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Cobordism of fibered knots and related topics

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Abstract.

This is a survey article on the cobordism theory of non-spherical knots studied in [BM, B2, BS1, BMS, BS2, BS3]. Special emphasis is put on fibered knots.

We first recall the classical results concerning cobordisms of spherical knots. Then we give recent results on cobordisms of simple fibered $(2n-1)$-knots for $n \geq 2$ together with relevant examples. We discuss the Fox-Milnor type relation and show that the usual spherical knot cobordism group modulo the subgroup generated by the cobordism classes of fibered knots is infinitely generated for odd dimensions. The pull back relation on the set of knots is also discussed, which is closely related to the cobordism theory of knots via the codimension two surgery theory. We also present recent results on cobordisms of surface knots in S^4 and 4-dimensional knots in S^6. Finally we give some open problems related to the subject.

CONTENTS

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§1. Introduction

1.1. History

In the early fifties Rohlin [Rh1] and Thom [Th] studied the *cobordism groups of manifolds.* At the 1958 International Congress of Mathematicians in Edinburgh, René Thom received a Fields Medal for his development of cobordism theory. Then, Fox and Milnor [FM1, FM2] were the first to study *cobordism of knots*, i.e., cobordism of embeddings of the circle S^1 into the 3-sphere S^3. Knot cobordism is slightly different from the general cobordism, since its definition is more restrictive. After Fox and Milnor, Kervaire [K1] and Levine [L2] studied embeddings of the n-sphere S^n (or homotopy n-spheres) into the $(n+2)$-sphere S^{n+2}, and gave classifications of such embeddings up to cobordism for $n \geq 2$. Moreover, Kervaire defined group structures on the set of cobordism classes of n-spheres embedded in S^{n+2}, and on the set of concordance classes of embeddings of S^n into S^{n+2}. The structures of these groups for $n \geq 2$ were clarified by Kervaire [K1], Levine [L2, L3] and Stoltzfus [Sf].

Note that embeddings of spheres were studied only in the codimension two case, since in the PL category Zeeman [Ze] proved that all such embeddings in codimension greater than or equal to three are unknotted, and Stallings [Sg] proved that it is also true in the topological category (here, one needs to assume the locally flatness condition), provided that the ambient sphere has dimension greater than or equal to five. In the smooth category Haefliger [Ha] proved that a cobordism of spherical knots in codimension greater than or equal to three implies isotopy.

Milnor [M3] showed that, in a neighborhood of an isolated singular point, a complex hypersurface is homeomorphic to the cone over the *algebraic knot* associated with the singularity. Hence, the embedded topology of a complex hypersurface around an isolated singular point is given by the algebraic knot, which is a special case of a *fibered knot.* After Milnor's work, the class of fibered knots has been recognized as an important class of knots to study. Usually algebraic knots are not homeomorphic to spheres, and this motivated the study of embeddings of general manifolds (not necessarily homeomorphic to spheres) into spheres in codimension two. Moreover, in the beginning of the seventies, Lê [Lê] proved that isotopy and cobordism are equivalent for 1-dimensional algebraic knots. Lê proved this for the case of *connected* (or spherical) algebraic 1-knots, and the generalization to arbitrary algebraic 1-knots follows easily (for details, see §4). About twenty years later, Du Bois and Michel [DM] gave the first examples of algebraic spherical knots that are cobordant but are not isotopic. These examples motivated the classification of fibered knots up to cobordism.

1.2. Contents

This article is organized as follows. In §2 we give several definitions related to the cobordism theory of knots. *Seifert forms* associated with knots are also introduced. In §3 we review the classifications of (simple) spherical $(2n-1)$-knots with $n \geq 2$ up to isotopy and up to cobordism. In §4 we review the properties of algebraic 1-knots and present the classification theorem of algebraic 1-knots up to cobordism due to Lê [Lê]. In §5 we present the classifications of simple fibered $(2n-1)$-knots with $n \geq 3$ up to isotopy and up to cobordism. The classification up to cobordism is based on the notion of the *algebraic cobordism.* In order to clarify the definition of algebraic cobordism, we give several explicit examples. We also explain why this relation might not be an equivalence relation on the set of bilinear forms defined on free $\mathbf{Z}$-modules of finite rank. A classification of 3-dimensional simple fibered knots up to cobordism is given in §6. In §7 we recall the Fox-Milnor type relation on the Alexander polynomials of cobordant knots. As an application, we show that the usual spherical knot cobordism group modulo the subgroup generated by the cobordism classes of fibered knots is infinitely generated for odd dimensions. In §8 we present several examples of knots with interesting properties in view of the cobordism theory of knots. In §9 we define the pull back relation for knots which naturally arises from the viewpoint of the codimension two surgery theory. We illustrate several results on pull back relations for fibered knots using some explicit examples. Some results for even dimensional knots are given in §10, where we explain recent results about embedded surfaces in S^4 and embedded

4-manifolds in S^6. Finally in §11, we give several open problems related to the cobordism theory of non-spherical knots.[1]

With all the results collected in this paper, we have classifications of knots up to cobordism in every dimension, except for the classical case of one dimensional knots and the case of three dimensional knots. In the latter two cases, a complete classification still remains open until now.

Throughout the article, we shall work in the smooth category unless otherwise specified. All the homology and cohomology groups are understood to be with integer coefficients. The symbol "$\cong$" denotes an (orientation preserving) diffeomorphism between (oriented) manifolds, or an appropriated isomorphism between algebraic objects.

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§2. Several definitions

Since our aim is to study cobordisms of codimension two embeddings of general manifolds, not necessarily homeomorphic to spheres, we define the following.

Definition 2.1. Let K be a closed n-dimensional manifold embedded in the $(n+2)$-dimensional sphere S^{n+2}. We suppose that K is $([n/2]-1)$-connected, where for $a \in \mathbf{R}$, $[a]$ denotes the greatest integer not exceeding a. (We adopt the convention that a space is (-1)-connected if it is not empty.) Equivalently, we suppose that K is

$(k-2)$-connected if $n = 2k-1$ and $k \geq 2$, or
$(k-1)$-connected if $n = 2k$ and $k \geq 1$.

When K is orientable, we further assume that it is oriented.[2] Then we call K or its (oriented) isotopy class an *n-knot*, or simply a *knot.*

An n-knot K is *spherical* if K is

(1) diffeomorphic to the standard n-sphere S^n for $n \leq 4$, or
(2) a homotopy n-sphere for $n \geq 5$.

Remark 2.2. We adopt the above definition of a spherical knot for $n \leq 4$ in order to avoid the difficulty related to the smooth Poincaré conjecture in dimensions three and four.

[1] A "non-spherical manifold" in this article refers to a general manifold which may not necessarily be a homotopy sphere.

[2] In this article, we always assume that n-knots are oriented if $n \neq 2$.

Note that we impose the connectivity condition on the embedded submanifold in Definition 2.1. This is motivated by the following reasons. First, a knot associated with an isolated singularity of a complex hypersurface satisfies the above connectivity condition as explained below. Second, if we assume that K is $[n/2]$-connected, then K is necessarily a homotopy sphere so that K is spherical at least for $n \neq 3, 4$. Third, the connectivity condition on K technically helps to perform certain embedded surgeries and this simplifies the arguments in various situations.

Remark 2.3. For the case of $n = 1$, i.e., for the classical knot case, a 1-knot in our sense is usually called a "link", and a connected (or spherical) 1-knot is usually called a "knot".

As mentioned in §1, Definition 2.1 is motivated by the study of the topology of isolated singularities of complex hypersurfaces. More precisely, let $f\colon \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function germ with an isolated singularity at the origin. If $\varepsilon > 0$ is sufficiently small, then $K_f = f^{-1}(0) \cap S^{2n+1}_\varepsilon$ is a $(2n-1)$-dimensional manifold which is naturally oriented, where S^{2n+1}_ε is the sphere in $\mathbf{C}^{n+1}$ of radius ε centered at the origin. Furthermore, its (oriented) isotopy class in $S^{2n+1}_\varepsilon = S^{2n+1}$ does not depend on the choice of ε (see [M3]). We call K_f the *algebraic knot* associated with f. Since the pair $(D^{2n+2}_\varepsilon, f^{-1}(0) \cap D^{2n+2}_\varepsilon)$ is homeomorphic to the cone over the pair $(S^{2n+1}_\varepsilon, K_f)$, the algebraic knot completely determines the local embedded topological type of $f^{-1}(0)$ near the origin, where D^{2n+2}_ε is the disk in $\mathbf{C}^{n+1}$ of radius ε centered at the origin.

In [M3], Milnor proved that algebraic knots associated with isolated singularities of holomorphic function germs $f\colon \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ are $(2n-1)$-dimensional closed, oriented and $(n-2)$-connected submanifolds of the sphere S^{2n+1}. This means that algebraic knots are in fact knots in the sense of Definition 2.1. Moreover, the complement of an algebraic knot K_f in the sphere S^{2n+1} admits a fibration over the circle S^1, and the closure of each fiber is a compact $2n$-dimensional oriented $(n-1)$-connected submanifold of S^{2n+1} which has K_f as boundary. This motivates the following definition.

Definition 2.4. We say that an oriented n-knot K is *fibered* if there exists a smooth fibration $\phi\colon S^{n+2} \smallsetminus K \to S^1$ and a trivialization $\tau\colon N(K) \to K \times D^2$ of a closed tubular neighborhood $N(K)$ of K in S^{n+2} such that $\phi|_{N(K) \smallsetminus K}$ coincides with $\pi \circ \tau|_{N(K) \smallsetminus K}$, where $\pi\colon K \times (D^2 \smallsetminus \{0\}) \to S^1$ is the composition of the projection to the second factor and the obvious projection $D^2 \smallsetminus \{0\} \to S^1$. Note that then the closure of each fiber of ϕ in S^{n+2} is a compact $(n+1)$-dimensional oriented manifold whose boundary coincides with K. We shall often call the closure of each fiber simply a *fiber*.

Furthermore, we say that a fibered n-knot K is *simple* if each fiber of ϕ is $[(n-1)/2]$-connected.

Note that an algebraic knot is always a simple fibered knot.

Let us now recall the classical definition of Seifert forms of odd dimensional oriented knots, which were first introduced in [Se] and play an important role in the study of knots.

First of all, for every oriented n-knot K with $n \geq 1$, there exists a compact oriented $(n+1)$-dimensional submanifold V of S^{n+2} having K as boundary. Such a manifold V is called a *Seifert manifold* associated with K.

For the construction of Seifert manifolds (or Seifert surfaces) associated with 1-knots, see [Rl], for example.

For general dimensions, the existence of a Seifert manifold associated with a knot K can be proved by using the obstruction theory as follows. It is known that the normal bundle of a closed orientable manifold embedded in a sphere in codimension two is always trivial (see [MS, Corollary 11.4], for example). Let $N(K) \overset{\tau}{\cong} K \times D^2$ be a closed tubular neighborhood of K in S^{n+2}, and $\Phi \colon \partial N(K) \overset{\cong}{\to} K \times S^1 \overset{pr_2}{\to} S^1$ the composite of the restriction of τ to the boundary of $N(K)$ and the projection pr_2 to the second factor. Using the exact sequence

$$H^1(S^{n+2} \setminus \operatorname{Int} N(K)) \to H^1(\partial N(K)) \\ \to H^2(S^{n+2} \setminus \operatorname{Int} N(K), \partial N(K)),$$

associated with the pair $(S^{n+2} \setminus \operatorname{Int} N(K), \partial N(K))$, we see that the obstruction to extending Φ to $\widetilde{\Phi} \colon S^{n+2} \setminus \operatorname{Int} N(K) \to S^1$ lies in the cohomology group

$$H^2(S^{n+2} \setminus \operatorname{Int} N(K), \partial N(K)) \cong H_n(S^{n+2} \setminus \operatorname{Int} N(K)).$$

By Alexander duality we have

$$H_n(S^{n+2} \setminus \operatorname{Int} N(K)) \cong H^1(K),$$

which vanishes if $n \geq 4$, since K is simply connected for $n \geq 4$. When $n \leq 3$, we can show that by choosing the trivialization τ appropriately, the obstruction in question vanishes. Therefore, a desired extension $\widetilde{\Phi}$ always exists. Now, for a regular value y of $\widetilde{\Phi}$, the manifold $\widetilde{\Phi}^{-1}(y)$ is a submanifold of S^{n+2} with boundary being identified with $K \times \{y\}$ in

$K \times S^1$. The desired Seifert manifold associated with K is obtained by gluing a small collar $K \times [0, 1]$ to $\widetilde{\Phi}^{-1}(y)$.

When K is a fibered knot, the closure of a fiber is always a Seifert manifold associated with K.

Definition 2.5. We say that an n-knot is *simple* if it admits an $[(n-1)/2]$-connected Seifert manifold.

Now let us recall the definition of Seifert forms for odd dimensional knots.

Definition 2.6. Suppose that V is a compact oriented $2n$-dimensional submanifold of S^{2n+1}, and let G be the quotient of $H_n(V)$ by its $\mathbf{Z}$-torsion. The *Seifert form* associated with V is the bilinear form $A\colon G \times G \to \mathbf{Z}$ defined as follows. For $(x, y) \in G \times G$, we define $A(x, y)$ to be the linking number in S^{2n+1} of ξ_+ and η, where ξ and η are n-cycles in V representing x and y respectively, and ξ_+ is the n-cycle ξ pushed off V into the positive normal direction to V in S^{2n+1}.

By definition a *Seifert form* associated with an oriented $(2n-1)$-knot K is the Seifert form associated with F, where F is a Seifert manifold associated with K. A matrix representative of a Seifert form with respect to a basis of G is called a *Seifert matrix.*

Remark 2.7. Some authors define $A(x, y)$ to be the linking number of ξ and η_+ instead of ξ_+ and η, where η_+ is the n-cycle η pushed off V into the positive normal direction to V in S^{2n+1}. There is no essential difference between such a definition and ours. However, one should be careful, since some formulas may take different forms.

Remark 2.8. For codimension two embeddings between general manifolds, similar invariants have been constructed by Cappell-Shaneson [CS1] and Matsumoto [Mt1, Mt2] (see also [St]). These invariants arose as obstructions for certain codimension two surgeries.

Let us illustrate the above definition in the case of the trefoil knot. Let us consider the Seifert manifold V associated with this knot as depicted in Fig. 1, where "+" indicates the positive normal direction. Note that $\operatorname{rank} H_1(V) = 2$. We denote by ξ and η the 1-cycles which represent the generators of $H_1(V)$. Then, with the aid of Fig. 1, we see that the Seifert matrix for the trefoil knot is given by

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that a Seifert matrix is not symmetric in general. When A is a Seifert matrix associated with a Seifert manifold $V \subset S^{2n+1}$ of a

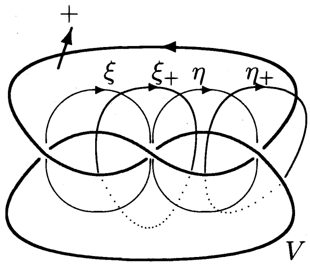

Fig. 1. Computing a Seifert matrix for the trefoil knot

$(2n-1)$-knot $K = \partial V$, the matrix $S = A + (-1)^n A^T$ is the matrix of the intersection form for V with respect to the same basis, where A^T denotes the transpose of A (for example, see [D]).

When a knot is fibered, its Seifert form associated with a fiber is always unimodular by virtue of Alexander duality (see [Kf]). In the following, for a fibered $(2n-1)$-knot, we use the Seifert form associated with a fiber unless otherwise specified.

Furthermore, when a $(2n-1)$-knot is simple, we consider an $(n-1)$-connected Seifert manifold associated with this knot unless otherwise specified.

Let us now focus on the cobordism classes of knots.

Definition 2.9. Two n-knots K_0 and K_1 in S^{n+2} are said to be *cobordant* if there exists a properly embedded $(n+1)$-dimensional manifold X of $S^{n+2} \times [0, 1]$ such that

(1) X is diffeomorphic to $K_0 \times [0, 1]$, and
(2) $\partial X = (K_0 \times \{0\}) \cup (K_1 \times \{1\})$

(see Fig. 2). The manifold X is called a *cobordism* between K_0 and K_1. When the knots are oriented, we say that K_0 and K_1 are *oriented cobordant* (or simply *cobordant*) if there exists an oriented cobordism X between them such that $\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})$, where $-K_0$ is obtained from K_0 by reversing the orientation.

In Fig. 2 the manifold $X \cong K_0 \times [0, 1]$, embedded in $S^{n+2} \times [0, 1]$, and its boundary $(K_0 \times \{0\}) \cup (K_1 \times \{1\})$, embedded in $(S^{n+2} \times \{0\}) \cup (S^{n+2} \times \{1\})$, are drawn by solid curves and black dots respectively, and the levels $S^{n+2} \times \{t\}$, $t \in (0, 1)$, are drawn by dotted curves.

Recall that a manifold with boundary Y embedded in a manifold X with boundary is said to be *properly embedded* if $\partial Y = \partial X \cap Y$ and Y is transverse to ∂X.

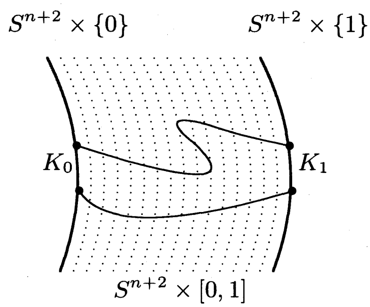

Fig. 2. A cobordism between K_0 and K_1

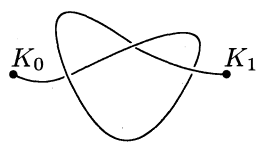

Fig. 3. A cobordism which is not an isotopy

It is clear that isotopic knots are always cobordant. However, the converse is not true in general, since the manifold $X \cong K_0 \times [0, 1]$ can be knotted in $S^{n+2} \times [0, 1]$ as depicted in Fig. 3. For explicit examples, see §8.

We also introduce the notion of *concordance* for embedding maps as follows.

Definition 2.10. Let K be a closed n-dimensional manifold. We say that two embeddings $f_i \colon K \to S^{n+2}$, $i = 0, 1$, are *concordant* if there exists a proper embedding $\Phi \colon K \times [0, 1] \to S^{n+2} \times [0, 1]$ such that $\Phi|_{K \times \{i\}} = f_i \colon K \times \{i\} \to S^{n+2} \times \{i\}$, $i = 0, 1$.

Note that an embedding map $\varphi \colon Y \to X$ between manifolds with boundary is said to be *proper* if $\partial Y = \varphi^{-1}(\partial X)$ and Y is transverse to ∂X.

Recall that for a simple $(2n-1)$-knot K with an $(n-1)$-connected Seifert manifold V, we have the following exact sequence

$$(2.1)\qquad 0 \to H_n(K) \to H_n(V) \xrightarrow{S_*} H_n(V, K) \to H_{n-1}(K) \to 0,$$

where the homomorphism S_* is induced by the inclusion. Let

$$\widetilde{\mathfrak{P}}\colon H_n(V, K) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Z}}(H_n(V), \mathbf{Z})$$

be the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism. Set $S = A + (-1)^n A^T$ and let $S^*\colon H_n(V) \to \mathrm{Hom}_{\mathbf{Z}}(H_n(V), \mathbf{Z})$ be the adjoint of S, where A is the Seifert form associated with V. Then we see easily that the homomorphisms S_* and S^* are related together by $S^* = \widetilde{\mathfrak{P}} \circ S_*$.

Cobordant knots are diffeomorphic. Hence, to have a cobordism between two given knots, we need to have topological informations about the knots as abstract manifolds. Since a simple fibered $(2n-1)$-knot is the boundary of the closure of a fiber, which is an $(n-1)$-connected Seifert manifold associated with the knot, by considering the above exact sequence (2.1) we can use the kernel and the cokernel of the homomorphism S^* to get topological data of the knot. Note that in the case of spherical knots, these considerations are not necessary, since S_* and S^* are isomorphisms.

§3. Spherical knots

In this section, let us briefly review the case of spherical knots, which was studied mainly by Kervaire and Levine.

The Seifert form is the main tool to study cobordisms of odd dimensional spherical knots. In [L4] Levine described the possible modifications on Seifert forms of a spherical simple knot corresponding to alterations of Seifert manifolds as follows.

An *enlargement* A' of a square integral matrix A is defined as follows:

$$A' = \begin{pmatrix} A & \mathcal{O} & \mathcal{O} \\ \alpha & 0 & 0 \\ \mathcal{O}^T & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \beta & \mathcal{O} \\ \mathcal{O}^T & 0 & 1 \\ \mathcal{O}^T & 0 & 0 \end{pmatrix},$$

where $\mathcal{O}$ is a column vector whose entries are all 0, and α (resp. β) is a row (resp. column) vector of integers. In this case, we also call A a *reduction* of A'.

Two square integral matrices are said to be S-*equivalent* if they are related each other by enlargement and reduction operations together with the congruence. We also say that two integral bilinear forms defined on free $\mathbf{Z}$-modules of finite rank are *S-equivalent* if so are their matrix representatives.

Levine [L4] proved

Theorem 3.1. *For $n \geq 2$, two spherical simple $(2n-1)$-knots are isotopic if and only if they have S-equivalent Seifert forms.*

Remark 3.2. For spherical simple $(2n-1)$-knots, we have another algebraic invariant, called the *Blanchfield pairing*, which is closely related to the Seifert form (see [Ke1, T]). In fact, it is known that giving an S-equivalence class of a Seifert form is equivalent to giving an isomorphism class of a Blanchfield pairing.

Kervaire showed that the set C_n of cobordism classes of spherical n-knots has a natural group structure. The group operation is given by the connected sum and the inverse of a knot K is given by its mirror image with reversed orientation $-K^!$. We say that an n-knot $K \subset S^{n+2}$ is *null-cobordant* if it is cobordant to the trivial knot, i.e., if there exists an $(n+1)$-disk D^{n+1} properly embedded in the $(n+3)$-disk D^{n+3} such that $\partial D^{n+1} = K \subset S^{n+2} = \partial D^{n+3}$. Note that the neutral element of C_n is the class of null-cobordant n-knots.

In the case of spherical $(2n-1)$-knots Kervaire and Levine used the following notion for integral bilinear forms.

Definition 3.3. Let $A\colon G \times G \to \mathbf{Z}$ be an integral bilinear form defined on a free $\mathbf{Z}$-module G of finite rank. The form A is said to be *Witt associated to* 0 if the rank m of G is even and there exists a submodule M of rank $m/2$ in G such that M is a direct summand of G and A vanishes on M. Such a submodule M is called a *metabolizer* for A.

The following theorem was proved by Levine [L2] (see also [K2]).

Theorem 3.4. *For $n \geq 2$, a spherical $(2n-1)$-knot is null-cobordant if and only if its Seifert form is Witt associated to* 0.

Remark 3.5. For Blanchfield pairing (see Remark 3.2), there is also a notion of "null-cobordism", and we have a result similar to Theorem 3.4 (see [Ke2]).

For two spherical $(2n-1)$-knots K_0 and K_1 with Seifert forms A_0 and A_1 respectively, the oriented connected sum $K = K_0\sharp(-K_1^!)$ has $A = A_0 \oplus (-A_1)$ as the Seifert form associated with the oriented connected sum along the boundaries of the Seifert manifolds associated with K_0 and $-K_1^!$, where $-K_1^!$ denotes the mirror image of K_1 with reversed orientation. Hence, as a consequence of Theorem 3.4, we have that two spherical knots K_0 and K_1 are cobordant if and only if the form $A = A_0 \oplus (-A_1)$ is Witt associated to 0. In this case we sometimes say that A_0 and A_1 are *Witt equivalent.*

For $\varepsilon = \pm 1$, let $C^{\varepsilon}(\mathbf{Z})$ be the set of all Witt equivalence classes of integral bilinear forms A defined on free $\mathbf{Z}$-modules of finite rank such that $A + \varepsilon A^T$ is unimodular (for the notation, we follow [K2]). It can be shown that $C^{\varepsilon}(\mathbf{Z})$ has a natural abelian group structure, where the addition is defined by the direct sum. Then we have the following.

Theorem 3.6 (Levine [L2]). *Let $\Phi_n : C_{2n-1} \to C^{(-1)^n}(\mathbf{Z})$ be the (well-defined) homomorphism induced by the Seifert form. Then Φ_n is an isomorphism for $n \geq 3$. For $n = 2$, Φ_2 is a monomorphism whose image $C^{+1}(\mathbf{Z})^0$ is a specified subgroup of $C^{+1}(\mathbf{Z})$ of index 2. For $n = 1$, $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is merely an epimorphism.*

Furthermore, Levine [L3] showed the following (see also Remark 7.4).

Theorem 3.7. *For $\varepsilon = \pm 1$, we have*

$$C^{\varepsilon}(\mathbf{Z}) \cong \mathbf{Z}_2^{\infty} \oplus \mathbf{Z}_4^{\infty} \oplus \mathbf{Z}^{\infty}, \tag{3.1}$$

where the right hand side is the direct sum of countably many (but infinite) copies of the cyclic groups $\mathbf{Z}$, $\mathbf{Z}_2$ and $\mathbf{Z}_4$.

Note that the right hand side of (3.1) is *not* an unrestricted direct sum, i.e., each element of the group is a linear combination of *finitely many* elements corresponding to the generators of the factors.

Remark 3.8. Michel [Mc] showed that for $n \geq 1$, spherical algebraic $(2n-1)$-knots have infinite order in C_{2n-1}, provided that the associated holomorphic function germ has an isolated singularity at the origin and is not non-singular. Note, however, that they are not independent. See Remark 4.2.

For $n = 1$, $\Phi_1 : C_1 \to C^{-1}(\mathbf{Z})$ is far from being an isomorphism. The non-triviality of the kernel of this epimorphism was first shown by Casson-Gordon [CG]. The classification of spherical 1-knots up to cobordism is still an open problem. Moreover, for spherical 1-knots, we have also the important notion of a *ribbon knot* (see, for example, [Rl]). Ribbon knots are null-cobordant. It is still an open problem whether the converse is true or not.

For even dimensions, we have the following vanishing theorem.

Theorem 3.9 (Kervaire [K1]). *For all $n \geq 1$, C_{2n} vanishes.*

Let $\widetilde{C}_n$ be the group of concordance classes of embeddings into S^{n+2} of

(1) the n-dimensional standard sphere S^n for $n \leq 4$, or
(2) homotopy n-spheres for $n \geq 5$.

In [K1] Kervaire showed that the natural surjection $\mathfrak{i}\colon \widetilde{C}_n \to C_n$ is a group homomorphism.

Let us denote by Θ_n the group of h-cobordism classes of smooth oriented homotopy n-spheres, and by bP_{n+1} the subgroup of Θ_n consisting of the h-cobordism classes represented by homotopy n-spheres which bound compact parallelizable manifolds [KM]. Then we have the following

Theorem 3.10 (Kervaire [K1]). *For $n \leq 5$ we have $\widetilde{C}_n \cong C_n$, and for $n > 6$ we have the short exact sequence*

$$0 \to \Theta_{n+1}/bP_{n+2} \to \widetilde{C}_n \xrightarrow{\mathfrak{i}} C_n \to 0.$$

Note that for $n \geq 4$, Θ_{n+1}/bP_{n+2} is a finite abelian group. For details, see [KM].

§4. Cobordism of algebraic 1-knots

As has been pointed out in the previous section, the classification of 1-knots up to cobordism is still unsolved. However, for algebraic 1-knots, a classification is known as follows.

Consider an algebraic 1-knot K associated with a holomorphic function germ $f\colon \mathbf{C}^2, 0 \to \mathbf{C}, 0$ of two variables with an isolated critical point at the origin. Note that K is naturally oriented. Let us further assume that K is spherical. Then it is known that K is an iterated torus knot [Br]. An *iterated torus knot* is a knot obtained from a torus knot by an iteration of the cabling operation (for example, see [Rl]). Furthermore, the relevant operations are always "positive" cablings, which is peculiar to algebraic knots.

For a knot, the fundamental group of its complement in the ambient sphere is called the *knot group.* In [Z1] Zariski explicitly gave generators and relations of the knot group of a spherical algebraic 1-knot. When two spherical algebraic 1-knots are isotopic, they have isomorphic knot groups. Although the converse is not true for general spherical (not necessarily algebraic) 1-knots, it was proved that two spherical algebraic 1-knots with isomorphic knot groups are isotopic (see [Bu1, Z1, Re, Lê]). Furthermore, Burau [Bu1] proved that two spherical algebraic 1-knots with the same Alexander polynomial are isotopic. For a definition of the Alexander polynomial, see §7. It is known that the Alexander polynomial of a spherical 1-knot is determined by its knot group (see, for example, [CF]).

For general algebraic 1-knots which are not necessarily spherical, the following is known. Let $K = K_1 \cup K_2 \cup \cdots \cup K_s$ and $L = L_1 \cup L_2 \cup \cdots \cup L_t$ be algebraic 1-knots, where K_i, $1 \leq i \leq s$, and L_j, $1 \leq j \leq t$, are components of K and L respectively. Then K and L are isotopic if and only if $s = t$, K_i is isotopic to L_i, $1 \leq i \leq s$, and the linking number of K_i and K_j coincides with that of L_i and L_j for $i \neq j$, after renumbering the indices if necessary (for example, see [Re]). It is also known that the multi-variable Alexander polynomial classifies algebraic 1-knots [Bu2, Re, Y].

As to the classification of algebraic 1-knots up to cobordism, we have the following result due to Lê [Lê]. Let K and L be two cobordant spherical algebraic 1-knots. Let us denote their Alexander polynomials by $\Delta_K(t)$ and $\Delta_L(t)$ respectively, where we normalize them so that their degree 0 terms are positive. In [FM2], Fox and Milnor proved that then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that $\Delta_K(t)\Delta_L(t) = t^d f(t) f(1/t)$, where d is the degree of $f(t)$ (for details, see §7 of the present survey). Using this, one can conclude that the product of the Alexander polynomials of two cobordant spherical algebraic 1-knots is a square in $\mathbf{Z}[t]$. In fact, Lê [Lê] proved that two cobordant spherical algebraic 1-knots have the same Alexander polynomial, and hence the following holds.

Theorem 4.1 ([Lê]). *Two cobordant spherical algebraic 1-knots are isotopic.*

For general (not necessarily spherical) algebraic 1-knots, since the linking numbers between the components are cobordism invariants, we see that the same conclusion as in Theorem 4.1 holds also for the general case of not necessarily spherical algebraic 1-knots.

Remark 4.2. It has been shown that the images of the cobordism classes of spherical algebraic 1-knots by $\Phi_1 \colon C_1 \to C^{-1}(\mathbf{Z})$ are not independent. An explicit example is given in [LM].

§5. Cobordism of simple fibered $(2n-1)$-knots

In this section, we will give a classification of simple fibered $(2n-1)$-knots up to cobordism for $n \geq 3$.

Let us first recall that Durfee [D] and Kato [Kt] independently proved an analogue of Theorem 3.1 for (not necessarily spherical) simple fibered knots as follows. Recall that Seifert forms associated with simple fibered knots are unimodular.

Theorem 5.1. *For $n \geq 3$, there is a one-to-one correspondence between the isotopy classes of simple fibered $(2n-1)$-knots in S^{2n+1} and the isomorphism classes of integral unimodular bilinear forms, where the correspondence is given by the Seifert form.*

Note that isomorphism classes of integral bilinear forms correspond to congruence classes of integral square matrices.

The study of cobordism of (not necessarily spherical) odd dimensional simple fibered knots cannot be done by a direct generalization of the results proved by Kervaire and Levine for spherical $(2n-1)$-knots with $n \geq 2$, since we have to consider the topological data contained in the kernel and the cokernel of the intersection form of the fiber (see the exact sequence (2.1)).

For $n \geq 3$, Du Bois and Michel [DM] constructed the first examples of spherical algebraic $(2n-1)$-knots which are cobordant but are not isotopic. Hence, algebraic knots of dimension greater than or equal to five do not have the nice behavior of algebraic 1-knots, since the notion of cobordism and isotopy are distinct.

Moreover, there exist plenty of examples of knots, not necessarily spherical nor algebraic, which are cobordant but are not isotopic for any dimension. For example, in the case of dimension one, the square knot, which is the connected sum of the right hand and the left hand trefoil knots, is cobordant but is not isotopic to the trivial knot. (For more explicit examples, see §8.)

Using Seifert forms, we have a complete characterization of cobordism classes of simple fibered knots as follows (see [BM, B1, B3]).

Theorem 5.2 ([BM]). *For $n \geq 3$, two simple fibered $(2n-1)$-knots are cobordant if and only if their Seifert forms are algebraically cobordant.*

The definition of algebraically cobordant forms will be given later in this section.

Remark 5.3. Related results had been obtained by Vogt [V1, V2], who proved that if two simple (not necessarily fibered) $(2n-1)$-knots, $n \geq 3$, are cobordant, then their Seifert forms are Witt equivalent and satisfy certain properties which are weaker than the algebraic cobordism. Conversely, if two simple $(2n-1)$-knots, $n \geq 3$, with torsion free homologies have algebraically cobordant Seifert forms, then they are cobordant.

In Theorem 5.2 the condition on the integer n is only used to prove the sufficiency, and we have the following theorem which is valid for all odd dimensions.

Theorem 5.4 ([BM]). *For $n \geq 1$, two cobordant simple fibered $(2n-1)$-knots have algebraically cobordant Seifert forms.*

Furthermore, the following holds for (not necessarily fibered) simple knots.

Theorem 5.5 ([BM]). *For $n \geq 3$, two simple $(2n-1)$-knots are cobordant if their Seifert forms associated with $(n-1)$-connected Seifert manifolds are algebraically cobordant.*

To define the algebraic cobordism, we first need to fix some notations and definitions. Let $\mathcal{A}$ be the set of all bilinear forms defined on free $\mathbf{Z}$-modules of finite rank. Set $\varepsilon = (-1)^n$. Let $A\colon G \times G \to \mathbf{Z}$ be a bilinear form in $\mathcal{A}$. We denote by A^T the transpose of A, by S the ε-symmetric form $A + \varepsilon A^T$ associated with A, by $S^*\colon G \to G^*$ the adjoint of S with G^* being the dual $\mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{Z})$ of G, and by $\overline{S}\colon \overline{G} \times \overline{G} \to \mathbf{Z}$ the ε-symmetric non-degenerate form induced by S on $\overline{G} = G/\operatorname{Ker} S^*$. For a submodule M of G, we denote by $\overline{M}$ the image of M in $\overline{G}$ by the natural projection map. A submodule M of a free $\mathbf{Z}$-module G of finite rank is said to be *pure* if G/M is torsion free, or equivalently if M is a direct summand of G. For a submodule M of a free $\mathbf{Z}$-module G of finite rank, we denote by $M^\wedge$ the smallest pure submodule of G which contains M.

Definition 5.6 ([BM]). Let $A_i\colon G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms in $\mathcal{A}$. Set $G = G_0 \oplus G_1$, $A = A_0 \oplus (-A_1)$, $S_i = A_i + \varepsilon A_i^T$ and $S = A + \varepsilon A^T$. We say that A_0 is *algebraically cobordant* to A_1 if there exist a metabolizer M for A in the sense of Definition 3.3 with $\overline{M}$ pure in $\overline{G}$, an isomorphism $\psi\colon \operatorname{Ker} S_0^* \to \operatorname{Ker} S_1^*$, and an isomorphism $\theta\colon \mathrm{Tors}(\operatorname{Coker} S_0^*) \to \mathrm{Tors}(\operatorname{Coker} S_1^*)$ which satisfy the following two conditions:

$$\text{(c1)} \qquad M \cap \operatorname{Ker} S^* = \{(x, \psi(x))\colon x \in \operatorname{Ker} S_0^*\} \subset \operatorname{Ker} S_0^* \oplus \operatorname{Ker} S_1^* = \operatorname{Ker} S^*,$$

$$\text{(c2)} \qquad \begin{aligned} d(S^*(M)^\wedge) &= \{(y, \theta(y))\colon y \in \mathrm{Tors}(\operatorname{Coker} S_0^*)\} \\ &\subset \mathrm{Tors}(\operatorname{Coker} S_0^*) \oplus \mathrm{Tors}(\operatorname{Coker} S_1^*) \\ &= \mathrm{Tors}(\operatorname{Coker} S^*), \end{aligned}$$

where d is the quotient map $G^* \to \operatorname{Coker} S^*$ and "Tors" means the torsion subgroup.

In the above situation, we also say that A_0 and A_1 are *algebraically cobordant with respect to ψ and θ.*

Recall that the knot cobordism is an equivalence relation. Furthermore, any unimodular matrix can be realized as a Seifert matrix associated with a simple fibered $(2n-1)$-knot, $n \geq 3$. Therefore, Theorem 5.2 implies the following

Theorem 5.7. *Algebraic cobordism is an equivalence relation on the set of unimodular forms.*

Example 5.8. In [BM, Theorem 1], it is claimed that the algebraic cobordism is an equivalence relation on the whole set of integral bilinear forms $\mathcal{A}$. However, this may be not true as explained below.

Let us consider the three matrices

$$A_0 = \begin{pmatrix} 0 & 4 & -2 & -3 \\ -4 & 0 & -2 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & -1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 4 & 1 & 2 \\ -4 & 0 & 1 & -2 \\ -1 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 4 & -6 & 1 \\ -4 & 0 & -2 & -1 \\ 6 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

which are given in [V2, p. 45]. We identify A_i with the corresponding bilinear form $A_i \colon G_i \times G_i \to \mathbf{Z}$ with $G_i \cong \mathbf{Z}^4$, $i = 0, 1, 2$. Set

$$\begin{aligned}
m_1 &= (0, 0, 1, 0, 0, \ \ 0, -2, \ \ 0) \in G_0 \oplus G_1, \\
m_2 &= (0, 1, 0, 2, 0, \ \ 0, \ \ 0, -1) \in G_0 \oplus G_1, \\
m_3 &= (1, 0, 0, 0, 1, \ \ 0, \ \ 0, \ \ 0) \in G_0 \oplus G_1, \\
m_4 &= (0, 1, 0, 0, 0, \ \ 1, \ \ 0, \ \ 0) \in G_0 \oplus G_1, \\
n_1 &= (0, 0, 2, 0, 0, -1, -1, \ \ 0) \in G_1 \oplus G_2, \\
n_2 &= (0, 0, 0, 1, 0, \ \ 0, \ \ 0, \ \ 2) \in G_1 \oplus G_2, \\
n_3 &= (1, 0, 0, 0, 1, \ \ 0, \ \ 0, \ \ 0) \in G_1 \oplus G_2, \\
n_4 &= (0, 1, 0, 0, 0, \ \ 1, \ \ 0, \ \ 0) \in G_1 \oplus G_2.
\end{aligned}$$

Then we see that the subgroup generated by m_1, m_2, m_3, m_4 of $G_0 \oplus G_1$ gives a metabolizer for $A_0 \oplus (-A_1)$, and that the subgroup generated by n_1, n_2, n_3, n_4 of $G_1 \oplus G_2$ gives a metabolizer for $A_1 \oplus (-A_2)$. Furthermore, it is easy to check that A_i and A_{i+1} are algebraically cobordant for $\varepsilon = +1$ with respect to the "identity"

$$\mathbf{Z} \oplus \mathbf{Z} \oplus 0 \oplus 0 = \operatorname{Ker} S_i^* \to \operatorname{Ker} S_{i+1}^* = \mathbf{Z} \oplus \mathbf{Z} \oplus 0 \oplus 0,$$

$i = 0, 1$, where $S_i = A_i + A_i^T$, $i = 0, 1, 2$.

However, in [V2] it is shown that A_0 and A_2 are not algebraically cobordant with respect to the "identity".

In the proof given in [BM, pp. 38–39], it is shown that if A_i and A_{i+1} are algebraically cobordant with respect to ψ_i, $i = 0, 1$ (see Definition 5.6 (c1)), then A_0 and A_2 are algebraically cobordant with respect to $\psi_1 \circ \psi_0$. So, this contradicts Vogt's result mentioned above.

In fact, in general we may not have the direct sum decomposition $G_i = \operatorname{Ker} S_i^* \oplus T_i$, $i = 0, 1, 2$, mentioned in the proof given in [BM, p. 39].

Presumably, the above example would show that the algebraic cobordism is not an equivalence relation on the set of general (not necessarily unimodular) integral bilinear forms defined on free $\mathbf{Z}$-modules of finite rank. Since the relation introduced by Vogt [V2] and that of Definition 5.6 are slightly different, we do not know at present if the relation of algebraic cobordism is an equivalence relation or not.

Remark 5.9. For general forms which are not necessarily unimodular, we can consider the equivalence relation generated by the algebraic cobordism, called the *weak algebraic cobordism.* Then by using Theorem 5.5,[3] we can show that if two simple $(2n-1)$-knots, $n \geq 3$, have weakly algebraically cobordant Seifert forms with respect to $(n-1)$-connected Seifert manifolds, then they are cobordant.

Furthermore, we can prove the following. A simple $(2n-1)$-knot is said to be *C-algebraically fibered* if its Seifert form is algebraically cobordant to a unimodular form (see [BS1]). Then, two simple C-algebraically fibered $(2n-1)$-knots, $n \geq 3$, are cobordant if and only if their Seifert forms are weakly algebraically cobordant. We do not know if this is true for all simple $(2n-1)$-knots, $n \geq 3$.

Let A_i be Seifert forms associated with $(n-1)$-connected Seifert manifolds V_i of simple $(2n-1)$-knots K_i, $i = 0, 1$, and S_i^* the adjoint of the intersection form of V_i. Since we have the exact sequence

$$\begin{aligned} 0 = H_{n+1}(V_i, K_i) \to H_n(K_i) \to H_n(V_i) \xrightarrow{S_i^*} H_n(V_i, K_i) \\ \to H_{n-1}(K_i) \to H_{n-1}(V_i) = 0 \end{aligned}$$

[3]Here, we also need the fact that every form in $\mathcal{A}$ can be realized as the Seifert form of a simple $(2n-1)$-knot.

associated with the pair (V_i, K_i), where we identify $H_n(V_i, K_i)$ with the dual of $H_n(V_i)$ (see (2.1)), $\operatorname{Ker} S_i^*$ and $\operatorname{Coker} S_i^*$ are naturally identified with $H_n(K_i)$ and $H_{n-1}(K_i)$ respectively.

As remarked before, in the case of a spherical knot K we have $H_n(K) = H_{n-1}(K) = 0$, and the intersection form is an isomorphism. Hence the algebraic cobordism for Seifert forms associated with *spherical* simple knots is reduced to the Witt equivalence, and Theorem 5.2 follows from the classical result of Kervaire and Levine (see Theorem 3.4 and the paragraph just after Remark 3.5).

In order to clarify the relation of algebraic cobordism, we present here several examples.

Example 5.10. (1) Let us consider any integral bilinear form A in $\mathcal{A}$ such that $A + \varepsilon A^T$ is unimodular. Then, $A \oplus (-A)$ is always algebraically cobordant to the zero form.

(2) Let us consider the integral bilinear forms A_0 and A_1 represented by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$$

respectively, which are given in [K2, p. 93]. Then it is easy to check that the subgroup of $\mathbf{Z}^4$ generated by $(3, 1, 3, 0)^T$ and $(0, 1, 2, 1)^T$ is a metabolizer for $A_0 \oplus (-A_1)$. Since $A_i - A_i^T$ are unimodular, $i = 0, 1$, we see that A_0 and A_1 are algebraically cobordant for $\varepsilon = -1$. Note that A_0 and A_1 are not congruent to each other.

(3) The following example is a generalization of those given in [BMS]. Let us consider the two matrices

$$A_0 = \begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix},$$

which are identified with the corresponding integral bilinear forms, where p and q are odd integers with $1 \leq p < q$. Note that they are both unimodular and

$$S_0 = A_0 + \varepsilon A_0^T = S_1 = A_1 + \varepsilon A_1^T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

where $\varepsilon = -1$. Let us show that A_0 and A_1 are algebraically cobordant in the sense of Definition 5.6 for $\varepsilon = -1$.

Let r be the greatest common divisor of p and q and set $p = rp'$ and $q = rq'$. Furthermore, set $m = (q', 0, p', 0)^T$ and $m' = (0, p', 0, q')^T$. Then it is easy to see that the submodule M of $\mathbf{Z}^4$ generated by m and

m' constitutes a metabolizer for $A = A_0 \oplus (-A_1)$. Since $S_0 = S_1$ are non-degenerate, we have only to verify condition (c2) of Definition 5.6.

Set $S = S_0 \oplus (-S_1) = A - A^T$. Let $S^*\colon \mathbf{Z}^4 \to \mathbf{Z}^4$, $S_0^*\colon \mathbf{Z}^2 \to \mathbf{Z}^2$ and $S_1^*\colon \mathbf{Z}^2 \to \mathbf{Z}^2$ be the adjoints of S, S_0 and S_1 respectively. It is easy to see that $\operatorname{Coker} S_0^* = \operatorname{Coker} S_1^*$ is naturally identified with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Furthermore, we have

$$S^*(m) = m^T S = (0,\, 2q',\, 0,\, -2p') \quad \text{and}$$
$$S^*(m') = (m')^T S = (-2p',\, 0,\, 2q',\, 0).$$

Therefore, $S^*(M)^\wedge$, the smallest direct summand of $\mathbf{Z}^4$ containing $S^*(M)$, is the submodule of $\mathbf{Z}^4$ generated by $(0,\, q',\, 0,\, -p')$ and $(-p',\, 0,\, q',\, 0)$. Hence, for the natural quotient map $d\colon \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we have

$$d(S^*(M)^\wedge) = \{(x,\, x)\colon x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\},$$

since $\operatorname{Im} S_i^*$ is generated by $(2,\, 0)$ and $(0,\, 2)$, $i = 0,\, 1$, and $\operatorname{Im} S^*$ is generated by $(2,\, 0,\, 0,\, 0)$, $(0,\, 2,\, 0,\, 0)$, $(0,\, 0,\, 2,\, 0)$ and $(0,\, 0,\, 0,\, 2)$. Therefore, we conclude that the unimodular matrices A_0 and A_1 are algebraically cobordant.

Note that A_0 and A_1 are not congruent, since there exists an element $x \in \mathbf{Z}^2$ such that $x^T A_0 x = p^2$, while such an element does not exist for A_1.

Let us give a sketch of the proof of Theorem 5.2. Let $K_0 = \partial F_0$ and $K_1 = \partial F_1$ be two simple fibered $(2n-1)$-knots with $n \geq 3$ with fibers F_0 and F_1 respectively. Denote by A_0 and A_1 the Seifert forms associated with F_0 and F_1 respectively.

To prove the necessity in Theorem 5.2, we first suppose that $K_0 \subset S^{2n+1} \times \{0\}$ and $K_1 \subset S^{2n+1} \times \{1\}$ are cobordant. Then we see that the union of the cobordism and the fibers bound a compact oriented $(2n+1)$-dimensional manifold W embedded in $S^{2n+1} \times [0,\, 1]$ by using the obstruction theory as in §2. Using the kernel of the homomorphism induced by the inclusion $F_0 \cup F_1 \to W$, we can construct a metabolizer for $A_0 \oplus (-A_1)$ which fulfills all the conditions in the definition of algebraic cobordism. (For this we need to have that A_0 and A_1 are unimodular, which is guaranteed since K_0 and K_1 are fibered.) We refer to [BM] for details.

For sufficiency we suppose that A_0 and A_1 are algebraically cobordant with respect to a metabolizer M. We consider F_i to be embedded in $S^{2n+1} \times \{i\}$, $i = 0,\, 1$, and denote by F the connected sum $F = F_0 \sharp F_1$ embedded in $S^{2n+1} \times [0,\, 1]$. Note that we naturally have

$H_n(F) = H_n(F_0) \oplus H_n(F_1)$. Then, since $n \geq 3$, we can show that one can perform embedded surgeries on the connected sum of Seifert manifolds in $S^{2n+1} \times [0, 1]$ so that we obtain a simply connected submanifold X of $S^{2n+1} \times [0, 1]$ with $\partial X = (K_0 \times \{0\}) \coprod (K_1 \times \{0\})$ and $H_*(X, K_i) = 0$ for $i = 0, 1$. According to Smale's h-cobordism theorem [Sm2, M2] we have $X \cong K_0 \times [0, 1]$, and thus X gives a cobordism between K_0 and K_1. This is where we need to have $(2n-1)$-dimensional knots with $n \geq 3$, since the h-cobordism theorem is valid only for $\dim X \geq 6$.

The crucial point in the proof is to see that the technical conditions imposed on the metabolizer in Definition 5.6 give a strategy to perform the right embedded surgeries. For details, see [BM, B3].

§6. 3-Dimensional knots

In this section, we deal with 3-dimensional knots.[4] This case is much more difficult than that of higher dimensional knots, since the dimension of the Seifert manifolds associated with 3-knots is equal to four. The topology of 4-dimensional manifolds is exceptional, and the usual technics like the Whitney trick [W2] used in the case of higher dimensional manifolds are not available any more.

The algebraic cobordism of Seifert forms is a necessary condition for the existence of a cobordism between two simple fibered $(2n-1)$-knots for all $n \geq 1$ (see Theorem 5.4). Furthermore, two isotopic simple fibered $(2n-1)$-knots have isomorphic Seifert forms for all $n \geq 1$ (for example, see [D, Kt, S1]). However, it is known that there exist 3-dimensional simple fibered knots which are abstractly diffeomorphic and have isomorphic Seifert forms but which are not isotopic (see Example 6.1 below). This shows that the one-to-one correspondence between the isotopy classes of knots and the isomorphism classes of Seifert forms stated in Theorem 5.1 does not hold for $n = 2$. In fact, these fibered 3-knots are even not cobordant (see Remark 6.7). Hence, for 3-dimensional knots, isotopy classes and cobordism classes must be characterized by new equivalence relations. Isotopy classes of 3-knots were studied in [S1, S2, S4] (see also [Hi]). For cobordism classes we will define a new equivalence relation. For this we need to use Spin structures on manifolds.

Recall that a Spin *structure* on a manifold X means the homotopy class of a trivialization of $TX \oplus \varepsilon^N$ over the 2-skeleton $X^{(2)}$ of X, where TX denotes the tangent bundle and ε^N is a trivial vector bundle of dimension N sufficiently large. Note that X admits a Spin structure if and only if its second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbf{Z}_2)$ vanishes

[4] In the following, all 3-knots will be oriented.

and that if it admits, then the set of all Spin structures on X is in one-to-one correspondence with $H^1(X; \mathbf{Z}_2)$.

Let K be an oriented 3-knot, with a Seifert manifold V, embedded in S^5. Then K has a natural normal 2-framing $\nu = (\nu_1, \nu_2)$ in S^5 such that the first normal vector field ν_1 is obtained as the inward normal vector field of $K = \partial V$ in V. The homotopy class of this 2-framing does not depend on the choice of the Seifert manifold V. Then K carries a tangent 3-framing on its 2-skeleton $K^{(2)}$ such that the juxtaposition with the above 2-framing gives the standard framing of S^5 restricted to $K^{(2)}$ up to homotopy. This means that K carries a natural Spin structure, which is determined uniquely up to homotopy. Furthermore, this Spin structure coincides with that induced from the Seifert manifold V, which is endowed with the natural Spin structure induced from S^5.

In the case of 3-knots, Spin structures must be considered as the following example shows.

Example 6.1. Let K_0 and K_1 be the simple fibered 3-knots which are abstractly diffeomorphic to $S^1 \times \Sigma_g$, constructed in [S4, Proposition 3.8], where Σ_g is the closed connected orientable surface of genus $g \geq 2$. They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between K_0 and K_1 which preserves their Spin structures. Consequently they are not isotopic.

In order to study cobordisms of 3-knots, we will use some results valid only for 3-dimensional manifolds without torsion on the first homology group. Hence, we define

Definition 6.2 ([BS1]). We say that a 3-knot K is *free* if $H_1(K)$ is torsion free over $\mathbf{Z}$.

Moreover, for free knots we do not need to consider condition (c2) in the definition of the algebraic cobordism (see Definition 5.6), which simplifies the argument.

Definition 6.3 ([BS1]). Consider two simple 3-knots K_0 and K_1. Let A_0 and A_1 be the Seifert forms of K_0 and K_1 respectively with respect to 1-connected Seifert manifolds. We say that the pairs (K_0, A_0) and (K_1, A_1) are Spin *cobordant* (for simplicity, we also say that the Seifert forms A_0 and A_1 are Spin *cobordant*) if there exists an orientation preserving diffeomorphism $h: K_0 \to K_1$ such that

(1) h preserves their Spin structures,
(2) A_0 and A_1 are algebraically cobordant with respect to $h_*: H_2(K_0) \to H_2(K_1)$ and $h_*|_{\operatorname{Tors} H_1(K_0)}: \operatorname{Tors} H_1(K_0) \to \operatorname{Tors} H_1(K_1)$, where we identify $H_2(K_i)$ and $H_1(K_i)$ with $\operatorname{Ker} S_i^*$ and $\operatorname{Coker} S_i^*$

respectively (see the exact sequence (2.1)) and $S_i = A_i + A_i^T$, $i = 0, 1$.

Note that if K_0 and K_1 are free 3-knots, then we do not need to consider condition (c2) of Definition 5.6 and hence the isomorphism $h_*|_{\mathrm{Tors}\, H_1(K_0)}$ in the above definition.

In [BS1] we proved the following.

Theorem 6.4. *Two simple fibered free* 3*-knots are cobordant if and only if their Seifert forms with respect to* 1*-connected fibers are* Spin *cobordant.*

Remark 6.5. Note that in the case of homology 3-spheres embedded in S^5, the corresponding result had been obtained in [S3].

Since the cobordism for knots is an equivalence relation, the Spin cobordism is an equivalence relation on the set of Seifert forms of simple fibered free 3-knots with respect to 1-connected Seifert manifolds.

Let us show that the Spin cobordism is a necessary condition for the existence of a knot cobordism between given two simple fibered 3-knots. Let K_0 and K_1 be two cobordant simple fibered 3-knots with fibers F_0 and F_1 respectively. Denote by $X \cong K_0 \times [0, 1]$ a submanifold of $S^5 \times [0, 1]$ which gives a cobordism between K_0 and K_1, and set $N = F_0 \cup X \cup (-F_1)$. By classical obstruction theory as described in §2, we see that the closed oriented 4-manifold $N \subset S^5 \times [0, 1]$ is the boundary of a compact oriented 5-dimensional submanifold W of $S^5 \times [0, 1]$. Using a normal 2-framing of X in $S^5 \times [0, 1]$ induced from the inward normal vector field along $N = \partial W$ in W, we see that the diffeomorphism h between K_0 and K_1 induced by X preserves their Spin structures.

Moreover, in [BM], it has been shown that the two forms A_0 and A_1, associated with the fibers, are algebraically cobordant with respect to $h_*\colon H_2(K_0) \to H_2(K_1)$ and $h_*|_{\mathrm{Tors}\, H_1(K_0)}\colon \mathrm{Tors}\, H_1(K_0) \to \mathrm{Tors}\, H_1(K_1)$.

Finally we get the following result, in which the knots may not necessarily be free.

Proposition 6.6 ([BS1]). *If two simple fibered* 3*-knots are cobordant, then their Seifert forms with respect to* 1*-connected fibers are* Spin *cobordant.*

Remark 6.7. In Example 6.1 above, the Seifert forms of K_0 and K_1 are algebraically cobordant, but are not Spin cobordant. Hence they cannot be cobordant by Proposition 6.6 (or Theorem 6.4). Example 6.1 shows that Spin structures are essential in the theory of cobordism of 3-knots as well.

We have another example as follows.

Example 6.8. Let P be a non-trivial orientable S^1-bundle over the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(P)$ is not torsion free in general. For every positive integer n, let $K_1, K_2, \ldots, K_n$ be the simple fibered 3-knots constructed in [S4, Theorem 3.1] which are all abstractly diffeomorphic to P. They have the property that their fibers are all diffeomorphic and their Seifert forms are isomorphic to each other, but any such isomorphism restricted to $H_2(K_i)$ cannot be realized by a diffeomorphism. Thus, the Seifert forms of K_i, $i = 1, 2, \ldots, n$, are algebraically cobordant to each other, but are not Spin cobordant. Hence they are not cobordant by Proposition 6.6, which is valid also for non-free simple fibered 3-knots.

Using the 5-dimensional stable h-cobordism theorem due to Lawson [La] and Quinn [Q] together with Boyer's work [Bo], we also have the following theorem, in which the 3-knots are simple and free, but may not be fibered.

Theorem 6.9 ([BS1]). *Consider two simple free* 3*-knots in* S^5*. If their Seifert forms with respect to* 1*-connected Seifert manifolds are* Spin *cobordant, then they are cobordant.*

The proof of the above theorem is very technical and complicated, and we refer to [BS1] for details. Finally Proposition 6.6 and Theorem 6.9 imply Theorem 6.4.

Remark 6.10. Some of the results in [BS1] depend on the possibly erroneous hypothesis that the algebraic cobordism is an equivalence relation on the whole set of integral bilinear forms. However, all the results are valid if we replace the algebraic cobordism with the weak algebraic cobordism as introduced in Remark 5.9 and the Spin cobordism with the equivalence relation generated by the Spin cobordism.

§7. Fox-Milnor type relation

In [FM2] Fox and Milnor showed that the Alexander polynomials of two cobordant 1-knots should satisfy a certain property. In this section, we explain this property for odd dimensional knots and present an application to the cobordism classes of spherical fibered knots.

In the following, for a polynomial $f(t) \in \mathbf{Z}[t]$, we set

$$f^*(t) = t^d f(t^{-1}),$$

where d is the degree of $f(t)$. We say that a polynomial $f(t) \in \mathbf{Z}[t]$ is *symmetric* if $f^*(t) = \pm t^a f(t)$ for some $a \in \mathbf{Z}$.

Let K be either a spherical $(2n-1)$-knot or a simple $(2n-1)$-knot with Seifert matrix A. As mentioned before, we still assume that A is associated with an $(n-1)$-connected Seifert manifold when K is simple. Then the polynomial

$$\Delta_K(t) = \det(tA + (-1)^n A^T)$$

is called the *Alexander polynomial* of K (see [Al, L1]). It is known to be an isotopy invariant of K up to a multiple of $\pm t^a$, $a \in \mathbf{Z}$. For fibered knots, we use (unimodular) Seifert matrices with respect to fibers so that the Alexander polynomial is well-defined up to a multiple of ± 1 and has leading coefficient ± 1. Note that the Alexander polynomial of a knot is always symmetric.

The following relation is called the *Fox-Milnor type relation* (for proofs, see [L2, BM], for example).

Proposition 7.1. *Let K_0 and K_1 be two $(2n-1)$-knots which are both spherical or both simple. If they are cobordant, then there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that*

$$\Delta_{K_0}(t)\Delta_{K_1}(t) = \pm t^a f(t) f^*(t) \tag{7.1}$$

for some $a \in \mathbf{Z}$.

For example, in [DM], Du Bois and Michel showed that the algebraic knots constructed in [Sz] are in fact not cobordant by exploiting the Fox-Milnor type relation.

Let us show that the above relation, although very simple, gives us a lot of information on the cobordism of knots.

Let us recall that C_n denotes the cobordism group of spherical n-knots. Let us denote by F_n the subgroup of C_n generated by the cobordism classes of fibered knots. Note that F_n coincides with the set of all cobordism classes which contain a fibered knot.

Then we can prove the following proposition by using the Fox-Milnor type relation. Although it might be implicit in the works of Levine [L2, L3], Kervaire [K2] and Stoltzfus [Sf], here we give a detailed proof in order to clarify how to apply the Fox-Milnor type relation.

Proposition 7.2. *The group C_n/F_n is infinitely generated if n is odd.*

Proof. Set $n = 2k-1$. We have only to prove that $(C_n/F_n) \otimes \mathbf{Z}_2$ contains $\mathbf{Z}_2^\infty$.

First we consider the case where k is odd. For each positive integer p, set $\Delta_p(t) = pt^2 + (1-2p)t + p$. Note that $\Delta_p(t)$ is irreducible over $\mathbf{Z}$.

According to Levine (see [L2]), there exists a simple spherical $(2k-1)$-knot K_p in S^{2k+1} whose Alexander polynomial $\Delta_{K_p}(t)$ is equal to $\Delta_p(t)$. Let $[K_p]$ denote the class in $(C_n/F_n)\otimes \mathbf{Z}_2 = (C_n/F_n)/2(C_n/F_n) = C_n/(F_n+2C_n)$ represented by K_p. In order to show that $(C_n/F_n)\otimes \mathbf{Z}_2$ contains $\mathbf{Z}_2^\infty$, we have only to show that $\{[K_p]\}_{p\geq 2}$ are linearly independent over $\mathbf{Z}_2$.

Suppose that $K_{p_1}\sharp K_{p_2}\sharp\cdots\sharp K_{p_\ell}$ is cobordant to $L\sharp L\sharp L'$, where $p_1, p_2, \ldots, p_\ell$ are distinct positive integers with $p_i \geq 2$, L is a spherical $(2k-1)$-knot, and L' is a spherical fibered $(2k-1)$-knot. Then by Proposition 7.1 we have

$$\Delta_{K_{p_1}}(t)\Delta_{K_{p_2}}(t)\cdots\Delta_{K_{p_\ell}}(t)\Delta_L(t)^2\Delta_{L'}(t) = \pm t^a f(t)f^*(t)$$

for some $a\in\mathbf{Z}$ and $f(t)\in\mathbf{Z}[t]$.

Since $\Delta_{K_{p_i}}(t)$ are irreducible and symmetric, each $\Delta_{K_{p_i}}(t)$ should appear an even number of times in the irreducible decomposition of $f(t)f^*(t)$. Therefore, $\Delta_{K_{p_i}}(t)$ should divide $\Delta_{L'}(t)$, since $\Delta_{K_{p_1}}(t)$, $\Delta_{K_{p_2}}(t), \ldots, \Delta_{K_{p_\ell}}(t)$ are pairwise relatively prime.

On the other hand, since L' is fibered, its Seifert matrix is unimodular and hence $\Delta_{L'}(t)$ has leading coefficient ± 1. This is a contradiction, since the leading coefficient of $\Delta_{K_{p_i}}(t)$ is equal to $p_i\geq 2$.

Therefore, $\{[K_p]\}_{p\geq 2}\subset (C_n/F_n)\otimes\mathbf{Z}_2$ are linearly independent over $\mathbf{Z}_2$.

When k is even, by considering the polynomial $\widetilde{\Delta}_p(t) = pt^4-(2p-1)t^2+p$, $p\geq 2$, instead of $\Delta_p(t)$ in the above argument, we get the desired conclusion. This completes the proof. Q.E.D.

Remark 7.3. The above polynomials $\Delta_p(t)$ and $\widetilde{\Delta}_p(t)$ were used by Kervaire in [K1, Théorème III.12] for showing that C_{2k-1} is infinitely generated.

Remark 7.4. When k is even, every degree two symmetric polynomial which arises as the Alexander polynomial of a $(2k-1)$-knot is reducible. In fact, in [L2], it is mentioned that such a polynomial should be of the form

$$a(a+1)t^2-(2a(a+1)+1)t+a(a+1) = (at-(a+1))((a+1)t-a).$$

The degree two symmetric polynomial constructed in [L3, p. 109] for $\varepsilon = 1$ is also reducible, and it seems that the proof of Theorem 3.7 (or [L3, Theorem, p. 108]) given there should appropriately be modified.

§8. Examples

In this section, we review some examples constructed in [B2, BMS, BS1].

First we construct non-spherical 3-knots which are cobordant but are not isotopic.

Example 8.1 ([BS1]). A *stabilizer* is a simple fibered spherical 3-knot whose fiber F is diffeomorphic to $(S^2 \times S^2)\sharp(S^2 \times S^2) \setminus \operatorname{Int} D^4$. Such a stabilizer does exist (see [S2, §4]). Moreover, we denote by K_S a stabilizer with Seifert matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

with respect to a basis of $H_2(F)$ denoted by a_1, a_2, a_3, a_4 (see [S1, p. 600] or [S4, §10]).

Since A is not congruent to the zero form, K_S is a non-trivial 3-knot.

Moreover, the submodule generated by a_1 and a_3 is a metabolizer for A, and one can perform embedded surgeries on the two cycles a_1 and a_3, represented by two embedded 2-spheres in F. The result of this embedded surgery in D^6 is a 4-dimensional disk properly embedded in D^6 with K_S as boundary. Thus K_S is null-cobordant, i.e., it is cobordant to the trivial spherical 3-knot.

Then consider any simple fibered 3-knot K which is not spherical. The two simple fibered 3-knots $K\sharp K_S$ and K are not isotopic, since the ranks of the second homology groups of their fibers are distinct. However, these knots are cobordant.

In the following example, we construct non-spherical simple fibered $(2n-1)$-knots with $n \geq 3$ which are cobordant but are not isotopic. These knots are constructed using algebraic knots.

Example 8.2 ([B2]). Let K_i, with $i = 0, 1$, be the spherical algebraic $(2n-1)$-knots, $n \geq 3$, associated with the isolated singularity at 0 of the polynomial functions $h_i \colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defined by

$$h_i(x_0, x_1, \ldots, x_n) = g_i(x_0, x_1) + x_2^p + x_3^q + \sum_{k=4}^{n} x_k^2$$

with

$$g_0(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{s+6} - 4x_1 x_0^{(s+9)/2}\Big) \Big((x_0^2 - x_1^5)^2 - x_1^{r+10} - 4x_0 x_1^{(r+15)/2}\Big),$$

and

$$g_1(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{r+14} - 4x_1 x_0^{(r+17)/2}\Big) \Big((x_0^2 - x_1^5)^2 - x_1^{s+2} - 4x_0 x_1^{(s+7)/2}\Big),$$

where $s \geq 11$, $s \neq r + 8$, s and r are odd, and p and q are distinct prime numbers which do not divide the product $330(30+r)(22+s)$ (see [DM, p. 166]). Note that the algebraic knots K_i associated with h_i are spherical for $i = 0, 1$. It has been shown in [DM] that the algebraic knots K_0 and K_1 are cobordant but are not isotopic.

Now let L be the algebraic $(2n-1)$-knot associated with the isolated singularity at 0 of the polynomial function $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defined by

$$f(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{n} x_k^2.$$

Note that L is not spherical.

Let us consider the connected sums $L_i = K_i \sharp L$, $i = 0, 1$, which are simple fibered $(2n-1)$-knots. Then in [B2] it has been shown that L_0 and L_1 are cobordant but are not isotopic.

Note that according to [A, Theorem 4, p. 117], the knots L_0 and L_1, which are connected sums of two algebraic knots, are not algebraic.

Let K be a knot. A *stabilization* of K is the operation of taking the connected sum $K \sharp K_S$ for some null-cobordant spherical knot K_S. As the above examples show, stabilization is a natural way to construct knots that are cobordant but are not isotopic. We have other types of constructions as follows.

Example 8.3. The matrices given in Example 5.10 (2) give two spherical simple $(2n-1)$-knots with $n \geq 3$ odd which are cobordant but are not isotopic. Similarly, the matrices given in Example 5.10 (3) give two simple fibered non-spherical $(2n-1)$-knots with $n \geq 3$ odd which are cobordant but are not isotopic.

§9. Pull back relation for knots

For cobordisms of non-spherical knots, Yukio Matsumoto asked the following question.

($\mathcal{Q}$) *If two non-spherical knots (of sufficiently high dimension) are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots? In other words, consider the action of the spherical knot cobordism group on the set of cobordism classes of codimension two embeddings of manifolds of a fixed simple homotopy type into a sphere. Then, is the action transitive?*

According to the codimension two surgery theory [Mt2], the answer to the above question is affirmative provided that the two non-spherical knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres. This motivates the following definition.

Definition 9.1 ([BMS]). Let K_0 and K_1 be oriented m-knots in S^{m+2}. We say that K_0 is a *pull back* of K_1 if there exists a degree one smooth map $g\colon S^{m+2} \to S^{m+2}$ with the following properties:

(1) g is transverse to K_1,
(2) $g^{-1}(K_1) = K_0$,
(3) $g|_{K_0}\colon K_0 \to K_1$ is an orientation preserving simple homotopy equivalence.

In this case, we write $K_0 \succ K_1$. We say that two m-knots are *pull back equivalent* if they are equivalent with respect to the equivalence relation generated by the pull back relation.

The following properties are direct consequences of the previous definition.

(1) $K \succ K$ for any m-knot K.
(2) $K_0 \succ K_1$ and $K_1 \succ K_2$ imply $K_0 \succ K_2$ for any m-knots K_0, K_1 and K_2.
(3) $K_0 \succ K_1$ and $K_0' \succ K_1'$ imply $K_0 \sharp K_0' \succ K_1 \sharp K_1'$ for any m-knots K_0, K_0', K_1 and K_1'.

Furthermore, if we restrict ourselves to spherical m-knots, then it is not difficult to see that the *trivial m-knot K_U* is the minimal element, i.e., $K \succ K_U$ for every spherical m-knot K, where K_U is defined to be the isotopy class of the boundary of an $(m+1)$-dimensional disk embedded in S^{m+2}.

Here are some basic results on the pull back relation for simple fibered $(2n-1)$-knots, $n \geq 3$.

Theorem 9.2 ([BMS]). *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. If $K_0 \succ K_1$ and $K_1 \succ K_0$, then K_0 is isotopic to K_1. In other words, the relation "$\succ$" defines a partial order for simple fibered $(2n-1)$-knots in S^{2n+1} for $n \geq 3$.*

Theorem 9.3 ([BMS]). *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then $K_0 \succ K_1$ if and only if there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is isotopic to the connected sum $K_1 \sharp \Sigma$.*

Remark 9.4. For $n = 1$, Theorem 9.3 does not hold[5]. Let K_1 be a non-trivial spherical prime fibered 1-knot in S^3 and K_0 a spherical prime satellite fibered 1-knot with companion K_1, where their fibering structures are compatible. Then we can show that $K_0 \succ K_1$. However, K_0 is not isotopic to the connected sum $K_1 \sharp \Sigma$ for any non-trivial 1-knot Σ. Note that such a construction does not give a counter example to Theorem 9.3 for $n \geq 3$, since such a satellite knot in higher dimensions is always a connected sum by virtue of Theorem 5.1.

Let K_0 and K_1 be two simple fibered $(2n-1)$-knots with $n \geq 3$. By Theorem 9.3 if K_0 is pull back equivalent to K_1, then they are cobordant after taking connected sums with some spherical knots. In the following proposition, we show that the converse is not true in general.

Proposition 9.5 ([BMS]). *For every odd integer $n \geq 3$, there exists a pair (K_0, K_1) of simple fibered $(2n-1)$-knots with the following properties:*

(1) *the knots K_0 and K_1 are cobordant, but*
(2) *the knots K_0 and K_1 are not pull back equivalent.*

Proof. Let us consider the two matrices A_0 and A_1 given in Example 5.10 (3).

By Theorem 5.1, there exists a simple fibered $(2n-1)$-knot K_i which realizes A_i as its Seifert form with respect to the fiber, $i = 0, 1$. By Theorem 5.5, K_0 and K_1 are cobordant.

Let us now show that K_0 and K_1 are not pull back equivalent. By Theorem 9.3, we have only to show that for any spherical simple fibered $(2n-1)$-knots Σ_0 and Σ_1 in S^{2n+1}, $K_0 \sharp \Sigma_0$ is never isotopic to $K_1 \sharp \Sigma_1$.

Since $K_i \sharp \Sigma_i$ is a fibered knot, we can consider the monodromy on the n-th homology group of the fiber, $i = 0, 1$. Let us denote by H_i the monodromy matrix of $K_i \sharp \Sigma_i$ and by $\widetilde{A}_i$ its Seifert matrix with respect

[5]The authors are indebted to Shicheng Wang for the construction in this remark.

to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for K_i and that for Σ_i. It is known that $H_i = (-1)^{n+1}\widetilde{A}_i^{-1}\widetilde{A}_i^T$ (for example, see [D]). Therefore, we have

$$H_0 = \begin{pmatrix} -1 & 0 \\ 2p^2 & -1 \end{pmatrix} \oplus H_0' \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 \\ 2q^2 & -1 \end{pmatrix} \oplus H_1',$$

where H_i' is the monodromy matrix of Σ_i, $i = 0, 1$.

Let us consider $\operatorname{Ker}((I + H_i)^2)$, where I is the unit matrix, $i = 0, 1$. Since Σ_i are spherical knots, the monodromy matrices H_i' cannot have eigenvalue -1. Therefore, $\operatorname{Ker}((I + H_i)^2)$ corresponds exactly to the homology of the fiber of K_i.

Suppose that $K_0 \sharp \Sigma_0$ is isotopic to $K_1 \sharp \Sigma_1$. Then the Seifert form of $K_0 \sharp \Sigma_0$ restricted to $\operatorname{Ker}((I + H_0)^2)$ should be isomorphic to that of $K_1 \sharp \Sigma_1$ restricted to $\operatorname{Ker}((I + H_1)^2)$. This means that A_0 should be congruent to A_1. However, as we saw in Example 5.10 (3), this is a contradiction. Thus, we conclude that K_0 and K_1 are not pull back equivalent. Q.E.D.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition (see [BMS, V2]).

Proposition 9.6. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with fibers F_0 and F_1 respectively, $n \geq 3$. For $i = 0, 1$, we denote by $I(K_i)$ the image of the homomorphism $H_n(K_i) \to H_n(F_i)$ induced by the inclusion. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

In the following example we give a pair of diffeomorphic knots for which their connected sums with any spherical knots are never cobordant. This answers question $(\mathcal{Q})$ mentioned at the beginning of this section negatively.

Example 9.7 ([BMS]). Let us consider the following unimodular matrices:

$$A_0 = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^{n+1} & 0 & 0 & 1 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}.$$

Then, for every integer $n \geq 3$, there exist simple fibered $(2n-1)$-knots K_i in S^{2n+1} whose Seifert matrices are given by A_i, $i = 0, 1$. Note that if we denote their fibers by F_i, $i = 0, 1$, then F_1 is orientation preservingly diffeomorphic to $F_0 \sharp (S^n \times S^n)$. In particular, K_0 and K_1 are orientation preservingly diffeomorphic to each other.

It is easy to verify that the Seifert form restricted to $I(K_1)$ is the zero form, while it is not zero for K_0. Hence, by Proposition 9.6, $K_0 \sharp \Sigma_0$ is never cobordant to $K_1 \sharp \Sigma_1$ for any spherical (not necessarily simple or fibered) knots Σ_0, Σ_1.

Note that for this example, we have $H_{n-1}(K_i) \cong \mathbf{Z} \oplus \mathbf{Z}$, $i = 0, 1$.

Let us give another kind of an example together with an argument using the Alexander polynomial.

Example 9.8 ([BMS]). Let us consider the unimodular matrices

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and their associated simple fibered $(2n-1)$-knots K_i, $i = 0, 1$, with $n \geq 4$ even. As in Example 9.7 we see that K_0 and K_1 are orientation preservingly diffeomorphic to each other.

Now, suppose that for some spherical $(2n-1)$-knots Σ_i, $i = 0, 1$, $K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$. We may assume that Σ_0 and Σ_1 are simple. The Alexander polynomials of K_0 and K_1 are given by

$$\Delta_{K_0}(t) = \det(tA_0 + A_0^T) = t^2 + t + 1$$

and

$$\Delta_{K_1}(t) = \det(tA_1 + A_1^T) = -(t^4 + t^3 - t^2 + t + 1)$$

respectively. Both of these polynomials are irreducible over $\mathbf{Z}$. If $K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$, then by Proposition 7.1, we must have a Fox-Milnor type relation

$$\Delta_{K_0}(t)\Delta_{\Sigma_0}(t)\Delta_{K_1}(t)\Delta_{\Sigma_1}(t) = \pm t^a f(t) f^*(t) \tag{9.1}$$

for some $a \in \mathbf{Z}$ and $f(t) \in \mathbf{Z}[t]$, where $\Delta_{\Sigma_i}(t)$ denotes the Alexander polynomial of Σ_i, $i = 0, 1$.

Note that we have $|\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3$ and $|\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1$. Since $\Delta_{K_0}(t)$ is irreducible of degree 2, and $\Delta_{K_1}(t)$ is irreducible of degree 4, the relation (9.1) leads to a contradiction.

Hence, $K_0 \sharp \Sigma_0$ is not cobordant to $K_1 \sharp \Sigma_1$ for any spherical (not necessarily simple or fibered) $(2n-1)$-knots Σ_0, Σ_1. In this example we have $H_{n-1}(K_i) \cong \mathbf{Z}_3$, for $i = 0, 1$.

§10. Even dimensional knots

In this section, we study cobordism classes of non-spherical $2n$-knots for $n = 1, 2$.

Recall that in [K1] Kervaire showed that C_{2n}, the cobordism group of spherical $2n$-knots in S^{2n+2}, is trivial for all $n \geq 1$. In particular, any two such knots are cobordant. For $n \geq 3$, Vogt [V1, V2] showed that two $2n$-knots in S^{2n+2} are cobordant if and only if they have the same n-th Betti number. Note that the technics used by Vogt are only available for $2n \geq 6$, since it is difficult to perform embedded surgeries in low dimensions, and the h-cobordism theorem is not available for low dimensions.

10.1. Cobordism of surfaces in S^4

In [K1] Kervaire proved that a $2n$-sphere embedded in $S^{2n+2} = \partial(D^{2n+3})$ is the boundary of a $(2n+1)$-disk properly embedded in D^{2n+3}. This implies that C_{2n} is trivial.

Although there is no group structure on the set of cobordism classes of non-spherical 2-knots, we have a similar result. In fact we show that any connected, closed and orientable surface embedded in S^4 is the boundary of an orientable handlebody properly embedded in the disk D^5. When the surface is non-orientable, it is the boundary of a non-orientable handlebody properly embedded in D^5 if and only if the Euler number of the normal bundle vanishes.

Recall that the normal Euler number of an orientable surface embedded in S^4 always vanishes (see [MS]). Let us recall the definition of the normal Euler number of a closed non-orientable surface M embedded in S^4, where S^4 is considered to be oriented. (Throughout this section, we use the letter "M" for $2n$-knots rather than "K", since the letter "K" will be used for another purpose.) The tubular neighborhood N of M may be regarded as a normal disk bundle over M. Let $p\colon \widetilde{M} \to M$ be the orientation double cover of M. Consider the induced bundle $\widetilde{N}$ over $\widetilde{M}$ so that we have the commutative diagram

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\widetilde{p}} & N \\ \downarrow & & \downarrow \\ \widetilde{M} & \xrightarrow{p} & M. \end{array}$$

We orient $\widetilde{N}$ so that the induced map $\widetilde{p}\colon \widetilde{N} \to N$ preserves the orientations. The *normal Euler number* $e(M)$ of the surface M is then defined by $e(M) = (\widetilde{M} \cdot \widetilde{M})/2$, where $\widetilde{M} \cdot \widetilde{M}$ denotes the self-intersection number of $\widetilde{M}$ in $\widetilde{N}$, which is always even.

Let us denote by N_g the closed connected non-orientable surface of non-orientable genus g. For a closed connected non-orientable surface $M \cong N_g$ embedded in S^4, it is known that $e(M) \in \{-2g, 4-2g, 8-2g, \ldots, 2g\}$. Furthermore, all the values in the set can be realized as the normal Euler number of some N_g embedded in S^4 (see [W1, Ms, Km]).

In [BS2] we characterized those closed connected surfaces embedded in S^4 which are the boundary of a handlebody properly embedded in D^5. For this purpose, we need to use Pin^- structures on manifolds.

A Pin^- *structure* on a manifold X is the homotopy class of a trivialization of $TX \oplus \det TX \oplus \varepsilon^N$ over the 2-skeleton $X^{(2)}$ of X, where TX denotes the tangent bundle, $\det TX$ denotes the orientation line bundle, and ε^N is a trivial vector bundle of dimension N sufficiently large. A Pin^- structure is equivalent to a Spin structure when X is orientable.

When M is a closed surface embedded in S^4, there is a canonical Pin^- structure defined on M. More precisely, since M is characteristic, i.e., as a submanifold of S^4 it represents the $\mathbf{Z}_2$ homology class dual to the second Stiefel-Whitney class of S^4, there exists a unique Spin structure on $S^4 \setminus M$ which cannot be extended to any normal 2-disk of M. This Spin structure on $S^4 \setminus M$ induces a unique Pin^- structure on M (see [KT1]).

We denote by H_g the orientable handlebody of dimension three which is obtained by gluing g orientable 1-handles to a 0-handle. The boundary of H_g is the closed connected orientable surface of genus g, denoted by Σ_g. Furthermore, we denote by I_g the non-orientable handlebody of dimension three which is obtained by gluing g non-orientable 1-handles to a 0-handle. Then the boundary of I_g is identified with N_{2g}. In the following we will denote by K_g the handlebody H_g or I_g.

Definition 10.1 ([BS2]). Let M be a closed connected surface embedded in S^4. Suppose that M has genus g if M is orientable and $2g$ if M is non-orientable. Let $\psi \colon \partial K_g \to M$ be a diffeomorphism. We say that ψ is Pin^- *compatible* if the Pin^- structure on ∂K_g induced by ψ extends through K_g.

When M is oriented, there always exists a compact oriented 3-dimensional submanifold V of S^4 such that $\partial V = M$ as oriented manifolds (see, for example, [E]). Such a manifold V is again called a *Seifert manifold* associated with M (see the definition of Seifert manifolds associated with odd dimensional knots in §2). When M is non-orientable, a compact 3-dimensional submanifold V of S^4 with $\partial V = M$ is also called a Seifert manifold. Such a (non-orientable) Seifert manifold exists for

M if and only if $e(M) = 0$ (see [GL, Km]). When a surface M admits a Seifert manifold V, the unique Spin structure on S^4 induces a Pin^- structure on V and this induces a Pin^- structure on M, which coincides with the Pin^- structure described above (see [Fi]).

In [BS2] we proved the following theorem.

Theorem 10.2. *Let M be a closed connected surface embedded in $S^4 = \partial D^5$, and $\psi\colon \partial K_g \to M$ a diffeomorphism, where K_g denotes the 3-dimensional handlebody with g 1-handles. Then, there exists an embedding $\widetilde{\psi}\colon K_g \to D^5$ with $\widetilde{\psi}|_{\partial K_g} = \psi$ if and only if $e(M) = 0$ and ψ is* Pin^- *compatible.*

Remark 10.3. Since every closed connected 3-dimensional manifold admits a Heegaard splitting of genus $g \geq 0$, as a consequence of Theorem 10.2 we have a new proof of Rohlin's theorem [Rh2] on the existence of an embedding of an arbitrary closed 3-dimensional manifold into $\mathbf{R}^5$ (see also [Wl, WZ] and [GM, p. 90]). For details, see [BS2].

Let us give a sketch of a proof of Theorem 10.2. First, it is easy to see that the vanishing of $e(M)$ and the Pin^- compatibility of ψ are necessary conditions. The proof of the sufficiency is based on embedded surgeries inside the disk D^5 on a Seifert manifold V of M. To do that we start with the abstract closed 3-manifold $V' = V \cup_\psi K_g$ obtained by attaching V and K_g along their boundaries by using ψ. Since the 3-dimensional cobordism group Ω_3^{Spin} (resp. $\Omega_3^{\mathrm{Pin}^-}$) of Spin (resp. Pin^-) manifolds is trivial (see [M1], [K1, Lemme III.7, p. 265], [GM, p. 91], [MK] or [Ki] for Ω_3^{Spin}, and [ABP, KT1, KT2] for $\Omega_3^{\mathrm{Pin}^-}$), there exists a compact (oriented if so is M) Pin^- 4-manifold W such that $\partial W = V'$ as (oriented) Pin^- manifolds. Let f be a Morse function $f\colon W \to [0, 1]$ which extends the projection to the second factor $\partial W = (V \times \{0\}) \cup_\psi (\partial K_g \times [0, 1]) \cup (K_g \times \{1\}) \to [0, 1]$. Note that f can be chosen so that all its critical values lie in the interval $(\varepsilon, 1-\varepsilon)$ for $\varepsilon > 0$ small enough. Moreover, we may assume that the critical points have index 1, 2 or 3.

Consider the handlebody decomposition of W associated with this Morse function. We can remove handles of index 1 and 3 using modifications described by Wallace in [Wc], respecting the Pin^- structure. Then we get a new (oriented) Pin^- manifold W' such that $\partial W = \partial W'$. Since the handlebody decomposition of the manifold W' has only handles of index 2, we can attach the handles to $V \times [0, 1]$ inside D^5 to get an embedding of W' into D^5. Finally we have a proper embedding of $K_g \cong (\partial K_g \times [0, 1]) \cup (K_g \times \{1\}) \subset \partial W'$ into the disk D^5 such that $\partial K_g = M$.

As a corollary to Theorem 10.2 we have

Corollary 10.4 ([BS2]). *Let M be a closed connected surface embedded in $S^4 = \partial D^5$. Then there exists a 3-dimensional handlebody embedded in D^5 such that its boundary coincides with M if and only if $e(M) = 0$.*

Using Theorem 10.2, we can characterize cobordism classes of closed connected surfaces embedded in S^4 as follows.

Theorem 10.5 ([BS2]). *Let M_0 and M_1 be two closed connected surfaces embedded in S^4. Then they are cobordant if and only if they are diffeomorphic as abstract manifolds and have the same normal Euler number.*

Remark 10.6. The above theorem in the orientable case is proved by Ogasa [O], although his proof is slightly different from ours explained below.

When two closed connected surfaces embedded in S^4 are cobordant, it is clear that they are diffeomorphic as abstract manifolds and have the same normal Euler number (for details, see [BS2]). Thus we have the necessity in Theorem 10.5.

For the sufficiency, start with two closed connected surfaces M_0 and M_1 in S^4 which are diffeomorphic as abstract manifolds and have the same normal Euler number. In the following, we consider the case where M_0 and M_1 are non-orientable of non-orientable genus g. (For the orientable case, the proof is similar. For details, see [BS2].)

By changing M_0 and M_1 by isotopies, we may assume that for a 4-disk D^4 in S^4, we have $M_0 \cap D^4 = M_1 \cap D^4 = D^2$ and (D^4, D^2) is the standard disk pair. Set $\Delta = (S^4 \setminus \operatorname{Int} D^4) \times [0, 1] \cong D^5$ and

$$\widetilde{M} = (M_0 \setminus \operatorname{Int} D^2) \cup (\partial D^2 \times [0, 1]) \cup (M_1 \setminus \operatorname{Int} D^2) = M_0^! \sharp M_1 \subset \partial\Delta,$$

where $M_0^!$ denotes the mirror image of M_0. Since $e(M_0) = e(M_1)$, we have $e(\widetilde{M}) = 0$. Furthermore, one can prove that there exists a Pin$^-$ compatible diffeomorphism between $\partial((N_g \setminus \operatorname{Int} D^2) \times [0, 1]) \cong \partial I_g$ and $\widetilde{M}$ which sends $(N_g \setminus \operatorname{Int} D^2) \times \{i\}$ diffeomorphically onto $M_i \setminus \operatorname{Int} D^2$.

According to Theorem 10.2 we can embed I_g in Δ so that $M_0^! \sharp M_1 = \partial I_g$. The cobordism between M_0 and M_1 is then obtained by gluing back $D^4 \times [0, 1]$ to Δ and by replacing $I_g \cong (N_g \setminus \operatorname{Int} D^2) \times [0, 1]$ by $N_g \times [0, 1]$.

As a consequence of Theorem 10.5 we have that two closed connected orientable surfaces embedded in S^4 are cobordant if and only if they have the same genus. Hence, the monoid of cobordism classes of closed connected orientable surfaces embedded in S^4 is isomorphic to the monoid of non-negative integers $\mathbf{Z}_{\geq 0}$.

Let us consider non-orientable surfaces. First note that by adding the cobordism class of an embedding of S^2 into S^4 to the associative groupoid (or the associative magma or the semigroup) of cobordism classes of closed connected non-orientable surfaces embedded in S^4, we get a monoid denoted by $\mathfrak{N}$. We can also describe the monoid structure of $\mathfrak{N}$ as follows. Let $\mathbf{R}P^2_+$ (resp. $\mathbf{R}P^2_-$) be the projective plane standardly embedded in S^4 with normal Euler number being equal to $+2$ (resp. -2) (see [HK]). For a pair of non-negative integers (k, l) such that $k + l \geq 1$, let $M_{k,l}$ be the non-orientable surface embedded in S^4 obtained by taking the connected sum of k copies of $\mathbf{R}P^2_+$ and l copies of $\mathbf{R}P^2_-$. Then we have $e(M_{k,l}) = 2(k-l)$ and the genus of $M_{k,l}$ is equal to $k+l$. Hence, the set of non-orientable surfaces $\{M_{k,l}\colon k, l \in \mathbf{Z}, k, l \geq 0, k + l \geq 1\}$ constitutes a complete set of representatives of the cobordism classes of closed connected non-orientable surfaces embedded in S^4. Therefore, $\mathfrak{N}$ is isomorphic to the monoid of pairs of non-negative integers $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$. If we denote by $[M]$ the cobordism class of a closed connected non-orientable surface M embedded in S^4, and by $g(M)$ the genus of M, then the isomorphism $\mathfrak{N} \to \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ is given by mapping $[M]$ to

$$\left(\frac{2g(M) + e(M)}{4}, \frac{2g(M) - e(M)}{4}\right).$$

10.2. Concordance of embeddings of a surface

In this subsection, we consider the concordance classification of embeddings of closed connected surfaces into S^4. For the definition of the concordance, see Definition 2.10.

Examining the proof of Theorem 10.5 carefully, we see that the following characterization of concordant embeddings of surfaces into S^4 holds.

Theorem 10.7 ([BS2]). *Let Σ be a closed connected surface. Two embeddings of Σ into S^4 are concordant if and only if the* Pin^- *structures induced by these embeddings coincide and the normal Euler numbers of these embeddings coincide.*

When the knots are spherical of dimension two, the notions of cobordism and concordance coincide with each other, since every diffeomorphism of S^2 which preserves the orientation is isotopic to the identity [Sm1]. However, when $g \geq 1$, for an arbitrary embedding $f\colon \Sigma_g \to S^4$ there exists an orientation preserving diffeomorphism $h\colon \Sigma_g \to \Sigma_g$ which does not preserve the Pin^- structure induced by f. Therefore, the embeddings $f \circ h$ and f are not concordant. This means that contrary to the spherical case, the notions of cobordism and concordance differ for orientable surfaces of genus $g \geq 1$.

β	g: odd	g: even
0	0	$2^{(g-2)/2}(2^{(g-2)/2}+1)$
1	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0
2	0	2^{g-2}
3	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
4	0	$2^{(g-2)/2}(2^{(g-2)/2}-1)$
5	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
6	0	2^{g-2}
7	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0

Table 1. Number of Pin^-structures on the non-orientable surface N_g with Brown invariant $\beta \in \mathbf{Z}_8$

The group of orientation preserving diffeomorphisms of a closed connected oriented surface acts transitively on the set of Pin^- structures with trivial Brown invariant (see, for example, [BS2]). This set is naturally identified with the set of Spin structures with trivial Arf invariant, since the surface is assumed to be orientable. This implies that the number of concordance classes of embeddings of a closed connected oriented surface is equal to the number of Spin structures with trivial Arf invariant on this surface. According to [J] this number is equal to $2^{g-1}(2^g+1)$, where g is the genus of the surface. If we denote by ω_g the number of concordance classes of embeddings of Σ_g, then we have $\omega_g = 2^{g-1}(2^g+1)$.

Let us denote by ν_g the number of concordance classes of embeddings of the closed connected non-orientable surface N_g of non-orientable genus g. According to [Ms, Km], the set of possible normal Euler numbers for such embeddings coincides with $\{-2g, 4-2g, 8-2g, \dots, 2g\}$. Hence, we have

$$\nu_g = \sum_{i=0}^{g} \nu_{g,\,-2g+4i},$$

where $\nu_{g,\,-2g+4i}$ denotes the number of concordance classes of embeddings of N_g into S^4 with normal Euler number equal to $-2g+4i$. Moreover, according to [KT1, Theorem 6.3], $\nu_{g,\,-2g+4i}$ is equal to the number of Pin^- structures with Brown invariant equal to $-g+2i$ modulo 8. Such numbers can be calculated as in Table 1 (see [DP]).

Using the values given in Table 1, we get

$$\nu_g = \begin{cases} 2^{g-2}(g+1) & \text{if } g \text{ is odd,} \\ 2^{g-2}(g+1) + 2^{(g-2)/2} & \text{if } g \text{ is even.} \end{cases}$$

10.3. Cobordism of 4-knots

In the study of cobordism of embeddings of even dimensional manifolds, the only case which remains to be studied is the case of 4-dimensional manifolds embedded in S^6. In [BS3] we proved the following

Theorem 10.8. *Let M be a closed simply connected 4-dimensional manifold. Then all the embeddings of M into S^6 are concordant.*

In particular, two 4-knots in S^6, i.e., two closed simply connected 4-dimensional manifolds embedded in S^6, are (oriented) cobordant if and only if they are abstractly (orientation preservingly) diffeomorphic to each other.

One can prove Theorem 10.8 by imitating the proofs of Theorems 10.2 and 10.5, and the proof is based essentially on Kervaire's original idea [K1].

Remark 10.9. It is known that a closed connected orientable 4-dimensional manifold M can be embedded in S^6 if and only if it is Spin and its signature vanishes (see [CS2]). If in addition M is simply connected, then it can be embedded in S^6 if and only if it is homeomorphic to a connected sum of some copies of $S^2 \times S^2$ by the homeomorphism classification of closed simply connected 4-dimensional manifolds due to Freedman [Fr].

Remark 10.10. By Park [P], for any sufficiently large odd integer m, there exist infinitely many smooth manifolds which are all homeomorphic to the connected sum of m copies of $S^2 \times S^2$ but which are not diffeomorphic to each other. Let us denote by $\mathfrak{O}_4$ the monoid of (oriented) cobordism classes of closed simply connected 4-manifolds embedded in S^6, and by $\mathbf{Z}_{\geq 0}$ the monoid of non-negative integers. Then the homomorphism $\mathfrak{O}_4 \to \mathbf{Z}_{\geq 0}$ which associates to a 4-knot one half of its second Betti number is an epimorphism. The above result of Park shows that this homomorphism is far from being an isomorphism. Compare this with the result of Vogt [V1, V2]: the corresponding homomorphism $\mathfrak{O}_{2n} \to \mathbf{Z}_{\geq 0}$ for $n \geq 3$ is an isomorphism, where $\mathfrak{O}_{2n}$ denotes the monoid of (oriented) cobordism classes of $2n$-knots in S^{2n+2}.

Remark 10.11. When $n \neq 2$, for an arbitrary $2n$-knot M, its orientation reversal $-M$ is oriented cobordant to M. For $n = 2$, there exists a closed 4-dimensional manifold N homeomorphic to a connected sum of some copies of $S^2 \times S^2$ such that N is not oriented diffeomorphic to $-N$. In fact, by Kotschick [Ko2], every simply connected compact complex surface of general type which is Spin and has vanishing signature gives such an example. Such a complex surface has been constructed by Moishezon and Teicher [MT1, MT2, Ko1]. Hence, there exists a closed simply connected oriented 4-dimensional manifold embedded in S^6 which is not oriented cobordant to its orientation reversal.

§11. Open problems

To conclude this survey article, we would like to list some open problems.

Problem 11.1. In Definition 2.1, if we remove the connectivity condition on the embedded manifolds, then is it still possible to characterize their isotopy and cobordism classes?

Problem 11.2. Construct efficient invariants of algebraic cobordism.

Problem 11.3. Is the algebraic cobordism an equivalence relation on the whole set of integral bilinear forms?

See Theorem 5.7, Example 5.8, Remarks 5.9 and 6.10 for the above problem.

Problem 11.4. Is it true that two simple $(2n-1)$-knots, $n \geq 3$, are cobordant if and only if their Seifert forms associated with $(n-1)$-connected Seifert manifolds are weakly algebraically cobordant? In particular, is there a pair of two simple $(2n-1)$-knots, $n \geq 3$, which are cobordant, but whose Seifert forms are not (weakly) algebraically cobordant?

Note that for C-algebraically fibered simple knots, the above equivalence is true (see Remark 5.9).

Problem 11.5. Is the Spin cobordism of Seifert forms associated with non-free 3-knots a sufficient condition of cobordism?

Problem 11.6. Does Theorem 9.3 (a characterization of the pull back relation for simple fibered $(2n-1)$-knots) hold for $n = 2$?

As noted in Remark 9.4, the above characterization does not hold for $n = 1$.

Problem 11.7. Let us fix an oriented simple homotopy type (or an oriented diffeomorphism type) of manifolds, and consider the set of all embeddings of such manifolds into a sphere in codimension two. Then, does there exist a minimal element with respect to the pull back relation?

As mentioned in §9, for spheres, the trivial knot is such a minimal element.

Problem 11.8. Is C_n/F_n isomorphic to $\mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty \oplus \mathbf{Z}^\infty$ for odd n? Determine the group structure of F_n for odd n. Is F_n a direct summand of C_n?

Problem 11.9. Is the multiplicity of a complex holomorphic function germ at an isolated singular point a cobordism invariant of the associated algebraic knot?

This is known to be true for the case of algebraic 1-knots. See also [Z2].

Problem 11.10. Let us consider Brieskorn type polynomials of the form

$$z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}.$$

If two algebraic knots associated with Brieskorn type polynomials are cobordant, then do their exponents coincide?

A related result is obtained in [S3]. Note that the associated Seifert matrix has been explicitly determined (for example, see [Sk]). It is also known that two algebraic $(2n-1)$-knots associated with Brieskorn polynomials with the same Alexander polynomial have the same exponents [YS].

Problem 11.11. Two fibered n-knots in S^{n+2} are said to be *fibered cobordant* if there exists a cobordism $X \subset S^{n+2} \times [0, 1]$ between them whose complement $S^{n+2} \setminus X$ fibers over the circle in a sense similar to Definition 2.4. Is there a pair of two fibered knots which are cobordant but are not fibered cobordant?

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