} \le C_R(1+t)^{-\frac{n}{2p}-\frac{j}{2}}\|h\|_{W^{2,p}(H)}$$

for $j = 0, 1$ and

$$\begin{aligned} &\|\nabla^2 v(t,\cdot)\|_{L^p(C_R^+)} + \|\partial_t v(t,\cdot)\ |_{L^p(C_R^+)} \\ &\le C_{R,p,r}\left(\|\nabla^2 v(t,\cdot)\|_{L^r(C_R^+)} + \|\partial_t v(t,\cdot)\|_{L^r(C_R^+)}\right) \\ &\le C_{R,p,r}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})-1}\|h\|_{W^{2,p}(H)} \\ &\le C_{R,p}(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|h\|_{W^{2,p}(H)} \end{aligned} \tag{25}$$

for $t \geq 1$ and $1 < p < r < \infty$ where $\max(p,n) \leq r < \infty$. By (19), Poincare's inequality, (20) and (23), we have

(26) $\|\rho(t,\cdot)\|_{L^p(C_R^+)} \leq C\|\nabla\rho(t,\cdot)\|_{L^p(C_R^+)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|h\|_{W^{2,p}(H)}.$

Since $v(t,x) = \int_0^{x_n} \partial_n v(t,x',y_n)dy_n$ as follows from the fact that $v|_{x_n=0} = 0$, we obtain

(27) $$\|v(t,\cdot)\|_{L^p(C_R^+)} \leq R\|\nabla v(t,\cdot)\|_{L^p(C_R^+)},$$

which combined with (24) with $j = 1$ implies that

(28) $$\|v(t,\cdot)\|_{L^p(C_R^+)} \leq C_R(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|h\|_{W^{2,p}(H)}.$$

Summing up (24) - (26) and (28), we have shown that

(29) $$\begin{aligned}&\|v(t)\|_{W^{2,p}(C_R^+)} + \|\partial_t v(t,\cdot)\|_{L^p(C_R^+)} + \|\rho(t,\cdot)\|_{L^p(C_R^+)}\\ &\leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}.\end{aligned}$$

In order to estimate (u,π) in (18) we set

(30) $$w(t) = u(t) - \{\psi_{R-1}v(t) - \mathbb{B}[(\nabla\psi_{R-1}) : v(t)]\},$$

(31) $$\theta(t) = \pi(t) - \psi_{R-1}\rho(t).$$

It is easily observed that $(w(t),\theta(t))$ satisfies the equations:

$$\begin{aligned}\partial_t w(t) - \Delta w(t) + \nabla\theta(t) = K(t),\ \nabla\cdot w(t) = 0 \qquad &\text{in } \Omega\times(0,\infty),\\ w|_{\partial\Omega} = 0,&\\ w(0) = u(0) - (\psi_{R-1}v(0) - \mathbb{B}[(\nabla\psi_{R-1})\cdot v(0)])&\\ = g - (\psi_{R-1}h - \mathbb{B}[(\nabla\psi_{R-1})\cdot h]),&\end{aligned}$$

where

$$K(t) = 2\nabla\psi_{R-1} : \nabla v + (\Delta\psi_{R-1})v - (\partial_t - \Delta)\mathbb{B}[(\nabla\psi_{R-1})\cdot v] - (\nabla\psi_{R-1})\rho.$$

Noticing that supp $w(0) \subset \overline{B_R}$ by (3) and $w \in W^{2,p}(\Omega)$, we obtain $w \in D(A)\cap L^p_R(\Omega)$. Since $w(t) \in C^0([0,\infty); D(A)\cap L^p_R(\Omega))\cap C^1([0,\infty); J^p(\Omega))$, we can write

$$w(t) = T(t)w(0) - \int_0^t T(t-s)PK(s)ds.$$

We shall show the estimate of $w(t)$ by Theorem 2.2. (3) implies that supp $K(t) \subset \overline{D_R^+}$ and by (29) we see that

$$\|K(t)\|_{L^p} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}.$$

Therefore we obtain

$$\|w(t)\|_{W^{1,p}(\Omega_R)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{32}$$

for $t \geq 1$.

By (29)-(31) and (32) we obtain

$$\|u(t)\|_{W^{1,p}(\Omega_R)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}$$

for $t \geq 1$. In particular since $T(t)f = u(t-1)$, for $f \in J^p(\Omega)$ we have

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \qquad t \geq 2.$$

Q.E.D.

Remark 3.1. *We know that in the exterior domain, there holds the estimate:*

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}.$$

The reason why the decay rate in the perturbed half-space case is one half better than the one in the exterior domain case is that (27) *holds.*

Next, we can prove the following two lemmas in the analogue way to Hishida [19] and Iwashita [20].

Lemma 3.3. *Let $f \in J^p(\Omega)$. Then for $t \geq 2$ we have*

$$\|T(t)f\|_{L^q(\Omega\setminus\Omega_R)} \leq C_{p,q}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

for $1 < p \leq q \leq \infty$ and $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) < 1$. And we have

$$\|\nabla T(t)f\|_{L^p(\Omega\setminus\Omega_R)} \leq C_{p,R}t^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$ and $t \geq 2$.

Lemma 3.4. *For $0 < t \leq 2$, there exists a positive number $C_{p,q} = C(p,q,\Omega)$ such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq C_{p,q}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

for $1 < p \le q \le \infty$ *and* $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. *And we have*

$$\|\nabla T(t)f\|_{L^p(\Omega)} \le C_p t^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$.

Finally we shall show Theorem 2.1 by using Lemmas 3.3 and 3.4.

Proof of Theorem 2.1. By Lemmas 3.3 and 3.4 we have

$$\|T(t)f\|_{L^q(\Omega)} \le C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)} \tag{33}$$

for $1 < p \le q \le \infty$, $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$ and $t > 0$. We will remove the restriction of (33). To this end we choose $p_1, \dots, p_\ell$ in such a way that $p = p_1 < p_2 < \cdots < p_\ell = q$ and $\frac{n}{2}(\frac{1}{p_{j-1}} - \frac{1}{p_j}) < 1$ for $j = 2, 3, 4, \cdots, \ell$. Then by Lemma 3.4 we have

$$\begin{aligned}
\|T(t)f\|_{L^q(\Omega)} &= \left\| T\left(\frac{t}{\ell-1}\right) T\left(\frac{t}{\ell-1}\right) \cdots T\left(\frac{t}{\ell-1}\right) f \right\|_{L^q(\Omega)} \\
&\le C \prod_{j=2}^{\ell} \left(\frac{t}{\ell-1}\right)^{-\frac{n}{2}(\frac{1}{p_{j-1}}-\frac{1}{p_j})} \|f\|_{L^p(\Omega)} \\
&= Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)}.
\end{aligned}$$

Summing up we have obtained

$$\|T(t)f\|_{L^q(\Omega)} \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)} \tag{34}$$

for $1 < p \le q \le \infty$, $f \in J^p(\Omega)$ and $t > 0$.

Since for $\phi, \psi \in C_{0,\sigma}^\infty(\Omega)$ we have

$$|(T(t)\phi, \psi)| = |(\phi, T(t)\psi)| \le \|\phi\|_{L^1} \|T(t)\psi\|_{L^\infty} \le Ct^{-\frac{n}{2p'}} \|\phi\|_{L^1} \|\psi\|_{L^{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\begin{aligned}
&\|T(t)\phi\|_{L^p} \le Ct^{-\frac{n}{2}(1-\frac{1}{p})} \|\phi\|_{L^1}, \qquad 1 < p < \infty, \\
&\|T(t)\phi\|_{L^\infty} \le Ct^{-\frac{n}{2p}} \|T(t/2)\phi\|_{L^p} \le Ct^{-\frac{n}{2}} \|\phi\|_{L^1}.
\end{aligned}$$

Summing up we have obtained (6) for $1 \le p \le q \le \infty$, $f \in J^p(\Omega)$ and $t > 0$.

By using (6) and (7) with $p = q$ we have

$$\begin{aligned}\|\nabla T(t)f\|_{L^q(\Omega)} &= \left\|\nabla T\left(\frac{t}{2}\right)T\left(\frac{t}{2}\right)f\right\|_{L^q(\Omega)} \\ &\leq C_q t^{-\frac{1}{2}}\left\|T\left(\frac{t}{2}\right)f\right\|_{L^q(\Omega)} \leq C_q t^{-\frac{1}{2}}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}\end{aligned}$$

for $t > 0$ and $1 \leq p \leq q < \infty$.

We have completed the proof of Theorem 1.2. Q.E.D.

3.3. The Navier-Stokes flow

We shall consider the application of $L^p - L^q$ estimate to the Navier-Stokes equation. At first we begin to show Theorem 2.5.

Proof of Theorem 2.5. Employing the argument due to Kato [21] we can construct a unique global solution $u(t)$ of the integral equation

$$\text{(IE)} \qquad u(t) = T(t)a - \int_0^t T(t-\tau)P(u(\tau)\cdot\nabla u(\tau))\,d\tau,$$

provided that $\|a\|_{L^n} \leq \delta_0$, where $\delta_0 = \delta_0(\Omega, n)$ is some small positive constant. The solution $u(t)$ enjoys the estimates:

$$(35) \qquad \|u(t)\|_{L^r} \leq Ct^{-\frac{1}{2}+\frac{n}{2r}}\|a\|_n \qquad \text{for } n \leq r \leq \infty,$$

$$(36) \qquad \|\nabla u(t)\|_{L^r} \leq Ct^{-1+\frac{n}{2r}}\|a\|_n \qquad \text{for } n \leq r < \infty$$

for $t > 0$ together with the singular behavior

$$\begin{aligned}\|u(t)\|_{L^r} &= o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad &\text{for } \quad n < r \leq \infty, \\ \|\nabla u(t)\|_{L^r} &= o(t^{-1+\frac{n}{2r}}) \qquad &\text{for } \quad n \leq r < \infty\end{aligned}$$

as $t \to 0$. (35) and (36) implies the Hölder estimate:

$$(37) \quad \|u(t) - u(\tau)\|_{L^\infty} + \|\nabla u(t) - \nabla u(\tau)\|_{L^n} \leq C(t-\tau)^\theta \tau^{-\frac{1}{2}-\theta}\|a\|_{L^n}$$

for $0 < \tau < t$ and $0 < \theta < \frac{1}{2}$. Due to the Hölder estimate the solution $u(t)$ becomes actually a strong one of (NS) (see [33]). Furthermore, in the same way as in Hishida [19] we can obtain the decay properties (14) and (15) for $n \leqq r < \infty$. The proof is complete. Q.E.D.

In the same manner as in Hishida [19] we can prove Theorem 2.6. The key of his proof is to show the following Lemma 3.5. According to Hishida's argument [19] we can also prove Lemma 3.5:

Lemma 3.5. *Let $n \geq 2$ and $a \in L^1(\Omega) \cap J^n(\Omega)$. For any small $\varepsilon > 0$ there is a constant $\eta_* = \eta_*(\Omega, n, \varepsilon) \in (0, \delta]$ such that if $\|a\|_{L^n} \leq \eta_*$, then the solution $u(t)$ obtained in Theorem 2.5 satisfies*

$$\|u(t)\|_{L^{\frac{n}{n-1}}} \leq C(1+t)^{-\frac{1}{2}+\varepsilon},$$
$$\|u(t)\|_{L^{2n}} \leq Ct^{-\frac{1}{4}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon},$$
$$\|\nabla u(t)\|_{L^n} \leq Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon}$$

for $t > 0$.

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