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Quasiconformal mappings and minimal Martin boundary of p-sheeted unlimited covering surfaces of the once punctered Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type

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Dedicated to Professor Masakazu Shiba on his sixtieth birthday

Abstract.

Let R and R' be p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformal equivalent to each other. Then the cardinal numbers of minimal Martin boundaries of R and R' are same.

Let R be a 2-sheeted unlimited covering surface of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type and R' be an open Riemann surface. If R and R' are quasiconformal equivalent to each other and the set of branch points of R satisfies a condition, then the cardinal numbers of minimal Martin boundaries of R and R' are same.

§1. Introduction.

Let W be an open Riemann surface. We denote by Δ_1^W the minimal Martin boundary of W. In [8], it was showed that there exist open Riemann surfaces F and F' quasiconformally equivalent to each other such that F' possesses nonconstant positive harmonic functions although F does not possess nonconstant positive harmonic functions. This means that $\sharp\Delta_1^{F'} \geq 2$ although $\sharp\Delta_1^{F} = 1$, where $\sharp A$ stands for the cardinal

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number of a set A. Needless to say, the above F and F' are of *positive boundary*, i.e. F and F' admit the Green function (cf. e.g. [16]). However, in case open Riemann surfaces F and F' are of *null boundary* (i.e. not positive boundary), it does not seem to be known whether $\sharp\Delta_1^F = \sharp\Delta_1^{F'}$ or not if F and F' are quasiconformally equivalent to each other.

Consider two positive decreasing sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1} < a_n < b_n < 1$ and $\lim_{n\to\infty} a_n = 0$. Set $G = \hat{\mathbb{C}} \setminus (\{0\} \cup I)$, where $\hat{\mathbb{C}}$ is the extended complex plane, $I = \cup_{n=1}^{\infty} I_n$ and $I_n = [a_n, b_n]$. We take p copies $G_1, \cdots, G_p$ of G and join the upper edge of I_n on G_j with the lower edge of I_n on G_{j+1} (j mod p) for every n. Then we obtain a p-sheeted covering surface R of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ and say that R is of Heins type(cf. [4]).

In this paper, we are concerned with p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type. Consider p-sheeted unlimited covering surfaces R and R' of $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformally equivalent to each other. Then it seems to be valid that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$ (cf. [12], [10], [18]). The first purpose of this paper is to give an answer to this conjecture. Namely,

Theorem 1. *Let R and R' be p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformally equivalent to each other. Then it holds that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

Let R be a 2-sheeted unlimited covering surface of $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type with the projection π from R onto $\hat{\mathbb{C}} \setminus \{0\}$. We have the following.

Theorem 2. *Suppose that $b_n - b_{n+1} \approx 2^{-n}$, that is, there exists a constant $\alpha(> 1)$ with $\alpha^{-1}2^{-n} < b_n - b_{n+1} < \alpha 2^{-n}$ ($n \in \mathbb{N}$). Let R' be an open Riemann surface and f a quasiconformal mapping with $R' = f(R)$. Then it holds that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

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§2. Preliminaries.

In this section we consider as R a general p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$. Let Δ^R and Δ_1^R be as in §1, and π the projection map from R onto $\hat{\mathbb{C}} \setminus \{0\}$. Set $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$, $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$ and $R_0 = \pi^{-1}(\mathbb{D}_0)$. It is well-known that Δ^{R_0} and $\Delta_1^{R_0}$ are identified with $\Delta^R \cup \pi^{-1}(\partial\mathbb{D})$ and $\Delta_1^R \cup \pi^{-1}(\partial\mathbb{D})$, respectively, where $\partial\mathbb{D} = \{x \in \mathbb{C} \mid |x| = 1\}$. From now on we consider $\mathbb{D}_0$ (resp. R_0) in place of $\hat{\mathbb{C}} \setminus \{0\}$ (resp. R) since $\hat{\mathbb{C}} \setminus \{0\}$ (resp. R) does not admit the Green function. Let g_0 be the Green function on $\mathbb{D}$ with pole at 0.

Definition 2.1 (cf. [2]). We say that a subset E of $\mathbb{D}_0$ is *thin* at 0 if ${}^{\mathbb{D}}\widehat{\mathrm{R}}_{g_0}^{E} \neq g_0$, where ${}^{\mathbb{D}}\widehat{\mathrm{R}}_{g_0}^{E}$ is the balayage of g_0 relative to E on $\mathbb{D}$.

If E is a closed subset of $\mathbb{D}$, it is well-known that E is thin at 0 if and only if 0 is an irregular boundary point of $\mathbb{D} \setminus E$ in the sense of the Dirichlet problem.

The following lemma gives the quasiconformal invariance for thinness.

Lemma 2.1 (cf. [10],[18]). *Let M be a subdomain of $\mathbb{C}$ and φ a quasiconformal mapping from $\mathbb{C}$ onto $\mathbb{C}$. If ζ is an irregular boundary point of M in the sense of Dirichlet problem, $\varphi(\zeta)$ is an irregular boundary point of $\varphi(M)$ in the sense of Dirichlet problem.*

Definition 2.2. A subset U in $\mathbb{D}$ which contains 0 is said to be a *fine neighborhood* of 0 if $\mathbb{D} \setminus U$ is thin at 0.

Let k_ζ be the Martin function on R_0 with pole at $\zeta \in \Delta^R$. If we take a sequence $\{x_n\}$ in R_0 such that $\lim_{n\to\infty} x_n = \zeta$, we can give a definition of k_ζ by the following.

$$k_\zeta(z) = \lim_{n\to\infty} \frac{g_{x_n}(x)}{g_{x_n}(x_0)},$$

where x_0 is a fixed point in R_0. For details we refer to [3] and [5].

Definition 2.3. Let ζ be a point in Δ_1^R and E a subset of R_0. We say that E is *minimally thin* at ζ if ${}^{R_0}\widehat{\mathrm{R}}_{k_\zeta}^{E} \neq k_\zeta$.

Definition 2.4. Let ζ be a point in Δ_1^R and U a subset of R_0. We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of ζ if $R_0 \setminus U$ is minimally thin at ζ.

The following proposition gives the characterization of $\sharp\Delta_1^R$ in terms of minimal fine topology.

Proposition 2.1 ([11]). *Let $\mathcal{M}$ be the class of subdomains M of $\mathbb{D}_0$ such that $M \cup \{0\}$ is a fine neighborhood of $x = 0$. Then it holds that*

$$\sharp\Delta_1^R = \max_{M\in\mathcal{M}} n_R(M),$$

where $n_R(M)$ is the number of connected components of $\pi^{-1}(M)$ and π is the projection map from R onto $\hat{\mathbb{C}} \setminus \{0\}$.

§3. Proof of Theorem 1.

In this section we first consider as R a general p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}}\setminus\{0\}$. Let Δ^R and Δ_1^R be as in §1, and π the projection map from R onto $\hat{\mathbb{C}}\setminus\{0\}$. Let $\mathbb{D}$, $\mathbb{D}_0$, and R_0 be as in §2. The next proposition will play an important role for the proof of Theorem 1.

Proposition 3.1. *Let R' be an open Riemann surface and f a quasiconformal mapping with $R' = f(R)$. If $\sharp\Delta_1^R = p$, then $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

Proof. By Proposition 2.1 we find a subdomain M of $\mathbb{D}_0$ such that $\mathbb{D}_0 \setminus M$ is thin at 0, $\partial M \setminus \{0\}$ may consist of infinitely many Jordan curves and

$$\sharp\Delta_1^R = n_R(M),$$

where $n_R(M)$ is the number of connected components of $\pi^{-1}(M)$. By the assumption of this proposition $n_R(M) = p$. Let M_j $(j = 1, 2, \ldots, p)$ be components of $\pi^{-1}(M)$. Since each M_j is a 1-sheeted unlimited covering surface of M, it is easily seen that each M_j is considered as a replica of M. Let $g_x^{f(M_j)}$ $(j = 1, 2 \ldots, p)$ be the Green function on $f(M_j)$ with pole at x and ψ_j the inverse of $\pi|_M$ from $M \to M_j$. Denote by $\mu_{f\circ\psi_j}$ the complex dilatation of $f \circ \psi_j$ on M. Set

$$\mu_j = \begin{cases} \mu_{f\circ\psi_j} & \text{on } M \\ 0 & \text{on } \mathbb{C} \setminus M. \end{cases}$$

It is well-known that there exists a quasiconformal mapping f_j from $\mathbb{C}$ onto $\mathbb{C}$ with the complex dilatation μ_j (cf. e.g. [6]). Set $V_j = f_j(M)$. By Lemma 2.1 we find that $f_j(0)$ is an irregular boundary point of V_j in the sense of the usual Dirichlet problem since 0 is an irregular boundary point of M in the sense of the usual Dirichlet problem. On the other hand, the function $x' \mapsto g^{f(M_j)}_{f\circ\psi_j\circ f_j^{-1}(x')} \circ f \circ \psi_j \circ f_j^{-1}(y')$ $(y' \in V_j)$ is a positive harmonic function on $V_j \setminus \{y'\}$ since $f \circ \psi_j \circ f_j^{-1}$ is conformal. Hence, by [5, Theorem 10.16], there exists a positive fine limit $\mathcal{F} - \lim_{x'\to f_j(0)} g^{f(M_j)}_{f\circ\psi_j\circ f_j^{-1}(x')} \circ f \circ \psi_j \circ f_j^{-1}$. Denote by $g_0^{V_j}$ this limit function on V_j and set $g_0^{f(M_j)} = g_0^{V_j} \circ f_j \circ \psi_j^{-1} \circ f^{-1}$. We see that each $g_0^{f(M_j)}$ is a positive harmonic function on $f(M_j)$ since each $g_0^{V_j}$ is a positive harmonic function on V_j and $f_j \circ \psi_j^{-1} \circ f^{-1}$ is conformal. For $j = 1, 2, \ldots, p$ set

$$S_j(g_0^{f(M_j)})(x') = \inf_s s(x'),$$

where s runs over the space of positive superharmonic functions s on $f(R_0)$ satisfying $s \geq g_0^{f(M_j)}$ on $f(M_j)$. By Perron-Wiener-Brelot method we find that each $S_j(g_0^{f(M_j)})$ is a positive harmonic function on $f(R_0)$. Then the following inequality

$$(*) \qquad S_j(g_0^{f(M_j)}) - {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{S_j(g_0^{f(M_j)})} \geq g_0^{f(M_j)}$$

holds on $f(M_j)$. In fact, to prove the inequality $(*)$ note that

$${}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{S_j(g_0^{f(M_j)})} = H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$$

on $f(M_j)$, where $H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$ is the Dirichlet solution for $S_j(g_0^{f(M_j)})$ on $f(M_j)$ (cf. e.g. [3], [5]). By definition $S_j(g_0^{f(M_j)}) \geq g_0^{f(M_j)}$ on $f(M_j)$. Hence, by the definition of the Dirichlet solution in the sense of Perron-Wiener-Brelot,

$$S_j(g_0^{f(M_j)}) - g_0^{f(M_j)} \geq H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$$

on $f(M_j)$. Thus $(*)$ is proved.

We shall proceed the proof of this proposition. By [17, Theorem 3] it is known that $1 \leq \sharp\Delta_1^{R'} \leq p$. By the Martin representation theorem, there exist at most p minimal functions $h_{j,1}, h_{j,2}, \ldots, h_{j,p}$ on $f(R_0)$ with $S_j(g_0^{f(M_j)}) = h_{j,1} + h_{j,2} + \ldots + h_{j,p}$ on $f(R_0)$. Hence, by the above inequality $(*)$, we have

$$\begin{aligned} & h_{j,1} + h_{j,2} + \ldots + h_{j,p} \\ = \; & S_j(g_0^{f(M_j)}) \\ \geq \; & {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,1}+h_{j,2}+\ldots+h_{j,p}} + g_0^{f(M_j)} \\ > \; & {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,1}} + {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,2}} + \ldots + {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,p}} \end{aligned}$$

on $f(M_j)$. Therefore we find that there exists a minimal function h_j on $f(R_0)$ such that $h_j \neq {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_j}$. Hence, by the definition of minimal thinness, $f(R_0) \setminus f(M_j)$ is minimally thin at the minimal boundary point corresponding to h_j. Since $f(M_i) \cap f(M_j) = \emptyset \;\; (i \neq j)$, we find that $\sharp\Delta_1^{R'} = p$. □

Now we give the following result which Proposition 2.1 yields.

Theorem 3.1 (cf. [11]). *Let R be a p-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type. Then $\sharp\Delta_1^R = 1$ or p.*

Proof of Theorem 1. By Theorem 3.1 we have only to prove that $\sharp\Delta_1^{R'} = p$ if and only if $\sharp\Delta_1^R = p$. Since f^{-1} is a quasiconformal mapping from R' onto R, it is sufficient to prove that if $\sharp\Delta_1^R = p$, then $\sharp\Delta_1^{R'} = p$. Suppose that $\sharp\Delta_1^R = p$. By Proposition 3.1 $\sharp\Delta_1^{R'} = p$. We have the desired result. □

§4. Proof of Theorem 2.

By Proposition 3.1 we find that if $\sharp\Delta_1^R = 2$, $\sharp\Delta_1^{R'} = 2$. By [17, Theorem 3] it is known that $\sharp\Delta_1^{R'} = 1$ or 2. Hence, by Theorem 3.1, it is sufficient to prove that if $\sharp\Delta_1^{R'} = 2$, $\sharp\Delta_1^R = 2$. Suppose that $\sharp\Delta_1^{R'} = 2$. Set $\Delta_1^{R'} = \{\zeta_1', \zeta_2'\}$. Let $g_{\xi'}^{f(R_0)}$ be the Green function with pole at ξ' on $f(R_0)$. It is known that there exists $\lim_{y'\to\zeta_j'} g_{y'}^{f(R_0)}(x')(=: g'_{\zeta_j'}(x'))$ $(j = 1,2)$ and $g'_{\zeta_j'}$ $(j = 1,2)$ is the minimal harmonic function with pole at ζ_j' $(j = 1,2)$.

For $x \in R_0$ set

$$L = L_f = L_{x,f} = \begin{cases} \sum_{i,k=1}^2 \partial_k(J_f(x)(f'(x)^{-1}f'(x)^{-1*})_{k,i}\partial_i), \\ \qquad \text{(if there exist } f'(x) \text{ and } f'(x)^{-1}), \\ \sum_{i=1}^2 \partial_i^2, \\ \qquad \text{(elsewise)}, \end{cases}$$

where $J_f(x)$ (resp. $f'(x)$) is the Jacobian (resp. Jacobi matrix) of the mapping $(u(x), v(x))$ $(f = u + iv)$, $f'(x)^{-1}$ is the inverse of $f'(x)$ and $f'(x)^{-1*}$ is the transpose of $f'(x)^{-1}$. L is a elliptic second order partial differential operator of divergence type on R. Set $g_j^L(x) := g'_{\zeta_j'} \circ f(x)$ $(x \in R_0)$. We see that g_j^L $(j = 1,2)$ is a positive harmonic function on R_0 with respect to L. We recall the assumption that $b_n - b_{n+1} \approx 2^{-n}$, that is, there exists a constant $\alpha(> 1)$ with

$$\alpha^{-1}2^{-n} < b_n - b_{n+1} < \alpha 2^{-n} \quad (n \in \mathbb{N}).$$

For $r(> 0)$, set $C_r = \{|x| = r\}$, $B_r = \{|x| < r\}$, $\mathcal{C}_r = \pi^{-1}(C_r)$, and $\mathcal{B}_r = \pi^{-1}(B_r \setminus \{0\})$.

Suppose that there exist a constant $\alpha'(> 1)$ and a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ with $b_{n_l} - a_{n_l} > (\alpha')^{-1}2^{-n_l}$. Set $\mathcal{R}_l = \mathcal{B}_{(a_{n_l}+3b_{n_l})/4} \setminus Cl(\mathcal{B}_{(3a_{n_l}+b_{n_l})/4})$, where, for a set $E \subset R_0$, $Cl(E)$ stands for the closure of E with respect to the usual topology on R_0. By the assumption that $b_{n_l} - a_{n_l} > (\alpha')^{-1}2^{-n_l}$, $Mod(\mathcal{R}_l) \approx 1$, where $Mod(\mathcal{R}_l)$ stands for the logarithmic module of $\mathcal{R}_l$ (cf. [1]), and hence, by the quasiconformal invariance of logarithmic module (cf. [6], [15]), $Mod(f(\mathcal{R}_l)) \approx 1$. Since the cardinal number of connected components of $\mathcal{R}_l$ is equal to 1, that of $f(\mathcal{R}_l)$ is so. By [17, Theorem 3], we find that $\sharp\Delta_1^{R'} = 1$. This is a contradiction. Hence we may suppose that there exists a constant $\alpha''(>1)$, for every integer l, $a_l - b_{l+1} > (\alpha'')^{-1}2^{-l}$. Set $\mathcal{A} = \cup_{l=1}^{\infty}\mathcal{A}_l$ ($\mathcal{A}_l = \mathcal{B}_{(3a_l+b_{l+1})/4} \setminus Cl(\mathcal{B}_{(a_l+3b_{l+1})/4})$), where $Cl(\mathcal{B}_{(a_l+3b_{l+1})/4})$ is the closure of $\mathcal{B}_{(a_l+3b_{l+1})/4}$ with respect to the usual topology on R.

Lemma 4.1. *On* $\mathcal{A}$,

$$g_j^L(x) + g_j^L(\iota(x)) \approx \log\frac{1}{|\pi(x)|} \quad (j = 1, 2),$$

where ι *is the sheet exchange on* R.

Proof. Let $A_{l,k}$ ($k = 1, 2$) be connected components of $\mathcal{A}_l$. Then we have

$$(\sharp) \quad {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)} \leq {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,1})} + {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,2})} \leq 2\,{}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}.$$

Since ${}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}$ is a Green potential on $f(R_0)$ (cf. [3]), we can find the Radon measure $\mu_{l,j}$ ($j = 1, 2$) with

$$(\sharp\sharp) \quad {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,j})}(x') = \int_{Cl(f(A_{l,j}))} g_{x'}^{f(R_0)}\, d\mu_{l,j}.$$

By the fact that ${}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}(x') = 1$ for $x' \in f(\mathcal{B}_{(3a_l+b_{l+1})/4})$, letting x' be to ζ_j' in ($\sharp$), we have

$$1 \leq \int_{Cl(f(A_{l,1}))} g'_{\zeta_j'}\, d\mu_{l,1} + \int_{Cl(f(A_{l,2}))} g'_{\zeta_j'}\, d\mu_{l,2} \leq 2 \quad (j = 1, 2),$$

and hence

$$1 \leq \int_{Cl(A_{l,1})} g_j^L d(f^{-1})^*(\mu_{l,1}) + \int_{Cl(A_{l,2})} g_j^L d(f^{-1})^*(\mu_{l,2}) \leq 2 \quad (j = 1, 2),$$

where $(f^{-1})^*(\mu_{l,2})$ is the image measure of $\mu_{l,2}$ by f^{-1}. On the other hand, by the definition of capacitary potential, quasiconformal invariance of capacity(cf. [15, Theorem 10.10]), [9, Lemma 2.3], [1, Theorems 13C and 13D in Chap. IV] and [3, Satz 5.2 and Satz 7.2], we have

$$\begin{aligned}(f^{-1})^*\mu_{l,j}(Cl(A_{l,j})) &= \mu_{l,j}(f(Cl(A_{l,j}))) = cap(f(Cl(A_{l,j})), f(R_0)) \\ &\approx cap(Cl(A_{l,j}), R_0) \approx cap(\pi(Cl(\mathcal{A}_l)), \mathbb{D}_0) \\ &= cap(Cl(B_{(3a_l+b_{l+1})/4}), \mathbb{D}_0) \\ &= 2\pi/\log[4/(3a_l + b_{l+1})] \approx 1/l,\end{aligned}$$

where, for a subset E of an open Riemann surface F of positive boundary $cap(E,F)$ stands for the greenian capacity of E on F. Therefore, by Harnack's inequality with respect to L (cf. [13]), we have the desired result. $\square$

Set $D_I = \mathbb{D}_0 \setminus I$.

Lemma 4.2. *There exist components $\mathcal{D}_{I,j}$ $(j = 1,2)$ of $\pi^{-1}(D_I)$ such that*

$$g_j^L(x) \approx \log \frac{1}{|\pi(x)|} \quad (x \in \mathcal{A} \cap \mathcal{D}_{I,j},\ j = 1,2),$$

$$g_j^L(x) = o(\log \frac{1}{|\pi(x)|}) \ (\pi(x) \to 0, x \in \mathcal{A} \cap \mathcal{D}_{I,j+(-1)^{j-1}},\ j = 1,2).$$

Proof. Denote by $\mathcal{D}_{I,j}$ $(j = 1,2)$ components of $\pi^{-1}(D_I)$. Set $\mathcal{A}_{l,j} = \mathcal{A}_l \cap \mathcal{D}_{I,j}$. By Lemma 4.1 we may suppose that there exist subsequences $\{n_1\}$ and $\{n_2\}$ of $\mathbb{N} = \{n\}$ such that

(i) $\{n_1\} \cup \{n_2\} = \mathbb{N}$ and $\{n_1\} \cap \{n_2\} = \emptyset$;

(ii) $g_1^L(x) \approx \log \dfrac{1}{|\pi(x)|}$ $(x \in (\cup_{n_1} \mathcal{A}_{n_1,1}) \cup (\cup_{n_2} \mathcal{A}_{n_2,2})\ )$;

(iii) $g_1^L(x) = o(\log \dfrac{1}{|\pi(x)|})$ $(\pi(x) \to 0, x \in (\cup_{n_1} \mathcal{A}_{n_1,2}) \cup (\cup_{n_2} \mathcal{A}_{n_2,1})\ )$.

In fact, suppose the above does not hold. Then there exists a subsequence $\{n_3\}$ of $\mathbb{N} = \{n\}$ with

$$g_1^L(x) \approx \log \frac{1}{|\pi(x)|} \quad (x \in \cup_{n_3} \mathcal{A}_{n_3}\ ).$$

On the other hand, for any $\beta(> 0)$, $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta g'_{\zeta'_1}(x')\} \cup \{\zeta'_2\}$ is a minimal fine neighborhood of ζ'_2, because, on $\{g'_{\zeta'_2} > \beta g'_{\zeta'_1}\}$, ${}^{f(R_0)}\widehat{\mathrm{R}}_{g'_{\zeta'_2}}^{\{g'_{\zeta'_2} \le \beta g'_{\zeta'_1}\}} < g'_{\zeta'_2}$, by the fact that, on $f(R_0)$,

$${}^{f(R_0)}\widehat{\mathrm{R}}_{g'_{\zeta'_2}}^{\{g'_{\zeta'_2} \le \beta g'_{\zeta'_1}\}} \le {}^{f(R_0)}\widehat{\mathrm{R}}_{\beta g'_{\zeta'_1}}^{\{g'_{\zeta'_2} \le \beta g'_{\zeta'_1}\}} \le \beta g'_{\zeta'_1}.$$

Hence, by Lemma 4.1 and by the fact that $g_j^L = g'_{\zeta'_j} \circ f$, there exists a positive β_0 with $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta_0 g'_{\zeta'_1}(x')\} \subset f(R_0) \setminus f(\cup_{n_3} \mathcal{A}_{n_3})$. It is well-known that we can take a connected component G_1 of $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta_0 g'_{\zeta'_1}(x')\}$ such that $G_1 \cup \{\zeta'_2\}$ is a minimal fine neighborhood of ζ'_2 (cf. [14, Corollaire 2 in p.206]). This is a contradiction.

Suppose that both $\{n_1\}$ and $\{n_2\}$ are infinite sets. Let $\{m_1\}$ be a subsequence of $\{n_1\}$ with $m_1+1 \in \{n_2\}$. By (ii) we can find a positive constant κ_1 (>1) with

$$\kappa_1^{-1}\log\frac{1}{|\pi(x)|} \le g_1^L(x) \le \kappa_1\log\frac{1}{|\pi(x)|}\ (x \in (\cup_{n_1}\mathcal{A}_{n_1,1}) \cup (\cup_{n_2}\mathcal{A}_{n_2,2})).$$

By Harnack's inequality with respect to L, we can find a positive constant κ_2 (>1) with

$$(\kappa_1\kappa_2)^{-1}\log\frac{1}{|\pi(x)|} \le g_1^L(x) \le (\kappa_1\kappa_2)\log\frac{1}{|\pi(x)|}\ (x \in \cup_{m_1}\mathcal{A}_{m_1+1,1}).$$

On the other hand, by (iii), there exists an integer N_0 such that,

$$g_1^L(x) < (\kappa_1\kappa_2)^{-1}\log\frac{1}{|\pi(x)|}\ (x \in \cup_{m_1>N_0-1}\mathcal{A}_{m_1+1,1}).$$

This is a contradiction. Here, if necessary , by substituting $\mathcal{D}_{I,1}$ (resp. $\mathcal{D}_{I,2}$) for $\mathcal{D}_{I,2}$ (resp. $\mathcal{D}_{I,1}$), we have

$$(\flat 1)\quad g_1^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in \cup_n\mathcal{A}_{n,1}\ )$$

$$(\flat 2)\quad g_1^L(x) = o(\log\frac{1}{|\pi(x)|})\ (\pi(x)\to 0, x \in \cup_n\mathcal{A}_{n,2}\ ).$$

Repeating the same process for g_1^L as in obtaining ($\flat$1) and ($\flat$2), we have

$$(\flat' 1)\quad g_2^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in \cup_n\mathcal{A}_{n,2}\ )$$

$$(\flat' 2)\quad g_2^L(x) = o(\log\frac{1}{|\pi(x)|})\ (\pi(x)\to 0, x \in \cup_n\mathcal{A}_{n,1}\ )$$

or

$$(\flat'' 1)\quad g_2^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in \cup_n\mathcal{A}_{n,1}\ )$$

$$(\flat'' 2)\quad g_2^L(x) = o(\log\frac{1}{|\pi(x)|})\ (\pi(x)\to 0, x \in \cup_n\mathcal{A}_{n,2}\ ).$$

Suppose that the estimates ($\flat$"1) and ($\flat$"2) hold. By ($\flat$1) and ($\flat$"1), we find that $f(\cup_n\mathcal{A}_{n,1})$ is minimally thin at ζ_1'. In fact, there exists a positive constant β_0 such that $\beta_0 g_{\zeta_1'}^{f(R_0)} \le g_{\zeta_2'}^{f(R_0)}$ on $f(\cup_n\mathcal{A}_{n,1})$, that is, $f(\cup_n\mathcal{A}_{n,1}) \subset \{x' \in f(R_0) | \beta_0 g_{\zeta_1'}^{f(R_0)} \le g_{\zeta_2'}^{f(R_0)}\}$. Using the same argument as that in the former part of the proof of this lemma, we find that $\{x' \in$

$f(R_0)|\beta_0 g_{\zeta_1'}^{f(R_0)} \leq g_{\zeta_2'}^{f(R_0)}\}$ is minimally thin at ζ_1'. Hence $f(\cup_n \mathcal{A}_{n,1})$ is minimally thin at ζ_1'.
By ($\flat$2) we can prove that there exists a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ such that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at ζ_1'. This fact will be proved afterwards. Hence $f(\cup_l \mathcal{A}_{n_l})$ is minimally thin at ζ_1' because $f(\cup_n \mathcal{A}_{n,1})$ is minimally thin at ζ_1'. Since $[f(R_0) \setminus f(\cup_l \mathcal{A}_{n_l})] \cup \{\zeta_1'\}$ is a minimal fine neighborhood of ζ_1', we can take a connected component G_2 of $[f(R_0) \setminus f(\cup_l \mathcal{A}_{n_l})] \cup \{\zeta_1'\}$ such that $G_2 \cup \{\zeta_1'\}$ is a minimal fine neighborhood of ζ_1' (cf. [14, Corollaire 2 in p.206]). This is a contradiction. Hence we have the estimates ($\flat'$1) and ($\flat'$2).

We still remain to prove that there exists a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ such that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at ζ_1'. By ($\flat$2) we can take a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ with

$$g_{\zeta_1'}^{f(R_0)}(x') \leq \frac{n_l}{l^2} \ (x' \in f(\cup_l \mathcal{A}_{n_l,2})\ ).$$

From this estimate it follows that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at ζ_1'. In fact, we take a point x_0' be a point of $f(R_0) \setminus Cl(f(\cup_l \mathcal{A}_{n_l,2}))$. Then, by ($\sharp\sharp$) in Lemma 4.1, the definition of capacitary potential, and the same estimate for capacity as in the latter part of the proof of Lemma 4.1, we have

$$\begin{aligned}
0 \leq {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0)}}^{f(\cup_{l\geq m}\mathcal{A}_{n_l,2})}(x_0') &\leq \sum_{l=m}^{\infty} {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0)}}^{f(\mathcal{A}_{n_l,2})}(x_0') \\
&\leq \sum_{l=m}^{\infty} \frac{n_l}{l^2}\ {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_{n_l,2})}(x_0') \\
&\leq \sum_{l=m}^{\infty} \frac{n_l}{l^2} \int_{Cl(f(\mathcal{A}_{n_l,2}))} g_{x_0'}^{f(R_0)} d\mu_{n_l,2} \\
&\leq \alpha_0 \sum_{l=m}^{\infty} \frac{n_l}{l^2}\ \mu_{n_l,2}(Cl(f(\mathcal{A}_{n_l,2}))) \\
&\approx \sum_{l=m}^{\infty} \frac{n_l}{l^2}\ cap(Cl(f(\mathcal{A}_{n_l,2})), f(R_0)) \\
&\approx \sum_{l=m}^{\infty} \frac{n_l}{n_l l^2} \approx \sum_{l=m}^{\infty} \frac{1}{l^2} \to 0\ (m \to +\infty),
\end{aligned}$$

where, $\alpha_0 = \sup\{g_{x_0'}^{f(R_0)}(x') | x' \in Cl(f(\cup_l \mathcal{A}_{n_l,2}))\}$. Hence we have $\lim_{m\to+\infty} {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0)}}^{f(\cup_{l\geq m}\mathcal{A}_{n_l,2})}(x_0') = 0$. If m is sufficiently large,

$f(\cup_{l\geq m}\mathcal{A}_{n_l,2})$ is minimally thin at ζ_1'. Since $f(\cup_{l\leq m}\mathcal{A}_{n_l,2})$ is relatively compact, it is minimally thin at ζ_1'. Hence $f(\cup_l\mathcal{A}_{n_l,2})$ is minimally thin at ζ_1'.

The proof is herewith complete. □

For an integer l, take the bounded simply connected domain Q_l whose boundary in the closed polygonal line without self-intersections and which has four vertexes $((3a_l + b_{l+1})/4, (3a_l + b_{l+1})/32), ((3a_l + b_{l+1})/4, -(3a_l + b_{l+1})/32), ((a_{l-1} + 3b_l)/4, -(a_{l-1} + 3b_l)/32), ((a_{l-1} + 3b_l)/4, (a_{l-1} + 3b_l)/32)$ in positive cyclic order. Set $Q = \cup_{l=1}^{\infty}Q_l$ and $\mathcal{D}_{Q,j} = \mathcal{D}_{I,j}\backslash\pi^{-1}(Q)$ $(j = 1,2)$. By Lemma 4.2 and Harnack's inequality with respect to L, we find that

(1) there exists a positive constant κ_0 such that
$\frac{1}{\kappa_0}\log\frac{1}{|\pi(x)|} \leq g_j^L(x) \leq \kappa_0\log\frac{1}{|\pi(x)|}$ $(x \in \mathcal{D}_{Q,j})$ $(j = 1,2)$;

(2) $g_j^L(x) = o(\log\frac{1}{|\pi(x)|})$ $(\pi(x) \to 0, x \in \mathcal{D}_{Q,j+(-1)^{j-1}})$ $(j = 1,2)$.

Set $E_1' = \{x' \in f(R_0)|g'_{\zeta_1'}(x') > g'_{\zeta_2'}(x')\}$, $E_2' = \{x' \in f(R_0)|g'_{\zeta_1'}(x') < g'_{\zeta_2'}(x')\}$, and $E_3' = \{x' \in f(R_0)|g'_{\zeta_1'}(x') = g'_{\zeta_2'}(x')\}$. Set $E_3 = f^{-1}(E_3') = \{x \in R_0|g_1^L(x) = g_2^L(x)\}$ and $\gamma_j = \pi^{-1}(\partial Q)\cap\mathcal{D}_{I,j}$. By (1) and (2), we may suppose that there exists an integer N_1 such that, for any integer $n(\geq N_1)$, $E_3\cap\mathcal{B}_{(a_n+b_{n+1})/2} \subset \pi^{-1}(Q)$, $g_1^L > g_2^L$ on $\gamma_1\cap\mathcal{B}_{(a_n+b_{n+1})/2}$ and $g_1^L < g_2^L$ on $\gamma_2\cap\mathcal{B}_{(a_n+b_{n+1})/2}$. Hence, by the implicit function theorem, $E_3'\cap f(\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ consists of infinitely many connected components $E_{3,l}'(\subset f(\pi^{-1}(Q_l)), l \geq N_1 + 1)$ which are piecewise analytic closed curves because each $g'_{\zeta_j'}$ is harmonic on $f(R_0)$. Hence each $E_j'\cap f(\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ is a planar region, that is, each $E_j\cap\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2}$ is planar region. Set $K_j = E_j\cap\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2}$ and $E_{3,l} = f^{-1}(E_{3,l}')$. By Koebe's theorem and R. de Possel's theorem (cf. [20, Theorems IX.32 and IX.22], [19, Theorem 9-1]) there exist plane regions $\mathcal{E}_j$ $(j = 1,2)$ of $\mathbb{C}$ and conformal mappings ϕ_j $(j = 1,2)$ from K_j onto $\mathcal{E}_j$ $(j = 1,2)$ such that $\mathbb{C}\setminus\mathcal{E}_j$ $(j = 1,2)$ consist of infinitely many parallel segments $\ell_{j,l}$ to the real axis with

$$\ell_{j,l} = \begin{cases} \bigcap\left\{Cl(\phi_j(M)) \;\middle|\; \begin{array}{l} M \text{ is a subdomain of } E_j \text{ with} \\ Cl(M) \supset E_{3,l} \end{array}\right\}, & \text{for } l > N_1, \\ \bigcap\left\{Cl(\phi_j(M)) \;\middle|\; \begin{array}{l} M \text{ is a subdomain of } E_j \text{ with} \\ Cl(M) \supset \mathcal{C}_{(a_{N_1}+b_{N_1+1})/2}\cap\mathcal{D}_{I,j} \end{array}\right\}. & \text{for } l = N_1. \end{cases}$$

Set $\ell_j = \cap_{n \geq N_1+1} Cl(\cup_{l \geq n} \ell_{j,l})$ $(j = 1, 2)$.

Lemma 4.3. *Each ℓ_j is a singleton.*

Proof. Suppose that $\sharp \ell_j \geq 2$ $(j = 1, 2)$. We remark that each ℓ_j is connected. In fact, suppose that ℓ_j is disconnected. Let $\Lambda_{j,1}$ be a component of ℓ_j. Set $\Lambda_{j,2} = \ell_j \setminus \Lambda_{j,1}$. We can take two Jordan curves $\mathcal{C}_{j,1}$ and $\mathcal{C}_{j,2}$ in $\mathcal{E}_j$ such that, for $k = 1, 2$, each bounded region $G_{j,k,1}$ determined by $\mathcal{C}_{j,k}$ in $\mathbb{C}$ contains $\Lambda_{j,k}$, and that $Cl(G_{j,1,1}) \cap Cl(G_{j,2,1}) = \emptyset$. By the definition of $\Lambda_{j,k}$, each $G_{j,k,1}$ contains infinitely many $\ell_{j,l}$. Since $\pi \circ \phi_j^{-1}$ is continuous on $\mathcal{E}_j$ and $\mathcal{C}_{j,k}$ is a compact subset of $\mathcal{E}_j$, $\pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})$ is a compact subset of $\pi(K_j)$, and hence there exists uniquely a component $M_{j,k,1}$ of $\pi(K_j) \setminus \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})$ such that $Cl(M_{j,k,1})$ is a neighborhood of the origin. Denote by $M_{j,k,2}$ the union of component of $[\pi(K_j) \setminus \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})] \cup M_{j,k,1}$. It is easily seen that $Cl(M_{j,k,1})$ (resp. $Cl(M_{j,k,2})$) contains infinitely (resp. at most finitely) many components $\pi(E_{3,l})$ of $\pi(E_3 \cap \mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ because $\pi(E_{3,l}) \subset Q_l$ $(l \geq N_1 + 1)$. Let $G_{j,k,2}$ be unbounded regions determined by $\mathcal{C}_{j,k}$ in $\mathbb{C}$. We can prove that ($\natural$) $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \subset G_{j,k,1} \cap \mathcal{E}_j$ or ($\natural'$) $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \subset G_{j,k,2} \cap \mathcal{E}_j$. Suppose this fact does not hold, that is, $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,1} \cap \mathcal{E}_j \neq \emptyset$ and $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,2} \cap \mathcal{E}_j \neq \emptyset$. Then we can find points $\xi_{j,k,i} \in \phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,i} \cap \mathcal{E}_j$ $(i = 1, 2)$. Since $\pi(\phi_j^{-1}(\xi_{j,k,i})) \in M_{j,k,1}$ and $M_{j,k,1}$ is connected, we can find a curve C in $M_{j,k,1}$ which joins $\pi(\phi_j^{-1}(\xi_{j,k,1}))$ to $\pi(\phi_j^{-1}(\xi_{j,k,2}))$. From the definition of component it is easily seen that the lift of C in K_j by π meets $\phi_j^{-1}(C_{j,k})$ since $K_j \setminus \phi_j^{-1}(C_{j,k})$ has just two components $\phi_j^{-1}(G_{j,k,1} \cap \mathcal{E}_j)$ and $\phi_j^{-1}(G_{j,k,2} \cap \mathcal{E}_j)$. Hence $M_{j,k,1} \cap \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k}) \neq \emptyset$. This is a contradiction.
We may assume that ($\natural$) holds. For, if ($\natural'$) holds, repeating the same argument as in case that ($\natural$) holds, we arrive at a contradiction. By ($\natural$) $\phi_j(\pi^{-1}(M_{j,k,2}) \cap K_j) \supset G_{j,k,2} \cap \mathcal{E}_j$. Hence $G_{j,k,1}$ (resp. $G_{j,k,2}$) contains infinitely (resp. at most finitely) many $\ell_{j,l}$ because $Cl(M_{j,k,1})$ (resp. $Cl(M_{j,k,2})$) contains infinitely (resp. at most finitely) many components $\pi(E_{3,l})$ of $\pi(E_3 \cap \mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$. Since $G_{j,k,2} \supset G_{j,k+(-1)^{k-1},1}$, $G_{j,k+(-1)^{k-1},1}$ contains at most finitely many components of $\ell_{j,l}$. This is a contradiction. Thus we conclude that each ℓ_j is connected.

Since each ℓ_j is connected, by [5, Theorem 8.26], all points of ℓ_j $(j = 1, 2)$ are regular boundary points of $\mathcal{E}_j$ $(j = 1, 2)$. E'_j $(j = 1, 2)$ is minimally thin at $\zeta'_{j+(-1)^{j-1}}$, and hence E'_3 is minimally thin at ζ'_j $(j = 1, 2)$. By [14, Théorème 1 and Théorème 5], it is known that there exists a

Green potential $g_{\mu_j}(x') = \int g_{x'}^{f(R_0)} d\mu_j$ such that

$$g_{\mu_j}(\zeta_j') < +\infty,$$

$$\lim_{x' \to \Delta^{R'}, x' \in E_3'} g_{\mu_j}(x') = +\infty,$$

because $\lim_{x' \to \zeta_j'} g_x^{f(R_0)}(x') = g_x^{f(R_0)}(\zeta_j') < +\infty$. Since there exists an integer $N_2 (\geq N_1)$ such that $g_{\mu_j}(y') > 2g_{\mu_j}(\zeta_j')$ $(y' \in E_3' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))$, for every $x' \in f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$,

$$ {}^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E_3' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x') \leq \frac{g_{\mu_j}(x')}{2g_{\mu_j}(\zeta_j')}.$$

Hence

$$\liminf_{x'(\in E_j' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})) \to \zeta_j'} {}^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E_3' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x')$$

$$\leq \liminf_{x'(\in E_j' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})) \to \zeta_j'} \frac{g_{\mu_j}(x')}{2g_{\mu_j}(\zeta_j')}$$

$$= \liminf_{x' \to \zeta_j'} \frac{g_{\mu_j}(x')}{2g_{\mu_j}(\zeta_j')} = \frac{1}{2} < 1,$$

because each $E_j' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))$ is not minimally thin at ζ_j'. Hence there exists a sequence $\{x_{l,j}'\} (\subset E_j' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}),\ j = 1, 2)$ such that, for $j = 1, 2$,

$$\lim_{l \to \infty} x_{l,j}' = \zeta_j',$$

$$\lim_{l \to \infty} {}^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E_j' \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x_{l,j}') < 1.$$

Set

$$B_{N_2}^{(j)} = Int[Cl(\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))]\ (j = 1, 2),$$

where $Int[Cl(\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))]$ stands for the interior of the closure of $\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$ in $\mathbb{C}$. For $\phi_j \circ f^{-1}$ we define $L_{\phi_j \circ f^{-1}}$ as L_f in the first part of this section. The above inequality implies that there exist points $z_j \in \ell_j$ $(j = 1, 2)$ and sequences $\{z_{l,j}\}$ $(\subset \mathcal{E}_j \cap B_{N_2}^{(j)},\ j = 1, 2)$ such that, for $j = 1, 2$,

$$\lim_{l \to \infty} z_{l,j} = z_j\ (j = 1, 2),$$

$$\lim_{l \to \infty} {}^{B_{N_2}^{(j)}}\widehat{\mathrm{R}}_1^{\cup_{l > N_2} \ell_{l,j}, L_{\phi_j \circ f^{-1}}}(z_{l,j}) < 1,$$

where ${}^{B_{N_2}^{(j)}}\widehat{\mathrm{R}}_1^{\cup_{l>N_2}\ell_{l,j},L_{\phi_j\circ f^{-1}}}$ stands for the balayage of 1 relative to $\cup_{l>N_2}\ell_{l,j}$ on $B_{N_2}^{(j)}$ with respect to $L_{\phi_j\circ f^{-1}}$. Hence each z_j is an irregular boundary point of $\phi_j(E_j\cap\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$ with respect to $L_{\phi_j\circ f^{-1}}$. By [7, Theorem 9.1] and [5, Theorem 10.3], each z_j is an irregular boundary points of $\mathcal{E}_j$ in the usual sense. This is a contradiction. Therefore we have the desired result. □

Let N_1 be an integer as in the definition of ℓ_j. Let $g_\xi^{\mathcal{E}_j}$ be the Green function with pole at ξ (resp. x) on $\mathcal{E}_j$. By Lemma 4.3, for $j=1,2$, there exists a sequence $\{\xi_{j,n}\}$ in $\mathcal{E}_j$ such that $\lim_{n\to\infty}\xi_{j,n}=z_j$ and there exists $\lim_{n\to\infty} g_{\xi_{j,n}}^{\mathcal{E}_j}$ on $\mathcal{E}_j$. For $j=1,2$, set $g_{z_j}^{\mathcal{E}_j}=\lim_{n\to\infty} g_{\xi_{j,n}}^{\mathcal{E}_j}$ and $g_j=g_{z_j}^{\mathcal{E}_j}\circ\phi_j$. Each g_j is a positive harmonic function on K_j. For $j=1,2$, set

$$S_j(g_j)(x)=\inf_s s(x),$$

where s runs over the space of positive superharmonic functions s on R_0 satisfying $s\ge g_j$ on K_j. By Perron-Wiener-Brelot method each $S_j(g_j)$ is a positive harmonic function on R_0. Using the same argument as that in the proof of Theorem 1, we find that the following inequality

$$(**)\qquad S_j(g_j)-{}^{R_0}\widehat{\mathrm{R}}_{S_j(g_j)}^{R_0\setminus K_j}\ge g_j$$

holds on K_j $(j=1,2)$. Since $\sharp\Delta_1^R=1$ or 2 by means of [17, Theorem 3], by the Martin representation theorem, we find that there exist at most two minimal functions $h_{j,k}$ $(k=1,2)$ on R_0 with $S_j(g_j)=h_{j,1}+h_{j,2}$ on R_0. Hence, by the above inequality $(**)$, we have

$$\begin{aligned} h_{j,1}+h_{j,2}=S_j(g_j) &\ge {}^{R_0}\widehat{\mathrm{R}}_{h_{j,1}+h_{j,2}}^{R_0\setminus K_j}+g_j \\ &> {}^{R_0}\widehat{\mathrm{R}}_{h_{j,1}}^{R_0\setminus K_j}+{}^{R_0}\widehat{\mathrm{R}}_{h_{j,2}}^{R_0\setminus K_j}\end{aligned}$$

on K_j. Therefore we find that there exists a minimal function h_j $(j=1,2)$ on R_0 such that $h_j\ne {}^{R_0}\widehat{\mathrm{R}}_{h_j}^{R_0\setminus K_j}$. Hence, by the definition of minimal thinness, $R_0\setminus K_j$ is minimally thin at the minimal boundary point corresponding to h_j. Since $K_1\cap K_2=\emptyset$, we find that $\sharp\Delta_1^R=2$.

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