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Polarized K3 surfaces of genus thirteen

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A smooth complete algebraic surface S is *of type K3* if S is regular and the canonical class K_S is trivial. A *primitively polarized* K3 surface is a pair (S, h) of a K3 surface S and a primitive ample divisor class $h \in \mathrm{Pic}\, S$. The integer $g := \frac{1}{2}(h^2) + 1 \geq 2$ is called the *genus* of (S, h). The moduli space of primitively polarized K3 surfaces of genus g exists as a quasi-projective (irreducible) variety, which we denote by $\mathcal{F}_g$. As is well known a general polarized K3 surface of genus $2 \leq g \leq 5$ is a complete intersection of hypersurfaces in a weighted projective space: $(6) \subset \mathbf{P}(1112), (4) \subset \mathbf{P}^3, (2) \cap (3) \subset \mathbf{P}^4$ and $(2) \cap (2) \cap (2) \subset \mathbf{P}^5$.

In connection with the classification of Fano threefolds, we have studied the system of defining equations of the projective model $S_{2g-2} \subset \mathbf{P}^g$ and shown that a general polarized K3 surface of genus g is a complete intersection with respect to a homogeneous vector bundle $\mathcal{V}_{g-2}$ (of rank $g-2$) in a g-dimensional Grassmannian $G(n, r)$, $g = r(n-r)$, in a unique way for the following six values of g:

g	6	8	9	10	12	20
r	2	2	3	5	3	4
$\mathcal{V}_{g-2}$	$3\mathcal{O}_G(1) \oplus \mathcal{O}_G(2)$	$6\mathcal{O}_G(1)$	$\bigwedge^2 \mathcal{E} \oplus 4\mathcal{O}_G(1)$	$\bigwedge^4 \mathcal{E} \oplus 3\mathcal{O}_G(1)$	$3\bigwedge^2 \mathcal{E} \oplus \mathcal{O}_G(1)$	$3\bigwedge^2 \mathcal{E}$

Here $\mathcal{E}$ is the universal quotient bundle on $G(n, r)$. See [4] and [5] for the case $g = 6, 8, 9, 10$, [6, §5] for $g = 20$ and §3 for $g = 12$.

By this description, the moduli space $\mathcal{F}_g$ is birationally equivalent to the orbit space $H^0(G(n, r), \mathcal{V}_{g-2})/(PGL(n) \times Aut_{G(n,r)}\, \mathcal{V}_{g-2})$ and

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hence is unirational for these values of g. The uniqueness of the description modulo the automorphism group is essentially due to the *rigidity* of the vector bundle $E := \mathcal{E}|_S$. All the cohomology groups $H^i(sl(E))$ vanish.

A general member $(S,h) \in \mathcal{F}_g$ is a complete intersection with respect to the homogeneous vector bundle $8\mathcal{U}$ in the orthogonal Grassmannian $O\text{-}G(10,5)$ in the case $g = 7$ ([4]), and with respect to $5\mathcal{U}$ in $O\text{-}G(9,3)$ in the case 18 ([6]), where $\mathcal{U}$ is the homogeneous vector bundle on the orthogonal Grassmannian such that $H^0(\mathcal{U})$ is a half spinor representation U^{16}. Both descriptions are unique modulo the orthogonal group. Hence $\mathcal{F}_7$ and $\mathcal{F}_{18}$ are birationally equivalent to $G(8,U^{16})/PSO(10)$ and $G(5,U^{16})/SO(9)$, respectively. The unirationality of $\mathcal{F}_{11}$ is proved in [7] using a non-abelian Brill-Noether locus and the unirationality of $\mathcal{M}_{11}$, the moduli space of curves of genus 11.

In this article, we shall study the case $g = 13$ and show the following:

Theorem 1. *A general member $(S,h) \in \mathcal{F}_{13}$ is isomorphic to a complete intersection with respect to the homogeneous vector bundle*

$$\mathcal{V} = \bigwedge^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$$

of rank 10 *in the* 12*-dimensional Grassmannian $G(7,3)$, where $\mathcal{F}$ is the dual of the universal subbundle.*

Corollary *$\mathcal{F}_{13}$ is unirational.*

Remark 1. A general complete intersection (S,h) with respect to the homogeneous vector bundle $\bigwedge^4 \mathcal{F} \oplus S^2\mathcal{E}$ in the 10-dimensional Grassmannian $G(7,2)$ is also a primitively polarized K3 surface of genus 13. But (S,h) is not a general member of $\mathcal{F}_{13}$. In fact, S contains 8 mutually disjoint rational curves $R_1, \ldots, R_7$, which are of degree 3 with respect to h. This will be discussed elsewhere.

Unlike the known cases described above, the vector bundle $E = \mathcal{E}|_S$ in the theorem is not rigid. Hence the theorem does not give a birational equivalence between $\mathcal{F}_{13}$ and an orbit space. But E is *semi-rigid*, that is, $H^0(sl(E)) = 0$ and $\dim H^1(sl(E)) = 2$. Instead of $\mathcal{F}_{13}$ itself, the theorem gives a birational equivalence between the universal family over it and an orbit space.

Let $S \subset G(7,3)$ be a general complete intersection with respect to $\mathcal{V}$. Then S is the common zero locus of the two global sections of $\bigwedge^2 \mathcal{E}$ corresponding to general bivectors $\sigma_1, \sigma_2 \in \bigwedge^2 \mathbf{C}^7$ and one global section of $\bigwedge^3 \mathcal{F}$ corresponding to a general $\tau \in \bigwedge^3 \mathbf{C}^{7,\vee}$. The 2-dimensional subspace $P = \langle\sigma_1, \sigma_2\rangle \subset \bigwedge^2 \mathbf{C}^7$ is uniquely determined by S. Let $\overline{P \wedge P}$ be the subspace of $\bigwedge^3 \mathbf{C}^{7,\vee}$ corresponding to $P \wedge P \subset \bigwedge^4 \mathbf{C}^7$. Then $\mathbf{C}\mathcal{T}$ modulo $\overline{P \wedge P}$ is also uniquely determined by S. It is known that the natural action of $PGL(7)$ on $G(2, \bigwedge^2 \mathbf{C}^7)$ has an open dense orbit (Sato-Kimura[9, p. 94]). Hence we obtain the natural birational map

$$\psi : \mathbf{P}_*(\bigwedge^4 \mathbf{C}^7/(P \wedge P))/G \cdots \to \mathcal{F}_{13}, \tag{1}$$

which is dominant by the theorem, where G is the (10-dimensional) stabilizer group of the action at $P \in G(2, \bigwedge^2 \mathbf{C}^7)$.

Theorem 2. *For every general member $p = (S, h) \in \mathcal{F}_{13}$, the fiber of ψ at p is birationally equivalent to the moduli K3 surface $M_S(3, h, 4)$ of semi-rigid rank three vector bundles with $c_1 = h$ and $\chi = 3 + 4$.*

As is shown in [8], $\hat{S} := M_S(3, h, 4)$ carries a natural ample divisor class $\hat{h}$ of the same genus (=13) and $(S, h) \mapsto (\hat{S}, \hat{h})$ induces an automorphism of $\mathcal{F}_{13}$. (In fact, this is an involution.) Hence we have

Corollary *The orbit space $\mathbf{P}^*(\bigwedge^4 \mathbf{C}^7/(P \wedge P))/G$ is birationally equivalent to the universal family over $\mathcal{F}_{13}$, or the coarse moduli space of one pointed polarized K3 surfaces (S, h, x) of genus* 13.

Remark 2. 8 Kondō[3] proves that the Kodaira dimension of $\mathcal{F}_g$ is non-negative for the following 17 values:

$$g = 41, 42, 50, 52, 54, 56, 58, 60, 65, 66, 68, 73, 82, 84, 104, 118, 132.$$

The Kodaira dimension of $\mathcal{F}_{m^2(g-1)+1}$ is non-negative for these values of g and for every $m \geq 2$ since it is a finite covering of $\mathcal{F}_g$.

Notations and convention. Algebraic varieties and vector bundles are considered over the complex number field $\mathbf{C}$. The dual of a vector bundle (or a vector space) E is denoted by $E^\vee$. Its Euler-Poincarè characteristic $\sum_i (-)^i h^i(E)$ is denoted by $\chi(E)$. The vector bundles of traceless endomorphisms of E is denoted by $sl(E)$. For a vector space V, $G(V, r)$ is the Grassmannian of r-dimensional quotient spaces of V and $G(r, V)$ that of r-dimensional subspaces. The isomorphism class of $G(V, r)$ with $\dim V = n$ is denoted by $G(n, r)$. The projective spaces $G(V, 1)$ and $G(1, V)$ are denoted by $\mathbf{P}^*(V)$ and $\mathbf{P}_*(V)$, respectively. $\mathcal{O}_G(1)$ is the pull-back of the tautological line bundle by the Plücker embedding $G(V, r) \hookrightarrow \mathbf{P}^*(\bigwedge^r V)$.

§1. Vanishing

We prepare the vanishing of cohomology groups of homogeneous vector bundles on the Grassmannian $G(n,r)$, which is the quotient of $SL(n)$ by a parabolic subgroup P. The reductive part P_{red} of P is the intersection of $GL(r) \times GL(n-r)$ and $SL(n)$ in $GL(n)$. We take $\{(a_1, \cdots, a_r; a_{r+1}, \ldots, a_n) \mid \sum_1^n a_i = 0\} \subset \mathbf{Z}^n$ as root lattice and $\mathbf{Z}^n/\mathbf{Z}(1,1,\ldots,1)$ as the common weight lattice of $SL(n)$ and P_{red}. We take $\{e_i - e_{i+1} \mid 1 \le i \le n-1\}$ as standard root basis. The half of the sum of all positive roots is equal to

$$\delta = (n-1, n-3, n-5, \ldots, -n+3, -n+1)/2.$$

Let ρ be an irreducible representation of P_{red} and $w \in \mathbf{Z}^n/\mathbf{Z}(1,1,\ldots,1)$ its highest weight. We denote the homogeneous vector bundle on $G(n,r)$ induced from ρ by E_w. w is *singular* if a number appears more than once in $w + \delta$. If w is not singular and $w+\delta = (a_1, a_2, \ldots, a_n)$, then there is a unique (Grassmann) permutation $\alpha = \alpha_w$ such that $a_{\alpha(1)} > a_{\alpha(2)} > \cdots > a_{\alpha(n)}$. We denote the length of α_w, that is, the cardinality of the set $\{(i,j) \mid 1 \le i < j \le n,\ a_i < a_j\}$, by $l(w)$.

Theorem 3 (Borel-Hirzebruch[2]). (a) *If w is singular, then all the cohomology groups $H^i(G(n,r), \mathcal{E}_w)$ vanish.*

(b) *If w is not, then all the cohomology groups $H^i(G(n,r), \mathcal{E}_w)$ vanish except for one $i := l(w)$. Moreover, $H^{l(w)}(G(n,r), \mathcal{E}_\rho)$ is an irreducible representation of $SL(n)$ with highest weight*

$$(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(n)}) - \delta.$$

The dimension of this unique nonzero cohomology group is equal to $\prod_{1 \le i < j \le n} |a_i - a_j|/(j-i)$.

$l(w)$ is called the *index* of the homogeneous vector bundle E_w.

Example. In the following table, $-$ means that the weight w is singular and we put $s = n - r$.

weight w	homogeneous bundle $\mathcal{E}_w$	$l(w)$	$H^{l(w)}$
$(1,0,0,\dots,0,0;0,\dots,0,0)$	$\mathcal{E}$, universal quotient bundle	0	$\mathbf{C}^n$
$(0,0,0,\dots,-1,0;0,\dots,0,0)$	$\mathcal{E}^\vee$	-	
$(1,1,0,\dots,0,0;0,\dots,0,0)$	$\bigwedge^2\mathcal{E}$	0	$\bigwedge^2\mathbf{C}^n$
$(1,1,1,\dots,1,1;0,\dots,0,0)$	$\mathcal{O}_G(1)=\det\mathcal{E}=\det\mathcal{F}$	0	$\bigwedge^r\mathbf{C}^n$
$(0,0,0,\dots,0,0;-1,\dots,-1)$			
$(0,0,0,\dots,0,0;1,\dots,0,0)$	$\mathcal{F}^\vee$, universal subbundle	-	
$(0,0,0,\dots,0,0;0,\dots,0,-1)$	$\mathcal{F}$	0	$\mathbf{C}^{n,\vee}$
$(1,0,0,\dots,0,0;0,\dots,0,-1)$	$T_{G(n,r)}$, tangent bundle	0	$sl(\mathbf{C}^n)$
$(0,0,0,\dots,-1;1,0,\dots,0,0)$	$\Omega_{G(n,r)}$, cotangent bundle	1	$\mathbf{C}$
$(-s,-s,\dots,-s;r,r,\dots,r)$	$\mathcal{O}_G(-n)$, canonical bundle	rs	$\mathbf{C}$

We apply the theorem to the 12-dimensional Grassmannian $G(7,3)$.

Lemma 4. (a) *All cohomology groups of the homogeneous vector bundle $\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ on $G(7,3)$ vanish except for the following:*

i) $p=q=0$, $h^0(\mathcal{O}_G)=1$, and

ii) $p=6, q=4$, $h^{12}(\mathcal{O}_G(-7))=1$.

(b) *All cohomology groups of $\mathcal{O}_G(1)\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for the following:*

i) $p=q=0$, $h^0(\mathcal{O}_G(1))=35$,

ii) $p=1, q=0$, $h^0(2\mathcal{E})=2\cdot 7=14$, and

iii) $p=0, q=1$, $h^0(\mathcal{F})=7$.

(c) *All cohomology groups of $\mathcal{E}\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for $h^0(\mathcal{E})=7$ with $p=q=0$.*

(d) *All cohomology groups of $\mathcal{F}\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for $h^0(\mathcal{F})=7$ with $p=q=0$.*

(e) *All cohomology groups of $\bigwedge^2\mathcal{E}\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for the following:*

i) $p=q=0$, $h^0(\bigwedge^2\mathcal{E})=21$, and

ii) $p=1, q=0$, $h^0(\bigwedge^2\mathcal{E}\otimes(2\mathcal{E}(-1)))=2$.

(f) *All cohomology groups of $\bigwedge^3\mathcal{F}\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for the following:*

i) $p=q=0$, $h^0(\bigwedge^3\mathcal{F})=35$,

ii) $p=0, q=1$, $h^0(\bigwedge^3\mathcal{F}\otimes\mathcal{F}(-1))=1$, and

iii) $p=2, q=0$, $h^1(\bigwedge^3\mathcal{F}\otimes\bigwedge^2(2\mathcal{E}(-1)))=3h^1(\bigwedge^3\mathcal{F}\otimes\bigwedge^2\mathcal{E}^\vee)=3$.

(g) *All cohomology groups of $sl(\mathcal{E})\otimes\bigwedge^p(2\mathcal{E}(-1))\otimes\bigwedge^q(\mathcal{F}(-1))$ vanish except for $h^6=2$ with $p=3, q=2$.*

Proof. The following table describes the decomposition of $\bigwedge^p(2\mathcal{E}(-1))$ into indecomposable homogeneous vector bundles.

(2)

p	decomposition	weight w'	$w'+\delta'$
0	$\mathcal{O}_G$	$(0,0,0)$	$(3,2,1)$
1	$2\mathcal{E}(-1)$	$2(0,-1,-1)$	$(3,1,0)$
2	$3(\bigwedge^2\mathcal{E})(-2)$	$3(-1,-1,-2)$	$(2,1,-1)$,
	$\oplus S^2\mathcal{E}(-2)$	$\oplus(0,-2,-2)$	$(3,0,-1)$
3	$4\mathcal{O}_G(-2)$	$4(-2,-2,-2)$	$(1,0,-1)$,
	$\oplus 2sl(\mathcal{E})(-2)$	$\oplus 2(-1,-2,-3)$	$(2,0,-2)$
4	$3\mathcal{E}(-3)$	$3(-2,-3,-3)$	$(1,-1,-2)$,
	$\oplus(S^2\bigwedge^2\mathcal{E})(-4)$	$\oplus(-2,-2,-4)$	$(1,0,-3)$
5	$2(\bigwedge^2\mathcal{E})(-4)$	$2(-3,-3,-4)$	$(0,-1,-3)$
6	$\mathcal{O}_G(-4)$	$(-4,-4,-4)$	$(-1,-2,-3)$

Here $\delta'=(3,2,1)$ is the *head* of $\delta=(3,2,1;0,-1,-2,-3)$.

$\bigwedge^q(\mathcal{F}(-1))$ is indecomposable. The following lists its weight w'' and $w''+\delta''$, where $\delta''=(0,-1,-2,-3)$ is the *tail* of δ.

(3)

q	bundle	weight w''	$w''+\delta''$
0	$\mathcal{O}_G$	$(0,0,0,0)$	$(0,-1,-2,-3)$
1	$\mathcal{F}(-1)$	$(1,1,1,0)$	$(1,0,-1,-3)$
2	$(\bigwedge^2\mathcal{F})(-2)$	$(2,2,1,1)$	$(2,1,-1,-2)$
3	$(\bigwedge^3\mathcal{F})(-3)$	$(3,2,2,2)$	$(3,1,0,-1)$
4	$\mathcal{O}_G(-3)$	$(3,3,3,3)$	$(3,2,1,0)$

We prove (a), (f) and (g) applying Theorem 3. The other cases are similar.

(a) Take w' and w'' from the tables (2) and (3), respectively, and combine into $w=(w';w'')$. Then w is singular except for the two cases

$$w+\delta=(3,2,1;0,-1,-2,-3) \quad \text{with} \quad p=q=0$$

and

$$w+\delta=(-1,-2,-3;3,2,1,0) \quad \text{with} \quad p=6, q=4.$$

The indices $l(w)$ are equal to 0 and 12, respectively.

(f) The homogeneous vector bundle $\bigwedge^3 \mathcal{F} \otimes \bigwedge^q(\mathcal{F}(-1))$ decomposes in the following way:

(4)

q	weight w''	$w'' + \delta''$
0	$(0,-1,-1,-1)$	$(0,-2,-3,-4)$
1	$(1,0,0,-1) \oplus (0,0,0,0)$	$(1,-1,-2,-4), (0,-1,-2,-3)$
2	$(2,1,0,0) \oplus (1,1,1,0)$	$(2,0,-2,-3), (1,0,-1,-3)$
3	$(3,1,1,1) \oplus (2,2,1,1)$	$(3,0,-1,-2), (2,1,-1,-2)$
4	$(3,2,2,2)$	$(3,1,0,-1)$

Take w' and w'' from the table (2) and this table, respectively, and consider $w = (w'; w'')$. Then w is singular except for the following three cases.

i) $p = q = 0, w + \delta = (3,2,1;0,-2,-3,-4)$, $l(w) = 0$,
ii) $p = 0$, $q = 1, w + \delta = (3,2,1;0,-1,-2,-3)$, $l(w) = 0$, and
iii) $p = 2$, $q = 0, w + \delta = (2,1,-1;0,-2,-3,-4)$, $l(w) = 1$.

(g) The following table shows the indecomposable components of $sl(\mathcal{E}) \otimes \bigwedge^p(2\mathcal{E}(-1))$ which do not appear in that of $\bigwedge^p(2\mathcal{E}(-1))$.

(5)

p	weight w' other than Table (2)	$w' + \delta'$
0	$(1,0,-1)$	$(4,2,0)$
1	$2(1,-1,-2) \oplus 2(0,0,-2)$	$(4,1,-1), (3,2,-1)$
2	$4(0,-1,-3) \oplus (1,-2,-3)$	$(3,1,-2), (4,0,-2)$
3	$2(0,-2,-4) \oplus 2(-1,-1,-4)$	$(3,0,-3), (2,1,-3)$
	$\oplus 2(0,-3,-3)$	$(3,-1,-2)$
4	$(-1,-2,-5) \oplus 4(-1,-3,-4)$	$(2,0,-4), (2,-1,-3)$
5	$2(-2,-3,-5) \oplus 2(-2,-4,-4)$	$(1,-1,-4), (1,-2,-3)$
6	$(-3,-4,-5)$	$(0,-2,-4)$

Take w' and w'' from the table (2) and this table, respectively, and consider $w = (w'; w'')$. Then w is singular except for the case $w + \delta = (3,0,-3;2,1,-1,-2)$ with $p = 3$ and $q = 2$. The index is equal to 6. Q.E.D.

Let $S \subset G(7,3)$ be a complete intersection with respect to $\mathcal{V} = 2\bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$. The Koszul complex

$$\mathbf{K} : \mathcal{O}_G \longleftarrow \mathcal{V}^\vee \longleftarrow \bigwedge^2 \mathcal{V}^\vee \longleftarrow \cdots \longleftarrow \bigwedge^9 \mathcal{V}^\vee \longleftarrow \bigwedge^{10} \mathcal{V}^\vee \longleftarrow 0$$

gives a resolution of the structure sheaf $\mathcal{O}_S$. $\bigwedge^n \mathcal{V}^\vee$ is isomorphic to $\bigoplus_{p+q=n} \bigwedge^p(2\mathcal{E}(-1)) \otimes \bigwedge^q(\mathcal{F}(-1))$.

Proposition 5. (a) $H^0(S, \mathcal{O}_S) = \mathbf{C}, H^1(S, \mathcal{O}_S) = 0$.

(b) *The restriction map* $H^0(G(7,3), \mathcal{O}_G(1)) \longrightarrow H^0(S, \mathcal{O}_S(1))$ *is surjective,* $H^0(S, \mathcal{O}_S(1))$ *is of dimension* 14 *and* $H^1(S, \mathcal{O}_S(1)) = H^2(S, \mathcal{O}_S(1)) = 0$.

(c) *The restriction map* $H^0(G(7,3), \mathcal{E}) \longrightarrow H^0(S, E)$ *is an isomorphism and* $H^1(S, E) = H^2(S, E) = 0$.

(d) *The restriction map* $H^0(G(7,3), \mathcal{F}) \longrightarrow H^0(S, F)$ *is an isomorphism.*

(e) $H^0(G(7,3), \bigwedge^2 \mathcal{E}) \longrightarrow H^0(S, \bigwedge^2 E)$ *is surjective and the kernel is of dimension 2.*

(f) $H^0(G(7,3), \bigwedge^3 \mathcal{F}) \longrightarrow H^0(S, \bigwedge^3 F)$ *is surjective and the kernel is of dimension 4.*

(g) E *is simple and semi-rigid, that is,* $H^0(sl(E)) = 0$ *and* $h^1(sl(E)) = 2$.

Proof. We prove (a) and (f) as sample. Other cases are similar.

(a) The restriction map $H^0(G(7,3), \mathcal{O}_G) \longrightarrow H^0(S, \mathcal{O}_S)$ is surjective by the vanishing $H^1(\mathcal{V}^\vee) = H^2(\bigwedge^2 \mathcal{V}^\vee) = \cdots = H^{10}(\bigwedge^{10} \mathcal{V}^\vee) = 0$ and the exact sequence $0 \longleftarrow \mathcal{O}_S \longleftarrow \mathbf{K}$. $H^1(S, \mathcal{O}_S)$ vanishes since $H^1(\mathcal{O}_G) = H^2(\mathcal{V}^\vee) = \cdots = H^{11}(\bigwedge^{10} \mathcal{V}^\vee) = 0$.

(f) The restriction map is surjective by the vanishing $H^n(\bigwedge^3 \mathcal{F} \otimes \bigwedge^n \mathcal{V}^\vee)$ for $n = 1, \ldots, 10$ and the exact sequence

$$0 \longleftarrow \bigwedge^3 F \longleftarrow \bigwedge^3 \mathcal{F} \otimes \mathbf{K}.$$

The dimension of the kernel is equal to

$$h^0(\bigwedge^3 \mathcal{F} \otimes \mathcal{V}^\vee) + h^1(\bigwedge^3 \mathcal{F} \otimes \bigwedge^2 \mathcal{V}^\vee) = 1 + 3 = 4$$

since $H^{n-1}(\bigwedge^3 \mathcal{F} \otimes \bigwedge^n \mathcal{V}^\vee) = 0$ for $n = 3, \ldots, 10$. Q.E.D.

§2. Proof of Theorems 1 and 2

Let S be the zero locus $(s)_0$ of a general global section s of the homogeneous vector bundle $\mathcal{V} = \bigwedge^2 \mathcal{E} \oplus \bigwedge^2 \mathcal{E} \oplus \bigwedge^3 \mathcal{F}$ on the Grassmannian $G(7,3)$. Since $\mathcal{V}$ is generated by global sections, S is smooth by [6, Theorem 1.10], the Bertini type theorem for vector bundles. Since $r(\mathcal{V}) = 3 + 3 + 4 = \dim G(7,3) - 2$ and

$$\det \mathcal{V} \simeq \mathcal{O}_G(2) \otimes \mathcal{O}_G(2) \otimes \mathcal{O}_G(3) \simeq \det T_{G(7,3)},$$

S is of dimension two and the canonical line bundle is trivial. By (a) of Proposition 5, S is connected and regular. Hence S is a K3 surface. We denote the class of hyperplane section by h. Then, by (b) of

Proposition 5, we have $\chi(\mathcal{O}_S(h)) = 14$, which implies $(h^2) = 24$ by the Riemann-Roch theorem. Hence we obtain the rational map

$$\Psi : \mathbf{P}_* H^0(G(7,3), \mathcal{V}) \cdots \rightarrow \mathcal{F}'_{13} \quad s \mapsto ((s)_0, h)$$

to the moduli space $\mathcal{F}'_{13}$ of polarized K3 surfaces which are not necessarily primitive.

By (g) of Proposition 5, the vector bundle $E = \mathcal{E}|_S$ is simple. Let (S', h') be a small deformation of (S, h). Then there is a vector bundle E' on S' which is a deformation of E by Proposition 4.1 of [6]. E' enjoys many properties satisfied by E: E' is simple, generated by global sections, $h^0(E') = 7$, $\bigwedge^3 H^0(E') \longrightarrow H^0(\bigwedge^3 E')$ is surjective, etc. Therefore, E' embeds S' into $G(7,3)$ and S' is also a complete intersection with respect to $\mathcal{V}$. Hence the rational map Ψ is dominant onto an irreducible component of $\mathcal{F}'_{13}$ and Theorem 1 follows from the following:

Proposition 6. *The polarization h of (S, h), a complete intersection with respect to $\mathcal{V}$ in $G(7,3)$, is primitive.*

In the local deformation space of (S, h), the deformations (S', h')'s with Picard number one form a dense subset. More precisely, it is the complement of an infinite but countable union of divisors. Hence we have

Lemma 7. *There exists a smooth complete intersection S with respect to $\mathcal{V}$ whose Picard number is equal to one.*

Proof of Proposition 6. Since the assertion is topological it suffices to show it for one such (S, h). We take (S, h) as in this lemma. Assume that h is not primitive. Since $(h^2) = 24$, h is linearly equivalent to $2l$ for a divisor class l with $(l^2) = 6$. The Picard group $\operatorname{Pic} S$ is generated by l. By the Riemann-Roch theorem and the (Kodaira) vanishing, we have $h^0(\mathcal{O}_S(nl)) = 3n^2 + 2$ for $n \geq 1$.

Claim 1. $h^0(E(-l)) = 0$.

Assume the contrary. Then E contains a subsheaf isomorphic to $\mathcal{O}_S(nl)$ with $n \geq 1$. Since $h^0(\mathcal{O}_S(nl)) \leq h^0(E) = 7$, we have $n = 1$ and the quotient sheaf $Q = E/\mathcal{O}_S(l)$ is torsion free. Since $5 = h^0(\mathcal{O}_S(l)) < h^0(E) = 7$, we have $H^0(Q) \neq 0$. Since Q is of rank two and $\det Q \simeq \mathcal{O}_S(l)$, we have $\operatorname{Hom}(Q, \mathcal{O}_S(l)) \neq 0$, which contradicts (g) of Proposition 5.

Now we consider the vector bundle $M = (\bigwedge^2 E)(-l)$. By the claim and the Serre duality, we have $h^2(M) = \dim \operatorname{Hom}(M, \mathcal{O}_S) = h^0(E(-l)) = 0$. Hence we have $h^0(M) \geq \chi(M) = 4$. Take 4 linearly independent global sections of M and we consider the homomorphism $\varphi : 4\mathcal{O}_S \longrightarrow M$.

Claim 2. φ is surjective outside a finite set of points on S.

Let r be the rank of the image of φ. Since $\mathrm{Hom}(\mathcal{O}_S(l), M) = H^0(\bigwedge^2 E)(-h)) = H^0(E^\vee) = H^2(E)^\vee = 0$ by (c) of Proposition 5, we have $r \geq 2$. Since $\mathrm{Hom}(M, \mathcal{O}_S) = 0$, $r = 2$ is impossible. Hence we have $r = 3$. Since the image and M have the same determinant line bundle ($\simeq \mathcal{O}_S(l)$), the cokernel of φ is supported by a finite set of points.

The kernel of φ is a line bundle by the claim. It is isomorphic to $\mathcal{O}_S(-l)$. Hence we have the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-l) \longrightarrow 4\mathcal{O}_S \xrightarrow{\varphi} M.$$

Since $\chi(\mathrm{Coker}\,\varphi) = 3 < \chi(M)$, φ is not surjective. In fact, the cokernel is a skyscraper sheaf supported at a point. Tensoring $\mathcal{O}_S(l)$, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow 4\mathcal{O}_S(l) \xrightarrow{\varphi(l)} \bigwedge^2 E \longrightarrow \mathbf{C}(p) \longrightarrow 0.$$

$H^0(\varphi(l))$ is surjective since $h^0(4\mathcal{O}_S(l)) = 20$ and $h^0(\bigwedge^2 E) = 19$. But this contradicts (e) of Proposition 5. Q.E.D.

Proof of Theorem 2. Let $P = \langle \sigma_1, \sigma_2 \rangle$ be a general 2-dimensional subspace of $\bigwedge^2 \mathbf{C}^7$ and $X^6 \subset G(7,3)$ the common zero locus of the two global sections of $\bigwedge^2 \mathcal{E}$ corresponding to σ_1 and σ_2. A point q of $\mathbf{P}_*(\bigwedge^3 \mathbf{C}^{7,\vee}/\overline{P \wedge P})$ determines a global section of $\bigwedge^3 \mathcal{F}|_X$. We denote the zero locus by $S_q \subset X^6$.

$$\begin{array}{ccccc} S_q & \subset & X^6 & \subset & G(7,3) \\ \cap & & \cap & & \cap \\ \mathbf{P}^{13} & \subset & \mathbf{P}^{20} & \subset & \mathbf{P}^{34} \end{array}$$

The restriction of $\mathcal{E}$ to S_q is semi-rigid by (g) of Proposition 5. Let $\Xi^{31} \subset \mathbf{P}_*(\bigwedge^3 \mathbf{C}^{7,\vee}/\overline{P \wedge P})$ be the open subset consisting of points q such that S_q is a K3 surface and the restriction $\mathcal{E}|_{S_q}$ is stable with respect to h.

Lemma 8. *Ξ^{31} is not empty.*

Proof. Let (S, h) be as in Lemma 7 and put $E = \mathcal{E}|_S$. Then, by Proposition 6, $\mathrm{Pic}\, S$ is generated by h. Since $h^0(\mathcal{O}_S(h)) = 14 > h^0(E) = 7$, we have $\mathrm{Hom}(\mathcal{O}_S(nh), E) = 0$ for every integer $n \geq 1/3$. Since

$c_1(E) = h$ and since $\mathrm{Hom}(E, \mathcal{O}_S(nh)) = 0$ for every integer $n \le 1/3$, E is stable. Q.E.D.

The correspondence $q \mapsto \mathcal{E}|_{S_q}$ induces a morphism from a general fiber of $\Xi^{31}/G \cdots \to \mathcal{F}_{13}$ at $[S_q]$ to the moduli space $M_S(3, h, 4)$ of semi-rigid bundles. Conversely there exists a morphism from a non-empty open subset of $M_S(3, h, 4)$ to the fiber since a small deformation E' of $\mathcal{E}|_{S_q}$ gives an embedding of S_q into $G(7, 3)$ such that the image is a complete intersection with respect to $\mathcal{V}$.

Remark 3. By (f) of Proposition 5, $H^0(X^6, \bigwedge^3 \mathcal{F}|_X)$ is isomorphic to $\bigwedge^3 \mathbf{C}^{7,\vee}/\overline{P \wedge P}$. Hence the rational map ψ in (1) coincides with $\mathbf{P}_*(H^0(X^6, \bigwedge^3 \mathcal{F}|_X))/G \cdots \to \mathcal{F}_{13}$ induced by $s \mapsto (s)_0$.

§3. K3 surface of genus seven and twelve

We describe two cases $g = 7$ and 12 closely related with Theorems 1 and 2. The proofs are quite similar to the cases $g = 13$ and 18, respectively, and we omit them.

First a polarized K3 surface of genus 7 has the following description other than that in the orthogonal Grassmannian $O\text{-}G(5, 10)$:

Theorem 9. *A general polarized K3 surface (S, h) of genus 7 is a complete intersection with respect to the rank four homogeneous vector bundle $2\mathcal{O}_G(1) \oplus \mathcal{E}(1)$ in the 6-dimensional Grassmannian $G(5, 2)$.*

S is the common zero locus of two hyperplane sections H_1 and H_2 of $G(5, 2) \subset \mathbf{P}^9$ corresponding to $\sigma_1, \sigma_2 \in \bigwedge^2 \mathbf{C}^5$ and one global section s of $\mathcal{E}(1)$. The 2-dimensional subspace $P = \langle \sigma_1, \sigma_2 \rangle \subset \bigwedge^2 \mathbf{C}^5$ is uniquely determined by S and $X^4 = G(5, 2) \cap H_1 \cap H_2$ is a quintic del Pezzo fourfold. Let Q be the image of $\mathbf{C}^5 \otimes P$ by the natural linear map $\mathbf{C}^7 \otimes \bigwedge^2 \mathbf{C}^7 \longrightarrow H^0(\mathcal{E}(1))$. Then Q is of dimension 10 and we obtain the natural rational map

$$\mathbf{P}_*(H^0(\mathcal{E}(1))/Q)/G^8 = \mathbf{P}_*(H^0(\mathcal{E}(1)|_X))/G^8 \cdots \to \mathcal{F}_7 \tag{6}$$

as in the case $g = 13$, where G^8 is the general stabilizer group of the action $PGL(5) \curvearrowright G(2, \bigwedge^2 \mathbf{C}^5)$. $H^0(\mathcal{E}(1))$ is a 40-dimensional irreducible representation of $GL(5)$ by Theorem 3. The fiber of the map (6) at general (S, h) is a surface and birationally equivalent to the moduli K3 surface $M_S(2, h, 3)$ of semi-rigid rank two vector bundles with $c_1 = h$ and $\chi = 2 + 3$.

Secondly, in the 12-dimensional Grassmannian $G(7, 3)$, there is another type of K3 complete intersection other than Theorem 1.

Theorem 10. *A general member* $(S, h) \in \mathcal{F}_{12}$ *is a complete intersection with respect to* $\mathcal{V}_{10} = 3 \bigwedge^2 \mathcal{E} \oplus \mathcal{O}_G(1)$ *in* $G(7, 3)$.

S is the common zero locus of the three global sections of $\bigwedge^2 \mathcal{E}$ corresponding to general bivectors $\sigma_1, \sigma_2, \sigma_3 \in \bigwedge^2 \mathbf{C}^7$. The 3-dimensional subspace $N = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \bigwedge^2 \mathbf{C}^7$ is uniquely determined by S. The common zero locus X_N of the global sections of $\bigwedge^2 \mathcal{E}$ corresponding to N is a Fano threefold and is embedded into $\mathbf{P}^{13}$ anti-canonically. X_N's are parameterized by an open set Ξ^6 of the orbit space $G(3, \bigwedge^2 \mathbf{C}^7)/PGL(7)$. See [5] for other descriptions of X_N's and their moduli spaces. The moduli space $\mathcal{F}_{12}$ is birationally equivalent to a $\mathbf{P}^{13}$-bundle over this Ξ^6.

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