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tt^* geometry and mixed Hodge structures

Claus Hertling

tt^* geometry is a generalization of variation of Hodge structures (section 2). Also the nilpotent orbits of Schmid and the relation to polarized mixed Hodge structures generalize; part of this is still a conjecture (section 3). tt^* geometry turns up in the unfoldings of holomorphic functions with isolated singularities (section 4).

This short paper is an introduction and a survey. It gives definitions, results, conjectures and references, but no proofs. It follows closely a talk which was given at the conference on *Singularity theory and its applications* (MSJ-IRI2003) in Sapporo, Japan, on 16-25 September 2003.

§1. Motivation and history

An isolated hypersurface singularity comes equipped with a polarized mixed Hodge structure (PMHS) on the middle cohomology of a Milnor fiber [St]. If one considers a semiuniversal unfolding with base space M of such a singularity, one obtains a variation of PMHS's on a subspace of M, the μ-constant stratum. But in fact, the variation of PMHS's extends to a variation of a more general structure on the whole base space.

This structure is called tt^* geometry. The purpose of this paper is to define it and discuss it first in an abstract setting and then in the case of singularities.

tt^* geometry was established more than 10 years ago in the work of Cecotti and Vafa [CV1][CV2][CV3]. They considered moduli spaces of $N = 2$ supersymmetric field theories. A distinguished class of these field theories, the Landau-Ginzburg models, is closely related to singularities. Especially, the unfoldings of quasihomogeneous singularities were studied by Cecotti and Vafa. Their work deserves much more attention from the singularity community.

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tt^* geometry in the semisimple case turned up already in the work on holonomic quantum fields of Jimbo, Miwa, Mori, Sato ([JM] and references there, [CV2]).

Completely independently, a slightly weaker version of tt^* geometry was studied by Simpson [Si1][Si2][Si3] with the notion of harmonic bundles. But his techniques and results seem to be further away from the singularity case than the physicists' work. In his work the base spaces M are compact manifolds.

Sabbah [Sab4] greatly generalized the concepts of Simpson and proved with them a special case of a conjecture of Kashiwara [Ka].

Mochizuki [Mo1][Mo2] built on Sabbah's work and extended it. It seems [Mo2, Remark 1.7] that his results imply the general case of Kashiwara's conjecture.

The idea to generalize variations of Hodge structures in terms of meromorphic connections is also present in the work of Barannikov [Ba].

One way to describe tt^* geometry is in terms of a holomorphic vector bundle $H \to \mathbb{C} \times M$ with a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank 1 along $\{0\} \times M$, with a flat real structure and a flat pairing with certain properties [He2]. The case $M = \{pt\}$ is explained in section 2.

In the singularity case this is realized by a Fourier-Laplace transformation of the Gauss-Manin system of the unfolding parametrized by M, that means, essentially, by oscillating integrals [DS1][He2].

The same structure was used 20 years ago by K. Saito [SaK] and M. Saito [SaM] to establish on M Frobenius manifold structures. Now tt^* geometry enriches this with a real structure and a hermitian metric. I hope that the interplay of these structures will have many applications, for example on the moduli of singularities, on K. Saito's period maps, on two conjectures about the distribution of the spectral numbers ([He1, ch. 14] and [CV3, ch. 4.3]), on the relation to quantum cohomology and mirror symmetry.

Note added in proof: Conjecture 4.3 has now been proved by C. Sabbah in [Sab5, theorem 4.9]. The proof of most of theorem 3.8 has now been written up in [HS, chapters 6 and 9].

§2. Definitions

Definition 2.1. (a) A (TERP)-structure (Twistor Extension Real Pairing) of weight $w \in \mathbb{Z}$ is a tuple $(H \to \mathbb{C}, \nabla, H'_{\mathbb{R}}, P)$ with

$H \to \mathbb{C}$ a hol. vector bundle;

∇ a flat connection on $H|_{\mathbb{C}^*}$ with a pole of order ≤ 2 at 0;

$H'_{\mathbb{R}} \to \mathbb{C}^*$ a ∇-flat subbundle of $H|_{\mathbb{C}^*}$ of real vector spaces with $H_z = (H'_{\mathbb{R}})_z \oplus i(H'_{\mathbb{R}})_z$ for $z \in \mathbb{C}^*$;

P a $\mathbb{C}$-bilinear $(-1)^w$-symmetric nondegenerate ∇-flat pairing

$$P : H_z \times H_{-z} \to \mathbb{C} \quad \text{for } z \in \mathbb{C}^*$$

such that

$$P : (H_{\mathbb{R}})_z \times (H_{\mathbb{R}})_{-z} \to i^w\mathbb{R}$$

and such that

$$P : \mathcal{O}(H)_0 \times \mathcal{O}(H)_0 \to z^w\mathcal{O}_{\mathbb{C},0}$$

is nondegenerate.

This generalizes part of a Hodge structure in the following sense. Define $H' := H|_{\mathbb{C}^*}$ and $H^\infty :=$ { global flat manyvalued sections in H'}. It comes equipped with a real subspace $H^\infty_{\mathbb{R}} \subset H^\infty$, a monodromy operator $M_{mon} : H^\infty_{\mathbb{R}} \to H^\infty_{\mathbb{R}}$, and a pairing S (from P, see [He2, 7.2] for the definition). If the pole at 0 is logarithmic (i.e. a pole of order 1), then the pole corresponds to a decreasing M_{mon}-invariant filtration $F^\bullet$ on H^∞ (which encodes the growth at 0 of sections in H). In this sense the pole of order ≤ 2 at 0 generalizes the notion of a (Hodge) filtration $F^\bullet$ on H^∞.

In order to generalize the notion of the filtration $\overline{F^{w-\bullet}}$ in the case of a Hodge structure of weight w, one has to do the following. Define

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1, \quad z \to \frac{1}{\bar{z}}$$

and define a $\mathbb{C}$-antilinear map

$$\begin{aligned} \tau \ : \ & H_z \to H_{\gamma(z)} \qquad \text{for } z \in \mathbb{C}^* \\ & a \mapsto \nabla\text{-flat shift to } H_{\gamma(z)} \text{ of } \overline{z^{-w}a} \end{aligned}$$

(one takes the ∇-flat shift from z to $\gamma(z)$ along the path within $\mathbb{R}_{>0} \cdot z$). Then $\tau^2 = \mathrm{id}$. Glue $H \to \mathbb{C}$ and $\overline{\gamma^* H} \to \mathbb{P}^1 - \{0\}$ with τ to a bundle $\hat{H} \to \mathbb{P}^1$. It is a holomorphic bundle with a pole of order ≤ 2 at ∞. The pole at ∞ generalizes $\overline{F^{w-\bullet}}$.

The condition that the filtrations $F^\bullet$ and $\overline{F^{w-\bullet}}$ are opposite is generalized as follows.

Definition 2.1. (b) A $(TERP(w))$-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ is a (tr.TERP)-structure if $\hat{H} \to \mathbb{P}^1$ is a trivial bundle.

This generalizes the notion of a Hodge structure.

In the case of a (tr.TERP)-structure the fiber H_0 and the space $\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H}))$ are canonically isomorphic. By construction, the map τ acts on $\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H}))$ and induces a $\mathbb{C}$-antilinear involution

$$\kappa : H_0 \to H_0.$$

The pairing P gives rise to a $\mathbb{C}$-bilinear symmetric nondegenerate pairing g on H_0,

$$\begin{aligned} g \quad : \quad & H_0 \times H_0 \to \mathbb{C} \\ & (a, b) \mapsto z^{-w} P(\widetilde{a}, \widetilde{b}) \quad \mod z\mathcal{O}_{\mathbb{C},0} \end{aligned}$$

where $\widetilde{a}, \widetilde{b} \in \mathcal{O}(H)_0$ with $\widetilde{a}(0) = a, \widetilde{b}(0) = b$. Then define a hermitian pairing $h := g(., \kappa.)$ on H_0.

Definition 2.1. (c) A (tr.TERP)-structure is a (pos.def.tr.TERP)-structure if h is positive definite.

This generalizes the notion of a polarized Hodge structure (PHS).

Lemma 2.2. *Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a (tr.TERP)-structure.*
Then there exist endomorphisms $\mathcal{U} : H_0 \to H_0$ and $\mathcal{Q} : H_0 \to H_0$ such that

$$\nabla_{z\partial_z} = \frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa$$

on $\Gamma(\mathbb{P}^1, \mathcal{O}(H)) \cong H_0$.

In the case of a (pos.def.tr.TERP)-structure, $\mathcal{Q}$ is a hermitian endomorphism with real eigenvalues symmetric around 0. In the case of a PHS it corresponds to $\bigoplus_p (p - \frac{w}{2})\,\mathrm{id}\,|_{H^{p,w-p}}$. The physicists called $\mathcal{Q}$ *a new supersymmetric index* [CFIV].

One can also define the notion of a variation of (TERP)-structures [He2], in terms of a vector bundle $H \to \mathbb{C} \times M$ with a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank 1 along $\{0\} \times M$ (generalizing Griffiths transversality), a flat real subbundle $H'_{\mathbb{R}}$ and a flat pairing P.

If one then has at generic parameters (tr.TERP)-structures then h and $\mathcal{Q}$ vary real analytically in a most interesting way.

§3. Generalization of mixed Hodge structures

PMHS's correspond to nilpotent orbits of Hodge structures (theorem 3.2). Conjecture 3.7 below will generalize this correspondence to (TERP)-structures. First we review some facts on PMHS's [Sch][CKS].

Fix a reference PHS $(H^\infty, H^\infty_\mathbb{R}, S, F^\bullet)$ of weight w with Hodge filtration $F^\bullet$ and polarizing form S. The projective manifold

$$\check{D} := \{\text{filtrations } F^\bullet \subset H \mid \dim F^p = \dim F^p_0,\ S(F^p, F^{w+1-p}) = 0\}$$

contains as an open submanifold the classifying space for PHS's

$$D := \{F^\bullet \in \check{D} \mid F^\bullet \text{ is part of a PHS}\}.$$

Fix a nilpotent endomorphism $N : H^\infty_\mathbb{R} \to H^\infty_\mathbb{R}$ with $S(Na, b) + S(a, Nb) = 0$. It gives rise to a unique increasing filtration $W_\bullet$ on $H^\infty_\mathbb{R}$ with $N(W_l) \subset W_{l-2}$ and $N^l : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l}$ an isomorphism [Sch, Lemma 6.4].

Definition 3.1. (a) The tuple $(H^\infty, H^\infty_\mathbb{R}, S, N, F^\bullet)$ with $F^\bullet \in \check{D}$ is a *PMHS* of weight w if $F^\bullet Gr^W_k$ is a Hodge structure of weight k, if $N(F^p) \subset F^{p-1}$, and if the induced Hodge structure on the primitive subspace

$$P_{w+l} := \ker(N^{l+1} : Gr^W_{w+l} \to \mathrm{Gr}^W_{w-l-2})$$

is polarized by $S_l := S(., N^l.)$.

(b) The pair $(F^\bullet, N)$ with $F^\bullet \in \check{D}$ *gives rise to a nilpotent orbit* (of Hodge structures) if $N(F^p) \subset F^{p-1}$ and

$$e^{i\xi N} F^\bullet \in D \text{ for } \xi \in \mathbb{C} \text{ with } \mathrm{Re}\xi \gg 0.$$

Theorem 3.2. [Sch][CKS] *The pair $(F^\bullet, N)$ is part of a PMHS $\iff$ it gives rise to a nilpotent orbit.*

The inclusion $\Leftarrow$ is a main consequence [Sch, theorem 6.16] of Schmid's Sl_2-orbit theorem; the inclusion $\Rightarrow$ is proved in [CKS, corollary 3.13]. The theorem gives a very nice geometric characterization of PMHS's.

The definitions 3.3 and 3.6 and conjecture 3.7 will generalize definition 3.1 and theorem 3.2 to (TERP)-structures.

For $x \in \mathbb{R}_{>0}$ define $\pi_x : \mathbb{C} \to \mathbb{C},\ z \mapsto \frac{1}{x} z$.

Definition 3.3. A (TERP)-structure $(H, \nabla, H'_\mathbb{R}, P)$ *gives rise to a nilpotent orbit* if $\pi_x^*(H, \nabla, H'_\mathbb{R}, P)$ is a (pos.def.tr.TERP)-structure for $x \gg 0$. (Here $x \sim e^{\mathrm{Re}\xi}$ in definition 3.1 (b).)

The generalization of PMHS's is much more involved and requires a description of the Stokes structure of the order 2 pole at 0 of (H, ∇).

Consider a (TERP)-structure $(H, \nabla, H'_\mathbb{R}, P)$ of rank n with pole part $\mathcal{U} := [z\nabla_{z\partial_z}] : H_0 \to H_0$ with set of eigenvalues $\{u_1, ..., u_k\}$. In the

following we will always make the assumption, called
(No ramification): *The pair* $(\mathcal{O}(H)_0, \nabla)$ *is formally isomorphic to a sum*

$$\bigoplus_{i=1}^{k} e^{-u_i/z} \otimes \mathcal{R}_i,$$

where $\mathcal{R}_i$ *is a free* $\mathcal{O}_{\mathbb{C},0}$*-module with flat connection with regular singularity at 0.*

Define

$$\begin{aligned} H'^{*} &:= \text{dual bundle to } H' = H|_{\mathbb{C}^*}, \\ D^+ &:= \{z \in \mathbb{C}^* \mid -\frac{\pi}{2} < \arg z < \frac{\pi}{2} + \varepsilon \text{ (mod } 2\pi)\}, \quad (\varepsilon > 0 \text{ small}), \\ D^- &:= \{z \in \mathbb{C}^* \mid \frac{\pi}{2} < \arg z < \frac{3\pi}{2} + \varepsilon \text{ (mod } 2\pi)\}, \\ \mathcal{A}_{\pm}^{\leq 0} &:= \{f \in \mathcal{O}(D^{\pm}) \mid f \text{ has an asymptotic expansion of the type} \\ & \qquad \sum_{\mathrm{Re}(\alpha) \geq \alpha_0} \sum_{p} a_{\alpha,p} z^{\alpha} (\log z)^p\}. \end{aligned}$$

(See [Mal2, IV.3, page 61] for the definition of the sheaf $\mathcal{A}^{\leq 0}$.) Then the Stokes structure which distinguishes $(\mathcal{O}(H)_0, \nabla)$ in its formal equivalence class can be described by the following splittings (Birkhoff, Hukuhara, Turrittin, Jurkat, Sibuya, Deligne, Malgrange, ... [Mal1][Mal2, IV]):

$$\Gamma^{flat}(D^{\pm}, H'^{*}) = \bigoplus_{i=1}^{k} \Gamma_i^{\pm}$$

where

$$\Gamma_i^{\pm} := \{\gamma \in \Gamma^{flat}(D^{\pm}, H'^{*}) \mid \forall\, \omega \in \mathcal{O}(H)_0 \ \ \langle \omega, \gamma \rangle \in e^{-u_i/z} \cdot \mathcal{A}_{\pm}^{\leq 0}\}.$$

The (TERP)-structure is said to have *compatible* real structure and Stokes structure if

$$\Gamma_i^{\pm} = \mathbb{C} \cdot (\Gamma_i^{\pm} \cap \Gamma^{flat}(D^{\pm}, H_{\mathbb{R}}'^{*})).$$

Remark 3.4. In the singularity case (section 4), the oscillating integrals, more precisely, the Fourier-Laplace transform of the Gauss-Manin system of a function, gives rise to such a pair $(\mathcal{O}(H)_0, \nabla)$ (theorem 4.1). The u_i are the critical values of the function. $\Gamma_i^{\pm}$ is the space of complex linear combinations of the Lefschetz thimbles starting at critical points with the (common) critical value u_i. The functions $\langle \omega, \gamma \rangle$ are oscillating integrals. Here the compatibility of real structure and Stokes structure is trivial, because the Lefschetz thimbles are real. The Lefschetz thimbles provide even a $\mathbb{Z}$-lattice which is compatible with the Stokes structure.

Lemma 3.5. *(a) If real structure and Stokes structure are compatible, then $\mathcal{R}_i$ is a (TERP)-structure with regular singularity at 0.*

(b) [He2, 7.3] *It induces in a canonical way a tuple $(H^\infty_{(i)}, H^\infty_{\mathbb{R}(i)}, S_{(i)}, M_{s(i)}, N_{(i)}, F^\bullet_{(i)})$ with $M_{s(i)}$ and $N_{(i)}$ commuting semisimple and nilpotent endomorphisms of $H^\infty_{\mathbb{R}(i)}$ and $F^\bullet_{(i)}$ a decreasing filtration on $H^\infty_{(i)}$.*

Definition 3.6. A (TERP(w))-structure with (No ramification) is a *(mixed. TERP)-structure* if real structure and Stokes structure are compatible and if the regular singular pieces $\mathcal{R}_i$ induce PMHS's $(H^\infty_{(i)}, H^\infty_{\mathbb{R}(i)}, S_{(i)}, M_{s(i)}, N_{(i)}, F^\bullet_{(i)})$ of weight w.

Conjecture 3.7. *A (TERP(w))-structure with (No ramification) is a (mixed. TERP)-structure $\iff$ it induces a nilpotent orbit.*

Theorem 3.8. *(a) $\Rightarrow$ is true.*
Then for $x \to \infty$ the eigenvalues of $\mathcal{Q}$ in $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ tend to $\bigcup_i \text{Exponents}(\mathcal{R}_i) - \frac{w}{2}$ (definition of Exponents$(\mathcal{R}_i)$ *in* [He2, 7.3]*).*

(b) $\Leftarrow$ is true if the (TERP)-structure has a regular singularity (i.e. if its pole part $\mathcal{U}$ is nilpotent).

(c) $\Leftarrow$ is true if $\text{rk}\, H = 2$.

Part (a) in the case $\mathcal{U}$ semisimple is due to Dubrovin [Du, proposition 2.2], part (a) in the case $\mathcal{U}$ nilpotent is proved in [He2, 7.6]. The whole proof will appear elsewhere.

Some remarks concerning the proof:

(a) Case $\mathcal{U}$ nilpotent: [He2, theorem 7.20], using [CKS, corollary 3.13] and additional estimations.
Case $\mathcal{U}$ semisimple: [Du, proposition 2.2], the case of trivial Stokes structure ($\Gamma_i^+ = \Gamma_i^-$) is simple, the general case is rewritten as a Riemann boundary value problem and is solved with a singular integral equation.
General case: combination of both cases [HS, chapter 9]

(b) [HS, chapter 6], the proof uses [Mo2, theorem 12.1].

(c) The case $\text{rk}\, H = 2$ and $\mathcal{U}$ semisimple is considered implicitly in [IN] and is reduced there to the radial sinh-Gordon equation

$$(\partial_x^2 + \frac{1}{x}\partial_x)\ \alpha(x) = \sinh \alpha(x).$$

Nilpotent orbits correspond to real solutions without singularities for $x \to \infty$. These are analyzed in [MTW] and [IN].

Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a (TERP)-structure. It is also interesting to look at $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ for $x \to 0$. Theorem 3.9 below is the analogue for this limit to theorem 3.8 (a)+(b) in the case $\mathcal{U}$ nilpotent. Sabbah [Sab2] defined a tuple $(H^\infty, H^\infty_{\mathbb{R}}, S, M_s, N, F^\bullet_{Sabbah})$ by looking at the behaviour at $z = \infty$ of sections in $\Gamma(\mathbb{C}, \mathcal{O}(H))$ with moderate growth at $z = \infty$.

Theorem 3.9. *This tuple is a PMHS if and only if $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ is a (pos.def.tr.TERP)-structure for $x > 0$ close to 0.*

In that case, the eigenvalues of $\mathcal{Q}$ tend for $x \to 0$ to Exponents(*this PMHS*) $- \frac{w}{2}$.

This result and its proof are close to theorem 3.8 (a)+(b).

The case of (mixed.TERP(w))-structures with $\mathcal{U}$ semisimple is especially nice. Such (mixed.TERP(w))-structures are uniquely characterized by $w \in \mathbb{Z}$, the eigenvalues $u_1, ..., u_n$ of $\mathcal{U}$, and a Stokes matrix $S \in M(n \times n, \mathbb{R})$ with $S_{ij} = 0$ if $i > j$ and $S_{ii} = 1$. Any such data give rise to a (mixed.TERP(w))-structure.

Conjecture 3.10. *If $S + S^{tr}$ is positive definite then this is a (pos.def.tr.TERP)-structure for any $u_1, ..., u_n$.*

This conjecture is true in rank 2 because of [MTW][IN]. In the case of the Stokes matrices of the ADE singularities it would follow from conjecture 4.3.

If $S + S^{tr}$ is not positive definite then it depends on the values $u_1, ..., u_n$ whether the (mixed.TERP)-structure is a (pos.def.tr.TERP)-structure. Dubrovin's result [Du, proposition 2.2] says that it is a (pos.def.tr.TERP)-structure if $|u_i - u_j|$ is sufficiently big for all $i \neq j$.

§4. The case of singularities

We consider simultaneously the following two cases.
Case I: $f : (\mathbb{C}^{n+1} \to (\mathbb{C}, 0)$ a holomorphic function germ with an isolated singularity at 0 and Milnor number μ.
Case II: $f : Y \to \mathbb{C}$ a regular function on an affine manifold Y ($\dim Y = n+1$), such that f has isolated singularities and is M-tame (definition in [NS]); then

$$\mu = \sum_{x \in \, Crit(f)} \mu(f, x).$$

In both cases a semiuniversal unfolding F exists (cf. [DS1] for the meaning of this in case II),

$$\begin{aligned} F &: B \times M \to \mathbb{C}, \\ F_t &: B \times \{t\} \to \mathbb{C}, \ t \in M, \end{aligned}$$

with $F_0 = f$, with $M \subset \mathbb{C}^\mu$ a neighborhood of 0, and with (case I) B a small ball in $\mathbb{C}^{n+1}$, respectively (case II) $B = Y \cap$ (large ball in $\mathbb{C}^N$), where $Y \subset \mathbb{C}^N$ is a closed embedding.

Theorem 4.1. *In both cases one obtains on M a variation of (mixed.TERP)-structures of rank μ from the Fourier-Laplace transform of the Gauss-Manin system of F.*

Some remarks concerning the proof: The study of the Gauss-Manin system [SaK][SaM] and its Fourier-Laplace transform in terms of oscillating integrals [Ph1][Ph2] is classical in case I; part of it is reviewed in [He2, 8.1]. A very careful more algebraic treatment of the Fourier-Laplace transform and of the (TERP)-structures for both cases is given in [DS1]. In [DS2, ch. 6 + Appendix] this is connected with the Lefschetz thimbles and oscillating integrals.

For any fixed parameter t one obtains a (TERP)-structure $(\mathcal{O}(H_t)_0, \nabla)$. The dual bundle $H_t'^*$ is a bundle of linear combinations of Lefschetz thimbles. Evaluating holomorphic sections of H_t on flat sections of $H_t'^*$ gives oscillating integrals.

The compatibility of real structure and Stokes structure is trivial, because Lefschetz thimbles are real. That the local singularities come equipped with PMHS's via the regular singular pieces $\mathcal{R}_i$ is essentially due to Varchenko [Va] and Steenbrink [St][SchSt]. The polarizing form of the PMHS is discussed in [Loe, Cor. 3] and [He1, 10.5+10.6][He2, 7.2+8.1].

The nilpotent orbits of these (mixed.TERP)-structures (theorem 3.8 (a)) have a nice geometric meaning: for $x \in \mathbb{R}_{>0}$

$$\pi_x^*((TERP)(F_t) \cong (TERP)(x \cdot F_t).$$

The real 1-parameter unfoldings $\{x \cdot F_t \mid x \in \mathbb{R}_{>0}\}$ correspond to the orbits in M of $E + \overline{E}$ ($\sim x\partial_x$), where E is the Euler field on M. The flow of $E + \overline{E}$ on M corresponds to the renormalization group flow of the physicists [CV1][CV3].

Above, M is a (small or large) ball in $\mathbb{C}^\mu$. But in [He2, remark 8.5] it is shown that one can extend M to a manifold which is complete with respect to the flow of E and that the variation of (mixed.TERP)-structures extends to this manifold. A part of the following corollary was still a conjecture (1.4 and 8.3) in [He2].

Corollary 4.2. *(of theorem 3.8 (a) and theorem 4.1)*
Going sufficiently far along $E+\overline{E}$ in (this extension of) M, the (TERP)-structures are (pos.def.tr.TERP)-structures.

The following conjecture seems to be a theorem for the physicists in the case of functions which correspond to Landau-Ginzburg models.

Conjecture 4.3. *In case II, the (TERP)-structure of f is a (pos.def.tr.TERP)-structure.*

If the conjecture is true it would give a very distinguished class of (pos.def.tr.TERP)-structures. It would be comparable to the fact that the cohomology of compact Kähler manifolds carries Hodge structures. (One could speculate that the tameness of f at infinity replaces the compactness of Kähler manifolds.) It would give together with conjecture 3.7 and theorem 3.9 a good explanation for all the PMHS's associated to hypersurface singularities.

Consider for f as in case II the family of functions $x \cdot f : Y \to \mathbb{C}$, $x \in \mathbb{R}_{>0}$. Conjecture 4.3 is true for $x \gg 0$ because the (TERP)-structure of f is a (mixed.TERP)-structure and because of theorem 3.8 (a). The other way round, the conjectures 4.3 and 3.7 together would give a new proof that this is a (mixed.TERP)-structure.

Conjecture 4.3 is true for $x > 0$ close to 0 because of theorem 3.9 and because Sabbah's tuple $(H^\infty, H^\infty_\mathbb{R}, S, N, F^\bullet_{Sabbah})$ for the (TERP)-structure of f is a PMHS. In [Sab1] Sabbah proved that it is a MHS (see also [Sab2] for a more explicit statement). Recently (not yet available in march 2004) he proved that it is a PMHS.

For example, if $Y = \mathbb{C}^{n+1}$ and f is quasihomogeneous, then the deformations of weight < 1 are all M-tame functions and are parametrized by a space $\mathbb{C}^m$ (for some $m \leq \mu$). With conjecture 4.3 one would obtain a variation of (pos.def.tr.TERP)-structures on $\mathbb{C}^m$. This might be useful for Torelli problems or Schottky problems.

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Claus Hertling
Institut Élie Cartan (Mathématiques)
Université Henri Poincaré Nancy 1
B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail address: hertling@iecn.u-nancy.fr