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On Davies' conjecture and strong ratio limit properties for the heat kernel

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Abstract.

We study strong ratio limit properties and the exact long time asymptotics of the heat kernel of a general second-order parabolic operator which is defined on a noncompact Riemannian manifold.

§1. Introduction

Let P be a linear, second-order, elliptic operator defined on a noncompact, connected, C^3-smooth Riemannian manifold $\mathcal{M}$ of dimension d with a Riemannian measure $\mathrm{d}x$. Here P is an elliptic operator with real, Hölder continuous coefficients which in any coordinate system $(U; x_1, \ldots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x).$$

We assume that for each $x \in \mathcal{M}$ the real quadratic form $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j$ is positive definite. The formal adjoint of P is denoted by P^*. Denote the cone of all positive (classical) solutions of the equation $Pu=0$ in $\mathcal{M}$ by $\mathcal{C}_P(\mathcal{M})$. The *generalized principal eigenvalue* is defined by

$$\lambda_0 = \lambda_0(P, \mathcal{M}) := \sup\{\lambda \in \mathbb{R} \,:\, \mathcal{C}_{P-\lambda}(\mathcal{M}) \neq \emptyset\}.$$

Throughout this paper we always assume that $\lambda_0 \geq 0$ (actually, as it will become clear below, it is enough to assume that $\lambda_0 > -\infty$).

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We consider the parabolic operator L

$$Lu = u_t + Pu \qquad \text{on } \mathcal{M} \times (0,\infty). \tag{1.1}$$

We denote by $\mathcal{H}_P(\mathcal{M}\times(a,b))$ the cone of all nonnegative solutions of the equation $Lu=0$ in $\mathcal{M}\times(a,b)$. Let $k_P^{\mathcal{M}}(x,y,t)$ be the *minimal (positive) heat kernel* of the parabolic operator L in $\mathcal{M}$. If for some $x \neq y$

$$\int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau < \infty \qquad \left(\text{respect., } \int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau = \infty\right),$$

then P is said to be a *subcritical* (respect., *critical*) operator in $\mathcal{M}$ [18].

Recall that if $\lambda < \lambda_0$, then P–λ is subcritical in $\mathcal{M}$, and for $\lambda \leq \lambda_0$, we have $k_{P-\lambda}^{\mathcal{M}}(x,y,t) = \mathrm{e}^{\lambda t} k_P^{\mathcal{M}}(x,y,t)$. Furthermore, P is critical (respect., subcritical) in $\mathcal{M}$, if and only if P^* is critical (respect., subcritical) in $\mathcal{M}$. If P is critical in $\mathcal{M}$, then there exists a unique positive solution $\varphi \in \mathcal{C}_P(\mathcal{M})$ satisfying $\varphi(x_0) = 1$, where $x_0 \in \mathcal{M}$ is a fixed reference point. This solution is called the *ground state of the operator P in $\mathcal{M}$* [15, 18]. The ground state of P^* is denoted by φ^*. A critical operator P is said to be *positive-critical* in $\mathcal{M}$ if $\varphi^*\varphi \in L^1(\mathcal{M})$, and *null-critical* in $\mathcal{M}$ if $\varphi^*\varphi \notin L^1(\mathcal{M})$. In [15, 17] we proved:

Theorem 1.1. *Let $x, y \in \mathcal{M}$. Then*

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^{\mathcal{M}}(x,y,t) = \begin{cases} \dfrac{\varphi(x)\varphi^*(y)}{\int_{\mathcal{M}} \varphi(z)\varphi^*(z)\,\mathrm{d}z} & \textit{if } P-\lambda_0 \textit{ is positive-critical,} \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, for $\lambda < \lambda_0$, let $G_{P-\lambda}^{\mathcal{M}}(x,y) := \int_0^\infty k_{P-\lambda}^{\mathcal{M}}(x,y,\tau)\mathrm{d}\tau$ be the minimal (positive) Green function of the operator $P-\lambda$ on $\mathcal{M}$. Then

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^{\mathcal{M}}(x,y,t) = \lim_{\lambda \nearrow \lambda_0} (\lambda_0 - \lambda) G_{P-\lambda}^{\mathcal{M}}(x,y). \tag{1.2}$$

Having proved that $\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^{\mathcal{M}}(x,y,t)$ always exists, we next ask how fast this limit is approached. It is natural to conjecture that the limit is approached equally fast for different points $x, y \in \mathcal{M}$. Note that in the context of Markov chains, such an *(individual) strong ratio limit property* is in general not true [5]. The following conjecture was raised by E. B. Davies [7] in the selfadjoint case.

Conjecture 1.1. *Let $Lu = u_t + P(x,\partial_x)u$ be a parabolic operator which is defined on a Riemannian manifold $\mathcal{M}$. Fix a reference point $x_0 \in \mathcal{M}$. Then*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)} = a(x,y) \tag{1.3}$$

exists and is positive for all $x, y \in \mathcal{M}$.

The aim of the present paper is to discuss Conjecture 1.1 and closely related problems, and to obtain some results under minimal assumptions.

Remark 1.1. Theorem 1.1 implies that Conjecture 1.1 holds true in the positive-critical case. So, we may assume in the sequel that P **is not positive critical**. Also, Conjecture 1.1 does not depend on the value of λ_0, hence from now on, **we shall assume that** $\lambda_0 = 0$.

Remark 1.2. In the selfadjoint case, Conjecture 1.1 holds true if $\dim \mathcal{C}_P(\mathcal{M}) = 1$ [2, Corollary 2.7]. In particular, it holds true for a critical selfadjoint operator. Therefore, it would be interesting to prove Conjecture 1.1 at least under the assumption

$$\dim \mathcal{C}_P(\mathcal{M}) = \dim \mathcal{C}_{P^*}(\mathcal{M}) = 1, \tag{1.4}$$

which holds true in the critical case and in many important subcritical cases. Recently, Agmon [1] has obtained the exact asymptotics (in (x, y, t)) of the heat kernel for a periodic (non-selfadjoint) operator on $\mathbb{R}^d$. It follows from Agmon's results that Conjecture 1.1 holds true in this case. For a probabilistic interpretation of Conjecture 1.1, see [2].

Remark 1.3. Let $t_n \to \infty$. By a standard parabolic argument, we may extract a subsequence $\{t_{n_k}\}$ such that for every $x, y \in \mathcal{M}$ and $s < 0$

$$a(x, y, s) := \lim_{k\to\infty} \frac{k_P^{\mathcal{M}}(x, y, s + t_{n_k})}{k_P^{\mathcal{M}}(x_0, y_0, t_{n_k})} \tag{1.5}$$

exists. Moreover, $a(\cdot, y, \cdot) \in \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$. Note that in the selfadjoint case, the above is valid for all $s \in \mathbb{R}$, since (2.7) holds in selfadjoint case [7, Theorem 10].

Remark 1.4. The example constructed in [16, Section 4] shows a case where Conjecture 1.1 holds true on $\mathcal{M}$, while the limit function $a(x, y) = 1$ is not a λ_0-invariant positive solution. Compare this with [7, Theorem 25] and the discussion therein above Lemma 26. Note also that in general, the limit function $a(x, y)$ in (1.3) need not be a *product* of solutions of the equations $Pu = 0$ and $P^*u = 0$, as is demonstrated in [6], in the hyperbolic space, and in Example 4.2.

The outline of the rest of paper is as follows. In the next section we study the existence of the strong ratio limit for the heat kernel. It turns out that if this limit exists, then it equals 1. This implies that any limit solution $u(\cdot, y, s)$ of (1.5) is time independent and is a positive solution of the equation $Pu = 0$ in $\mathcal{M}$. In Section 3 we discuss the relationship between Conjecture 1.1 and the parabolic Martin compactification of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$, while in Section 4 we study the relation between this conjecture and the parabolic and elliptic *minimal* Martin boundaries. Finally, in Section 5 we study Conjecture 1.1 under the assumption that the uniform restricted parabolic Harnack inequality holds true.

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§2. Strong ratio properties

In the symmetric case the function $t \mapsto k_P^{\mathcal{M}}(x,x,t)$ is log-convex, and therefore, a polarization argument implies that $\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = 1$ for all $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$ [7]. In the nonsymmetric case we have:

Lemma 2.1. *For every $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$, we have that*

$$\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq 1 \leq \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} . \tag{2.1}$$

Similarly, for any $s > 0$

$$\liminf_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n\pm1)s)}{k_P^{\mathcal{M}}(x,y,ns)} \leq 1 \leq \limsup_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n\pm1)s)}{k_P^{\mathcal{M}}(x,y,ns)} . \tag{2.2}$$

In particular, if $\lim_{t\to\infty}[k_P^{\mathcal{M}}(x,y,t+s)/k_P^{\mathcal{M}}(x,y,t)]$ exists, it equals to 1.

Proof. We may assume that P is not positive-critical. Let $s < 0$. By Theorem 1.1 and the parabolic Harnack inequality we have

$$1 \leq \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq C(s,y). \tag{2.3}$$

Suppose that $\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = \ell > 1$. It follows that there exists $0 < q < 1$ and $T_s > 0$ such that

$$k_P^{\mathcal{M}}(x,y,t) < q k_P^{\mathcal{M}}(x,y,t+s) \qquad \forall t > T_s.$$

By induction and the Harnack inequality, we obtain that there exist $\mu < 0$ and $C > 0$ such that $k_P^{\mathcal{M}}(x,y,t) < C\mathrm{e}^{\mu t}$ for all $t > 1$, a contradiction to the assumption $\lambda_0 = 0$. Therefore, (2.1) is proved for $s < 0$, which in turn implies (2.1) also for $s > 0$. (2.2) can be proven similarly. □

Remark 2.1. The condition $\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \geq 1$ for $s > 0$ is sometimes called *Lin's condition* [11].

Corollary 2.1. *Let $x, y \in \mathcal{M}$. Suppose that*

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n+1)s)}{k_P^{\mathcal{M}}(x,y,ns)} \tag{2.4}$$

exists for every $s > 0$ (i.e., the ratio limit exists for every "skeleton" sequence of the form $t_n = ns$, where $n = 1, 2, \dots$ and $s > 0$). Then

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x,y,t)} = 1 \qquad \forall r \in \mathbb{R}. \tag{2.5}$$

Proof. By Lemma 2.1, the limit in (2.4) equals 1. By induction, $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,ns+r)}{k_P^{\mathcal{M}}(x,y,ns)} = 1$, where $r = qs$, and $q \in \mathbb{Q}$, which (by the continuity of a limiting solution) implies that it holds for $\forall r \in \mathbb{R}$. Hence, [9, Theorem 2] implies (2.5). $\square$

Remark 2.2. If there exist $x_0, y_0 \in \mathcal{M}$ and $0 < s_0 < 1$ such that

$$M(x_0,y_0,s_0) := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} < \infty, \tag{2.6}$$

then by the parabolic Harnack inequality, for all $x, y, z, w \in K \subset\subset \mathcal{M}$, $t > 1$, we have the following Harnack inequality of elliptic type:

$$k_P^{\mathcal{M}}(z,w,t) \le C_1 k_P^{\mathcal{M}}(x_0,y_0,t+\frac{s_0}{2}) \le C_2 k_P^{\mathcal{M}}(x_0,y_0,t-\frac{s_0}{2}) \le C_3 k_P^{\mathcal{M}}(x,y,t).$$

Similarly, (2.6) implies that for all $x, y \in \mathcal{M}$ and $r \in \mathbb{R}$:

$$\begin{aligned} 0 < m(x,y,r) := \liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x_0,y_0,t)} \le \\ \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x_0,y_0,t)} = M(x,y,r) < \infty. \end{aligned} \tag{2.7}$$

Lemma 2.2. (a) *The following assertions are equivalent:*

(i) *For each $x, y \in \mathcal{M}$ there exists a sequence $s_j \to 0$ of negative numbers such that for all $j \ge 1$, and $s = s_j$, we have*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = 1. \tag{2.8}$$

(ii) *The ratio limit in (2.8) exists for any $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$.*

(iii) *Any limit function $u(x,y,s)$ of the quotients $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ does not depend on s and has the form $u(x,y)$, where $u(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$ for every $y \in \mathcal{M}$ and $u(x,\cdot) \in \mathcal{C}_{P^*}(\mathcal{M})$ for every $x \in \mathcal{M}$.*

(b) *If one assumes further* (1.4), *then Conjecture* 1.1 *holds true.*

Proof. (a) By Lemma 2.1, if the limit in (2.8) exists, then it is 1.

(i) $\Rightarrow$ (ii). Fix $x_0, y_0 \in \mathcal{M}$, and take $s_0 < 0$ for which the limit (2.8) exists. It follows that any limit solution $u(x,y,s) \in \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ of a sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)}$ with $t_n \to \infty$ satisfies $u(x_0,y_0,s+s_0) = u(x_0,y_0,s)$ for all $s<0$. So, $u(x_0,y_0,\cdot)$ is s_0-periodic. It follow from our assumption and the continuity of u that $u(x_0,y_0,\cdot)$ is the constant function. Since this holds for all $x,y \in \mathcal{M}$ and u, it follows that (2.8) holds for any $x,y \in \mathcal{M}$ and $s \in \mathbb{R}$.

(ii) $\Rightarrow$ (iii). Fix $y \in \mathcal{M}$. By Remark 1.3, any limit function u of the sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ belongs to $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$. Since

$$\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)} = \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)}, \tag{2.9}$$

(2.8) implies that such a u does not depend on s. Therefore, $u = u(x,y)$, where $u(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$ and $u(x,\cdot) \in \mathcal{C}_{P^*}(\mathcal{M})$.

(iii) $\Rightarrow$ (i). Write

$$\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \frac{k_P^{\mathcal{M}}(x_0,x_0,t)}{k_P^{\mathcal{M}}(x,y,t)}. \tag{2.10}$$

Let $t_n \to \infty$ be a sequence such that the sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ converges to a solution in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$. By our assumption, we have

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = \lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = u(x,y) > 0,$$

which together with (2.10) implies (2.8) for all $s \in \mathbb{R}$.

(b) The uniqueness and (iii) imply that $\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \to \frac{u(x)u^*(y)}{u(x_0)u^*(x_0)}$, where $u \in \mathcal{C}_P(\mathcal{M})$ and $u^* \in \mathcal{C}_{P^*}(\mathcal{M})$, and Conjecture 1.1 holds. $\square$

Remark 2.3. Let $\mathcal{M} \subsetneq \mathbb{R}^d$ be a smooth domain and P and P^* be (up to the boundary) smooth operators. Denote by $\mathcal{C}_P^0(\mathcal{M})$ the cone of all functions in $\mathcal{C}_P(\mathcal{M})$ which vanish on $\partial\mathcal{M}$. Suppose that one of the conditions (i)–(iii) of Lemma 2.2 is satisfied. Clearly, for any fixed y any limit function $u(\cdot,y)$ of Lemma 2.2 belongs to the Martin boundary 'at infinity' which in this case is $\mathcal{C}_P^0(\mathcal{M})$. Therefore, Conjecture 1.1 holds true if the Martin boundaries 'at infinity' of P and P^* are one-dimensional. As a simple example, take $P = -\Delta$ and $\mathcal{M} = \mathbb{R}_+^d$.

Lemma 2.3. *Suppose that P is null-critical, and for each $x, y \in \mathcal{M}$ there exists a sequence $\{s_j\}$ of negative numbers such that $s_j \to 0$, and*

$$\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \geq 1 \tag{2.11}$$

for $s = s_j$, $j = 1, 2, \ldots$. Then Conjecture 1.1 holds true.

Proof. Let $u(x,y,s)$ be a limit function of a sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ and $s < 0$. By our assumption, $u(x,y,s+s_j) \geq u(x,y,s)$, and therefore, $u_s(x,y,s) \leq 0$ for all $s < 0$. Thus, $u(\cdot,y,s)$ (respect., $u(x,\cdot,s)$) is a positive supersolution of the equation $Pu = 0$ (respect., $P^*u = 0$) in $\mathcal{M}$. Since P is critical, it follows that $u(\cdot,y,s) \in \mathcal{C}_P(\mathcal{M})$ (respect., $u(x,\cdot,s) \in \mathcal{C}_{P^*}(\mathcal{M})$), and hence $u_s(x,y,s) = 0$. By the uniqueness, u equals to $\frac{\varphi(x)\varphi^*(y)}{\varphi(x_0)\varphi^*(x_0)}$, and Conjecture 1.1 holds true. □

Remark 2.4. Suppose that P is null-critical, and fix $x_0 \neq y_0$. Then using Theorem 1.1 and [14, Theorem 2.1] we have for $x \neq y$:

$$\textit{(i)}\ \lim_{t\to\infty} k_P^{\mathcal{M}}(x,y,t) = \lim_{t\to\infty} k_P^{\mathcal{M}}(x_0,y_0,t) = 0,$$

$$\textit{(ii)}\ \int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau = \int_0^\infty k_P^{\mathcal{M}}(x_0,y_0,\tau)\,\mathrm{d}\tau = \infty,$$

$$\textit{(iii)}\ \lim_{\lambda\nearrow 0} \frac{\int_0^\infty \mathrm{e}^{\lambda\tau} k_P^{\mathcal{M}}(x,y,\tau)\mathrm{d}\tau}{\int_0^\infty \mathrm{e}^{\lambda\tau} k_P^{\mathcal{M}}(x_0,y_0,\tau)\mathrm{d}\tau} = \lim_{\lambda\nearrow 0} \frac{G_{P-\lambda}^{\mathcal{M}}(x,y)}{G_{P-\lambda}^{\mathcal{M}}(x_0,y_0)} = \frac{\varphi(x)\varphi^*(y)}{\varphi(x_0)\varphi^*(y_0)}.$$

Therefore, Conjecture 1.1 would follow from a strong ratio Tauberian theorem if additional Tauberian conditions are satisfied (see, [3, 19]).

§3. The parabolic Martin boundary

The large time behavior of quotients of the heat kernel is obviously closely related to the parabolic Martin boundary (for the parabolic Martin boundary theory see [8]). Theorem 3.1 relates Conjecture 1.1 and the parabolic Martin compactification of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$.

Lemma 3.1. *Fix $y \in \mathcal{M}$. The following assertions are equivalent:*

(i) *For each $x \in \mathcal{M}$ there exists a sequence $s_j \to 0$ of negative numbers such that*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \tag{3.1}$$

exists for $s = s_j$, $j = 1, 2, \ldots$.

(ii) *Any parabolic Martin function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ corresponding to a Martin sequence $\{(y, -t_n)\}_{n=1}^\infty$, where $t_n \to \infty$, is time independent.*

Proof. Let $K_P^{\mathcal{M}}(x,y,s) = \lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y,t_n)}$ be such a Martin function. The lemma follows from the identity

$$\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y,t_n)} = \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x,y,t_n)} \, \frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,y,t_n)},$$

and Lemma 2.2. □

Theorem 3.1. *Assume that* (2.6) *holds true for some* $x_0, y_0 \in \mathcal{M}$, *and* $s_0 > 0$. *Then the following assertions are equivalent:*

(i) *Conjecture 1.1 holds true for a fixed* $x_0 \in \mathcal{M}$.

(ii)

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_1,y_1,t)} \tag{3.2}$$

exists, and the limit is positive for every $x, y, x_1, y_1 \in \mathcal{M}$.

(iii)

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(y,y,t)}, \quad and \quad \lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x,x,t)} \tag{3.3}$$

exist, and these ratio limits are positive for every $x, y \in \mathcal{M}$.

(iv) *For any* $y \in \mathcal{M}$ *there is a unique nonzero parabolic Martin boundary point* $\bar{y}$ *for the equation* $Lu = 0$ *in* $\mathcal{M} \times \mathbb{R}$ *which corresponds to any sequence of the form* $\{(y, -t_n)\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$, *and for any* $x \in \mathcal{M}$ *there is a unique nonzero parabolic Martin boundary point* $\bar{x}$ *for the equation* $u_t + P^*u = 0$ *in* $\mathcal{M} \times \mathbb{R}$ *which corresponds to any sequence of the form* $\{(x, -t_n)\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$.

Moreover, if Conjecture 1.1 holds true, then for any fixed $y \in \mathcal{M}$ *(respect.,* $x \in \mathcal{M}$*), the limit function* $a(\cdot, y)$ *(respect.,* $a(x,\cdot)$*) is a positive solution of the equation* $Pu = 0$ *(respect.,* $P^*u = 0$*). Furthermore, the Martin functions of part (iv) are time independent, and (2.8) holds for all* $x, y \in \mathcal{M}$ *and* $s \in \mathbb{R}$.

Proof. (i) ⇒ (ii) follows from the identity

$$\frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_1,y_1,t)} = \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \cdot \left(\frac{k_P^{\mathcal{M}}(x_1,y_1,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)}\right)^{-1}.$$

(ii) ⇒ (iii). Take $x_1 = y_1 = y$ and $x_1 = y_1 = x$, respectively.

(iii) ⇒ (iv). It is well known that the Martin compactification does not depend on the fixed reference point x_0. So, fix $y \in \mathcal{M}$ and take it also as a reference point. Let $\{-t_n\}$ be a sequence such that $t_n \to \infty$

and such that the Martin sequence $\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)}$ converges to a Martin function $K_P^{\mathcal{M}}(x,\bar{y},t)$. By our assumption, for any t we have

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t+t_n)} = \lim_{\tau\to\infty} \frac{k_P^{\mathcal{M}}(x,y,\tau)}{k_P^{\mathcal{M}}(y,y,\tau)} = b(x) > 0,$$

where b does not depend on the sequence $\{-t_n\}$. On the other hand,

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(y,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)} = K_P^{\mathcal{M}}(y,\bar{y},t) = f(t).$$

Since

$$\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)} = \frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t+t_n)} \cdot \frac{k_P^{\mathcal{M}}(y,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)},$$

we have

$$K_P^{\mathcal{M}}(x,\bar{y},t) = b(x)f(t).$$

By separation of variables, there exists a constant λ such that

$$Pb - \lambda b = 0 \quad \text{on } \mathcal{M}, \qquad f' + \lambda f = 0 \quad \text{on } \mathbb{R},\ f(0) = 1.$$

Since b does not depend on the sequence $\{-t_n\}$, it follows in particular, that λ does not depend on this sequence. Thus, $\lim_{\tau\to\infty} \frac{k_P^{\mathcal{M}}(y,y,t+\tau)}{k_P^{\mathcal{M}}(y,y,\tau)} = f(t) = \mathrm{e}^{-\lambda t}$. Lemma 2.1 implies that $\lambda = 0$. It follows that b is a positive solution of the equation $Pu = 0$, and

$$K_P^{\mathcal{M}}(x,\bar{y},t) = \lim_{\tau\to-\infty} \frac{k_P^{\mathcal{M}}(x,y,t-\tau)}{k_P^{\mathcal{M}}(y,y,-\tau)} = b(x). \tag{3.4}$$

The dual assertion can be proved similarly.

(iv) $\Rightarrow$ (i). Let $K_P^{\mathcal{M}}(x,\bar{y},t)$ be a Martin function, and $s_0 > 0$ such that $K_P^{\mathcal{M}}(x_0,\bar{y},s_0/2) > 0$. Consequently, $K_P^{\mathcal{M}}(x,\bar{y},s) > 0$ for $s \geq s_0$. Using the substitution $\tau = s + s_0$ we obtain

$$\lim_{\tau\to\infty} \frac{k_P^{\mathcal{M}}(x,y,\tau)}{k_P^{\mathcal{M}}(x_0,x_0,\tau)} = \lim_{s\to\infty} \left\{ \frac{k_P^{\mathcal{M}}(x,y,s+s_0)}{k_P^{\mathcal{M}}(y,y,s)} \times \right.$$

$$\left. \frac{k_P^{\mathcal{M}}(y,y,s)}{k_P^{\mathcal{M}}(x_0,y,s+2s_0)} \frac{k_P^{\mathcal{M}}(x_0,y,s+2s_0)}{k_P^{\mathcal{M}}(x_0,x_0,s+s_0)} \right\} = \frac{K_P^{\mathcal{M}}(x,\bar{y},s_0) K_{P^*}^{\mathcal{M}}(\overline{x_0},y,s_0)}{K_P^{\mathcal{M}}(x_0,\bar{y},2s_0)}.$$

The last assertion of the theorem follows from (3.4) and Lemma 2.2. □

§4. Minimal positive solutions

In this section we discuss the relation between Conjecture 1.1 and the parabolic and elliptic *minimal* Martin boundaries.

Remark 4.1. By the parabolic Harnack inequality for P^*, we have for each $0 < \varepsilon < 1$

$$k_P^{\mathcal{M}}(x, y_0, t-\varepsilon) \le C(y_0, \varepsilon) k_P^{\mathcal{M}}(x, y_0, t) \qquad \forall x \in \mathcal{M},\ t > 1. \tag{4.1}$$

Therefore, if $\{(y_0, t_n)\}$ is a nontrivial minimal Martin sequence with $t_n \to -\infty$, then one infers as in [10] that the corresponding minimal parabolic function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ is of the form $u(x,t) = \mathrm{e}^{-\lambda t} u_\lambda(x, y_0)$ with $\lambda \le 0$ and $u_\lambda \in \operatorname{exr} \mathcal{C}_{P-\lambda}(\mathcal{M})$, where $\operatorname{exr} \mathcal{C}$ is the set of extreme rays of a cone $\mathcal{C}$. If further, for some $x_0 \in \mathcal{M}$ and $s < 0$ one has

$$\liminf_{t \to \infty} \frac{k_P^{\mathcal{M}}(x_0, y_0, t+s)}{k_P^{\mathcal{M}}(x_0, y_0, t)} \ge 1, \tag{4.2}$$

then $\lambda = 0$, and consequently, u is also a minimal solution in $\mathcal{C}_P(\mathcal{M})$. Recall that in the selfadjoint case, the ratio *limit* in (4.2) equals 1.

Lemma 4.1. *Suppose that the ratio limit in* (2.8) *exists for all $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$. Let $a(x,y) := \lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$, where $t_n \to \infty$. If for some $y_0 \in \mathcal{M}$ the function $u(x) := a(x, y_0)$ is minimal in $\mathcal{C}_P(\mathcal{M})$, then $a(x,y) = u(x)v(y)$, where $v \in \mathcal{C}_{P^*}(\mathcal{M})$.*

Proof. Fix $y \in \mathcal{M}$ and $\varepsilon > 0$. By the parabolic Harnack inequality for P^* and Lemma 2.2, we have

$$\frac{k_P^{\mathcal{M}}(x, y, t-\varepsilon)}{k_P^{\mathcal{M}}(x_0, x_0, t)} \le C(y, \varepsilon) \frac{k_P^{\mathcal{M}}(x, y_0, t)}{k_P^{\mathcal{M}}(x_0, x_0, t)} \qquad \forall x \in \mathcal{M}. \tag{4.3}$$

Therefore, $a(x,y) \le C(y) u(x)$ which implies the claim. □

The following examples demonstrate that if Conjecture 1.1 holds true while (1.4) does not hold, then the limit function $a(\cdot, y)$ is typically a non-minimal solution in $\mathcal{C}_P(\mathcal{M})$.

Example 4.1. Consider a (regular) Benedicks domain $\mathcal{M} \subseteq \mathbb{R}^d$ such that the cone of positive harmonic functions which vanish on $\partial\mathcal{M}$ is of dimension two. By [6], Conjecture 1.1 holds true in this case, the limit function is not a product of two (separated) harmonic functions, and therefore, $a(\cdot, y)$ is not minimal in $\mathcal{C}_{-\Delta}(\mathcal{M})$ for any $y \in \mathcal{M}$.

Example 4.2. Consider a radially symmetric Schrödinger operator $H := -\Delta + V(|x|)$ on $\mathbb{R}^d$ with a bounded potential. Suppose that $\lambda_0 = 0$, and that the Martin boundary of H on $\mathbb{R}^d$ is homeomorphic to S^{d-1} (see [12]). Clearly, any Martin function corresponding to $\{(y_0, t_n)\}$ with $x_0 = y_0 = 0$ is radially symmetric. It follows that Davies' conjecture holds true for $x_0 = y = 0$, and the limit function is the normalized positive radial solution in $\mathcal{C}_H(\mathbb{R}^d)$. This solution is not minimal in $\mathcal{C}_H(\mathbb{R}^d)$. Thus, any limit function $u(\cdot, y)$ is not minimal in $\mathcal{C}_H(\mathbb{R}^d)$.

We conclude this section with some related problems. The following conjecture was posed by the author in [15, Conjecture 3.6].

Conjecture 4.1. *Suppose that P is a critical operator in $\mathcal{M}$, then the ground state φ is a minimal positive solution in the cone $\mathcal{H}_P(\mathcal{M} \times \mathbb{R})$.*

Note that if (2.11) holds true, then by Theorem 3.1, the ground state is a Martin function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R})$.

Example 4.3. Consider again the example in [16, Section 4]. In that example $-\Delta$ is subcritical in $\mathcal{M}$, $\lambda_0 = 0$, and (1.4) and Conjecture 1.1 hold true. Hence, $\mathbf{1}$ is a Martin function in $\mathcal{H}_{-\Delta}(\mathcal{M} \times \mathbb{R})$. On the other hand, $\mathbf{1} \in \operatorname{exr} \mathcal{C}_{-\Delta}(\mathcal{M})$ but $\mathbf{1} \notin \operatorname{exr} \mathcal{H}_{-\Delta}(\mathcal{M} \times \mathbb{R})$. So, Conjecture 4.1 cannot be extended to the subcritical "Liouvillian" case (see also [4]).

Thus, it would be interesting to study the following problem which was raised by Burdzy and Salisbury [4] for $P = -\Delta$ and $\mathcal{M} \subset \mathbb{R}^d$.

Question 4.1. *Assume that $\lambda_0 = 0$. Determine which minimal positive solutions in $\mathcal{C}_P(\mathcal{M})$ are minimal in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$.*

§5. Uniform Harnack inequality and Davies' conjecture

In this section we discuss the relationship between the parabolic Martin boundary of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$, the elliptic Martin boundaries of $\mathcal{C}_{P-\lambda}(\mathcal{M})$, $\lambda \le \lambda_0 = 0$, and Conjecture 1.1 under a certain assumption.

Definition 5.1. We say that the *uniform restricted parabolic Harnack inequality* (in short, (URHI)) holds in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ if for any $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon) > 0$ such that

$$u(x, t-\varepsilon) \le C u(x,t) \quad \forall (x,t) \in \mathcal{M} \times \mathbb{R}_- \text{ and } \forall u \in \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-). \tag{5.1}$$

It is well known that (URHI) holds true if and only if the *separation principle* (SP) holds true, that is, $u \ne 0$ is in $\operatorname{exr} \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ if and only if u is of the form $\mathrm{e}^{-\lambda t} v_\lambda(x)$, where $v_\lambda \in \operatorname{exr} \mathcal{C}_{P-\lambda}(\mathcal{M})$ [10, 13]. In particular, the answer to Question 4.1 is simple if (URHI) holds.

Lemma 5.1. (i) *Suppose that (URHI) holds true, then for any* $s<0$

$$\ell_+ := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \le 1 \qquad \textit{(Lin's condition).}$$

(ii) *Assume further that for some* $x_0, y_0 \in \mathcal{M}$ *and* $s_0<0$

$$\ell_- := \liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} \ge 1,$$

then any limit function $u(x,y,s)$ *of* $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)}$ *with* $t_n\to\infty$ *does not depend on* s, *and has the form* $u(x,y)$, *where* $u(\cdot,y)\in\mathcal{C}_P(\mathcal{M})$ *for every* $y\in\mathcal{M}$ *and* $u(x,\cdot)\in\mathcal{C}_{P^*}(\mathcal{M})$ *for every* $x\in\mathcal{M}$.

(iii) *If one assumes further (1.4), then Conjecture 1.1 holds true.*

Proof. (i) By (URHI), if $u\in \operatorname{exr}\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$, then $u(x,t)=\mathrm{e}^{-\lambda t}u_\lambda(x)$, where $\lambda\le 0$. Consequently, for every $u\in\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$

$$u(x,t+s)\le u(x,t) \qquad \forall (x,t)\in\mathcal{M}\times\mathbb{R}_-, \text{ and } \forall s<0, \tag{5.2}$$

and equality holds for some $s<0$ and $(x,t)\in\mathcal{M}\times\mathbb{R}_-$ if and only if $u\in\mathcal{C}_P(\mathcal{M})$. Clearly, (5.2) implies that

$$\ell_+ := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \le 1 \quad \forall x,y\in\mathcal{M} \text{ and } s<0,$$

which together with Lemma 2.1 implies $\ell_+=1$.

(ii) At the point (x_0,y_0,s_0) we have $\ell_-=\ell_+=1$, therefore,

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} = 1. \tag{5.3}$$

Consequently, for any sequence $t_k\to\infty$ satisfying

$$\lim_{k\to\infty} \frac{k_P^{\mathcal{M}}(x,y_0,t_k+\tau)}{k_P^{\mathcal{M}}(x_0,y_0,t_k)} = u(x,\tau) \quad \forall (x,\tau)\in\mathcal{M}\times\mathbb{R}_-,$$

we have $u(x_0,s_0)=u(x_0,2s_0)=1$, and therefore, $u\in\mathcal{C}_P(\mathcal{M})$. The other assertions of the lemma follow from Lemma 2.2. □

Remark 5.1. From the proof of Lemma 5.1 it follows that if (URHI) holds true, then a sequence $t_n\to\infty$ satisfies

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t_n+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)} = 1,$$

for some $x_0, y_0 \in \mathcal{M}$ and $s_0 \neq 0$ if and only if

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x,y,t_n)} = 1 \qquad \forall x,y \in \mathcal{M} \text{ and } s \in \mathbb{R}.$$

Corollary 5.1. *Suppose that (URHI) holds true, then there exists a sequence* $t_n \to \infty$ *such that* $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = a(x,y)$ *exists and is positive for all* $x, y \in \mathcal{M}$. *Moreover,* $a(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$, *and* $a(\cdot,y)$ *is a parabolic Martin function for all* $y \in \mathcal{M}$. *For each* $x \in \mathcal{M}$ *the function* $a(x,\cdot)$ *satisfies similar properties with respect to* P^*.

Proof. Take $s_0 \neq 0$ and $\{t_n\}$ such that $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t_n+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)} = 1$, and use Remark 5.1 and a standard diagonalization argument. □

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