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On Horn-Kapranov uniformisation of the discriminantal loci

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Abstract.

In this note we give a rational uniformisation equation of the discriminant loci associated to a non-degenerate affine complete intersection variety. To show this formula we establish a relation of the fibre-integral with the hypergeometric function of Horn and that of Gel'fand-Kapranov-Zelevinski.

§0. Introduction

In this note we give a concrete rational uniformisation equation for the discriminantal loci of non-degenerate affine complete intersection depending on deformation parameters.

First of all, let us fix the situation. For the complex varieties $X = \mathbf{C}^{\times N}$ and $S = \mathbf{C}^k$, we consider the mapping,

$$f : X \to S \tag{0.1}$$

such that $X_s := \{(x_1, \ldots, x_N) \in X; f_1(x) + s_1 = 0, \ldots, f_k(x) + s_k = 0\}$. Let $f_1(x), \ldots, f_k(x)$ be polynomials that define a non-degenerate complete intersection (CI) in the sense of Danilov-Khovanski [3] with the following specific form:

$$f_\ell(x) = x^{\vec{\alpha}_{1,\ell}} + \cdots + x^{\vec{\alpha}_{\tau_\ell,\ell}},\ 1 \leq \ell \leq k, \tag{0.2}$$

where $\vec{\alpha}_{i,\ell} \in (\mathbf{Z}_{\geq 0})^N$. Let n be the dimension of the variety X_0, $\dim X_0 = n \geq 0$. $W_s := \{(x_1, \ldots, x_N, y_1, \ldots, y_k) \in X \times (\mathbf{C})^k; y_1(f_1(x)$

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$+s_1) + \cdots + y_k(f_k(x) + s_k) = 0\}$. Then it is known that the discriminantal loci of X_s coincides with that of W_s. That is to say, the study of the discriminantal loci of a CI can be reduced to that of an hypersurface associated with the original CI in a special manner. This fact has been discovered by Arthur Cayley [5] and thus the method to reduce the geometric study of a CI to that of a hypersurface is named "Cayley trick" in general, even in contexts apart from the study of discriminantal loci (e.g. the description of the mixed Hodge structure of the former by means of the latter given by T. Terasoma, A. Mavlyutov [9] and others). Here we return to the initial spirit of Cayley who treated the question of the discriminantal loci.

The main idea is based on that of the paper [6] which states that the singular loci of the linear differential operators annihilating the fibre integrals of X_s coincide with the discriminantal loci of X_s. In the modern terminology of the A-hypergeometric functions (HGF), it is equivalent to say that A-discriminantal loci are singular loci for generalized A-HGF. This fact has been proven in [7] and we give a more precise description of the discriminantal loci by means of combinatorial data of the polynomial mapping f and the toric geometry of W_s (see Theorem 2.6).

Let us review the contents of the note in short. In §1 we recall some basic facts on the Cayley trick and Néron-Severi torus. In §2, we calculate the Mellin transform of the fibre integral in an explicit manner. Using a representation of the Mellin transform we show that fibre integral satisfies the Horn type system of differential equations (Theorem 2.4). From this expression of the Horn type system, we get the discriminantal loci as the boundary of a convergence domain of solutions to the system. In §3, we show that the fibre integral calculated in §2 is nothing but the quotient of the Gel'fand-Kapranov-Zelevinski generalized hypergeometric function (HGF) by the torus action. In §4 we give two computational examples: discriminantal loci for the D_4 type singularity and the simplest non-quasihomogeneous complete intersection.

Finally we remark that this note is an abridged version of some parts from [13] where one can find more details.

§1. Cayley trick and Néron-Severi torus

Throughout this section we keep the notation of §0. Further we introduce the following notations. Let $\mathbf{T}^m = (\mathbf{C}\setminus\{0\})^m = (\mathbf{C}^\times)^m$ be the complex algebraic torus of dimension m. We denote by $x^{\mathbf{i}}$ the monomial $x^{\mathbf{i}} := x_1^{i_1} \cdots x_N^{i_N}$ with multi-index $\mathbf{i} = (i_1, \ldots, i_N) \in \mathbf{Z}^N$, and by dx the $N-$volume form $dx := dx_1 \wedge \cdots \wedge dx_N$. We shall also use the notations $x^{\mathbf{1}} := x_1 \cdots x_N$, $y^\zeta = y_1^{\zeta_1} \cdots y_k^{\zeta_k}$, $s^{\mathbf{z}} = s_1^{z_1} \cdots s_k^{z_k}$ and $ds = ds_1 \wedge \cdots \wedge ds_k$

and their analogies for each variable. In this section we consider an extension of the mapping f to that defined from $\mathbf{P}_{\tilde{\Sigma}}$ to $\mathbf{C}^k$. We follow the construction by [2] and [9]. Let us define M as the dimension of a minimal ambient space so that we can quasihomogenize simultaneously the polynomials $(f_1(x), \dots, f_k(x))$ by multiplying certain terms by new variables:

$$x^{\mathbf{i}} \longmapsto x'_j x^{\mathbf{i}}, \quad j = 1, 2, \dots .$$

Let us denote by $(f_1(x, x'), \dots, f_k(x, x'))$ the new polynomials obtained in such a way. These polynomials are quasi-homogeneous with respect to certain weight system i.e. there exists a set of positive integers $(w_1, \dots, w_N, w'_1, \dots, w'_{M-N})$ such that their G.C.D. equals 1 and the following relation holds:

$$E(x,x')(f_\ell(x,x')) = p_\ell f_\ell(x,x') \quad \text{for} \quad \ell = 1, \dots, k,$$

where p_ℓ is some positive integer and

$$E(x, x') = \sum_{i=1}^{N} w_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{M-N} w'_j x'_j \frac{\partial}{\partial x'_j}, \tag{1.1}$$

E an Euler vector field.

Example. We modify the polynomial $f(x) = x_1^a + x_1x_2 + x_2^b$, with $a, b > 2$, $\mathrm{GCD}(a, b) = 1$, in adding a new variable x'_1 so that the new polynomial $f(x, x') = x_1^a + x'_1x_1x_2 + x_2^b$, becomes quasihomogeneous with respect to the weight system $(b, a, ab - a - b)$.

In general there are of course many choices of terms that we modify to realize the quasihomogeneiety.

From now on we will use the notation $X := (X_1, \dots, X_M) := (x_1, \dots, x_N, x'_1, \dots, x'_{M-N})$ and that of the polynomial $f_\ell(X) := f_\ell(x, x')$. If we introduce the Euler vector field,

$$E(X') = \sum_{i=1}^{N} w_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{M-N} w'_j x'_j \frac{\partial}{\partial x'_j} + X_{M+1} \frac{\partial}{\partial X_{M+1}},$$

we have the following relation:

$$E(X')(f_\ell(X) + X_{M+1}^{p_\ell} s_\ell) = p_\ell(f_\ell(X) + X_{M+1}^{p_\ell} s_\ell) \quad \text{for} \quad \ell = 1, \dots, k.$$

From now on we denote $X' := (X, X_{M+1})$. Let $\mathbf{M_Z}$ be an integer lattice of rang N and $\mathbf{N_Z}$ be its dual, $\mathbf{N_Z} = \mathrm{Hom}(\mathbf{M_Z}, \mathbf{Z})$. We denote by $\mathbf{M_R}$ (resp. $\mathbf{N_R}$) the natural extension of $\mathbf{M_Z}$ (resp. $\mathbf{N_Z}$) to its real space. Let us take $\vec{e}_1, \dots, \vec{e}_{M+1}$ a set of generators of one dimensional cones such that $\sum_{\ell=1}^{M+1} \mathbf{R}\vec{e}_\ell = \mathbf{N_R}$. We can define a simplicial fan Σ in $\mathbf{N_R}$ as a set of simplicial cones spanned by the above $\vec{e}_1, \dots, \vec{e}_{M+1}$. Our construction of the Euler vector field $E(X')$ correspond to the superstructure $\mathbf{N_R} \times \mathbf{N}'_\mathbf{R}$ with a basis of generators $\vec{\bar{e}}_{N+1}, \dots \vec{\bar{e}}_{M+1}$ such that

$$\sum_{i=1}^{N} w_i \vec{\bar{e}}_i + \sum_{j=1}^{M-N} w'_j \vec{\bar{e}}_j + \vec{\bar{e}}_{M+1} = 0.$$

Here we have $p_\mathbf{N}(\vec{\bar{e}}_j) = \vec{e}_j$ for the projection $p_\mathbf{N}\colon \mathbf{N_R} \times \mathbf{N}'_\mathbf{R} \to \mathbf{N_R}$. While the dimension of the vector space $\mathbf{N_R} \times \mathbf{N}'_\mathbf{R}$ must be minimal i.e. $\dim(\mathbf{N_R} \times \mathbf{N}'_\mathbf{R}) = M$.

We introduce a polynomial,

$$(1.2) \quad H(x, y) := y_1 f_1(x) + \cdots + y_k f_k(x) \in \mathbf{Z}[x_1, \dots, x_N, y_1, \dots, y_k],$$

in adding new variables $y_1, \dots, y_k$. Let $\vec{n}_1, \dots, \vec{n}_{M+k}$ be the elements of the set $\operatorname{supp}(H(x, y)) \subset \mathbf{Z}^{N+k}$. We define a simplicial rational fan $\tilde{\Sigma}$ in $\mathbf{R}^{N+k}$ as a set of simplicial cones generated by $\vec{n}_1, \dots, \vec{n}_{M+k}$. We consider the injective homomorphism

$$\varphi\colon \tilde{\mathbf{M}}_\mathbf{Z} \to \mathbf{Z}^{M+k},$$

for $\tilde{\mathbf{M}}_\mathbf{Z} = \mathbf{M_Z} \times \mathbf{Z}^k$, defined by

$$\varphi(\vec{\tilde{m}}) = (\langle \vec{\tilde{m}}, \vec{n}_1 \rangle, \dots, \langle \vec{\tilde{m}}, \vec{n}_{M+k} \rangle).$$

The cokernel of this mapping is a free abelian group,

$$Cl(\tilde{\Sigma}) = \mathbf{Z}^{M+k} / \varphi(\tilde{\mathbf{M}}_\mathbf{Z})$$

for which the following group can be defined

$$(1.3) \qquad \mathbf{D}(\tilde{\Sigma}) := \operatorname{Spec} \mathbf{C}[Cl(\tilde{\Sigma})].$$

As a matter of fact this group $\mathbf{D}(\tilde{\Sigma})$ is isomorphic to an algebraic torus $\mathbf{T}^{M-N}$. One can define the toric variety $\mathbf{P}_{\tilde{\Sigma}}$ associated to the affine space,

$$\mathbf{A}^{M+k} = \operatorname{Spec} \mathbf{C}[X_1, \dots, X_M, y_1, \dots, y_k].$$

To this end we proceed following way after the method initiated by M. Audin. Let $\hat{X}_\sigma := \prod_{1 \le i \le M,\, \vec{n}_i \notin \sigma} X_i \prod_{1 \le j \le k,\, \vec{n}_{M+j} \notin \sigma} y_j$, be a monomial defining a coordinate plane and the ideal

$$B(\tilde{\Sigma}) = \langle \hat{X}_\sigma; \sigma \in \tilde{\Sigma} \rangle \subset \mathbf{C}[X_1, \dots, X_M, y_1, \dots, y_k].$$

Let $Z(\tilde{\Sigma}) := \mathbf{V}(B(\tilde{\Sigma})) \subset \mathbf{A}^{M+k}$ be the variety defined by the ideal $B(\tilde{\Sigma})$. We construct the toric variety $\mathbf{P}_{\tilde{\Sigma}}$ as the quotient of $U(\tilde{\Sigma}) := \mathbf{A}^{M+k} \setminus Z(\tilde{\Sigma})$ by the group action $\mathbf{D}(\tilde{\Sigma})$:

$$\mathbf{P}_{\tilde{\Sigma}} = U(\tilde{\Sigma})/\mathbf{D}(\tilde{\Sigma}),$$

with $\dim \mathbf{D}(\tilde{\Sigma}) = M - N$, $\dim U(\tilde{\Sigma}) = M + k$.

Definition 1. This group $\mathbf{D}(\tilde{\Sigma}) \cong \mathbf{T}^{M-N}$ is called the Néron-Severi torus associated to the fan $\tilde{\Sigma}$.

We introduce the following polynomial (named phase function below),

$$F(X, s, y) := y_1(f_1(X) + s_1) + \cdots + y_k(f_k(X) + s_k), \tag{1.4}$$

that will play essential rôle in our further studies. In §3, we treat the following affine variety defined for (1.4):

$$Z_{F(x,\mathbf{1},\mathbf{1},y)+1} = \{(x, y) \in \mathbf{T}^{N+k}; F(x, \mathbf{1}, \mathbf{1}, y) + 1 = 0\}. \tag{1.5}$$

Further on we shall prepare several lemmata on combinatorics which are useful for the derivation of the discriminant loci equation. We denote by L the number of monomials in (X, s, y) that take part in the phase function (1.4) for (0.2). That is to say $L = \sum_{q=1}^{k}(\tau_q + 1)$ Here we introduce new variables $(T_1, \ldots, T_L) \in \mathbf{T}^L$ that satisfy the following relations,

$$T_1 = y_1 x^{\vec{\alpha}_{1,1}}, T_2 = y_1 x^{\vec{\alpha}_{2,1}}, \ldots, T_L = y_k s_k. \tag{1.6}$$

Each T_q represents the $q-$th monomial present in $F(x, \mathbf{1}, s, y)$ (see (2.3) below). We will use the following matrix $\mathsf{M}(A)$ whose column is a vertex of the Newton polyhedron $\Delta(F(x, \mathbf{1}, \mathbf{1}, y))$,

(1.7) $\mathsf{M}(A)$

$$:= \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \alpha_{111} & \cdots & \alpha_{\tau_1 11} & 0 & \alpha_{121} & \cdots & \alpha_{\tau_2 21} & 0 & \cdots & 0 & \alpha_{1k1} & \cdots & \alpha_{\tau_k k1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{11N} & \cdots & \alpha_{\tau_1 1N} & 0 & \alpha_{12N} & \cdots & \alpha_{\tau_2 2N} & 0 & \cdots & 0 & \alpha_{1kN} & \cdots & \alpha_{\tau_k kN} \end{bmatrix}$$

Further we assume that $\operatorname{rank}(\mathsf{M}(A)) = k + N$. We always assume the inequality $N + 2k \leq L$ for (0.2).

In this situation we can define a non-negative integer m as the minimal number of variables

$$x'' = (x'_1, \dots, x'_m) \tag{1.8}$$

to make the number of variables present in the expression (1.4) equal to L. That is to say $L = N + m + 2k$. For example, the relation (1.6) may be modified into the following form:

$$(1.6)' \quad T_1 = y_1 x'_1 x^{\vec{\alpha}_{1,1}},\ T_2 = y_1 x'_2 x^{\vec{\alpha}_{2,1}}, \cdots, \\ T_{L-1} = y_k x'_m x^{\vec{\alpha}_{\tau_k,k}},\ T_L = y_k s_k.$$

In other words, proper addition of new variables $x'' = (x'_1, \dots, x'_m)$ to $f_1(x), \dots, f_k(x)$ makes the polynomial $F(X, 0, y)$ quasihomogeneous. In this way we have

$$M = N + m. \tag{1.9}$$

Further we shall consider a simple parametrisation of the variety

$$Z_{F(X,s,y)} = \{(X, y) \in \mathbf{T}^{M+k}; F(X, s, y) = 0\}. \tag{1.10}$$

Namely we denote,

$$(1.11) \quad \Xi := {}^t(x_1, \dots, x_N, x'_1, \dots, x'_m, s_1, \dots, s_k, y_1, \dots, y_k),$$
$$(1.12) \quad \mathrm{Log}\, T := {}^t(\log T_1, \dots, \log T_L)$$
$$(1.13) \quad \mathrm{Log}\, \Xi := {}^t(\log x_1, \dots, \log x_N, \log x'_1, \dots, \log x'_m, \\ \log s_1, \dots, \log s_k, \log y_1, \cdots, \log y_k).$$

Then we have, for example, a linear equation equivalent to $(1.6)'$ that can be written down as follows,

$$(1.14) \quad \begin{aligned} \log T_1 &= \log y_1 + \log x'_1 + \langle \vec{\alpha}_{1,1}, \log x \rangle, \\ \log T_2 &= \log y_1 + \log x'_2 + \langle \vec{\alpha}_{2,1} \log x \rangle, \\ &\ \vdots \\ \log T_{L-1} &= \log y_k + \log x'_m + \langle \vec{\alpha}_{\tau_k,k}, \log x \rangle, \\ \log T_L &= \log y_k + \log s_k. \end{aligned}$$

Let us write down the relation between (1.12) and (1.13) by means of a matrix $\mathsf{L} \in \mathrm{End}(\mathbf{Z}^L)$,

$$\mathrm{Log}\, T = \mathsf{L} \cdot \mathrm{Log}\, X. \tag{1.15}$$

Below the columns $\vec{v}_i$ (resp. $\vec{w}_i$) of the matrix L (resp. L^{-1}) shall always be ordered in accordance with (1.11), (1.12), (1.13) unless otherwise stated.

For the polynomial mapping (0.2), the choice of monomials to be modified by supplementary variables is a bit delicate. Namely, we have to observe the following rules to avoid the degeneracy of the matrix L of the relation (1.15).

Lemma 1.1. *For* (0.2) *and* (1.8), *we get a non-degenerate matrix* L *if we observe the following rules*:

a. *For the fixed index* $q \in \{1, \ldots, k\}$, *it is necessary to choose at least one of monomials* $x^{\vec{\alpha}_{i,q}}$, $1 \leq i \leq \tau_q$ *that remains without modification.*

b. *For the fixed index* $j \in \{1, \ldots, N\}$ *it is necessary to choose at least one of monomials* $x^{\vec{\alpha}_{r,i}}$ *such that* $\alpha_{r,i,j} \neq 0$, $1 \leq i \leq k$, $1 \leq r \leq \tau_i$, *that remains without modification.*

We recall here the notion of non-degenerate hypersurface,

Definition 2. The hypersurface defined by a polynomial $g(x) = \sum_{\alpha \in \mathrm{supp}(g)} g_\alpha x^\alpha \in \mathbf{C}[x_1, \ldots, x_n]$ is said to be non-degenerate if and only if for any $\xi \in \mathbf{R}^n$ the following inclusion takes place,

$$\left\{x \in \mathbf{C}^n; x_1 \frac{\partial g^\xi}{\partial x_1} = \cdots = x_n \frac{\partial g^\xi}{\partial x_n} = 0\right\} \subset \{x \in \mathbf{C}^n; x_1 \cdots x_n = 0\}$$

where $g^\xi(x) = \sum_{\{\beta; \langle \beta, \xi \rangle \leq \langle \alpha, \xi \rangle, \text{ for all } \alpha \in \mathrm{supp}(g)\}} g_\alpha x^\alpha$. We call the CI X_0 for (0.2) non-degenerate if the hypersurface $Z_{F(x,1,0,y)+1}$ is non-degenerate.

The following is an easy consequence of the above Definition.

Proposition 1.2. *If the matrix* L *is non-degenerate, the hypersurface* $Z_{F(x,1,0,y)+1}$ *and the CI* X_0 *are non-degenerate in the sense of the* Definition 2.

§2. Horn's hypergeometric functions

From this section, we change the name of variables $x'' = (x'_1, \ldots, x'_m)$ into $s' := (s'_1, \ldots, s'_m)$. We use both of the notations $X = (x, x") = (x, s')$.

Let us consider the Leray's coboundary (see [14]) to define the fibre integral, $\gamma \subset H_N(\mathbf{T}^N \setminus \cup_{i=1}^k \{x \in \mathbf{T}^N: f_i(X) + s_i = 0\})$ such that $\Re(f_i(X) + s_i)|_\gamma < 0$. Further on central object of our study is the following fibre integral,

(2.1)
$$I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\, s') = \int_{\gamma} (f_1(x,\, s') + s_1)^{-\zeta_1 - 1} \cdots (f_k(x,\, s') + s_k)^{-\zeta_k - 1} x^{\mathbf{i}+\mathbf{1}} \frac{dx}{x^{\mathbf{1}}},$$

and its Mellin transform,

$$M^{\zeta}_{\mathbf{i},\gamma}(\mathbf{z},\, \mathbf{z}') := \int_{\Pi} s^{\mathbf{z}} s'^{\mathbf{z}'} I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\, s') \frac{ds}{s^{\mathbf{1}}} \wedge \frac{ds'}{s'^{\mathbf{1}}}, \tag{2.2}$$

for certain cycle Π homologous to $\mathbf{R}^{m+k}$ which avoids the singular loci of $I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\, s')$ (cf. [11]). After Definition 1 above, we understand that $s' \in \mathbf{D}(\tilde{\Sigma})$ is a variable on the Néron-Severi torus. Thus the fibre integral $I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\, s')$ is a ramified function on the torus $\mathbf{D}(\tilde{\Sigma}) \times \mathbf{T}^k$. It is useful to understand the calculus of the Mellin transform in connection with the notion of the generalized HGF in the sense of Mellin-Barnes-Pincherle [1], [10]. After this formulation, the classical HGF of Gauss can be expressed by means of the integral,

$${}_2F_1(\alpha,\, \beta,\, \gamma | s) = \frac{1}{2\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} (-s)^z \frac{\Gamma(z+\alpha)\Gamma(z+\beta)\Gamma(-z)}{\Gamma(z+\gamma)} dz, \qquad -\Re\alpha,\, -\Re\beta < z_0.$$

Next we modify the Mellin transform

$$\begin{aligned}
&M^{\zeta}_{\mathbf{i},\gamma}(\mathbf{z},\, \mathbf{z}') \\
&= c(\zeta) \int_{S_+^{k-1}(w'') \times \gamma^{\Pi}} \frac{x^{\mathbf{i}} \omega^{\zeta} s^{\mathbf{z}-\mathbf{1}} s'^{\mathbf{z}'-\mathbf{1}} dx \wedge \Omega_0(\omega) \wedge ds \wedge ds'}{(\omega_1(f_1(X) + s_1) + \cdots + \omega_k(f_k(X) + s_k))^{\zeta_1 + \cdots + \zeta_k + k}} \\
&= c(\zeta) \int_{\mathbf{R}_+} \sigma^{\zeta_1 + \cdots + \zeta_k + k} \frac{d\sigma}{\sigma} \int_{S_+^{k-1}(w'')} \omega^{\zeta} \Omega_0(\omega) \\
&\qquad \int_{\gamma} x^{\mathbf{i}} dx \int_{\Pi} s^{\mathbf{z}} s'^{\mathbf{z}'} e^{\sigma(\omega_1(f_1(X)+s_1) + \cdots + \omega_k(f_k(X)+s_k))} \frac{ds}{s^{\mathbf{1}}} \frac{ds'}{s'^{\mathbf{1}}},
\end{aligned}$$

with $c(\zeta) = \Gamma(\zeta_1 + \cdots + \zeta_k + k)/\big(\Gamma(\zeta_1 + 1) \cdots \Gamma(\zeta_k + 1)\big)$. Here we made use of notations

$$S_+^{k-1}(w'') = \Big\{ (\omega_1,\, \ldots,\, \omega_k) \colon \omega_1^{\mathbf{w}''/w_1''} + \cdots + \omega_k^{\mathbf{w}''/w_k''} = 1,\, \omega_\ell > 0 \quad \text{for all} \quad \ell,\, \mathbf{w}'' = \prod_{1 \le i \le k} w_i'' \Big\}$$

and $\Omega_0(\omega)$ the $(k-1)$ volume form on $S_+^{k-1}(w'')$,

$$\Omega_0(\omega) = \sum_{\ell=1}^{k}(-1)^{\ell} w''_{\ell}\omega_{\ell} d\omega_1 \wedge \overset{\ell}{\overset{\vee}{\cdots}} \wedge d\omega_k.$$

In the above transformation we used a classical interpretation of Dirac's delta function as a residue:

$$\int_{\gamma}\int_{\mathbf{R}_+} e^{y_j(f_j(X)+s_j)} y_j^{\zeta_j} dy_j \wedge dx = \Gamma(\zeta_j+1)\int_{\gamma}(f_j(X)+s_j)^{-\zeta_j-1}dx.$$

We introduce the notation $\gamma^{\Pi} := \cup_{(s,\, s')\in\Pi}((s,\, s'),\, \gamma)$. One shall not confuse it with the thimble of Lefschetz, because γ^{Π} is rather a tube without thimble. We will rewrite the last expression,

$$\int_{(\mathbf{R}_+)^k\times\gamma^{\Pi}} e^{\Psi(T)} x^{\mathbf{i}+\mathbf{1}} y^{\zeta+\mathbf{1}} s^{\mathbf{z}} s'^{\mathbf{z}'} \frac{dx}{x^{\mathbf{1}}}\wedge\frac{dy}{y^{\mathbf{1}}}\wedge\frac{ds}{s^{\mathbf{1}}}\wedge\frac{ds'}{s'^{\mathbf{1}}}$$

where

$$\Psi(T) = T_1(X,s,y) + \cdots + T_L(X,s,y) = F(X,s,y), \tag{2.3}$$

in which each term $T_i(X,s,y)$ stands for a monomial in variables (X,s,y) of the phase function (1.4). We transform the above integral into the following form,

$$\begin{aligned}
&\int_{(\mathbf{R}_+)^k\times\gamma^{\Pi}} e^{\Psi(T(X,\,s,\,y))} x^{\mathbf{i}+\mathbf{1}} s^{\mathbf{z}} s'^{\mathbf{z}'} y^{\zeta+\mathbf{1}} \frac{dx}{x^{\mathbf{1}}}\wedge\frac{dy}{y^{\mathbf{1}}}\wedge\frac{ds}{s^{\mathbf{1}}}\wedge\frac{ds'}{s'^{\mathbf{1}}}\\
&= (\det \mathsf{L})^{-1}\int_{\mathsf{L}_*(\mathbf{R}_+{}^k\times\gamma^{\Pi})} e^{\sum_{a\in I}T_a}\prod_{a\in I}T_a^{\mathcal{L}_a(\mathbf{i},\,\mathbf{z},\,\mathbf{z}',\,\zeta)}\bigwedge_{a\in I}\frac{dT_a}{T_a}\\
&= (-1)^{\zeta_1+\cdots+\zeta_k+k}(\det \mathsf{L})^{-1}\\
&\quad\cdot\int_{-\mathsf{L}_*(\mathbf{R}_+{}^k\times\gamma^{\Pi})} e^{-\sum_{a\in I}T_a}\prod_{a\in I}T_a^{\mathcal{L}_a(\mathbf{i},\,\mathbf{z},\,\mathbf{z}',\,\zeta)}\bigwedge_{a\in I}\frac{dT_a}{T_a}.
\end{aligned}\tag{2.4}$$

Here $\mathsf{L}_*(\mathbf{R}_+{}^k\times\gamma^{\Pi})$ means the image of the chain in $\mathbf{C}_X^M\times\mathbf{C}_s^k\times\mathbf{C}_y^k$ into that in $\mathbf{C}_T^L$ induced by the transformation (1.15). We define $-\mathsf{L}_*(\mathbf{R}_+{}^k\times\gamma^{\Pi}) = \{(-T_1,\, \dots,\, -T_L)\in\mathbf{C}^L; (T_1,\, \dots,\, T_L)\in\mathsf{L}_*(\mathbf{R}_+{}^k\times\gamma^{\Pi}),\, \Re T_a < 0,\, a\in[1,\, L]\}$. The second equality of (2.4) follows from Proposition 2.1, 3) below that can be proven in a way independent of the argument to derive (2.4). We will denote the set of columns and rows of the matrix L by I,

$$I := \{1,\cdots,L\}.$$

Here we remember the relation $L = N + m + 2k = M + 2k$.

The following notion helps us to formulate the result in a compact manner.

Definition 3. A meromorphic function $g(\mathbf{z}, \mathbf{z}')$ is called Δ-periodic for $\Delta \in \mathbf{Z}_{>0}$, if

$$g(\mathbf{z}, \mathbf{z}') = h(e^{2\pi\sqrt{-1}\frac{z_1}{\Delta}}, \dots, e^{2\pi\sqrt{-1}\frac{z_k}{\Delta}}, e^{2\pi\sqrt{-1}\frac{z'_1}{\Delta}}, \dots, e^{2\pi\sqrt{-1}\frac{z'_m}{\Delta}}),$$

for some rational function $h(\zeta_1, \dots, \zeta_{k+m})$.

For the simplicial CI (0.2) (i.e. we can construct $F(X, s, y)$ for which the matrix L is non-degenerate), we have the following statement.

Proposition 2.1. 1) *For any cycle*

$$\Pi \in H_{k+m}(\mathbf{T}^{k+m} \setminus S.S.I^{\zeta}_{x^{\mathbf{i}}, \gamma}(s, s'))$$

the Mellin transform (2.1) *can be represented as a product of* $\Gamma-$ *function factors up to a* $\Delta-$*periodic function factor* $g(\mathbf{z})$,

$$M^{\zeta}_{\mathbf{i}, \gamma}(\mathbf{z}, \mathbf{z}') = g(\mathbf{z}) \prod_{a\in I} \Gamma(\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)),$$

with

$$(2.5) \quad \mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = \frac{\sum_{j=1}^{N} A^a_j(i_j + 1) + \sum_{j=1}^{m} C^a_j z'_j + \sum_{\ell=1}^{k} (B^a_\ell z_\ell + D^a_\ell(\zeta_\ell + 1))}{\Delta}, \quad a \in I.$$

Here the following matrix $\Delta^{-1}\mathsf{T} = (\mathsf{L})^{-1}$ *has integer elements,*

$$(2.6) \quad {}^t\mathsf{T} = (A^a_1, \dots, A^a_N, C^a_1, \dots, C^a_m, B^a_1, \dots, B^a_k, D^a_1, \dots, D^a_k)_{1\le a\le L},$$

with $\mathrm{GCD}(A^a_1, \dots, A^a_N, C^a_1, \dots, C^a_m, B^a_1, \dots, B^a_k, D^a_1, \cdots, D^a_k) = 1$, *for all* $a \in [1, L]$. *In this way* $\Delta > 0$ *is uniquely determined. The coefficients of* (2.5) *satisfy the following properties for each index* $a \in I$:

a *Either* $\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = \frac{\Delta}{\Delta} z_\ell$, *i.e.* $A^a_1 = \cdots = A^a_N = 0$, $B^a_1 = \overset{\overset{\ell}{\vee}}{\cdots} = B^a_k = 0$, $B^a_\ell = 1$.

b *Or*

$$\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = \frac{\sum_{j=1}^{N} A^a_j(i_j + 1) + \sum_{j=1}^{m} C^a_j z'_j + \sum_{\ell=1}^{k} B^a_\ell(z_\ell - \zeta_\ell - 1)}{\Delta}$$

2) *For each fixed index* $1 \le \ell \le N$, $1 \le q \le k$, $1 \le j \le m$ *the following equalities take place*:

$$\sum_{a \in I} A^a_\ell = 0,\ \sum_{a \in I} B^a_q = 0,\ \sum_{a \in I} C^a_j = 0. \tag{2.7}$$

3) *The following relation holds among the linear functions* $\mathcal{L}_a$, $a \in I$:

$$\sum_{a \in I} \mathcal{L}_a(\mathbf{i},\, \mathbf{z},\, \mathbf{z}',\, \zeta) = \zeta_1 + \cdots + \zeta_k + k.$$

Proof. 1) First of all we recall the definition of the Γ-function,

$$\int_{C_a} e^{-T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = (1 - e^{2\pi i \sigma_a})\Gamma(\sigma_a),$$

for the unique non-trivial cycle C_a that turns around $T_a = 0$ with the asymptotes $\Re T_a \to +\infty$. We consider a transformation of the integral (2.4) induced by the change of cycle λ: $C_a \to \lambda(C_a)$ defined by the relation,

$$\int_{\lambda(C_a)} e^{-T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = \int_{C_a} e^{-T_a} (e^{2\pi\sqrt{-1}} T_a)^{\sigma_a} \frac{dT_a}{T_a}.$$

By the aid of this action the chain $\mathsf{L}_*(\mathbf{R}_+{}^k \times \gamma^{\Pi})$ turns out to be homologous to a chain,

$$\sum_{(j_1^{(\rho)},\dots,j_L^{(\rho)}) \in [1,\,\Delta]^L} m_{j_1^{(\rho)},\dots,j_L^{(\rho)}} \prod_{a=1}^{k} \lambda^{j_a^{(\rho)}}(\mathbf{R}_+) \prod_{a'=k+1}^{L} \lambda^{j_{a'}^{(\rho)}}(C_{a'}),$$

with $m_{j_1^{(\rho)},\dots,j_L^{(\rho)}} \in \mathbf{Z}$. This fact explains the appearance of the factor

$$\begin{aligned} g(\mathbf{z},\, \mathbf{z}') = & \sum_{(j_1^{(\rho)},\dots,j_L^{(\rho)}) \in [1,\,\Delta]^L} m_{j_1^{(\rho)},\dots,j_L^{(\rho)}} \\ & \cdot \prod_{a=1}^{k} e^{2\pi\sqrt{-1} j_a^{(\rho)} \mathcal{L}_a(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta)} \\ & \cdot \prod_{a'=k+1}^{L} e^{2\pi\sqrt{-1} j_{a'}^{(\rho)} \mathcal{L}_{a'}(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta)} (1 - e^{2\pi\sqrt{-1}\mathcal{L}_{a'}(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta)}) \end{aligned}$$

apart from the factors of type $\Gamma(\bullet)$.

In the sequel we analyze the $\Gamma-$ function factors that arise from the integral (2.4). To this end, we represent the matrix L (resp. L^{-1}) as a set of L columns properly ordered:

$$\mathsf{L} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_L),\ \mathsf{L}^{-1} = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_L), \quad \vec{w}_a = {}^t(w_{a,1}, \dots, w_{a,L}). \tag{2.8}$$

The interior product of vectors $(\mathbf{i}+\mathbf{1}, \mathbf{z}, \mathbf{z}', \zeta+\mathbf{1})$ and $\vec{w}_a$ defines the linear function in question:

$$\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = (\mathbf{i}+\mathbf{1}, \mathbf{z}, \mathbf{z}', \zeta+\mathbf{1}) \cdot \vec{w}_a. \tag{2.9}$$

The vector columns of L^{-1} are divided into 3 groups:

1 the columns with all formally non-zero elements.

2 with unique non-zero element $(=1)$ that produces z_i, $1 \le i \le k$ and z'_j, $1 \le j \le m$ in (2.9).

3 with the non-zero elements that produce a function linear in $\zeta+\mathbf{1}$, $\mathbf{i}+\mathbf{1}$ after (2.5).

In the further argument, only the first two groups of columns are important.

The column that corresponds to $\log s_i$ of L contains the unique non-zero element $(=1)$ at the position $\tau_1+\cdots+\tau_i+i$. Meanwhile the column of L that corresponds to the variable $\log x'_\ell$ consists also of an unique non-zero element $(=1)$ outside the positions $\tau_1+\cdots+\tau_i+i$, $(1 \le i \le k)$. Let us denote this correspondence by

$$\vec{v}_{\rho(i)} = {}^t(0, \dots, 0, \overset{\sigma(i)}{\overset{\vee}{\dots}}, 1, 0, \dots, 0),$$

that yields in L^{-1},

$$\vec{w}_{\sigma(i)} = {}^t(0, \dots, 0, \overset{\rho(i)}{\overset{\vee}{\dots}}, 1, 0, \dots, 0).$$

Here the mappings $\rho, \sigma\colon \{N+1, \dots, M+k\} \to I$ are injections that send the number of columns corresponding to the variables s, x' to the total set of indices I. We divide the columns of L^{-1} into k groups $\Lambda_1, \dots, \Lambda_k \subset I$ each of which corresponds to $\Lambda_b = \{\tau_1 + \cdots + \tau_{b-1} + b, \cdots, \tau_1 + \cdots + \tau_b + b\} \subset I$. For this group, one can claim following assertions. a) The column

$$\vec{v}_{M+k+b} = {}^t(0, \dots, 0, 0, \dots, 0, \overset{\tau_1+\cdots+\tau_{b-1}+b}{\overset{\vee}{1}}, 1, \dots, \dots, 1, \dots, \overset{\tau_1+\cdots+\tau_b+b}{\overset{\vee}{1}}, 0, \dots, 0),$$

with τ_b+1, $(1 \leq b \leq k)$ non-zero elements $(=1)$. b) For the vectors $\vec{w}_a$ of the case **1** above,

$$\sum_{a\in\Lambda_b} w_{a,j} = 0 \quad \text{if} \quad j \neq M+k+b,\ 1 \leq b \leq k, \tag{2.10}$$

and there exists another vector of the same group Λ_b that satisfies:

$$w_{\sigma(i),\,j} = \delta_{\rho(i),\,j}, \tag{2.11}$$

where $\delta_{\cdot,\,*}$ is the Kronecker delta symbol. The vector (2.11) corresponds to the group **2**.

Thus the columns of the group **2** (resp. **1**) give rise to the linear functions of the group **b** (resp. **a**).

2) The 1-st, ... , $M+k$-th vector rows of the matrix L^{-1} are orthogonal to the vectors $\vec{v}_{M+k+1}, \cdots, \vec{v}_{M+2k}$ above. This means the relations (2.7).

3) The statement can be deduced from 2). Q.E.D.

In view of the Proposition 2.1, we introduce the subsets of indices $a \in \{1, 2, \dots, M\}$ as follows.

Definition 4. The subset $I_q^+ \subset \{1, 2, \dots, k\}$ (resp. I_q^-, I_q^0) consists of the indices a such that the coefficient B_q^a of $\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)$ (2.5) is positive (resp. negative, zero). Analogously we define the subset $J_r^+ \subset \{1, 2, \dots, m\}$ (resp. J_r^-, J_r^0) that consists in such indices a that the coefficient C_r^a of $\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)$ is positive (resp. negative, zero).

To assure the convergence of the Mellin inverse transform of $M^\zeta_{\mathbf{i},\gamma}(\mathbf{z}, \mathbf{z}')$ from (2.1) in a properly chosen angular sector in the variables $(s, s') \in \mathbf{C}^{k+m}$, we shall verify that the Mellin transform $M^\zeta_{\mathbf{i},\gamma}(\mathbf{z}, \mathbf{z}')$ admits the following estimation modulo multiplication by a Δ-periodic function $g(\mathbf{z}, \mathbf{z}')$.

$$|M^\zeta_{\mathbf{i},\gamma}(\mathbf{z}, \mathbf{z}')| < C_{\mathbf{i}} \exp(-\epsilon|\operatorname{Im} z|) \quad \text{while}$$
$$\operatorname{Im} z \to \infty, \text{ in a sector of aperture} < 2\pi.$$

for certain $\epsilon > 0$,

Here we remember an elementary lemma for the integral:

$$\int_{z_0-i\infty}^{z_0+i\infty} s^z g(z) \prod_{j=1}^{\nu} \frac{\Gamma(z+\alpha_j)}{\Gamma(z+\rho_j)} dz. \tag{2.12}$$

Lemma 2.2. *If one chooses one of the following functions $g^{+}(z)$ (resp. $g^{-}(z)$) in terms of $g(\mathbf{z}, \mathbf{z}')$, then the integrand of (2.12) is exponentially decaying as $\operatorname{Im} z$ tends to ∞ within the sector $0 \leq \arg z < 2\pi$, (resp. $-\pi \leq \arg z < \pi$.)*

$$g^{\pm}(z) = 1 + e^{\pm 2\pi i \beta_\nu} \prod_{j=1}^{\nu} \frac{\sin 2\pi(z+\alpha_j)}{\sin 2\pi(z+\rho_j)},$$

with $\beta_\nu = -1 + \sum_{j=1}^{\nu}(\rho_j - \alpha_j)$

Proof. It is enough to recall

$$\prod_{j=1}^{\nu} \frac{\Gamma(x+iy+\alpha_j)}{\Gamma(x+iy+\rho_j)} \rightarrow \text{const.}|y|^{-(\beta_\nu+1)}$$

while $y \rightarrow \pm\infty$. Here we used the formula of Binet:

$$\log\Gamma(z+a) = \log\Gamma(z) + a\log z - \frac{a-a^2}{2z} + \mathcal{O}(|z|^{-2})$$

if $|z| >> 1$, The factor $|s^{-(x+iy)}| = r^{-x}e^{\theta y}$, for $s = re^{i\theta}$ gives the exponentially decreasing contribution in each cases. Q.E.D.

Let us introduce a simplified notation,

$$\mathcal{L}_j(z) = A_{j1}z_1 + A_{j2}z_2 + \cdots + A_{jk}z_k + A_{j0},\ 1 \leq j \leq p,$$
$$\mathcal{M}_j(z) = B_{j1}z_1 + B_{j2}z_2 + \cdots + B_{jk}z_k + B_{j0},\ 1 \leq j \leq r.$$

Lemma 2.3. *The sufficient conditions so that*

$$\int_{\check{\Pi}} s^{\mathbf{z}} g(z) \frac{\prod_{j=1}^{p}\Gamma(\mathcal{L}_j(z))}{\prod_{j=1}^{r}\Gamma(\mathcal{M}_j(z))} dz_1 \wedge \cdots \wedge dz_k \tag{2.13}$$

defines a polynomially increasing function with $g(z)$ a properly chosen Δ-periodic function (including the infinity ∞) are the following.

i) *For every $i > 0$*

$$\sum_{j=1}^{p} A_{j,i} = \sum_{j=1}^{r} B_{j,i}$$

ii) *The real number*

$$\alpha = \min_{z \in S^{k-1}} \left(\sum_{j=1}^{p} |\mathcal{L}_j(z) - A_{j0}| - \sum_{j=1}^{r} |\mathcal{M}_j(z) - B_{j0}| \right)$$

is non negative.

To see the exponential decay property of the integrand, one shall make reference to Nörlund's trick [10]. Further we apply the Stirling's formula on the asymptotic behaviour of the Γ-function (Whittaker-Watson, Chapter XII, Example 44).

If we apply this lemma to our integral, we see that there exists a cycle $\check{\Pi}$ such that

$$(2.14)\quad I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s') := \int_{\check{\Pi}} g(\mathbf{z},\,\mathbf{z}')\frac{\prod_{a\in I_q^+\cup I_q^0}\Gamma(\mathcal{L}_a(\mathbf{i},\,\mathbf{z},\,\mathbf{z}',\,\zeta))}{\prod_{\bar{a}\in I_q^-}\Gamma(1-\mathcal{L}_{\bar{a}}(\mathbf{i},\,\mathbf{z},\,\mathbf{z}',\,\zeta))}s^{-\mathbf{z}}s'^{-\mathbf{z}'}d\mathbf{z}\wedge d\mathbf{z}',$$

with a Δ-periodic function $g(\mathbf{z},\,\mathbf{z}')$ rational with respect to $e^{2\pi\sqrt{-1}\mathcal{L}_a(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta)}$, $a\in I$. Here we remember the relation $e^{\pi\sqrt{-1}z}\Gamma(z)\Gamma(1-z)=\pi/(1-e^{-2\pi\sqrt{-1}z})$. Thus we get the theorem on the Horn type system.

Theorem 2.4. *The integral $I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s')$ satisfies the hypergeometric system of Horn type as follows:*

$$(2.15)_1\quad L_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'}s,\,s',\,\zeta)I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s') := \left[P_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)-s_q^{\Delta}Q_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)\right]I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s')=0,\ 1\le q\le k$$

with

$$(2.15)_2\quad P_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)=\prod_{a\in I_q^+}\prod_{j=0}^{B_q^a-1}\left(\mathcal{L}_a(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right),$$

$$(2.15)_3\quad Q_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)=\prod_{\bar{a}\in I_q^-}\prod_{j=0}^{-B_q^{\bar{a}}-1}\left(\mathcal{L}_{\bar{a}}(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right),$$

where I_q^+, I_q^-, $1\le q\le k$ are the sets of indices defined in Definition 4.

$$(2.15)_4\quad L'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,s,\,s',\,\zeta)I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s') := \left[P'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)-s_r'^{\Delta}Q'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)\right]I^{\zeta}_{x^{\mathbf{i}},\gamma}(s,\,s')=0,\ 1\le q\le k$$

$$(2.15)_5\quad P'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)=\prod_{a\in J_r^+}\prod_{j=0}^{C_r^a-1}\left(\mathcal{L}_a(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right)$$

$$(2.15)_6 \qquad Q'_{r,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \prod_{\bar{a}\in J_r^-} \prod_{j=0}^{-C_r^{\bar{a}}-1} \big(\mathcal{L}_{\bar{a}}(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\big).$$

where J_r^+, J_r^-, $1 \le r \le m$ *are the sets of indices defined in the* Definition 4. *The degree of two operators* $P_{q,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)$, $Q_{q,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta)$ *are equal. Namely,*

$$(2.16) \qquad \begin{aligned} &\deg P_{q,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) \\ &= \sum_{a\in I_q^+} B_q^a = -\sum_{\bar{a}\in I_q^-} B_q^{\bar{a}} = \deg Q_{q,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta). \end{aligned}$$

Analogously,

$$\deg P'_{r,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \sum_{a\in J_r^+} C_r^a = -\sum_{\bar{a}\in J_r^-} C_r^{\bar{a}} = \deg Q'_{r,\,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta).$$

The proof is mainly based on the Proposition 2.1. To deduce (2.15) from the Mellin transform $M^{\zeta}_{\mathbf{i},\,\gamma}(\mathbf{z},\,\mathbf{z}')$ we use the following well known recurrence relation:

$$\begin{aligned} &\Gamma\Big(\frac{\alpha(n+\Delta)}{\Delta}+\zeta\Big) \\ &= \Gamma\Big(\frac{\alpha n}{\Delta}+\zeta\Big)\Big(\frac{\alpha n}{\Delta}+\zeta\Big)\Big(\frac{\alpha n}{\Delta}+1+\zeta\Big)\cdots\Big(\frac{\alpha n}{\Delta}+\alpha-1+\zeta\Big), \end{aligned}$$

if $\alpha > 0$ a positive integer.

$$\begin{aligned} &\Gamma\Big(\frac{\alpha(n+\Delta)}{\Delta}+\zeta\Big) \\ &= \Gamma\Big(\frac{\alpha n}{\Delta}+\zeta\Big)\Big(\frac{\alpha n}{\Delta}+\zeta-1\Big)^{-1}\Big(\frac{\alpha n}{\Delta}+\zeta-2\Big)^{-1}\cdots\Big(\frac{\alpha n}{\Delta}+\zeta+\alpha\Big)^{-1}, \end{aligned}$$

if $\alpha < 0$ a negative integer.

The evident compatibility (i.e. integrability) of the above system $(2.15)_*$ in the sense of Ore-Sato ([12]) can be formulated like the following cocycle condition. To state the proposition we introduce the notation $\mathbf{z}+\Delta e_r = (z_1,\,\dots,\,z_{r-1},\,z_r+\Delta,\,z_{r+1},\,\dots,\,z_k)$.

Proposition 2.5. *The rational expression*

$$(2.17) \qquad R_q(\mathbf{z},\,\mathbf{z}') = \frac{P_{q,\,\mathbf{i}}(\mathbf{z},\,\mathbf{z}',\,\zeta)}{Q_{q,\,\mathbf{i}}(\mathbf{z}+\Delta e_q,\,\mathbf{z}',\,\zeta)},$$

defined for the operators $(2.15)_2$, $(2.15)_3$ *satisfies the following relation:*

$$\begin{aligned}(2.18)\qquad & R_q(\mathbf{z}+\Delta e_r,\, \mathbf{z}')R_r(\mathbf{z},\, \mathbf{z}') \\ &= R_r(\mathbf{z}+\Delta e_q,\, \mathbf{z}')R_q(\mathbf{z},\, \mathbf{z}'), \quad q,\, r = 1,\, \dots,\, k.\end{aligned}$$

Similarly for

$$(2.19)\qquad R'_\kappa(\mathbf{z},\, \mathbf{z}') = \frac{P'_{\kappa,\,\mathbf{i}}(\mathbf{z},\, \mathbf{z}',\, \zeta)}{Q'_{\kappa,\,\mathbf{i}}(\mathbf{z},\, \mathbf{z}'+\Delta e'_\kappa,\, \zeta)},$$

satisfies the following relation:

$$\begin{aligned}(2.20)\qquad & R'_\kappa(\mathbf{z},\, \mathbf{z}'+\Delta e'_\rho)R'_\rho(\mathbf{z},\, \mathbf{z}') \\ &= R'_\rho(\mathbf{z},\, \mathbf{z}'+\Delta e'_\kappa)R'_\kappa(\mathbf{z},\, \mathbf{z}'), \quad \kappa,\, \rho = 1,\, \dots,\, m.\end{aligned}$$

Remark 1. *As $m = \dim \mathbf{D}(\tilde{\Sigma})$ (see (1.3)), one can consider that the above system $(2.15)_*$ is defined on $\mathbf{T}^k \times \mathbf{D}(\tilde{\Sigma})$ for $\mathbf{D}(\tilde{\Sigma})$: the Néron-Severi torus associated to the fan $\tilde{\Sigma}$.*

We introduce here the main object of our study: the discriminantal loci of the CI defined by the polynomials $f_1(x,\, s')+s_1,\, \dots,\, f_k(x, s')+s_k$.

$$(2.21)\quad D_{s,\,s'} := \left\{(s,\, s') \in \mathbf{T}^{k+m};\ \begin{matrix} f_1(x,\, s')+s_1 \\ = \cdots \\ = f_k(x,\, s')+s_k \\ = 0 \end{matrix},\quad \operatorname{rank}\begin{pmatrix} \operatorname{grad}_x f_1(x,\, s') \\ \vdots \\ \operatorname{grad}_x f_k(x,\, s') \end{pmatrix} < k, \quad \text{for certain } x \in \mathbf{T}^N\right\}.$$

As it is easy to see [5], $D_{s,\,s'}$ coincides with the discriminantal loci of $F(x,\, s',\, s,\, y)$.

Let us define the Δ-th roots of rational functions associated with the linear functions (2.5) as follows.

$$(2.22)\quad \psi_q(\mathbf{z},\, \mathbf{z}') = \left(\frac{\prod_{a\in I_q^+}(\sum_{\ell=1}^k B_\ell^a z_\ell + \sum_{j=1}^m C_j^a z'_j)^{B_q^a}}{\prod_{\bar{a}\in I_q^-}(\sum_{\ell=1}^k B_\ell^{\bar{a}} z_\ell + \sum_{j=1}^m C_j^{\bar{a}} z'_j)^{-B_q^{\bar{a}}}}\right)^{\frac{1}{\Delta}}, \quad 1 \le q \le k,$$

$$(2.23)\quad \phi_r(\mathbf{z},\, \mathbf{z}') = \left(\frac{\prod_{a\in J_r^+}(\sum_{\ell=1}^k B_\ell^a z_\ell + \sum_{j=1}^m C_j^a z'_j)^{C_r^a}}{\prod_{\bar{a}\in J_r^-}(\sum_{\ell=1}^k B_\ell^{\bar{a}} z_\ell + \sum_{j=1}^m C_j^{\bar{a}} z'_j)^{-C_r^{\bar{a}}}}\right)^{\frac{1}{\Delta}}, \quad 1 \le r \le m.$$

$$\begin{aligned}(2.24)\quad & h\colon \mathbf{C}^{k+m}\setminus\{0\} \to (\mathbf{C}^\times)^{k+m}, \\ & (\mathbf{z},\, \mathbf{z}') \to (\psi_1(\mathbf{z},\, \mathbf{z}'),\, \dots,\, \psi_k(\mathbf{z},\, \mathbf{z}'),\, \phi_1(\mathbf{z},\, \mathbf{z}'),\, \dots,\, \phi_m(\mathbf{z},\, \mathbf{z}')).\end{aligned}$$

By virtue of the property (2.7), the rational function $\psi_q(\mathbf{z}, \mathbf{z}')^\Delta$ (resp. $\phi_r(\mathbf{z}, \mathbf{z}')^\Delta$) is of weight zero with respect to the variables $(\mathbf{z}, \mathbf{z}')$ and thus it is possible to consider the mapping h defined on $\mathbf{C}P^{k+m-1}$ instead of $\mathbf{C}^{k+m}$.

Let $\Delta_f(s, s')$ be a polynomial that defines the discriminantal loci $D_{s, s'}$ without multiplicity.

Theorem 2.6. *The image of $h\colon \mathbf{CP}^{k+m-1} \to (\mathbf{C}^\times)^{k+m}$ is identified with the discriminantal loci $D_{s, s'}$ if we choose a proper Δ-th branch in the equations* (2.2), (2.3).

Proof. From the system of equations (2.15) we see that $D_{s,s'}$ is contained in the set:

$$\begin{aligned}(2.25)\qquad \nabla_{s, s'} := \{(s, s') \in \mathbf{T}^{k+m};&\\ &\sigma(L_{q,-1})(s\xi, s'\xi', s, s', -1) = 0,\ 1 \le q \le k,\\ &\sigma(L'_{r,-1})(s\xi, s'\xi', s, s', -1) = 0,\ 1 \le r \le m\\ &\qquad\qquad\text{for some}(\xi, \xi') \in \mathbf{T}^{k+m}\}.\end{aligned}$$

here we use the notation

$$(s\xi, s'\xi') = (s_1\xi_1, \ldots, s_k\xi_k, s'_1\xi'_1, \ldots, s'_m\xi'_m).$$

The existence of $(\xi, \xi') \in \mathbf{T}^{k+m}$ in (2.25) is equivalent to the existence of $(\mathbf{z}, \mathbf{z}') = (s\xi, s'\xi') \in \mathbf{T}^{k+m}$. Thus the set $\nabla_{s, s'}$ admits a representation,

$$\left\{ (s, s') \in \mathbf{T}^{k+m};\ \begin{aligned} s_q^\Delta &= \frac{P_{q,-1}(\mathbf{z}, \mathbf{z}', -1)}{Q_{q,-1}(\mathbf{z}, \mathbf{z}', -1)}, \quad 1 \le q \le k,\\ (s'_r)^\Delta &= \frac{P'_{r,-1}(\mathbf{z}, \mathbf{z}', -1)}{Q'_{q,-1}(\mathbf{z}, \mathbf{z}', -1)}, \quad 1 \le r \le m \end{aligned} \right\}.$$

While after Theorem 2.1, a) and Remark 2.4 of [7], this set $\nabla_{s, s'}$ coincides with $D_{s, s'}$ if $\Delta = 1$. As for the case $\Delta > 1$, it is natural to consider the Δ-covering $\tilde{h}$ of the mapping h,

$$\tilde{h}\colon \mathbf{CP}^{k+m-1} \to (\widetilde{\mathbf{C}^\times})^{k+m},$$

while the branch of the image of h shall be specified in a proper way. To do that we remark that $h(\mathbf{CP}^{k+m-1}) \subset \nabla_{s, s'}$ where the difference $\nabla_{s, s'} \setminus h(\mathbf{CP}^{k+m-1})$ consists of the divisors that arise from the Δ-branching effect $\tilde{h}(\mathbf{CP}^{k+m-1})$. In considering $D_{s, s'}$ we shall discard the superfluous Δ-branching effect $\tilde{h}(\mathbf{CP}^{k+m-1}) \setminus h(\mathbf{CP}^{k+m-1})$. Q.E.D.

The mapping (2.24) is nothing but the inverse mapping of the logarithmic Gauss map;

$$
\begin{aligned}
&D_{s,\,s'} \to \mathbf{C}P^{k+m-1},\\
(s,\, s') \to\ &\Big(s_1\frac{\partial}{\partial s_1}\Delta_f(s,\, s')\colon \cdots : s_k\frac{\partial}{\partial s_k}\Delta_f(s,\, s')\\
&s_1'\frac{\partial}{\partial s_1'}\Delta_f(s,\, s')\colon \cdots : s_m'\frac{\partial}{\partial s_m'}\Delta_f(s, s')\Big).
\end{aligned}
$$

This is a direct consequence of the cocycle property (2.18), (2.20) of the operators $L_{q,\,\mathrm{i}}(\vartheta_s,\, \vartheta_{s'},\, s,\, s',\, \zeta)$ and $L'_{r,\,\mathrm{i}}(\vartheta_s,\, \vartheta_{s'},\, s,\, s',\, \zeta)$, see [7], Theorem 2.1, b).

§3. *A*-Hypergeometric function of Gel'fand-Kapranov-Zelevinski

Let us consider the set of polynomials with deformation parameter coefficients $(a_{0,\,1},\, \dots,\, a_{\tau_k,\,k})$ associated to the polynomial system (0.2),

$$
\bar{f}_\ell(x,\, \mathbf{a}) = a_{1,\,\ell}x^{\vec{\alpha}_{1,\,\ell}} + \cdots + a_{\tau_\ell,\,\ell}x^{\vec{\alpha}_{\tau_\ell,\,\ell}} + a_{0,\,\ell}.\ 1 \le \ell \le k. \tag{3.1}
$$

For the sake of simplicity we will further make use of the notation $\mathbf{a} := (a_{0,\,1},\, \dots,\, a_{\tau_k,\,k}) \in \mathbf{T}^L$. We consider the Leray coboundary $\partial\gamma_{\mathbf{a}}$ of a cycle $\gamma_{\mathbf{a}} \in H_n(X_{\mathbf{a}},\, \mathbf{Z})$ of the CI $X_{\mathbf{a}} = \{x \in \mathbf{T}^N; \bar{f}_1(x,\, \mathbf{a}) = \cdots = \bar{f}_k(x,\, \mathbf{a}) = 0\}$.

Then we can define the *A*-hypergeometric function $\Phi^{\zeta}_{x^{\mathbf{i}},\,\gamma_{\mathbf{a}}}(a_{0,\,1},\, \dots,\, a_{\tau_k,\,k})$ introduced by Gel'fand-Zelevinski-Kapranov [4] associated to the polynomials,

$$
\begin{aligned}
f_\ell(x) &= x^{\vec{\alpha}_{1,\,\ell}} + \cdots + x^{\vec{\alpha}_{\tau_\ell,\,\ell}}, \quad 1 \le \ell \le k,\\
x^{\mathbf{i}} &= x_1^{i_1}\cdots x_N^{i_N}, \quad x^{\vec{\alpha}_{j,\,\ell}} = x_1^{\alpha_{j,\,\ell,\,1}}\cdots x_N^{\alpha_{j,\,\ell,\,N}}.
\end{aligned}
$$

Namely it is defined as a kind of multiple residue along $X_{\mathbf{a}}$,

$$
\Phi^{\zeta}_{x^{\mathbf{i}},\,\gamma_{\mathbf{a}}}(a_{0,\,1},\, \dots,\, a_{\tau_k,\,k}) := \int_{\partial\gamma_{\mathbf{a}}}\prod_{\ell=1}^{k}\bar{f}_\ell(x,\, \mathbf{a})^{-\zeta_\ell-1}x^{\mathbf{i}+\mathbf{1}}\frac{dx}{x^{\mathbf{1}}}. \tag{3.2}
$$

We impose here the non-degeneracy condition of the Definition 2 for the complete intersection X_s after the procedure described in §1.

In the sequel we consider a lattice $\Lambda \subset \mathbf{Z}^L$ of L-vectors defined by the system of following linear equations:

$$\sum_{i=0}^{\tau_q} b(j,\, q,\, \nu) = 0,\ 1 \leq q \leq k,$$

$$\sum_{q=1}^{k} \sum_{j=1}^{\tau_q} \alpha_{jq\ell} b(j,\, q,\, \nu) = 0,\ 1 \leq \ell \leq N.$$

Here we denoted by $(b(0,1,\nu),\ldots,b(\tau_1,1,\nu),b(0,2,\nu),\ldots,b(\tau_2,2,\nu),\ldots,b(\tau_k,\, k,\, \nu))$, $1 \leq \nu \leq m+k$, a $\mathbf{Z}$ basis of Λ.

For the subset $\mathbf{K} \subset \{(0,\, 1),\, \ldots,\, (k,\, \tau_k)\}$ such that the columns $\vec{m}_{j,\,q}(A)$, $(j,\, q) \in \mathbf{K}$ of the matrix $\mathsf{M}(A)$ (1.7) span $\mathbf{R}^{N+k}$ over $\mathbf{R}$ and $|\mathbf{K}| = N+k$ we define the set of indices (a generalisation of the Frobenius' method) after [4],

$$\begin{aligned}\Pi((\zeta+\mathbf{1},\, \mathbf{i}+\mathbf{1}),\, \mathbf{K}) = \{((\lambda(0,\, 1,\, \nu),\, \ldots,\, \lambda(\tau_1,\, 1,\, \nu),\\ \ldots,\, \lambda(\tau_k,\, k,\, \nu))\}_{1\leq \nu \leq |\det(\vec{m}_{j,\,q}(A))_{(j,\,q)\in\mathbf{K}}|},\end{aligned}$$

which satisfy the following system of equations,

$$\sum_{j=0}^{\tau_\nu} \lambda(j,\, q,\, \nu) + \zeta_q + 1 = 0,\ 1 \leq q \leq k,$$

$$\sum_{q=1}^{k} \sum_{j=1}^{\tau_q} \alpha_{jq\ell} \lambda(j,\, q,\, \nu) - (i_\ell + 1) = 0,\ 1 \leq \ell \leq N.$$

Let T be a triangulation of the Newton polyhedron $\Delta(F(x,\, \mathbf{1},\, \mathbf{1},\, y)+1)$ for $F(x,\, \mathbf{1},\, \mathbf{1},\, y)$ of (1.4) after the definition [4], 1.2. Here we impose that $\lambda(j,\, q,\, \nu) \in \mathbf{Z}$ for $(j,\, q) \notin \mathbf{K}$. Let $\mathbf{K}_1, \mathbf{K}_2 \in T$ be two different simplices of the triangulation T. We suppose that $\vec{\lambda}(\nu_p) := (\lambda(0,\, 1,\, \nu_p),\, \ldots,\, \lambda(k,\, \tau_k,\, \nu_p)) \in \Pi((\zeta+\mathbf{1},\, \mathbf{i}+\mathbf{1}),\, \mathbf{K}_p)$, $\lambda(j,\, q,\, \nu_p) \in \mathbf{Z}$ for $(j,\, q) \notin \mathbf{K}_p$, $(p = 1,\, 2)$ with $1 \leq \nu_p \leq |\det(\vec{m}_\rho(A))_{\rho\in\mathbf{K}_p}|$. We introduce the condition of T-non-resonance on $(\zeta+\mathbf{1},\, \mathbf{i}+\mathbf{1})$

$$(3.3)\quad (\lambda(0,\, 1,\, \nu_1),\, \ldots,\, \lambda(k,\, \tau_k,\, \nu_1)) \not\equiv (\lambda(0,\, 1,\, \nu_2),\, \ldots,\, \lambda(k,\, \tau_k,\, \nu_2)) \mod \Lambda,$$

for any pair $\vec{\lambda}(\nu_p) = (\lambda(0,\, 1,\, \nu_p),\, \ldots,\, \lambda(k,\, \tau_k,\, \nu_p)) \in \Pi((\zeta+\mathbf{1},\, \mathbf{i}+\mathbf{1}),\mathbf{K}_p)$, $p = 1,\, 2$. An adaptation of Theorem 3 [4] to our situation can be formulated as follows.

Theorem 3.1. 1) *The A-HGF* $\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_a}(\mathbf{a})$ *satisfies the following system of equations.*

$$(3.4)\qquad \Big(\sum_{j=0}^{\tau_q} a_{ji}\frac{\partial}{\partial a_{ji}} + \zeta_q + 1\Big)\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_a}(\mathbf{a}) = 0,\ 1 \le q \le k,$$

$$\Big(\sum_{1\le q\le k,\,1\le j\le \tau_q} \alpha_{jq1}a_{jq}\frac{\partial}{\partial a_{jq}} - (i_1+1)\Big)\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_a}(\mathbf{a}) = \cdots$$

$$= \Big(\sum_{1\le q\le k,\,1\le j\le \tau_q} \alpha_{jqN}a_{jq}\frac{\partial}{\partial a_{jq}} - (i_N+1)\Big)\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_a}(\mathbf{a}) = 0,$$

$$\Big(\prod_{\{(j,q);b(j,q,\nu)>0\}} \Big(\frac{\partial}{\partial a_{jq}}\Big)^{b(j,q,\nu)} - \prod_{\{(j,q);b(j,q,\nu)<0\}} \Big(\frac{\partial}{\partial a_{jq}}\Big)^{-b(j,q,\nu)}\Big)$$

$$\cdot\,\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_a}(\mathbf{a}) = 0,\ 1 \le \nu \le L-(k+N).$$

2) *The dimension of solutions of the system above at a generic point* $\mathbf{a} \in \mathbf{T}^L$ *is equal to*

$$(N+k)!\,\mathrm{vol}_{N+k}\,\Delta(F(x,\mathbf{1},\mathbf{1},y)+1) = |\chi(Z_{F(x,\mathbf{1},\mathbf{1},y)})|$$

if the T-non-resonant condition (3.3) *is satisfied.*

In the sequel we shuffle the variables $\mathbf{a} = (a_{0,1},\dots,a_{\tau_k,k})$ in accordance with the order of their appearance and we define anew the indexed parameters $a_1 = a_{1,1},\dots,a_{\tau_1} = a_{\tau_1,1}, a_{\tau_1+1} = a_{0,1},\dots,a_{L-1} = a_{\tau_k,k}, a_L = a_{0,k}$. Let us introduce notations analogous to (1.14),

$$(3.5)\qquad \Xi(A) := {}^t(\log X_1,\dots,\log X_N,\log a_1,\dots,\log a_L,\log U_1,\dots,\log U_k).$$

$$\log T_1 = \langle \vec{\alpha}_{1,1},\log X\rangle + \log a_1 + \log U_1,$$

$$\vdots$$

$$\log T_{\tau_1} = \langle \vec{\alpha}_{1,\tau_1},\log X\rangle + \log a_{\tau_1} + \log U_1,$$

$$\vdots$$

$$\log T_L = \log a_L + \log U_k.$$

We consider the equation

$$\mathsf{L}(A)\cdot \mathrm{Log}\,\Xi(A) = \mathsf{L}\cdot\mathrm{Log}\,\Xi,$$

where the matrix $\mathsf{L}(A)$ is constructed as follows. The columns $\vec{\ell}_i(A) = \vec{v}_i$, $1 \le i \le N$ with vectors $\vec{v}_i$ defined like the column of the matrix L in

(1.15). For the columns of number $N+1$ to $N+L$

$$(\vec{\ell}_{N+1}(A), \ldots, \vec{\ell}_{N+L}(A)) = \mathrm{id}_L .$$

The columns

$$\vec{\ell}_{N+L+j}(A) = {}^t(\overbrace{0, \ldots, 0, 0,}^{\tau_1+\cdots+\tau_{j-1}+j-1} \overbrace{1, 1, \ldots, 1,}^{\tau_j+1} 0, \ldots, 0), \quad 1 \le j \le k,$$

the matrix $\mathsf{L}(A)$ is obtained after implementation of the matrix id_L into the transposed matrix ${}^t\mathsf{M}(A)$ between the k-th and the $(k+1)$-th column up to necessary permutations necessary after the implementation.

Proposition 3.2. *There exists a cycle $\gamma_{\mathbf{a}}$ such that the following equality holds for the integral defined in* (3.2),

$$\Phi^{\zeta}_{x^{\mathbf{i}},\gamma_{\mathbf{a}}}(\mathbf{a}) = B^{\zeta}_{\mathbf{i}}(\mathbf{a}) I^{\zeta}_{x^{\mathbf{i}},\gamma}(s(\mathbf{a}), s'(\mathbf{a})), \tag{3.6}$$

here

$$s_\ell(\mathbf{a}) = \prod_{j=1}^{L} a_j^{w_{j,\,N+\ell}}, \quad 1 \le \ell \le k,$$

$$s'_\rho(\mathbf{a}) = \prod_{j=1}^{L} a_j^{w_{j,\,N+k+\rho}}, \quad 1 \le \rho \le m,$$

$$B^{\zeta}_{\mathbf{i}}(\mathbf{a}) = \prod_{\ell=1}^{N}\Big(\prod_{j=1}^{L} a_j^{w_{j,\,\ell}}\Big)^{i_\ell+1} \prod_{\nu=1}^{k}\Big(\prod_{j=1}^{L} a_j^{w_{j,\,N+k+m+\nu}}\Big)^{\zeta_\nu+1}.$$

The exponents $w_{j,\ell}$ are determined by the following relation,

$$\mathsf{L}^{-1}\cdot\mathsf{L}(A) \tag{3.7}$$

$$= \begin{bmatrix} 1 & \cdots & 0 & w_{1,1} & \cdots & w_{L,1} & 0 & & \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & w_{1,N} & \cdots & w_{L,N} & 0 & & \\ 0 & \cdots & 0 & w_{1,N+1} & \cdots & w_{L,N+1} & 0 & & \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & w_{1,N+k+m} & \cdots & w_{L,N+k+m} & 0 & & \\ 0 & \cdots & 0 & w_{1,N+k+m+1} & \cdots & w_{L,N+k+m+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & w_{1,L} & \cdots & w_{L,L} & 0 & \cdots & 1 \end{bmatrix},$$

that has been essentially introduced in (2.8). *The transition of the cycle* $\gamma(a)$ *to* γ *is controlled by the transformations,*

$$X_i = \Big(\prod_{j=1}^{L} a_j^{w_{j,\,i}}\Big)^{-1} \cdot x_i.$$

Proof. It is enough to remark the following property,

$$x^{\mathbf{i}+\mathbf{1}} y^{\zeta+\mathbf{1}} \frac{dx}{x^{\mathbf{1}}} \wedge \frac{dy}{y^{\mathbf{1}}} = B_{\mathbf{i}}^{\zeta}(\mathbf{a}) x^{\mathbf{i}+\mathbf{1}} U^{\zeta+\mathbf{1}} \frac{dx}{x^{\mathbf{1}}} \wedge \frac{dU}{U^{\mathbf{1}}}.$$

Q.E.D.

One can thus conclude (at least locally on the chart $a_j \neq 0$ for $j \in I$, $|I| = k+m$) A-HGF of GZK (3.2) is expressed by means of a fibre integral annihilated by the Horn system (2.15). One can find a similar statement in [7] where Kapranov restricts himself to a power series expansion of the solution to (3.2).

Corollary 3.3. *The dimension of the solution space of the system* (3.3) *at the generic point is equal to* $|\chi(Z_{F(x,\,\mathbf{1},\,\mathbf{1},\,y)})|$ *if the* T*-non-resonance condition* (3.3) *is satisfied.*

Proof. We shall consider the convex hull of vectors that correspond to the vertices of the Newton polyhedron of the polynomial $y_1(f_1(x) + 1) + \cdots + y_k(f_k(x)+1)$. That is to say

$$\begin{aligned}(\vec{\alpha}_{1,\,1},\ &1,\ 0,\ \dots,\ 0),\ \dots,\ (\vec{\alpha}_{\tau_1,\,1},\ 1,\ 0,\ \dots,\ 0),\\ &(\vec{\alpha}_{1,\,2},\ 0,\ 1,\ 0,\ \dots,\ 0),\ \cdots,\ (\vec{\alpha}_{\tau_k,\,k},\ 0,\ \dots,\ 0,\ 1) \in \mathbf{Z}^{N+k}.\end{aligned}$$

They are located on the hyperplane $\zeta_1 + \cdots + \zeta_k = 1$. Thus it is possible to measure $(N+k-1)$ dimensional volume

$$(N+k-1)!\,\mathrm{vol}_{N+k-1}(\Delta(F(x,\,\mathbf{1},\,\mathbf{1},\,y))$$

that is equal to $(N+k)!\,\mathrm{vol}_{N+k}(\Delta(F(x,\,\mathbf{1},\,\mathbf{1},\,y)+1)$. The Euler characteristic admits the following expression

$$\begin{aligned}|\chi(Z_{F(x,\,\mathbf{1},\,\mathbf{1},\,y)})| &= \sum_p |\det \mathsf{M}_{\mathbf{K}_p}|\\ &= (N+k-1)!\,\mathrm{vol}_{N+k-1}\big(\Delta(F(x,\,\mathbf{1},\,\mathbf{1},\,y))\big),\end{aligned}$$

after Khovanski [8]. Q.E.D.

We define the A-discriminantal loci $\nabla^0_{\mathbf{a}}$ in $\mathbf{T}^L$ like following,

$$(3.8) \quad \nabla^0_{\mathbf{a}} = \left\{ \mathbf{a} \in \mathbf{T}^L;\ \begin{matrix} \bar{f}_1(x,\, \mathbf{a}) \\ = \cdots \\ = \bar{f}_k(x,\, \mathbf{a}) \\ = 0 \end{matrix},\quad \operatorname{rank} \begin{pmatrix} \operatorname{grad}_x \bar{f}_1(x,\, \mathbf{a}) \\ \vdots \\ \operatorname{grad}_x \bar{f}_k(x,\, \mathbf{a}) \end{pmatrix} < k \right\}.$$

As it is seen from (3.7) the uniformisation equations (2.22), (2.23) give rise to an uniformisation of A-discriminantal loci $\nabla^0_{\mathbf{a}}$ without Δ-branching effect.

Corollary 3.4. *We have the following relations among* $\mathbf{a} \in \mathbf{T}^L$ *located on the discriminantal loci* $\nabla^0_{\mathbf{a}}$,

$$(3.9)_1 \qquad \prod_{j=1}^{L} \left(\frac{a_j}{\mathcal{L}_j(-\mathbf{1},\, \mathbf{z},\, \mathbf{z}',\, -\mathbf{1})} \right)^{B^q_j} = 1, \quad 1 \le q \le k,$$

$$(3.9)_2 \qquad \prod_{j=1}^{L} \left(\frac{a_j}{\mathcal{L}_j(-\mathbf{1},\, \mathbf{z},\, \mathbf{z}',\, -\mathbf{1})} \right)^{C^r_j} = 1, \quad 1 \le r \le m.$$

This allows us to express $\nabla^0_{\mathbf{a}}$ *by means of the deformation parameters* $(\mathbf{z},\, \mathbf{z}') \in \mathbf{CP}^{k+m-1}$ *and* $\mathbf{a}' \in \mathbf{T}^{L-k}/\mathbf{D}(\Sigma) \cong \mathbf{T}^{L-(k+m)}$.

§4. Examples

4.1. Deformation of D_4.

Let us consider the versal deformation of D_4 singularity of the following form,

$$(4.1) \qquad f(x,\, s_0,\, s_1,\, s_2,\, s_3) = x_1^3 + x_1 x_2^2 + s_3 x_1^2 + s_2 x_1 + s_1 x_2 + s_0.$$

By means of the resultant calculus on computer, we get a defining equation of the discriminantal loci as follows,

$$\begin{aligned} (4.2) \quad \Delta_f(s) = {} & 1024 s_1^6 (432 s_0^4 + 64 s_1^6 + 576 s_0^2 s_1^2 s_2 + 128 s_1^4 s_2^2 \\ & + 64 s_0^2 s_2^3 + 64 s_1^2 s_2^4 + 192 s_0 s_1^4 s_3 - 288 s_0^3 s_2 s_3 \\ & - 320 s_0 s_1^2 s_2^2 s_3 - 24 s_0^2 s_1^2 s_3^2 - 144 s_1^4 s_2 s_3^2 - 16 s_0^2 s_2^2 s_3^2 \\ & - 16 s_1^2 s_2^3 s_3^2 + 64 s_0^3 s_3^3 + 72 s_0 s_1^2 s_2 s_3^3 + 27 s_1^4 s_3^4). \end{aligned}$$

This is a polynomial with quasihomogeneous weight 24 if we assign to the variables $(x_1,\, x_2; s_0,\, s_1,\, s_2,\, s_3)$ the weights $(1,\, 1; 3,\, 2,\, 2,\, 1)$. Here we remark that $s_1 = 0$ branch of the discriminantal locus $D_s = \{s \in \mathbf{C}^3; \Delta_f(s) = 0\}$ corresponds to the deformation of A_2 singularity.

On the other hand, our Theorem 2.6 states that the uniformisation equation of the discriminantal loci for the deformation (i.e. torus action quotient of the deformation parameter space $(s_0, s_1, 0, s_3)$ on the chart $s_3 \neq 0$),

$$f(x, s_0, s_1, 0, 1) = x_1^3 + x_1 x_2^2 + x_1^2 + s_1 x_2 + s_0,$$

has the following form,

$$\text{(4.3)} \qquad \begin{aligned} s_0 &= -\frac{z_2(3z_1+4z_2)^2}{4(2z_1+3z_2)^3}, \\ s_1 &= \left(-\frac{z_1(3z_1+4z_2)^3}{4(2z_1+3z_2)^4}\right)^{1/2}. \end{aligned}$$

If we eliminate the variables (z_1, z_2) from the expressions (4.3), we get an equation

$$64s_0^3 + 432s_0^4 - 24s_0^2s_1^2 + 27s_1^4 + 192s_0s_1^4 + 64s_1^6 = 0.$$

We recall here that our method requires that the expression $yf(x, s)$ contains so much terms as the variables in it. The reason why the value $(s_2, s_3) = (0, 1)$ has been chosen is of purely technical character. In substituting the special value $(0, 1)$ for (s_2, s_3) in (4.2) we get,

$$\frac{\Delta_f(s_0, s_1, 0, 1)}{1024s_1^6} = 64s_0^3 + 432s_0^4 - 24s_0^2s_1^2 + 27s_1^4 + 192s_0s_1^4 + 64s_1^6.$$

4.2. Deformation of a non-quasihomogeneous complete intersection.

Let us consider the following pair of polynomials that define a non-degenerate complete intersection X_s in $\mathbf{C}^2$,

$$\text{(4.4)} \qquad f_1 = x_1^3 + x_2^2 + s_1, \;\; f_2 = x_1^2 + x_2^3 + s_2.$$

The discriminant of this CI in $\mathbf{C}^2$ can be calculated as follows,

$$\text{(4.5)} \quad \begin{aligned} &(s_1^3+s_2^2)^3(s_2^3+s_1^2)^3(800000 + 387420489s_1^5 - 43740000s_1s_2 + \\ &+4384388 25s_1^2s_2^2 + 387420489s_1^3s_2^3 + 387420489s_2^5). \end{aligned}$$

Evidently the fibres corresponding to the parameter values on the divisor $(s_1^3 + s_2^2)^3(s_2^3 + s_1^2)^3 = 0$ are contained in $\{(x_1, x_2) \in \mathbf{C}^2; x_1x_2 = 0\}$. Thus the discriminant of CI $X_s \cap \mathbf{T}^2$ is given by the third factor of

(4.5). After Theorem 2.6, we can find an uniformisation equation of the discriminantal loci D_s,

$$
(4.6)\qquad
\begin{aligned}
s_1 &= -\left(\frac{(4z_1+6z_2)^4(5z_1)^5(6z_1+4z_2)^6}{(9z_1+6z_2)^9(6z_1+9z_2)^6}\right)^{1/5},\\
s_2 &= -\left(\frac{(4z_1+6z_2)^6(5z_2)^5(6z_1+4z_2)^4}{(9z_1+6z_2)^6(6z_1+9z_2)^9}\right)^{1/5}.
\end{aligned}
$$

If we eliminate the variables (z_1, z_2) from the expressions (4.6), we get an equation of ∇_s,

$$
\begin{aligned}
(800000 + 387420489s_1^5 - 43740000s_1s_2 + 438438825s_1^2s_2^2 \\
+ 387420489s_1^3s_2^3 + 387420489s_2^5)R(z_1, z_2),
\end{aligned}
$$

where $R(z_1, z_2)$ is a polynomial whose Newton polyhedron is contained in a four sided rectilinear figure with vertices (0, 0), (20, 0), (12, 12), (0, 20). This factor contains the image of $\tilde{h}(\mathbf{CP}^1)$ outside of D_s.

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