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The Helmholtz decomposition in Sobolev and Besov spaces

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Abstract.

This paper is concerned with the Helmholtz decomposition of vector fields on bounded domains and exterior domains, and provides the decompositions in Sobolev spaces and Besov spaces of order in certain interval. In particular, the decompositions in homogeneous Besov spaces $\dot{B}^s_{1,q}$ and $\dot{B}^s_{\infty,q}$ are given.

§ Introduction.

Let n be an integer such that $n \geq 2$. We are concerned with the initial boundary value problem for the nonstationary Navier-Stokes equation in the Sobolev spaces and the Besov spaces of negative order on the domain Ω in $\mathbb{R}^n$. For the equation in the whole spaces $\mathbb{R}^n$ there are many works. See Kato and Ponce [8] for the Sobolev spaces, and for Cannone and Planchon [4] and Kozono and Yamazaki [10] for the Besov spaces, and Koch and Tataru [9] for the derivatives of the elements of BMO. More detailed references are given in [15]. These results are based on the fact that the Helmholtz decomposition in the whole spaces can be described by the Riesz transformations and hence its property in real analysis is well-known.

For general domains, much less is known. Grubb [6] first considered this problem with the Neumann boundary conditions in the Sobolev spaces of order $n/p - 1$, which can be negative for $p > n$ on bounded domains by duality argument. Then Amann [1] considered this problem with the Dirichlet boundary condition in the Besov spaces on bounded domains, exterior domains and half spaces. Later on, Grubb [7] considered the problem for $s > 1/p - 2$ with the Dirichlet boundary condition.

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Amann [1], among others, obtained a unique time-global solution as the limit of the Leray-Hopf solutions with initial data in $L^2 \cap \dot{B}^{n/p-1}_{p,\infty}$ in $\dot{B}^{n/p-1}_{p,\infty}$. However, only initial data in the closure of $L^2 \cap \dot{B}^{n/p-1}_{p,\infty}$ can be treated in this way, and this closure does not coincide with $\dot{B}^{n/p-1}_{p,\infty}$ in general.

In order to consider this problem in a different way, we first establish the Helmholtz decomposition in the Sobolev spaces and the Besov spaces. Namely, we generalize the Helmholtz decomposition in L^p-spaces obtained by Fujiwara and Morimoto [5] for bounded domains, and by Miyakawa [11] and Simader and Sohr [13] for exterior domains. By using this decomposition we can define the Stokes operator in these spaces directly and cover the cases which is not treated by previous works, which will be done in forthcoming papers.

As is stated by [4], [10] and [1], the Navier-Stokes initial value problem is time-globally well-posed if initial data is small in the homogeneous Besov space $\dot{B}^{n/p-1}_{p,\infty}$ for $p \in (n, \infty)$, but similar results with initial data in the inhomogeneous space $B^{n/p-1}_{p,\infty}$ is hardly possible for unbounded domains. Hence we treat homogeneous spaces as well as inhomogeneous spaces. Moreover, in order to consider initial data with no decay property, we cover the case $p = \infty$.

The paper [13] employed the characterization of the weak solution of the Neumann problem by variational methods, which is based on the reflexivity of the function spaces. Since this method is not applicable to treat the Besov spaces for $p = 1$ and $p = \infty$, we employ the concrete expression of the solution employed in [5] and [11], and provide a new estimate of the solution.

This paper is organized as follows. In Section 1 we introduce the function spaces. In Section 2 we define the normal trace of solenoidal vector fields. Then the main theorem is stated in Section 3. In Section 4 we give estimates of solutions to the Neumann problem, and the main theorem is proved in Section 5.

§1. Function Spaces.

We first introduce the function spaces on the whole space $\mathbb{R}^n$. First, for $p \in (1, \infty)$ and $s \in \mathbb{R}$, we define the Sobolev space $H^s_p(\mathbb{R}^n)$ and the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ by

$$H^s_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' \;\middle|\; \|f \,|H^s_p\| = \|\mathcal{F}^{-1}[\langle\xi\rangle^s \mathcal{F}[f](\xi)]\|_p < \infty \right\},$$
$$\dot{H}^s_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'/\mathcal{P} \;\middle|\; \left\|f \,\middle|\dot{H}^s_p\right\| = \|\mathcal{F}^{-1}[|\xi|^s \mathcal{F}[f](\xi)]\|_p < \infty \right\}$$

respectively, where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, $\|\cdot\|_p$ denotes the standard L^p-norm, and $\mathcal{S}'$ and $\mathcal{P}$ denote the set of tempered distributions and the set of polynomials respectively.

Next, let $\chi(t)$ be a monotone-decreasing smooth function on $[0,+\infty)$ such that $\chi(t) \equiv 1$ on $[0,1]$ and $\chi(t) \equiv 0$ on $[2,\infty)$, and put $\varphi_j(\xi) = \chi\left(2^{-j}|\xi|\right) - \chi\left(2^{1-j}|\xi|\right)$ for $j \in \mathbb{Z}$ and $\Phi(\xi) = \chi(|\xi|)$. Now, for p, $q \in [1,\infty]$ and $s \in \mathbb{R}$, we define the Besov space $B^s_{p,q}(\mathbb{R}^n)$ and the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by

$$B^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' \,\middle|\, \|f\,|B^s_{p,q}\| = \left\|\mathcal{F}^{-1}\left[\Phi(\xi)\mathcal{F}[f](\xi)\right]\right\|_p + \left\|\left\{2^{js}\left\|\mathcal{F}^{-1}\left[\varphi_j(\xi)\mathcal{F}[f](\xi)\right]\right\|_p\right\}_{j=1}^{\infty}\,\middle|\,\ell^q\right\| < \infty \right\},$$

$$\dot{B}^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'/\mathcal{P} \,\middle|\, \left\|f\,\middle|\dot{B}^s_{p,q}\right\| = \left\|\left\{2^{js}\left\|\mathcal{F}^{-1}\left[\varphi_j(\xi)\mathcal{F}[f](\xi)\right]\right\|_p\right\}_{j=-\infty}^{\infty}\,\middle|\,\ell^q\right\| < \infty \right\}$$

respectively. Then we have the inclusion relations $B^s_{p,q_1}(\mathbb{R}^n) \subset B^s_{p,q_2}(\mathbb{R}^n)$ for $q_1 < q_2$, $B^s_{p,p}(\mathbb{R}^n) \subset H^s_p(\mathbb{R}^n) \subset B^s_{p,2}(\mathbb{R}^n)$ for $p \in (1,2]$, $B^s_{p,2}(\mathbb{R}^n) \subset H^s_p(\mathbb{R}^n) \subset B^s_{p,p}(\mathbb{R}^n)$ for $p \in [2,\infty)$. We also have the Sobolev embedding $B^s_{p,q}(\mathbb{R}^n) \subset B^{s-n/p+n/r}_{r,q}(\mathbb{R}^n)$ for $1 \le p < r \le \infty$. If p, $q < \infty$, the space $C_0^\infty(\mathbb{R}^n)$ is dense in the spaces $H^s_p(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$. If $s < n/p$, the same results hold also for homogeneous spaces. For more detailed property of these spaces, see Triebel [14, Chapter 2]. Furthermore, put

$$B^s_{p,\infty-}(\mathbb{R}^n) = \left\{ f \in B^s_{p,\infty}(\mathbb{R}^n) \,\middle|\, \lim_{j\to+\infty} 2^{js}\left\|\mathcal{F}^{-1}\left[\varphi_j(\xi)\mathcal{F}[f](\xi)\right]\right\|_p = 0 \right\},$$

$$\dot{B}^s_{p,\infty-}(\mathbb{R}^n) = \left\{ f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n) \,\middle|\, \lim_{j\to\pm\infty} 2^{js}\left\|\mathcal{F}^{-1}\left[\varphi_j(\xi)\mathcal{F}[f](\xi)\right]\right\|_p = 0 \right\}$$

respectively. Then the spaces $B^s_{p,\infty-}(\mathbb{R}^n)$ and $\dot{B}^s_{p,\infty-}(\mathbb{R}^n)$ are closed subspaces of $B^s_{p,\infty}(\mathbb{R}^n)$ and $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ respectively. Moreover, in the case $p < \infty$, the space $B^s_{p,\infty-}(\mathbb{R}^n)$ coincides with the closure of the space $C_0^\infty(\mathbb{R}^n)$ in $B^s_{p,\infty}(\mathbb{R}^n)$.

In the case $s < n/p$ we can (and do in the sequel) identify the spaces $\dot{H}^s_p(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$ with subspaces of $\mathcal{S}'$. (See Bourdaud [3].) In this way the space $\dot{B}^s_{p,\infty-}(\mathbb{R}^n)$ coincides with the closure of the space $C_0^\infty(\mathbb{R}^n)$ in $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ provided $p < \infty$ and $n/p - n < s < n/p$.

If $s > 0$, we have $H_p^s(\mathbb{R}^n) = \dot{H}_p^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{B}_{p,q}^s(\mathbb{R}^n)$. On the other hand, if $s < 0$, we have $H_p^s(\mathbb{R}^n) = \dot{H}_p^s(\mathbb{R}^n) + L^p(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n) = L^p(\mathbb{R}^n) + \dot{B}_{p,q}^s(\mathbb{R}^n)$. Furthermore, for $p \in (1,\infty)$ we have $\left(H_p^s(\mathbb{R}^n)\right)' = H_{p'}^{-s}(\mathbb{R}^n)$ and $\left(\dot{H}_p^s(\mathbb{R}^n)\right)' = \dot{H}_{p'}^{-s}(\mathbb{R}^n)$ with $p' = p/(p-1)$, and for p, $q \in [1,\infty)$ we have $\left(B_{p,q}^s(\mathbb{R}^n)\right)' = B_{p',q'}^{-s}(\mathbb{R}^n)$ and $\left(\dot{B}_{p,q}^s(\mathbb{R}^n)\right)' = \dot{B}_{p',q'}^{-s}(\mathbb{R}^n)$ with $p' = p/(p-1)$ and $q' = q/(q-1)$. For $p \in [1,\infty)$, we also have $\left(B_{p,\infty-}^s(\mathbb{R}^n)\right)' = B_{p',1}^{-s}(\mathbb{R}^n)$ and $\left(\dot{B}_{p,\infty-}^s(\mathbb{R}^n)\right)' = \dot{B}_{p',1}^{-s}(\mathbb{R}^n)$.

The following lemma provides a function space whose dual space coincides with $B_{1,1}^{-s}(\mathbb{R}^n)$,

Lemma 1.1. *Suppose that $s > 0$, and let $\hat{B}_{\infty,\infty}^s(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ in $B_{\infty,\infty}^s(\mathbb{R}^n)$. Then we have $\left(\hat{B}_{\infty,\infty}^s(\mathbb{R}^n)\right)' = B_{1,1}^{s}(\mathbb{R}^n)$.*

Proof. Let $C_b(\mathbb{R}^n)$ be a set of continuous functions $u(x)$ on $\mathbb{R}^n$ such that $\lim_{|x|\to\infty} u(x) = 0$. For a nonnegative integer k, let $C_b^k(\mathbb{R}^n)$ be a set of functions $u(x)$ such that $\left(\partial^{|\alpha|}u/\partial x^\alpha\right)(x) \in C_b(\mathbb{R}^n)$ holds for every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le k$. Then we have $B_{\infty,1}^k(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n) \subset B_{\infty,\infty}^k(\mathbb{R}^n)$. Now choose $k > 0$ and every $\theta \in (0,1)$ so that $s = \theta k$. Then we have the inclusion relations

$$
\begin{aligned}
B_{\infty,\infty}^{\theta k}(\mathbb{R}^n) = \left(B_{\infty,1}^0(\mathbb{R}^n), B_{\infty,1}^k(\mathbb{R}^n)\right)_{\theta,\infty} &\subset \left(C_b(\mathbb{R}^n), C_b^k(\mathbb{R}^n)\right)_{\theta,\infty} \\
&\subset \left(B_{\infty,\infty}^0(\mathbb{R}^n), B_{\infty,\infty}^k(\mathbb{R}^n)\right)_{\theta,\infty} = B_{\infty,\infty}^{\theta k},
\end{aligned}
$$

which yields $\left(C_b(\mathbb{R}^n), C_b^k(\mathbb{R}^n)\right)_{\theta,\infty} = B_{\infty,\infty}^{\theta k}(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $C_b(\mathbb{R}^n)$ and $C_b^k(\mathbb{R}^n)$, the space $\hat{B}_{\infty,\infty}^{\theta k}(\Omega)$ coincides with the closure of $C_b^k(\mathbb{R}^n)$ in $B_{\infty,\infty}^{\theta k}(\mathbb{R}^n)$. Hence it follows from Bergh and Löfström [2]3.7, Remark that

$$
\left(\hat{B}_{\infty,\infty}^{\theta k}(\mathbb{R}^n)\right)' = \left(\left(C_b(\mathbb{R}^n)\right)', \left(C_b^k(\mathbb{R}^n)\right)'\right)_{\theta,1}. \tag{1.1}
$$

On the other hand, we have $\left(C_b(\mathbb{R}^n)\right)' = M(\mathbb{R}^n)$, where $M(\mathbb{R}^n)$ denotes the set of Radon measures on $\mathbb{R}^n$ with bounded total variation. In view of the Radon-Nikodym theorem and the Fubini theorem, we have the inclusion relation

$$
B_{1,1}^0(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \subset M(\mathbb{R}^n) = \left(C_b(\mathbb{R}^n)\right)' \subset B_{1,\infty}^0(\mathbb{R}^n).
$$

From this fact we have

$$B^{-k}_{1,1}(\mathbb{R}^n) \subset \big(C^k_b(\mathbb{R}^n)\big)' \subset B^{-k}_{1,\infty}(\mathbb{R}^n).$$

These two inclusion relations imply

$$\begin{aligned} B^{-\theta k}_{1,1}(\mathbb{R}^n) = \big(B^0_{1,1}(\mathbb{R}^n), B^{-k}_{1,1}(\mathbb{R}^n)\big)_{\theta,1} &\subset \Big(\big(C_b(\mathbb{R}^n)\big)', \big(C^k_b(\mathbb{R}^n)\big)'\Big)_{\theta,1} \\ &\subset \big(B^0_{1,\infty}(\mathbb{R}^n), B^{-k}_{1,\infty}(\mathbb{R}^n)\big)_{\theta,1} = B^{-\theta k}_{1,1}(\mathbb{R}^n). \end{aligned}$$

From this relation and (1.1) we obtain the conclusion. Q.E.D.

Next, for a domain $\Omega \subset \mathbb{R}^n$, put

$$H^s_p(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \exists \tilde{u} \in H^s_p(\mathbb{R}^n) \text{ such that } \tilde{u}|_\Omega = u\}$$

with norm $\|u \,|H^s_p\| = \min\big\{ \|\tilde{u}|\, H^s_p\| \mid \tilde{u}|_\Omega = u\big\}$, where $\mathcal{D}'(\Omega)$ denotes the set of the distributions on Ω. The spaces $\dot{H}^s_p(\Omega)$, $B^s_{p,q}(\Omega)$ and $\dot{B}^s_{p,q}(\Omega)$ are defined similarly. If Ω is bounded, we have the identities $\dot{H}^s_p(\Omega) = H^s_p(\Omega)$ and $\dot{B}^s_{p,q}(\Omega) = B^s_{p,q}(\Omega)$. For a bounded $C^{2,1}$ hypersurface Γ in $\mathbb{R}^n$ and $s \in \mathbb{R}$ such that $|s| < 1$, we can define the Besov space $B^s_{p,q}(\Gamma)$ on Γ by way of the local coordinates.

Let Ω be either a bounded domain or an exterior domain with $C^{2,1}$ boundary Γ. If $1 < p < \infty$ and $1/p < s < 2$, we can define for every $u \in H^s_p(\Omega)$ the trace $\gamma_\Gamma u \in B^{s-1/p}_{p,p}(\Gamma)$, and the mapping $u \mapsto \gamma_\Gamma u$ is bounded from $\dot{H}^s_p(\Omega)$ to $B^{s-1/p}_{p,p}(\Gamma)$. If $1 \le p \le \infty$ and $1/p < s < 2$, we can define for every $u \in \dot{B}^s_{p,q}(\Omega)$ the trace $\gamma_\Gamma u \in B^{s-1/p}_{p,q}(\Gamma)$, and the mapping $u \mapsto \gamma_\Gamma u$ is bounded from $\dot{B}^s_{p,q}(\Omega)$ to $B^{s-1/p}_{p,q}(\Gamma)$. On the other hand, for every $u \in B^s_{p,q}(\Gamma)$ with $s \in (0, 1 - 1/p)$, there exists $U \in B^{s+1/p}_{p,q}(\Omega)$ such that $\gamma_\Gamma U = u$ and that $\Big\|U \,\Big|B^{s+1/p}_{p,q}(\Omega)\Big\| \le C\big\|u \,|B^s_{p,q}(\Omega)\big\|$. Furthermore, if $q = p \in (1,\infty)$, we can choose U satisfying $U \in H^{s+1}_p(\Omega)$ and $\Big\|U \,\Big|H^{s+1/p}_p(\Omega)\Big\| \le C\big\|u \,|B^s_{p,p}(\Omega)\big\|$ as well. (See Triebel [14, Section 2.9].)

Finally, for $p \in (1,\infty)$ and $q \in [1,\infty]$, we define the Lorentz space $L^{p,q}(\Omega)$ by

$$L^{p,q}(\Omega) = \Bigg\{ f(x) \in L^1_{\text{loc}}(\Omega) \,\Bigg|\, \|u\|_{p,q} = \left(\int_0^\infty \Big(s\mu(\{x \in \Omega \mid |u(x)| > s)^{1/p}\Big)^q \frac{ds}{s}\right)^{1/q} < \infty \Bigg\}.$$

Then $L^{p,p}(\Omega) = L^p(\Omega)$, and we have the real interpolation property $(L^{p_0,q_0}(\Omega), L^{p_1,q_1}(\Omega))_{\theta,q} = L^{p,q}(\Omega)$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ provided $p_0 \neq p_1$. For more detailed property of these spaces, See Bergh and Löfström [2, Chapter 5].

§2. Normal traces.

We start with the following lemma concerning the duality between the function spaces on general domains.

Lemma 2.1. *Suppose that $1 < p < \infty$ and $-1 + 1/p < s < 1/p$, and put $p' = p/(p-1)$. Then we have the following assertions:*

(1) *We have $\left(H_p^s(\Omega)\right)' = H_{p'}^{-s}(\Omega)$ and $\left(\dot{H}_p^s(\Omega)\right)' = \dot{H}_{p'}^{-s}(\Omega)$.*

(2) *Suppose that $1 \leq q < \infty$. Then we have $\left(B_{p,q}^s(\Omega)\right)' = B_{p',q'}^{-s}(\Omega)$ and $\left(\dot{B}_{p,q}^s(\Omega)\right)' = \dot{B}_{p',q'}^{-s}(\Omega)$.*

(3) *We have $\left(B_{p,\infty-}^s(\Omega)\right)' = B_{p',1}^{-s}(\Omega)$ and $\left(\dot{B}_{p,\infty-}^s(\Omega)\right)' = \dot{B}_{p',1}^{-s}(\Omega)$.*

Proof. Triebel [14, Theorem 2.10.2.2, Remark 2.10.2.3] imply that $\left(H_p^s(\Omega)\right)' = H_{p'}^{-s}(\Omega)$ and that $\left(B_{p,q}^s(\Omega)\right)' = B_{p',q'}^{-s}(\Omega)$ for $1 \leq q < \infty$.

Next, put $r = pn/(n-ps)$. If $s \geq 0$, we have the inequality $p \leq r \leq p/(1-ps) < \infty$, and it follows that $\dot{H}_p^s(\Omega) \subset L^r(\Omega)$ and $\dot{B}_{p,q}^s(\Omega) \subset L^{r,q}(\Omega)$. On the other hand, if $s < 0$, we have the inequality $1 < p/(1-ps) < r < p$ and the equality $n/r' = n/p' + s$. This implies that the spaces $\dot{H}_{p'}^{-s}(\Omega)$ and $\dot{B}_{p',q'}^{-s}(\Omega)$ are densely embedded into $L^{r'}(\Omega)$ and $L^{r',q'}(\Omega)$ respectively, where q' is replaced by $\infty-$ if $q = 1$. These facts imply the inclusion relations $L^r(\Omega) \subset \dot{H}_p^s(\Omega)$ and $L^{r,q}(\Omega) \subset \dot{B}_{p,q}^s(\Omega)$. Hence, in both cases, we can argue in the same way as before to conclude that $\left(\dot{H}_p^s(\Omega)\right)' = \dot{H}_{p'}^{-s}(\Omega)$ and $\left(\dot{B}_{p,q}^s(\Omega)\right)' = \dot{B}_{p',q'}^{-s}(\Omega)$ for $1 \leq q < \infty$. We thus established Assertions (1) and (2). Finally, in view of Triebel [14, Theorem 2.6.1, (c)], we can prove Assertion (3) in the same way. Q.E.D.

The following lemma can be proved in the same way.

Lemma 2.2. *For $s \in (0,1)$, we have the identities $\left(B_{1,q}^s(\Omega)\right)' = B_{\infty,q'}^{-s}(\Omega)$ and $\left(\dot{B}_{1,q}^s(\Omega)\right)' = \dot{B}_{\infty,q'}^{-s}(\Omega)$ for $1 \leq q < \infty$, and $\left(B_{1,\infty-}^s(\Omega)\right)' = B_{\infty,1}^{-s}(\Omega)$ and $\left(\dot{B}_{1,\infty-}^s(\Omega)\right)' = \dot{B}_{\infty,1}^{-s}(\Omega)$.*

By virtue of these lemmas we can prove the following theorem. In the sequel let n denote the outer normal vector of Γ.

Theorem 2.3. *Suppose that $p \in [1,\infty]$ and $1/p-1 < s < 1/p$, and assume that $r \in [pn/(p+n), p]$ satisfies $1/r < \min\{0,s\}+1$. Then we have the following assertions:*

(1) *Suppose that $p \neq 1, \infty$. If $u \in H^s_p(\Omega)$ and $\operatorname{div} u \in H^s_r(\Omega)$, then we have $n \cdot \gamma_\Gamma u \in B^{s-1/p}_{p,p}(\Gamma)$ with the estimate*

$$\left\| n \cdot \gamma_\Gamma u \,\middle|\, B^{s-1/p}_{p,p} \right\| \leq C \left(\left\| u \,\middle|\, H^s_p \right\| + \left\| \operatorname{div} u \,\middle|\, H^s_r \right\| \right). \tag{2.1}$$

(2) *Suppose that $q \in [1,\infty]$. If $u \in B^s_{p,q}(\Omega)$ and $\operatorname{div} u \in B^s_{r,q}(\Omega)$, then we have $n \cdot \gamma_\Gamma u \in B^{s-1/p}_{p,q}(\Gamma)$ with the estimate*

$$\left\| n \cdot \gamma_\Gamma u \,\middle|\, B^{s-1/p}_{p,q} \right\| \leq C \left(\left\| u \,\middle|\, B^s_{p,q} \right\| + \left\| \operatorname{div} u \,\middle|\, B^s_{r,q} \right\| \right). \tag{2.2}$$

These statements hold as well if we replace some of the spaces $H^s_p(\Omega)$, $H^s_r(\Omega)$, $B^s_{p,q}(\Omega)$ and $B^s_{r,q}(\Omega)$ by $\dot{H}^s_p(\Omega)$, $\dot{H}^s_r(\Omega)$, $\dot{B}^s_{p,q}(\Omega)$ and $\dot{B}^s_{r,q}(\Omega)$ respectively.

Proof. First, fix a bounded domain U in $\mathbb{R}^n$ such that $\Gamma \subset U$. Let φ be a smooth function on Γ. We define $n \cdot \gamma_\Gamma u$ as a distribution on Γ by the formula

$$\langle \varphi, n \cdot \gamma_\Gamma u \rangle_\Gamma = \int_\Omega \nabla \cdot (\Phi u)\, dx = \langle \Phi, \operatorname{div} u \rangle_\Omega + \langle \nabla \Phi, u \rangle_\Omega, \tag{2.3}$$

where $\Phi \in C^\infty_0(U)$ satisfies $\gamma_\Gamma \Phi = \varphi$. Then it is easy to verify that $\langle \varphi, n \cdot \gamma_\Gamma u \rangle_\Gamma$ is defined independently of the choice of Φ. Furthermore, for every p and q, we can choose a constant $C_{p,q}$ such that, for every φ we can choose Φ so that

$$\left\| \Phi \,\middle|\, H^{1-s}_{p'} \right\| + \left\| \Phi \,\middle|\, B^{1-s}_{p',p'} \right\| \leq C \left\| \varphi \,\middle|\, B^{1/p-s}_{p',p'} \right\|. \tag{2.4}$$

We first show the estimate of Assertion (1). The divergence theorem yields the estimate

(2.5)

$$\begin{aligned} \left| \langle \varphi, n \cdot \gamma_\Gamma u \rangle_\Gamma \right| &\leq \left| \langle \varphi, \operatorname{div} u \rangle_\Omega \right| + \left| \langle \nabla \varphi, u \rangle_\Omega \right| \\ &\leq C \left(\left\| \varphi \,\middle|\, H^{-s}_{r'} \right\| \left\| \operatorname{div} u \,\middle|\, H^s_r \right\| + \left\| \nabla \varphi \,\middle|\, H^{-s}_{p'} \right\| \left\| u \,\middle|\, H^s_p \right\| \right). \end{aligned}$$

Since $1-1/r < s < 1/p \leq 1/r$, we have the duality $(H^s_r(\Omega))' = H^{-s}_{r'}(\Omega)$. Moreover, since $p' \leq r' < \infty$ and since $r' \leq p'n/(n-p')$ if $p' < n$, we have the inequality $\|\varphi\,|H^{-s}_{r'}\| \le C\left\|\nabla\varphi\,\middle|H^{-s}_{p'}\right\|$ with a constant C depending on the domain U. Substituting this inequality and (2.4) into (2.5), we conclude that

$$|\langle\varphi, n\cdot\gamma_\Gamma u\rangle_\Gamma| \le C\left\|\varphi\,\middle|B^{1/p-s}_{p',p'}\right\|\left(\|\mathrm{div}\,u\,|H^s_r\| + \|u\,|H^s_p\|\right).$$

Since $\left(B^{1/p-s}_{p',p'}(\Gamma)\right)' = B^{s-1/p}_{p,p}(\Gamma)$, we have $n\cdot\gamma_\Gamma u \in B^{s-1/p}_{p,p}(\Gamma)$ with the estimate (2.1).

By using the estimate

$$|\langle\varphi, n\cdot\gamma_\Gamma u\rangle_\Gamma| \le C\left(\left\|\varphi\,\middle|B^{-s}_{r',q'}\right\|\|\mathrm{div}\,u\,|B^s_{r,q}\| + \left\|\nabla\varphi\,\middle|B^{-s}_{p',q'}\right\|\|u\,|B^s_{p,q}\|\right),$$

Assertion (2) in the case $q = p > 1$ can be proved in the same way. Assertion (2) in the case $q = p = 1$ can also be proved from the equality $\left(\hat{B}^{1-s}_{\infty,\infty}(\Gamma)\right)' = B^{1-s}_{1,1}(\Gamma)$, which can be proved in the same way as Lemma 1.1. Here $\hat{B}^{1-s}_{\infty,\infty}(\Gamma)$ denotes the closure of $C^2(\Gamma)$ in $B^{1-s}_{\infty,\infty}(\Gamma)$. Finally, Assertion (2) in the general case follows from Assertion (2) for $q = p$ and real interpolation. Q.E.D.

In particular, if $u \in H^s_p(\Omega)$ with $1/p - 1 < s < 1/p$ satisfies $\mathrm{div}\,u = 0$, we can consider the normal trace $n\cdot\gamma_\Gamma \in B^{s-1/p}_{p,p}(\Gamma)$, and similar facts holds for other function spaces.

§3. Main Result.

Our main result of bounded domains is the following theorem.

Theorem 3.1. *Suppose that Ω is either a bounded domain or an exterior domain with $C^{2,1}$ boundary Γ. Then we have the following:*

(1) *Suppose that $1 < p < \infty$ and $1/p - 1 < s < 1/p$, and put*

$$\dot{H}^s_{\sigma,p}(\Omega) = \left\{u \in \left(\dot{H}^s_p(\Omega)\right)^n \,\middle|\, \mathrm{div}\,u = 0 \text{ in } \Omega, n\cdot\gamma_\Gamma u = 0\right\} \tag{3.1}$$

and

$$\dot{G}^s_p(\Omega) = \left\{u \in \left(\dot{H}^s_p(\Omega)\right)^n \,\middle|\, \exists G \text{ such that } u = \nabla G\right\}. \tag{3.2}$$

Then we have the topological direct sum decomposition

$$\left(\dot{H}^s_p(\Omega)\right)^n = \dot{H}^s_{\sigma,p}(\Omega) \oplus \dot{G}^s_p(\Omega), \tag{3.3}$$

(2) *Under the same assumption as Assertion* (1), *put*

$$H^s_{\sigma,p}(\Omega) = \left\{ u \in \left(H^s_p(\Omega)\right)^n \mid \operatorname{div} u = 0 \ \textit{in}\ \Omega, n \cdot \gamma_\Gamma u = 0 \right\} \tag{3.4}$$

and

$$G^s_p(\Omega) = \left\{ u \in \left(H^s_p(\Omega)\right)^n \mid \exists G \ \textit{such that}\ u = \nabla G \right\}. \tag{3.5}$$

Then we have the topological direct sum decomposition

$$\left(H^s_p(\Omega)\right)^n = H^s_{\sigma,p}(\Omega) \oplus G^s_p(\Omega). \tag{3.6}$$

(3) *Suppose that* $p,\ q \in [1, \infty]$ *and* $1/p - 1 < s < 1/p$, *and put*

$$\dot{B}^s_{\sigma,p,q}(\Omega) = \left\{ u \in \left(\dot{B}^s_{p,q}(\Omega)\right)^n \,\middle|\, \operatorname{div} u = 0 \ \textit{in}\ \Omega, n \cdot \gamma_\Gamma u = 0 \right\} \tag{3.7}$$

and

$$\dot{G}^s_{p,q}(\Omega) = \left\{ u \in \left(\dot{B}^s_{p,q}(\Omega)\right)^n \,\middle|\, \exists G \ \textit{such that}\ u = \nabla G \right\}. \tag{3.8}$$

Then we have the topological direct sum decomposition

$$\left(B^s_{p,q}(\Omega)\right)^n = B^s_{\sigma,p,q}(\Omega) \oplus G^s_{p,q}(\Omega). \tag{3.9}$$

(4) *In addition to the assumption in Assertion* (3), *assume moreover that* Ω *is bounded, or that* $1 < p < \infty$. *Put*

$$B^s_{\sigma,p,q}(\Omega) = \left\{ u \in \left(B^s_{p,q}(\Omega)\right)^n \mid \operatorname{div} u = 0 \ \textit{in}\ \Omega, n \cdot \gamma_\Gamma u = 0 \right\} \tag{3.10}$$

and

$$G^s_{p,q}(\Omega) = \left\{ u \in \left(B^s_{p,q}(\Omega)\right)^n \mid \exists G \ \textit{such that}\ u = \nabla G \right\}. \tag{3.11}$$

Then we have the topological direct sum decomposition

$$\left(B^s_{p,q}(\Omega)\right)^n = B^s_{\sigma,p,q}(\Omega) \oplus G^s_{p,q}(\Omega). \tag{3.12}$$

(5) *Let* $\dot{P}_{p,s}$ *denote the projection operator on* $\dot{H}^s_{\sigma,p}(\Omega)$ *associated with the topological direct sum decomposition* (3.3), *and let* $P_{p,s}$ *denote the projection operator on* $H^s_{\sigma,p}(\Omega)$ *associated with the topological direct sum decomposition* (3.6). *Moreover, let* $\dot{P}_{p,q,s}$ *denote the projection operator on* $\dot{B}^s_{\sigma,p,q}(\Omega)$ *associated with the topological direct sum decomposition* (3.9), *and let* $P_{p,q,s}$ *denote the projection operator on* $B^s_{\sigma,p,q}(\Omega)$ *associated with the topological direct sum decomposition* (3.12). *Then these projection operators are identical on the intersection of the domains.*

(6) *We have the identities* $\left(\dot{P}_{p,s}\right)' = \dot{P}_{p',s}$ *and* $(P_{p,s})' = P_{p',s}$ *for* $p \in (1,\infty)$ *with* $p' = p/(p-1)$, *and the identities* $\left(\dot{P}_{p,q,s}\right)' = \dot{P}_{p',q',s}$ *and* $(P_{p,q,s})' = P_{p',q',s}$ *for* p, $q \in [1,\infty)$ *with* p' *as above and* $q' = q/(q-1)$.

Remark 3.1. If Ω is bounded, we have $\dot{H}^s_p(\Omega) = H^s_p(\Omega)$ and $\dot{B}^s_{p,q}(\Omega) = B^s_{p,q}(\Omega)$. Hence Assertion (2) and Assertion (4) are identical with Assertion (1) and Assertion (3) respectively.

§4. Estimate of the Neumann Problem.

We start with the following lemma concerning real interpolation relations which will be used in the estimate.

Lemma 4.1. *Suppose that* Ω *is either a bounded domain or an exterior domain with* $C^{2,1}$ *boundary* Γ, *and that* s_0, $s_1 \in (1/p-1, 1/p)$ *satisfy* $s_0 \neq s_1$. *Then we have the relations*

$$\left(B^{s_0}_{p,q_0}(\Omega), B^{s_1}_{p,q_1}(\Omega)\right)_{\theta,q} = B^{(1-\theta)s_0+\theta s_1}_{p,q}(\Omega),$$
$$\left(\dot{B}^{s_0}_{p,q_0}(\Omega), B^{s_1}_{p,q_1}(\Omega)\right)_{\theta,q} = \dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\Omega)$$

for every q_0, $q_1 \in [1,\infty]$. *If* $p \in (1,\infty)$, *we also have*

$$\left(H^{s_0}_p(\Omega), H^{s_1}_p(\Omega)\right)_{\theta,q} = B^{(1-\theta)s_0+\theta s_1}_{p,q}(\Omega),$$
$$\left(\dot{H}^{s_0}_p(\Omega), \dot{H}^{s_1}_p(\Omega)\right)_{\theta,q} = \dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\Omega).$$

This lemma follows from Triebel [14, Theorem 2.10.4.1] and the proof of Lemma 2.1.

We now state the required estimate. Let Ω be a domain with $C^{2,1}$-boundary Γ. For $p \in [1,\infty]$ and $s \in (1/p-1, 1/p)$, put

$$X^s_p(\Omega) = \begin{cases} \dot{B}^s_{1,1}(\Omega) & \text{for } p = 1, \\ \dot{H}^s_p(\Omega) & \text{for } 1 < p < \infty, \\ \dot{B}^s_{\infty,\infty}(\Omega) & \text{for } p = \infty. \end{cases} \tag{4.1}$$

Then we have the following theorem on the Neumann problem

$$\Delta h(x) = \operatorname{div} f(x) \text{ in } \Omega, \tag{4.2}$$
$$n \cdot \big(f(x) - \nabla h(x)\big) = 0 \qquad \text{on } \Gamma. \tag{4.3}$$

Theorem 4.2. *Suppose that Ω is either a bounded domain or an exterior domain with $C^{2,1}$ boundary Γ. Then, for every p and s as above, there exists a constant $C = C_{n,p,s,\Omega}$ such that, for every $f(x) \in \left(X^s_p(\Omega)\right)^n$, there exists a solution $h(x)$ of the problem (4.2)–(4.3), uniquely modulo constants, and we have the estimate $\|\nabla h \,|X^s_p\| \le C\|f\,|X^s_p\|$.*

Proof. Choose $\tilde{f}(x) \in X^s_p(\mathbb{R}^n)$ such that $\tilde{f}|_\Omega = f$ and that $\left\|\tilde{f}\,\middle|\, X^s_p\right\| \le 2\|f\,|X^s_p\|$. Furthermore, if Ω is an exterior domain, we choose $\tilde{f}$ so that $\int_K \tilde{f}\,dx = 0$, where $K = \mathbb{R}^n \setminus \Omega$. Then put $h_1 = E * \tilde{f}$, where E denotes the fundamental solution of Δ on $\mathbb{R}^n$. Then we have the estimate

$$\|\nabla h_1\,|X^s_p\| \le C\|h_1\,|X^{s+1}_p\| \le C\left\|\tilde{f}\,\middle|\, X^s_p\right\|, \tag{4.4}$$

and the equality $\operatorname{div}(f - \nabla h_1) = \operatorname{div} f - \Delta h_1 = 0$ on Ω. It follows from Theorem 2.3 that the normal trace

$$g(x) = n \cdot \gamma_\Gamma(f - \nabla h_1) \in B^{s-1/p}_{p,p}(\Gamma)$$

is well-defined. Furthermore, it satisfies the estimate

$$\left\|g\,\middle|B^{s-1/p}_{p,p}\right\| \le C\left(\|f\,|X^s_p\| + \|\nabla h_1\,|X^s_p\|\right) \le C\left\|\tilde{f}\,|X^s_p\right\| \tag{4.5}$$

and the equality

$$\int_\Gamma g(x)\,ds(x) = \int_\Omega \operatorname{div}\{f(x) - \nabla h_1(x)\}dx = 0.$$

Hence the Neumann problem

$$\begin{aligned} \Delta\tilde{h}(x) &= 0 \qquad \text{in } \Omega, \\ n \cdot \nabla\tilde{h}(x) &= -g(x) \text{ on } \Gamma, \end{aligned}$$

has a solution $\tilde{h}(x)$, uniquely modulo constants. This solution is given by the formula $\tilde{h}(x) = h_2(x) + h_3(x)$, where

$$\begin{aligned} h_2(x) &= -\int_\Gamma E(x-y)g(y)\,ds(y), \\ h_3(x) &= -\int_\Gamma r(x,y)g(y)\,ds(y). \end{aligned} \tag{4.6}$$

Here $r(x,y)$ is a function on $\Omega \times \Omega$ satisfying $\Delta_x r(x,y) = 0$ on Ω and $n \cdot \nabla_x \big(r(x,y) + E(x-y)\big) \equiv 0$ on Γ. This function also satisfies $r(x,y) = r(y,x)$ and

$$|\nabla_y r(x,y)| \le C|x-y|^{1-n} \left\| g \,\middle|\, \dot{B}^{s-1/p}_{p,p} \right\| \tag{4.7}$$

with a constant C independent of $\tilde{f}$. (See Mizohata [12, Theorem 8.6] in the case $n = 3$.) Furthermore, we have the identity

$$h_3(x) = -\int_\Gamma n \cdot \nabla E(x-y)\tilde{h}(y)\, ds(y) \tag{4.8}$$

In order to obtain an explicit estimates of h_2 from (4.6), we recall the definition of the normal trace. We have

$$\begin{aligned} h_2(x) &= \langle E(x-\cdot), g\rangle_\Gamma = \langle \nabla_y E(x-\cdot), -f + \nabla h_1\rangle_\Omega \\ &= \int_\Omega \nabla_y E(x-y)\big(-f(y) + \nabla h_1(y)\big) dy \end{aligned} \tag{4.9}$$

and

$$h_3(x) = \int_\Omega \nabla_y r(x,y)\big(-f(y) + \nabla h_1(y)\big) dy. \tag{4.10}$$

For bounded Ω, the equality (4.10), the estimates (4.5), (4.7) and the elliptic regularity theory imply

$$\|\nabla h_3 \,|X^s_p\| \le C \left\| g \,\middle|\, \dot{B}^{s-1/p}_{p,p} \right\| \le C \left\| \tilde{f} \,\middle|\, X^s_p \right\|. \tag{4.11}$$

We turn to the estimate of $h_2(x)$. If $1 \le p < \infty$, let $\psi(x)$ denote the zero extension of $-f(x) + \nabla h_1(x)$ on $\mathbb{R}^n$. Then the denseness of the space $C_0^\infty(\Omega)$ in X^s_p implies that $\psi(x) \in X^s_p(\mathbb{R}^n)$ with the estimate

$$\|\psi \,|X^s_p\| \le C \left(\|f \,|X^s_p\| + \|\nabla h_1 \,|X^s_p\| \right) \le C \left\| \tilde{f} \,\middle|\, X^s_p \right\|.$$

It follows from $h_2 = \nabla E * \psi$ that

$$\|\nabla h_2 \,|X^s_p\| \le C \|h_3 \,|X^{s+1}_p\| \le C\|\psi \,|X^s_p\| \le \left\| \tilde{f} \,\middle|\, X^s_p \right\|. \tag{4.12}$$

Suppose that $p = \infty$. We can choose a neighborhood U_0 of Γ and a $C^{1,1}$-diffeomorphism $F : \Gamma \times (-1,1) \to U_0$ such that $F(x,0) = x$ and $(\partial F/\partial t)(x,0) = n(x)$ for every $x \in \Gamma$. Next, for every $j = 1, 2, 3, \ldots$, let $\omega_j(t)$ be a smooth function on $(-1,1)$ such that $0 \le \omega_j(t) \le 1$, $\omega_j(t) = 1$

for $t \leq -2^{-j}$ and $\omega_j(t) = 0$ for $t \geq -2^{j+1}$, and let $\chi_j(x)$ denote the $C^{1,1}$-functions on U_0 such that the identity $\chi_j\big(F(x,t)\big) = \omega_j(t)$ holds for every $x \in \Gamma$ and every $t \in (-1,1)$. Then we can extend $\chi_j(x)$ as a $C^{1,1}$-function on the whole space $\mathbb{R}^n$ by putting

$$\begin{cases} \chi_j(x) = 1 & \text{for } x \in \Omega \setminus U_0, \\ \chi_j(x) = 0 & \text{for } x \in \mathbb{R}^n \setminus (\Omega \cup U_0). \end{cases}$$

Namely, we have

$$\lim_{j\to\infty} \chi_j(x) = \begin{cases} 1 & (x \in \Omega), \\ 0 & (x \notin \Omega). \end{cases}$$

Let Ψ be an arbitrary element of $\left(\dot{B}^{-s}_{1,1}(\mathbb{R}^n)\right)^n$. Then we have

$$v(y) = \int_{\mathbb{R}^n} \nabla E(x-y) \operatorname{div} \Psi(x)\, dx \in \left(\dot{B}^{-s}_{1,1}(\mathbb{R}^n)\right)^n.$$

We next verify that the sequence $\{\chi_j v\}_{j=1}^{\infty}$ converges in $\left(\dot{B}^{-s}_{1,1}(\mathbb{R}^n)\right)^n$ as $j \to \infty$. Indeed, the restriction $v|_\Omega$ belongs to $\left(\dot{B}^{-s}_{1,1}(\Omega)\right)^n$. Hence, for every $\varepsilon > 0$, there exists $\varphi \in \left(C_0^\infty(\Omega)\right)^n$ such that $\left\|v|_\Omega - \varphi \,\middle|\, \dot{B}^{-s}_{1,1}\right\| < \varepsilon$. For this φ there exists a positive integer j_0 such that $\chi_j(x) \equiv 1$ holds on $\operatorname{supp}\varphi$ for every $j \geq j_0$. It follows that $\chi_j(x)\varphi(x) \equiv \varphi(x)$ for $j \geq j_0$. From this fact we conclude that, for every $j \geq j_0$, the function $\chi_j v \in \left(\dot{B}^{-s}_{1,1}(\mathbb{R}^n)\right)^n$ satisfies the estimate

$$\left\|\chi_j v - \varphi \,\middle|\, \dot{B}^{-s}_{1,1}\right\| = \left\|\chi_j(v - \varphi) \,\middle|\, \dot{B}^{-s}_{1,1}\right\| < C\varepsilon$$

with a constant C independent of v and φ. Since $\varepsilon > 0$ is arbitrary, this implies that the sequence $\{\chi_j v\}_{j=1}^{\infty}$ is a Cauchy sequence in $\left(\dot{B}^{-s}_{1,1}(\mathbb{R}^n)\right)^n$. By using this fact, we define $\nabla h_2 \in X^s_\infty(\mathbb{R}^n)$ by

$$\begin{aligned} \langle \Psi, \nabla h_2 \rangle_\Omega &= -\int_{\mathbb{R}^n} \operatorname{div} \Psi(x) h_2(x)\, dx \\ &= -\lim_{j\to\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla E(x-y) \operatorname{div} \Psi(x)\, dx\, \psi_j(y)\, dy \\ &= -\lim_{j\to\infty} \int_{\mathbb{R}^n} v(y)\psi_j(y)\, dy, \end{aligned}$$

where $\psi_j(y) = \chi_j(y)\big(-f(y) + \nabla h_1(y)\big)$. That is,

$$\nabla h_2 = \text{weak-}* \lim_{j\to\infty} \nabla \int_{\mathbb{R}^n} \nabla E(x-y)\psi_j(y)dy.$$

From this fact we obtain

$$\big\|\nabla h_2 \,\big|X_p^s\,\big\| \le C \liminf_{j\to\infty} \|\psi_j \,|X_\infty^s\,\| \le C\|-f + \nabla h_1 \,|X_\infty^s\,\|.$$

Hence we have (4.12) also in the case $p = \infty$.

Putting $h(x) = h_1(x) + h_2(x) + h_3(x)$ and summing up (4.4), (4.11) and (4.12) we obtain the conclusion for bounded Ω.

We next consider the case that Ω is an exterior domain. Since the functions h_1 and h_2 are written as the integral on the whole space or its weak-$*$ limit, it follows that the estimates (4.4) and (4.12) hold also in this case. Hence it suffices to show (4.11). To this end choose a positive number $R > 0$ such that $\Omega \cap \{x \in \mathbb{R}^n \mid |x| < R\} = \mathbb{R}^n$, and put $U = \{x \in \Omega \mid |x| < 3R\}$ and $V = \{x \in \mathbb{R}^n \mid |x| > 2R\}$. Then $\big\|\nabla h_3 \,\big|X_p^s(U)\,\big\|$ can be estimated in the same way as in the case where Ω is bounded, and hence $\big\|\nabla h_3 \,\big|X_p^s(U)\,\big\| \le C\Big\|\tilde{f}\Big|\, X_p^s\Big\|$. Hence it suffices to estimate $\big\|\nabla h_3 \,\big|X_p^s(V)\,\big\|$. For this purpose we employ the identity (4.8). We first show that, for every $k = 1, 2, \ldots$, there exists a constant C_k such that

$$\big|\nabla^k h_3(x)\big| \le C_k \Big\|\tilde{h}\Big|\, B_{p,p}^{s+1-1/p}(\Gamma)\Big\| \frac{1}{1+|x|^{n-1+k}} \tag{4.13}$$

holds for every $x \in V$. Indeed, since $\Gamma \in \{x \in \mathbb{R}^n \mid |x| \le R\}$, we have $|x-y| \ge |x|/2$ for $x \in V$ and $y \in \Gamma$. Hence, differentiating both sides of (4.10) and observing that $|x| \ge 2R$ on V, we obtain (4.13).

From this fact and the estimate

$$\Big\|\tilde{h}\Big|\, B_{p,p}^{s+1-1/p}(\Gamma)\Big\| \le C\left(\big\|\nabla h_2 \,\big|X_p^s\,\big\| + \big\|\nabla h_3 \,\big|X_p^s(U)\,\big\|\right) \le \Big\|\tilde{f}\Big|\, X_p^s\Big\|$$

we conclude that $\nabla h_2 \in \dot{B}_{1,\infty}^s(V)$ with the estimate

$$\Big\|\nabla h_3 \,\Big|\dot{B}_{1,\infty}^s\Big\| \le C_s \Big\|\tilde{f}\Big|\, X_p^s\Big\| \tag{4.14}$$

for every $s \ge 1$. Next, suppose that $0 < s < 1$. Then we have
(4.15)

$$|\nabla h_3(x+y) - \nabla h_3(x)| = |y \cdot \nabla^2 h_3(x+\theta y)| \le C_1 \Big\|\tilde{f}\Big|\, X_p^s\Big\| \frac{|y|}{1+\big|x+\theta y\big|^{n+1}}$$

with some $\theta \in (0,1)$. If $|y| \le |x|/2$, we have $|x+\theta y| \ge |x| - |y| \ge |x|/2$. Substituting this estimate into (4.15) we obtain

$$\begin{aligned}(4.16)\quad \frac{|\nabla h_3(x+y) - \nabla h_3(x)|}{|y|^s} &\le C_1 \left\| \tilde{f} \,\middle|\, X_p^s \right\| \frac{|y|^{1-s}}{1+(|x|/2)^{n+1}} \\ &\le C \left\| \tilde{f} \,\middle|\, X_p^s \right\| \frac{1}{|x|^s + |x|^{n+s}}.\end{aligned}$$

Next, if $|y| \ge |x|/2$, we have $|x+y| \le |x| + |y| \le 3|y|$. It follows that

$$\begin{aligned}(4.17)\quad \frac{|\nabla h_3(x+y) - \nabla h_3(x)|}{|y|^s} &\le \frac{|\nabla h_3(x+y)| + |\nabla h_3(x)|}{|y|^s} \\ &\le C_0 \left\| \tilde{f} \,\middle|\, X_p^s \right\| \left(\frac{1}{(1+|x+y|^n)|y|^s} + \frac{1}{(1+|x|^n)|y|^s} \right) \\ &\le C_0 \left\| \tilde{f} \,\middle|\, X_p^s \right\| \left(\frac{1}{|x+y|^s + |x+y|^{n+s}} + \frac{1}{|x|^s + |x|^{n+s}} \right).\end{aligned}$$

In view of (4.16) and (4.17), we have

$$\begin{aligned}&\int_V \frac{|\nabla h_3(x+y) - \nabla h_3(x)|}{|y|^s}\, dx \\ &\le C \left\| \tilde{f} \,\middle|\, X_p^s \right\| \int_{\mathbb{R}^n} \left(\frac{1}{|x|^s + |x|^{n+s}} + \frac{1}{|x+y|^s + |x+y|^{n+s}} \right) dx \\ &\le C \left\| \tilde{f} \,\middle|\, X_p^s \right\|.\end{aligned}$$

This implies that (4.14) holds also for $s \in (0,1)$. By Lemma 4.1 and the Sobolev embedding theorem we conclude that $\nabla h_3 \in \dot{B}^s_{p,1}$ for every $s > n/p - n$. This implies $\nabla h_3 \in X_p^s(V)$ with $\left\| \nabla h_3 \,\middle|\, X_p^s \right\| \le C \left\| \tilde{f} \,\middle|\, X_p^s \right\|$ for every $p \in [1,\infty]$ and $s \in (1/p - 1, 1/p)$. Q.E.D.

§5. Proof of the Main Theorem.

First we derive the decomposition in X_p^s as in the previous section from Theorem 4.2. For every $f(x) \in \left(X_p^s(\Omega)\right)^n$, let $h(x)$ be the solution of the problem (4.2)–(4.3), and put $Pf(x) = f(x) - \nabla h(x)$. Then we have

$$\begin{aligned}\operatorname{div} Pf(x) &= \operatorname{div} f(x) - \Delta h(x) = 0 \qquad \text{in } \Omega, \\ n \cdot \gamma_\Gamma Pf(x) &= n \cdot \gamma_\Gamma \big(f(x) - \nabla h(x)\big) = 0 \text{ on } \Gamma.\end{aligned}$$

It follows that $(1-P)f \in \dot{G}^s_p(\Omega)$ and $Pf \in \dot{H}^s_{p,\sigma}(\Omega)$ for $p \in (1,\infty)$, and $(1-P)f \in \dot{G}^s_{p,p}(\Omega)$ and $Pf \in \dot{B}^s_{p,p,\sigma}(\Omega)$ for $p = 1, \infty$. Moreover, the mapping P is continuous. Assertion (5) follows immediately from the fact that the construction above is independent of p and s. Hence, applying Lemma 4.1 we conclude Assertions (1) and (3).

Next, assume that $p \in (1,\infty)$ and $1/p-1 < s < 1/p$. Then we have

$$H^s_p(\Omega) = \dot{H}^s_p(\Omega) \cap H^0_p(\Omega) \text{ for } s \geq 0,$$
$$H^s_p(\Omega) = \dot{H}^s_p(\Omega) + H^0_p(\Omega) \text{ for } s \leq 0.$$

For $p = 1, \infty$ and $s \in (1/p-1, 1/p)$, we have $B^s_{p,p}(\Omega) = \dot{B}^s_{p,p}(\Omega)$ provided Ω is bounded. Hence, if $1 < p < \infty$ or if Ω is bounded, Theorem 4.2 holds also for

$$X^s_p(\Omega) = \begin{cases} B^s_{1,1}(\Omega) & \text{for } p = 1, \\ H^s_p(\Omega) & \text{for } 1 < p < \infty, \\ B^s_{\infty,\infty}(\Omega) & \text{for } p = \infty. \end{cases}$$

instead of (4.1). Applying Lemma 4.1 we conclude Assertions (2) and (4).

It remains only to show Assertion (6). For p, $q \in [1,\infty)$ and $s \in (1/p-1, 1/p)$, put $p' = p/(p-1)$ and $q' = q/(q-1)$, and consider the following settings:

$$(5.1) \quad X = \left(\dot{H}^s_p(\Omega)\right)^n, \quad Y = \left(\dot{H}^{-s}_{p'}(\Omega)\right)^n, P = \dot{P}_{p,s}, \quad Q = \dot{P}_{p',-s},$$
$$(5.2) \quad X = \left(\dot{B}^s_{p,q}(\Omega)\right)^n, Y = \left(\dot{B}^{-s}_{p'}(\Omega)\right)^n, P = \dot{P}_{p,q,s}, Q = \dot{P}_{p',q',-s},$$
$$(5.3) \quad X = \left(H^s_p(\Omega)\right)^n, \quad Y = \left(H^{-s}_{p'}(\Omega)\right)^n, \quad P = P_{p,s}, \quad Q = P_{p',-s},$$
$$(5.4) \quad X = \left(B^s_{p,q}(\Omega)\right)^n, \quad Y = \left(B^{-s}_{p',q'}(\Omega)\right)^n, P = P_{p,q,s}, Q = P_{p',q',-s}.$$

Here we assume $p > 1$ in (5.1) and (5.3), and we assume $p > 1$ or Ω is bounded in (5.4). It suffices to show that $Q = P'$ in each of the settings above. Namely, it suffices to show the identity $\langle f, Qg\rangle_\Omega = \langle Pf, g\rangle_\Omega$ for every $f \in X$ and every $g \in Y$.

Suppose that $f \in (1-P)X$ and $g \in QY$. Then we have $\operatorname{div} g = 0$ and $n \cdot \gamma_\Gamma g = 0$. On the other hand, let $\{f_j\}_{j=1}^\infty$ be a sequence in $\left(C_0^\infty(\Omega)\right)^n$ such that $f_j \to f$ in X as $j \to \infty$. Let h_j be the solution of the problem (4.2)–(4.3) with f replaced by f_j. Then we have $\nabla h_j = (1-P)f_j \to (1-P)f = f$ in X. It follows that

$$\langle f, g\rangle_\Omega = \lim_{j\to\infty} \langle \nabla h_j, g\rangle_\Omega = \lim_{j\to\infty} \left(\langle h_j, n\cdot\gamma_\Gamma g\rangle_\Gamma - \langle h_j, \operatorname{div} g\rangle_\Omega\right) = 0,$$

where the integration by parts is justified because h_j can be approximated by functions in $C_0^\infty(\mathbb{R}^n)$ in $\tilde{X}$, where

$$\tilde{X} = \begin{cases} \dot{H}_p^{s+1}(\mathbb{R}^n) & \text{in the case (5.1)}, \\ \dot{B}_{p,q}^{s+1}(\mathbb{R}^n) & \text{in the case (5.2)}, \\ H_p^{s+1}(\mathbb{R}^n) & \text{in the case (5.3)}, \\ B_{p,q}^{s+1}(\mathbb{R}^n) & \text{in the case (5.4)}. \end{cases}$$

Hence

$$\langle (1-P)f, Qg\rangle_\Omega = 0 \text{ for every } f \in X, g \in Y. \tag{5.5}$$

Next, suppose that $f \in PX$ and $g \in (1-Q)Y$. Then we can write $g = \nabla\psi$ with some ψ. Next, let $\{f_j\}_{j=1}^\infty$ be a sequence in $\left(C_0^\infty(\Omega)\right)^n$ such that $f_j \to f$ in X as $j \to \infty$. Let h_j be the solution of the problem (4.2)–(4.3) with f replaced by f_j, and put $\varphi_j = Pf_j = f_j - \nabla h_j \in X$. Then we have $\varphi_j \in PX$ and $\varphi_j \to Pf = f$ in X as $j \to \infty$. This implies $\operatorname{div}\varphi_j \to \operatorname{div} f = 0$ and $n\cdot\gamma_\Gamma\varphi_j \to n\cdot\gamma_\Gamma f = 0$ as $j \to \infty$. It follows that

$$\langle f, g\rangle_\Omega = \lim_{j\to\infty}\langle \varphi_j, \nabla\psi\rangle_\Omega = \lim_{j\to\infty}\left(\langle n\cdot\gamma_\Gamma\varphi_j, \psi\rangle_\Gamma - \langle \operatorname{div}\varphi_j, \psi\rangle_\Omega\right) = 0,$$

where the integration by parts is justified since φ_j can be approximated by functions in $C_0^\infty(\mathbb{R}^n)$ in X. Hence

$$\langle Pf, (1-Q)g\rangle_\Omega = 0 \text{ for every } f \in X, g \in Y. \tag{5.6}$$

The identities (5.5) and (5.6) imply the identity $\langle Pf, g\rangle_\Omega = \langle f, Qg\rangle_\Omega$ for every $f \in X$ and $g \in Y$, as required. This completes the proof of Assertion (6).

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