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Invariants of combinatorial line arrangements and Rybnikov's example

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Abstract.

Following the general strategy proposed by G.Rybnikov, we present a proof of his well-known result, that is, the existence of two arrangements of lines having the same combinatorial type, but non-isomorphic fundamental groups. To do so, the Alexander Invariant and certain invariants of combinatorial line arrangements are presented and developed for combinatorics with only double and triple points. This is part of a more general project to better understand the relationship between topology and combinatorics of line arrangements.

One of the main subjects in the theory of hyperplane arrangements is the relationship between combinatorics and topological properties. To be precise, one has to make the following distinction: for a given hyperplane arrangement $\mathcal{H} \subset \mathbb{P}^n$, one can study the topological type of the pair $(\mathbb{P}^n, \mathcal{H})$ or the topological type of the complement $\mathbb{P}^n \setminus \mathcal{H}$. For the first concept we will use the term *relative topology* of $\mathcal{H}$, whereas for the second one we will simply say *topology* of $\mathcal{H}$. It is clear that if two hyperplane arrangements have the same relative topology, then they have the same topology, but the converse is not known. For $n = 2$, topology, relative topology and combinatorics are also related via graph manifolds with the boundary of a compact regular neighbourhood of $\mathcal{H}$, see [14, 24].

In a well-known and very cited unpublished paper [26], G. Rybnikov found an example of two line arrangements L_1 and L_2 in the complex

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projective plane $\mathbb{P}^2$ having the same combinatorics but different topology. A better understanding of this paper has been the aim of several works since then ([7, 20, 8, 22]).

The most common way to prove that two topologies of line arrangements are different is to check that the fundamental groups of their complements are not isomorphic. This is usually not done directly, but by calculating invariants of the fundamental group, mostly *borrowed* from invariants of links, such as Alexander polynomials ([25] and references there for links, [16] for algebraic curves), character (or characteristic) varieties ([13] for links, [17] for algebraic curves), Alexander invariants and Chen groups ([12, 19, 28] for links, [15, 11, 6, 20, 8, 23] for line arrangements) just to mention a few (both invariants and publications). In [26], Rybnikov uses central extensions of Chen groups in order to study the relative topology of line arrangements (and the fundamental groups of their complements); in this work, we use truncations of the Alexander Invariant by the $\mathfrak{m}$-adic filtration, where $\mathfrak{m}$ is the augmentation ideal; such truncations were studied by L.Traldi in [27] for links.

Recently, the authors of this work have provided an example of two line arrangements with different relative topologies (see [2]). The contribution of [2] is that it refers to real arrangements, that is, arrangements that admit real equations for each line (note that Rybnikov's example does not admit real equations).

The proof proposed by Rybnikov has two steps. Let $G_i := \pi_1(\mathbb{P}^2 \setminus \bigcup L_i)$, $i = 1, 2$.

(R1) Recall that the homology of the complement of a hyperplane arrangement depends only on combinatorics. This way, one can identify the abelianization of G_1 and G_2 with an Abelian group H combinatorially determined. Rybnikov proves that no isomorphisms exist between G_1 and G_2 that induce the identity on H. In particular, this result proves that both arrangements have different relative topologies. The reason can be outlined as follows: any automorphism of the combinatorics of Rybnikov's arrangement can be obtained from a diffeomorphism of $\mathbb{P}^2$, thus inducing an automorphism of fundamental groups. Since any homeomorphism of pairs $(\mathbb{P}^2, \bigcup L_i)$ defines an automorphism of the combinatorics of $\bigcup L_i$, after composition one can assume that any homeomorphism of pairs induces the identity on H. The strategy rests on the study of the first terms of the Lower Central Series (LCS), which coincide with the first terms of the series producing Chen groups. Since L_1 and L_2 are constructed using the MacLane arrangement L_ω

(see Example 1.7), it is enough to study, by some combinatorial arguments, the LCS of L_ω with an extra structure (referred to as an ordered arrangement). Although this part is explained in [26, Section 3], computations are hard to verify.

(R2) The second step is essentially combinatorial. The main point is to truncate the LCS of G_i such that the quotient K depends only on the combinatorics. Rybnikov proposes to prove that an automorphism of K induces the identity on H (up to sign and automorphisms of the combinatorics). This proof is only outlined in [26, Proposition 4.2]. It is worth pointing out that such a result cannot be expected for any arrangement. Also [26, Proposition 4.3] needs some explanation of its own. The main difference between relative topology and topology of the complement in terms of isomorphisms of the fundamental group is that homeomorphisms of pairs induce isomorphisms that send meridians to meridians, whereas homeomorphisms of the complement can induce any kind of isomorphism, and even if we know that the isomorphism induces the identity on homology, this is not enough to claim that meridians are sent to meridians.

The aim of our work is to follow the idea behind Rybnikov's work and, using slightly different techniques, provide detailed proofs of his result. This is part of a more general project by the authors that aims to better understand the relationship between topology and combinatorics of line arrangements.

The following is a more detailed description of the layout of this paper. In Section 1, the more relevant definitions are set, as well as a description of Rybnikov's and MacLane's combinatorics. Sections 2 and 3 provide a proof of Step (R1). In order to do so, we propose a new approach related to Derived Series, which is also useful in the study of Characteristic Varieties and the Alexander Invariant. The Alexander Invariant of a group G, with a fixed isomorphism $G/G' \approx \mathbb{Z}^r$, is the quotient G'/G'' considered as a module over the ring $\Lambda := \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, which is the group algebra of $\mathbb{Z}^r$. Using the truncated modules $\Lambda/\mathfrak{m}^j$, the problem is reduced to solving a system of linear equations. Note that other ideal could be used instead of $\mathfrak{m}$. Section 4 is devoted to the study of combinatorial properties of a line arrangement which ensure that any automorphism of the fundamental group of the complement *essentially* induces the identity on homology (that is, the analogous of [26, Proposition 4.2]). This is an interesting question that can be applied to general line arrangements. For the sake of simplicity, we only present our progress on line arrangements with double and triple points. This provides a proof for the second step (R2).

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§1. Settings and Definitions

In this section, some standard facts about line combinatorics and ordered line combinatorics will be described. Special attention will be given to MacLane and Rybnikov's line combinatorics.

Definition 1.1. A *combinatorial type* (or simply a *(line) combinatorics*) is a couple $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a finite set and $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$, satisfying that:

(1) For all $P \in \mathcal{P}$, $\#\mathcal{P} \geq 2$;
(2) For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, $\exists! P \in \mathcal{P}$ such that $\ell_1, \ell_2 \in P$.

An *ordered combinatorial type* $\mathscr{C}^{\text{ord}}$ is a combinatorial type where $\mathcal{L}$ is an ordered set.

Notation 1.2. Given a combinatorial type $\mathscr{C}$, the multiplicity m_P of $P \in \mathcal{P}$ is the number of elements $L \in \mathcal{L}$ such that $P \in L$; note that $m_P \geq 2$. The *multiplicity of* a combinatorial type is the number $1 - \#\mathcal{L} + \sum_{P \in \mathcal{P}}(m_P - 1)$.

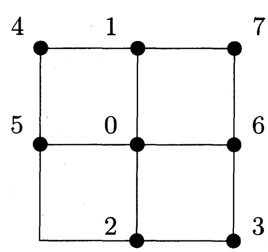

Fig. 1. Ordered MacLane *lines* in $\mathbb{F}_3^2$

Example 1.3 (MacLane's combinatorics). Let us consider the 2-dimensional vector space on the field $\mathbb{F}_3$ of three elements. Such a plane contains 9 points and 12 lines, 4 of which pass through the origin. Consider $\mathcal{L} = \mathbb{F}_3^2 \setminus \{(0,0)\}$ and $\mathcal{P}$, the set of lines in $\mathbb{F}_3^2$ (as a subset of

$\mathcal{P}(\mathcal{L})$). This provides a combinatorial type structure $\mathscr{C}_{\mathrm{ML}}$ that we will refer to as *MacLane's combinatorial type.* Figure 1 represents an ordered MacLane's combinatorial type.

Definition 1.4. Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorial type. We say a complex line arrangement $\mathcal{H} := \ell_0 \cup \ell_1 \cup ... \cup \ell_r \subset \mathbb{P}^2$ is a *realization* of $\mathscr{C}$ if and only if there are bijections $\psi_1 : \mathcal{L} \to \{\ell_0, \ell_1, ..., \ell_r\}$ and $\psi_2 : \mathcal{P} \to \mathrm{Sing}(\mathcal{H})$ such that $\forall \ell \in \mathcal{H}, P \in \mathcal{P}$, one has $P \in \ell \Leftrightarrow \psi_1(\ell) \in \psi_2(P)$. If $\mathscr{C}^{\mathrm{ord}}$ is an ordered combinatorial type and the irreducible components of $\mathcal{H}$ are also ordered, we say $\mathcal{H}$ is an ordered realization if ψ_1 respects orders.

Notation 1.5. The space of all complex realizations of a line combinatorics $\mathscr{C}$ is denoted by $\Sigma(\mathscr{C})$. This is a quasiprojective subvariety of $\mathbb{P}^{\frac{r(r+3)}{2}}$, where $r := \#\mathscr{C}$. If $\mathscr{C}^{\mathrm{ord}}$ is ordered, we denote by $\Sigma^{\mathrm{ord}}(\mathscr{C}) \subset (\check{\mathbb{P}}^2)^r$ the space of all ordered complex realizations of $\mathscr{C}^{\mathrm{ord}}$.

There is a natural action of $\mathrm{PGL}(3;\mathbb{C})$ on such spaces. This justifies the following definition.

Definition 1.6. The *moduli space* of a combinatorics $\mathscr{C}$ is the quotient $\mathscr{M}(\mathscr{C}) := \Sigma(\mathscr{C})/\mathrm{PGL}(3;\mathbb{C})$. The *ordered moduli space* $\mathscr{M}^{\mathrm{ord}}(\mathscr{C})$ of an ordered combinatorics $\mathscr{C}^{\mathrm{ord}}$ is defined accordingly.

Example 1.7. Let us consider the MacLane line combinatorics $\mathscr{C}_{\mathrm{ML}}$. It is well known that such combinatorics has no real realization and that $\#\mathscr{M}(\mathscr{C}_{\mathrm{ML}}) = 1$, however $\#\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}}) = 2$. The following are representatives for $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$:

$$\begin{gathered} \ell_0 = \{x=0\} \quad \ell_1 = \{y=0\} \quad \ell_2 = \{x=y\} \quad \ell_3 = \{z=0\} \\ \ell_4 = \{x=z\} \quad \ell_5^{\pm} = \{z+\omega y = 0\} \quad \ell_6^{\pm} = \{z+\omega y = (\omega+1)x\} \\ \ell_7^{\pm} = \{(\omega+1)y + z = x\} \end{gathered} \tag{1}$$

where $\omega = e^{2\pi i/3}$.

We will refer to such ordered realizations as

$$L_\omega := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5^+, \ell_6^+, \ell_7^+\}$$

and

$$L_{\bar\omega} := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5^-, \ell_6^-, \ell_7^-\}.$$

Remark 1.8. Given a line combinatorics $\mathscr{C} = (\mathcal{L}, \mathcal{P})$, the automorphism group $\mathrm{Aut}(\mathscr{C})$ is the subgroup of the permutation group of $\mathcal{L}$ preserving $\mathcal{P}$. Let us consider an ordered line combinatorics $\mathscr{C}^{\mathrm{ord}}$. It is easily seen that $\mathrm{Aut}(\mathscr{C}^{\mathrm{ord}})$ acts on both $\Sigma^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}})$ and $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}})$. Note also that $\mathscr{M}(\mathscr{C}^{\mathrm{ord}}) \cong \mathscr{M}^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}})/\mathrm{Aut}(\mathscr{C}^{\mathrm{ord}})$.

Example 1.9. The action of $\operatorname{Aut}(\mathscr{C}_{\mathrm{ML}}) \cong \mathrm{PGL}(2,\mathbb{F}_3)$ on the moduli spaces is as follows: matrices of determinant $+1$ (resp. -1) fix (resp. exchange) the two elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$. Of course complex conjugation also acts on $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$ exchanging the two elements. From the topological point of view one has that:

- There exists a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ preserving orientations on both $\mathbb{P}^2$ and the lines. Such a homeomorphism does not respect the ordering.
- There exists a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ preserving orientations on $\mathbb{P}^2$, but not on the lines. Such a homeomorphism respects the ordering.

Also note that the subgroup of automorphisms that preserve the set $L_0 := \{\ell_0, \ell_1, \ell_2\}$ is isomorphic to Σ_3, since the vectors $(1,0)$, $(1,1)$ and $(1,2)$ generate $\mathbb{F}_3^2$. We will denote by L_+ and L_- the sets of 5 lines such that $L_\omega = L_0 \cup L_+$ and $L_{\bar\omega} = L_0 \cup L_-$. Since any transposition of $\{0,1,2\}$ in $\mathscr{C}_{\mathrm{ML}}$ produces a determinant -1 matrix in $\mathrm{PGL}(2,\mathbb{F}_3)$, one concludes from the previous paragraph that any transposition of $\{0,1,2\}$ induces a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ that exchanges L_ω and $L_{\bar\omega}$ as representatives of elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$ and globally fixes L_0.

Example 1.10 (Rybnikov's combinatorics). Let L_ω and $L_{\bar\omega}$ be ordered MacLane realizations as above, where $L_0 := \{\ell_0, \ell_1, \ell_2\}$. Let us consider a projective transformation ρ_ω (resp. $\rho_{\bar\omega}$) fixing the initial ordered set L_0 (that is, $\rho(\ell_i) = \ell_i$ $i = 0,1,2$) and such that $\rho_\omega L_\omega$ (resp. $\rho_{\bar\omega} L_{\bar\omega}$) and L_ω intersect each other only in double points outside the three common lines. Note that $\rho_\omega, \rho_{\bar\omega}$ can be chosen with real coefficients.

Let us consider the following ordered arrangements of thirteen lines: $R_{\alpha,\beta} = L_\alpha \cup \rho_\gamma L_\beta$, where $\alpha, \beta \in \{\omega, \bar\omega\}$ and $\gamma = \beta$ (resp $\bar\beta$) if $\alpha = \omega$ (resp. $\bar\omega$). They produce the following combinatorics $\mathscr{C}_{\mathrm{RYB}} := (\mathcal{R}, \mathcal{P})$ given by:

(2)

$$\mathcal{R} := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10}, \ell_{11}, \ell_{12}\}$$

$$\mathcal{P}_2 := \left\{ \begin{array}{llll} \{\ell_2,\ell_3\}, & \{\ell_0,\ell_7\}, & \{\ell_1,\ell_6\}, & \{\ell_4,\ell_5\}, \\ \{\ell_2,\ell_8\}, & \{\ell_0,\ell_{12}\}, & \{\ell_1,\ell_{11}\}, & \{\ell_9,\ell_{10}\}, \\ \{\ell_i,\ell_j\} & 3 \le i \le 7, & 8 \le j \le 12 & \end{array} \right\}$$

$$\mathcal{P}_3 := \left\{ \begin{array}{llll} \{\ell_0,\ell_1,\ell_2\}, & \{\ell_3,\ell_6,\ell_7\}, & \{\ell_0,\ell_5,\ell_6\}, & \{\ell_1,\ell_4,\ell_7\}, \\ \{\ell_1,\ell_3,\ell_5\}, & \{\ell_2,\ell_4,\ell_6\}, & \{\ell_2,\ell_5,\ell_7\}, & \{\ell_0,\ell_3,\ell_4\}, \\ \{\ell_8,\ell_{11},\ell_{12}\} & \{\ell_0,\ell_{10},\ell_{11}\}, & \{\ell_1,\ell_9,\ell_{12}\}, & \{\ell_1,\ell_8,\ell_{10}\}, \\ \{\ell_2,\ell_9,\ell_{11}\}, & \{\ell_2,\ell_{10},\ell_{12}\}, & \{\ell_0,\ell_8,\ell_9\} & \end{array} \right\}$$

$$\mathcal{P} := \mathcal{P}_2 \cup \mathcal{P}_3$$

Proposition 1.11. *The following combinatorial properties hold:*

(1) *The different arrangements $R_{\alpha,\beta}$ have the same combinatorial type $\mathscr{C}_{\mathrm{RYB}}$.*

(2) *The set of lines L_0 has the following distinctive combinatorial property: every line in L_0 contains exactly 5 triple points of the arrangement; the remaining lines only contain 3 triple points.*

(3) *For the other* 10 *lines we consider the equivalence relation generated by the relation of* sharing a triple point. *There are two equivalence classes which correspond to L_ε and $\rho L_{\varepsilon'}$, $\varepsilon, \varepsilon' = \pm$.*

By the previous remarks one can group the set $\mathcal{R}$ together in three subsets. One is associated with the set of lines L_0 (referred to as $\mathcal{R}_0$), and the other two are combinatorially indistinguishable sets ($\mathcal{R}_1$ and $\mathcal{R}_2$) such that $\mathcal{R}_0 \cup \mathcal{R}_1$ and $\mathcal{R}_0 \cup \mathcal{R}_2$ are MacLane's combinatorial types. Note that any automorphism of $\mathscr{C}_{\mathrm{RYB}}$ must preserve $\mathcal{R}_0$ and either preserve or exchange $\mathcal{R}_1$ and $\mathcal{R}_2$. Therefore, $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}}) \cong \Sigma_3 \times \mathbb{Z}/2\mathbb{Z}$. The following results are immediate consequences of the aforementioned remarks.

Proposition 1.12. *The following are (or induce) homeomorphisms between the pairs $(\mathbb{P}^2, \bigcup R_{\bar\omega,\bar\omega})$ and $(\mathbb{P}^2, \bigcup R_{\omega,\omega})$ (resp. $(\mathbb{P}^2, \bigcup R_{\omega,\bar\omega})$ and $(\mathbb{P}^2, \bigcup R_{\bar\omega,\omega})$) preserving the orientation of $\mathbb{P}^2$:*

(a) *Complex conjugation, which reverses orientations of the lines.*

(b) *A transposition in $\mathcal{R}_0$, which preserves orientations of the lines.*

We will refer to $R_{\omega,\omega}$ and $R_{\bar\omega,\bar\omega}$ (resp. $R_{\omega,\bar\omega}$ and $R_{\bar\omega,\omega}$) as a type $+$ (resp. type $-$) arrangements.

Proposition 1.13. *Any homeomorphism of pairs between a type $+$ and a type $-$ arrangement should lead (maybe after composing with complex conjugation) to an orientation-preserving homeomorphism of pairs between a type $+$ and a type $-$ arrangement.*

If such a homeomorphism existed, there should be an orientation-preserving homeomorphism of ordered MacLane arrangements of type L_ω and $L_{\bar\omega}$.

The purpose of the next section will be to prove that there is no orientation-preserving homeomorphism of ordered MacLane arrangements of type L_ω and $L_{\bar\omega}$.

§2. The truncated Alexander Invariant

Even though the Alexander Invariant can be developed for general projective plane curves, we will concentrate on the case of line arrangements. Let $\bigcup L \subset \mathbb{P}^2$ be a projective line arrangement where $L = \{\ell_0, \ell_1, ..., \ell_r\}$. Let us denote its complement $X := \mathbb{P}^2 \setminus \bigcup L$ and G its fundamental group. The derived series associated with this group is recursively defined as follows: $G^{(0)} := G$, $G^{(n)} := (G^{(n-1)})' = [G^{(n-1)}, G^{(n-1)}]$, $n \geq 1$, where G' is the derived subgroup of G, i.e. the subgroup generated by $[a,b] := aba^{-1}b^{-1}$, $a, b \in G$. Note that the consecutive quotients are Abelian. This property also holds for the lower central series defined as $\gamma_1(G) := G$, $\gamma_n(G) := [\gamma_{n-1}(G), G]$, $n \geq 1$. It is clear that $G^{(0)} = \gamma_1(G)$ and $G^{(1)} = \gamma_2(G)$.

Since $H_1(X) = G/G'$, one can consider the inclusion $G' \hookrightarrow G$ as representing the universal Abelian cover $\tilde{X}$ of X, where $\pi_1(\tilde{X}) = G'$, and therefore $H_1(\tilde{X}) = G'/G''$.

The group of transformations $H_1(X) = G/G' = \mathbb{Z}^r$ of the cover acts on G'. This results in an action by conjugation on $G'/G'' = H_1(\tilde{X})$, $G'' = G^{(2)}$:

$$\begin{array}{ccll} G/G' \times G'/G'' & \to & G'/G'' & \\ (g, [a,b]) & \mapsto & g * [a,b] \mod G'' & = [g,[a,b]] + [a,b], \end{array}$$

where $a * b := aba^{-1}$. This action is well defined since $g \in G'$ implies $g * [a,b] \equiv [a,b] \bmod G''$. Additive notation will be used for the operation in G'/G''.

This action endows the Abelian group G'/G'' with a G/G'-module structure, that is, a module on the group ring $\Lambda := \mathbb{Z}[G/G']$. If $x \in G$, then t_x denotes its class in Λ. For $i = 1, \ldots, r$, we choose $x_i \in G$ a meridian of ℓ_i in G; the class $t_i := t_{x_i} \in \Lambda$ does not depend on the particular choice of the meridian in ℓ_i. Note that $t_1, \ldots, t_r$ is a basis of $G/G' \cong \mathbb{Z}^r$ and therefore one can identify

$$\Lambda := \mathbb{Z}[G/G'] = \mathbb{Z}[t_1^{\pm 1}, ..., t_r^{\pm 1}]. \tag{3}$$

This module is denoted by M_L and is referred to as the *Alexander Invariant* of L. Since we are interested in oriented topological properties of $(\mathbb{P}^2, \bigcup L)$, the coordinates $t_1, \ldots, t_r$ are well defined.

Remark 2.1. The module structure of M_L is in general complicated. One of its invariants is the zero set of the Fitting ideals of the complexified Alexander Invariant of L, that is, $M_L^{\mathbb{C}} := M_L \otimes (\Lambda \otimes \mathbb{C})$. This sequence of invariants is called the sequence of characteristic varieties of L introduced by A. Libgober [17]. These are subvarieties of the torus $(\mathbb{C}^*)^r$; in fact, irreducible components of characteristic varieties are translated subtori [1].

Our approach in studying the structure of the Λ-module M_L is via the associated graded module by the augmentation ideal $\mathfrak{m} := (t_1 -$

$1, ..., t_r - 1)$. In order to do so, and to be able to do calculations, we need some formulæ on this module relating operations in G'/G''. For the sake of completeness, these formulæ are listed below. However, since they are straightforward consequences of the definitions, their proof will be omitted. The symbol " $\overset{2}{=}$ " means that the equality is considered in G'/G'':

Properties 2.2.

(1) $[x,p] \overset{2}{=} (t_x - 1)p \quad \forall p \in G'$,

(2) $[x_1 \cdot \ldots \cdot x_n, y_1 \cdot \ldots \cdot y_m] \overset{2}{=} \sum_{i=1}^{n}\sum_{j=1}^{m} T_{ij}[x_i, y_j]$, where $T_{ij} = \prod_{k=1}^{i-1} t_{x_k} \cdot \prod_{l=1}^{j-1} t_{y_l}$.

(3) $[p_1 \cdot \ldots \cdot p_n, x] \overset{2}{=} -(t_x - 1)(p_1 + \ldots + p_n) \quad \forall p_i \in G'$,

(4) $[p_x x, p_y y] \overset{2}{=} [x,y] + (t_x - 1)p_y - (t_y - 1)p_x \quad \forall p_x, p_y \in G'$.

(5) $[x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}, y_1^{\beta_1} \cdot \ldots \cdot y_m^{\beta_m}] \overset{2}{=} \sum_{i=1}^{n}\sum_{j=1}^{m} T_{ij}\Big([x_i, y_j] + \delta(i,j)\Big)$,

where

$$\delta(i,j) = (t_{y_j} - 1)[\alpha_i, x_i] - (t_{x_i} - 1)[\beta_j, y_j].$$

(6) *Jacobi relations:*

$$J(x,y,z) := (t_x - 1)[y,z] + (t_y - 1)[z,x] + (t_z - 1)[x,y] \overset{2}{=} 0.$$

Let us recall a well-known result on presentations of fundamental groups of line arrangements based on the celebrated Zariski-Van Kampen method for computing the fundamental group of the complement of an algebraic curve. Let us recall briefly a description of this method applied to $X := \mathbb{P}^2 \setminus \bigcup L$. For a more detailed exposition see [9, p.121] or [21].

Let $P_0 \in \ell_0 \setminus (\ell_1 \cup \ldots \cup \ell_r)$ and consider the pencil of lines in $\mathbb{P}^2$ based on P_0. This defines a locally trivial fibration outside a finite number of points $\Delta := \{a_0, a_1, ..., a_s\} \subset \mathbb{P}^1$, that is, a fibration $X \setminus \pi^{-1}(\Delta) \xrightarrow{\pi|} \mathbb{P}^1 \setminus \Delta$, where $a_0 = \pi(\ell_0)$ and $\pi^{-1}(a_j) = H_j \cap X$ such that H_j is a line passing through P_0 and a singular point of $\bigcup L$. Let $* \in \mathbb{P}^1 \setminus \Delta$ be a base point and choose $\{\gamma_1, ..., \gamma_s\}$ a set of meridians on $\pi_1(\mathbb{P}^1 \setminus \Delta; *)$ such that γ_j is a meridian of a_j and $\gamma_1 \cdot \ldots \cdot \gamma_s$ is the inverse of a meridian of a_0. Let $y_* \in \pi^{-1}(*)$ ($|y_*|$ *big enough*) and consider $x_i \in \pi_1(\pi^{-1}(*); y_*) =: \mathbb{F}$, a meridian of ℓ_i. The group $\pi_1(\mathbb{P}^1 \setminus \Delta; *)$ of the base acts on the group $\mathbb{F}$ of the fiber in such a way that $\gamma_j(x_i)$ is a conjugate of x_i. This action comes from a morphism $\pi_1(\mathbb{P}^1 \setminus \Delta; *) \to \mathbb{B}_r$ and the Artin action of $\mathbb{B}_r$ on the free group $\mathbb{F}$ with the list of generators $\bar{x} := (x_1, \ldots, x_r)$.

A straightforward consequence of the Zariski-Van Kampen method ([30, 29], and [31, Chapter VIII]) is that

$$\langle \bar{x};\ x_i = x_i^{\gamma_j},\ i = 1, ..., r,\ j = 1, ..., s\rangle$$

is a presentation of G.

Moreover, one can describe the action of each γ_j in more detail as follows. Let D_j be a small enough disk around a_j, and g_j a path from $*$ to $p_j \in \partial D_j$ such that $\gamma_j = g_j \cdot \partial D_j \cdot g_j^{-1}$. Let us work on $\pi_1(\pi^{-1}(p_j), y_*)$.

The action of ∂D_j on $\pi_1(\pi^{-1}(p_j), y_*)$ can be described in a suitable set of free generators $\bar{y}$ of $\pi_1(\pi^{-1}(p_j), y_*)$ as follows. Let $P := \ell_{i_1} \cap ... \cap \ell_{i_p}$ be a singular point of $\bigcup L$ of multiplicity p on H_j and let $y_{i_1}, ..., y_{i_p}$ meridians of the lines such that $Y_P := y_{i_1} \cdots y_{i_p}$ is homotopic to a meridian of P on H_j. In that case $y_{i_k}^{\partial D_j} = y_{i_k}^{Y_P}$. If $H_j \cap \ell_i$ is not a singular point $\bigcup L$ then $y_i^{\partial D_j} = y_i$.

The path g_j induces a natural isomorphism $\beta_j : \pi_1(\pi^{-1}(p_j), y_*) \to \mathbb{F}$ of Artin type between $\bar{y}$ and $\bar{x}$; β_j is induced by a pure braid associated to g_j and we will identify these groups via β_j. Let us denote by $\mathbb{F}_P$ the subgroup of $\mathbb{F}$ generated by $y_{i_1}, ..., y_{i_p}$. Since each y_{i_k} is a conjugate of x_{i_k}, one obtains the following.

Proposition 2.3. *The group G admits a presentation of the form $\langle \bar{x}; \bar{W}\rangle$, where $\bar{W} := \{W_1(\bar{x}), ..., W_m(\bar{x})\}$ $m \geq 0$, and $W_i(\bar{x}) \in \mathbb{F}'$, $\forall i = 1, ..., m$.*

Moreover, $\bar{W}$ consists of words of type

$$[y_{i_k}, Y_P] \in [\mathbb{F}_P, Y_P],\ k = 1, \ldots, p-1. \tag{4}$$

for every $P \in \ell_{i_1} \cap \ldots \cap \ell_{i_p}$ ordinary multiple point of $\bigcup L$ of multiplicity p not belonging to ℓ_0.

Remark 2.4. The difficult part of actually finding a presentation is the computation of the pure braids mentioned above. Effective methods have been constructed in several works [3, 7, 5].

Remark 2.5. The relations $[y_{i_k}, Y_P]$ can also be written in the form $[x_{i_k}, X_{P,k}]$, where $X_{P,k}$ is a product of conjugates of $x_{i_1}, \ldots, x_{i_p}$. Moreover, we may use other relations to *simplify* the elements $X_{P,k}$.

Definition 2.6. Any presentation $\langle \bar{x}; \bar{W}(\bar{x})\rangle$ of G as in Proposition 2.3 (or Remark 2.5) will be called a *Zariski presentation* of G. The free group $\mathbb{F} := \langle \bar{x}\rangle$ will be referred to as the *free group associated with* the given presentation.

Notation 2.7. Most of the following construction could be done on a broader variety of groups such as 2-formal, or 2-free groups, but since we want to apply this theory to a particular problem, we will only deal with Zariski presentations of groups of line arrangements.

For technical reasons it is important to consider the Alexander Invariant corresponding to the free group associated with a given presentation. Such a module will be denoted by $\tilde{M}_L$. The following is a standard presentation of the modules $\tilde{M}_L$ and $\tilde{M}_L$ in terms of a Zariski presentation of G.

The following is a presentation of the Alexander Invariant from a given Zariski presentation. For another presentation of the Alexander Invariant from the braid monodromy see [8, Theorem 5.3].

Proposition 2.8. *Let $\langle \bar{x}; \bar{W} \rangle$ be a Zariski presentation of G and let $\mathbb{F} := \langle \bar{x} \rangle$ be its associated free group, then the module $\tilde{M}_L$ admits a presentation $\tilde{\Gamma}/\mathcal{J}$, where*

$$\tilde{\Gamma} := \bigoplus_{1 \leq i < j \leq r} [x_i, x_j]\Lambda$$

and $\mathcal{J}$ is the submodule of $\tilde{\Gamma}$ generated by the Jacobi relations (Property 2.2(6))

$$J(i,j,k) := (t_i - 1)x_{jk} + (t_j - 1)x_{ki} + (t_k - 1)x_{ij}.$$

Moreover, the module M_L can be obtained as a quotient of $\tilde{M}_L$ as $\tilde{\Gamma}/(\mathcal{J} + \mathcal{W})$, where $\mathcal{W}$ is the submodule of $\tilde{\Gamma}$ generated by the relations $\bar{W}$.

Proof. First, the Reidemeister-Schreier method on $\mathbb{F}' \hookrightarrow \mathbb{F}$ can be used to obtain a system of generators and a generating system of relations of $\mathbb{F}'$. Let $\bar{\iota} = (i_1, ..., i_r) \in \mathbb{Z}^r$ and $\ell(\bar{\iota}) = \max\{k \mid i_k \neq 0\}$. Note that $x_{\bar{\iota}} := x_1^{i_1} \cdot \ldots \cdot x_r^{i_r}$ is a Reidemeister-Schreier system of representatives of $\mathbb{F}'/\mathbb{F}''$ (from now on, and to avoid ambiguities, we will write $\mathbb{F}'/\mathbb{F}''$ when referring to the group structure and $\tilde{M}_L$ when referring to the module structure). Hence the family

$$x[\alpha]_{\bar{\iota}} := x_{\bar{\iota}} x_\alpha x_{\bar{\iota}+e_\alpha}^{-1}, \quad \alpha = 1, ..., \ell(\bar{\iota}) - 1$$

(where e_α is such that $x_\alpha = x_{e_\alpha}$) represents a free system of generators of $\mathbb{F}'$. Our purpose now is to use the module structure in order to obtain a finite set of generators of $\mathbb{F}'/\mathbb{F}''$ as the module $\tilde{M}_L$. Let $\bar{\iota}_\alpha = (i_1, ..., i_{\alpha-1}, 0, ..., 0)$, where $\alpha = 1, ..., r$ and $\bar{\iota}_1 = (0, ..., 0)$. Note that:

a) $x[\alpha]_{\bar\iota} = [x_{\bar\iota}, x_\alpha][x_\alpha, x_{\bar\iota_\alpha}]$. Therefore, using Property 2.2(2), one has

$$x[\alpha]_{\bar\iota} \overset{2}{=} [x_{\bar\iota}, x_\alpha] - [x_{\bar\iota_\alpha}, x_\alpha] \overset{2}{=} \sum_{k=\alpha+1}^{r} T^{\bar\iota_k}[x_k^{i_k}, x_\alpha], \tag{5}$$

where $T^{\bar\iota} := t_1^{i_1} \cdots t_r^{i_r}$.

b) $[x_k^{i_k}, x_\alpha] \overset{2}{=} \dfrac{t_k^{i_k} - 1}{t_k - 1}[x_k, x_\alpha]$.

Hence, the module $\tilde{M}_L$ is generated by the elements $x_{ij} := [x_i, x_j]$, where $1 \leq i < j \leq r$. Let us define the following sets of elements in $\tilde{M}_L$:

$$\Gamma_1 := \{T^{\bar\iota} x_{jk} \mid \max\{i \mid i \text{ is a coordinate of } \bar\iota\} \leq j\}.$$

and

$$\Gamma_2 := \{T^{\bar\iota} x_{jk} \mid \max\{i \mid i \text{ is a coordinate of } \bar\iota\} > j\}.$$

Note that $\mathbb{F}'/\mathbb{F}''$ is generated by Γ_1. Moreover the elements in Γ_1 are independent, since

$$T^{\bar\iota} x_{jk} \overset{2}{=} x_{\bar\iota} x_j x_k x_j^{-1} x_k^{-1} x_{\bar\iota}^{-1} = (x_{\bar\iota} x_k x_j x_k^{-1} x_j^{-1} x_{\bar\iota}^{-1})^{-1}$$

$$= x[j]_{\bar\iota+e_k}^{-1} \overset{2}{=} -x[j]_{\bar\iota+e_k}.$$

Therefore the relations in the module $\tilde{M}_L$ come from rewriting the elements in Γ_2 in terms of the base Γ_1. In fact, it is enough to consider the elements in Γ_2 of the form $t_i x_{jk}$ where $i > j < k$. One has the following two situations:

(1) If $j < i \leq k$ then

$$t_i x_{jk} \overset{2}{=} x_i * x_{jk} = x_i x_j x_k x_j^{-1} x_k^{-1} x_i^{-1}$$

$$= (x_i x_j x_i^{-1} x_j^{-1})(x_j x_i x_k x_j^{-1} x_k^{-1} x_i^{-1}) =$$

$$= x[j]_{e_i} x[j]_{e_i+e_k}^{-1} \overset{2}{=} x[j]_{e_i} - x[j]_{e_i+e_k}.$$

Finally, applying (5),

$$t_i x_{jk} \overset{2}{=} x_{ij} - x_{ij} - t_i x_{kj} \overset{2}{=} t_i x_{jk}.$$

(2) If $j < k < i$, then

$$t_i x_{jk} \overset{2}{=} x_i * x_{jk} = x_i x_j x_k x_j^{-1} x_k^{-1} x_i^{-1} =$$

$$= (x_i x_j x_i^{-1} x_j^{-1})(x_j x_i x_k x_i^{-1} x_k^{-1} x_j^{-1})(x_j x_k x_i x_j^{-1} x_i^{-1} x_k^{-1})(x_k x_i x_k^{-1} x_i^{-1}) =$$

$$= x[j]_{e_i} x[k]_{e_j+e_i} x[j]^{-1}_{e_k+e_i} x[k]^{-1}_{e_i} \overset{2}{=} x[j]_{e_i} + x[k]_{e_j+e_i} - x[j]_{e_k+e_i} - x[k]_{e_i}.$$

Applying (5) one has

$$t_i x_{jk} \overset{2}{=} x_{ij} + t_j x_{ik} - x_{kj} - t_k x_{ij} - x_{ik} \overset{2}{=} x_{jk} - (t_k - 1)x_{ij} - (t_i - 1)x_{ki}.$$

which produces the Jacobi relation $J(j, k, i) = 0$.

The second statement follows from the abelianization of

$$\mathbb{W} \hookrightarrow \mathbb{F}' \twoheadrightarrow G',$$

where $\mathbb{W}$ is the normal subgroup of $\mathbb{F}$ generated by $\bar{W}$. Note that $\mathcal{W} = \mathbb{W}/(\mathbb{W} \cap \mathbb{F}')$ and hence $\mathcal{W}$ is generated by the projection of the system $\bar{W}$ in $\mathcal{W}$. Q.E.D.

Remark 2.9. Note that the expression (4) and Property 2.2(5) provide a method to rewrite the relations $\bar{W}$ as elements of $\tilde{\Gamma}$.

Example 2.10. As an example of how to obtain $\mathcal{W}$ note that, if the lines ℓ_i and ℓ_j in L intersect in a double point, then there is a relation in $\bar{W}$ of type $[x_i^{\alpha_i}, x_j^{\alpha_j}]$, where $\alpha_i, \alpha_j \in G$. Using Property 2.2(5), this relation can be written in $\tilde{M}_L$ as $x_{i,j} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] \in \mathcal{W}$.

Analogously, if the lines ℓ_i, ℓ_j and ℓ_k in L intersect at a triple point, one obtains relations in G of type $[x_i^{\alpha_i}, x_j^{\alpha_j} x_k^{\alpha_k}]$, where $\alpha_i, \alpha_j, \alpha_k \in G$, which can be rewritten in $\mathcal{W}$ as

$$x_{i,j} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] + t_j\big(x_{i,k} + (t_k - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_k, x_k]\big).$$

Let $\mathfrak{m}$ be the augmentation ideal in Λ associated with the origin, that is, the kernel of homomorphism of Λ-modules, $\varepsilon : \Lambda \to \mathbb{Z}$, $\varepsilon(t_i) := 1$, where $\mathbb{Z}$ has the trivial module structure.

One can consider the filtration on M_L associated with $\mathfrak{m}$, that is, $F^i M_L := \mathfrak{m}^i M_L$. The associated graded module $\operatorname{gr} M_L := \oplus_{i=0}^{\infty} \operatorname{gr}^i M_L$, where $\operatorname{gr}^i M_L := F^i M_L / F^{i+1} M_L$ is a graded module over $\operatorname{gr}_{\mathfrak{m}} \Lambda := \oplus_{i=0}^{\infty} F^i \Lambda / F^{i+1} \Lambda$.

Consider the rings $\Lambda_j := \Lambda/\mathfrak{m}^j$, obtained by taking the quotient of Λ by successive powers of the ideal $\mathfrak{m}$. This allows one to define truncations of the Alexander Invariant.

Definition 2.11. The Λ_j-module, $M_L^j := M_L \otimes_\Lambda \Lambda_j$ will be called the *j-th truncated Alexander Invariant of L.* The induced filtration is finite and will be denoted in the same way.

Example 2.12. From Example 2.10, it is easily seen that the relations in M_L^2 coming from double and triple points are as follows.

(1) If ℓ_i and ℓ_j intersect at a double point one has:

$$x_{ij} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] = 0, \tag{6}$$

$$(t_k - 1)x_{ij} = 0. \tag{7}$$

(2) If ℓ_i, ℓ_j and ℓ_k intersect at a triple point one has:

$$\begin{aligned} x_{i,j} + t_j x_{i,k} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] \\ +(t_k - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_k, x_k] = 0, \end{aligned} \tag{8}$$

$$(t_m - 1)x_{i,j} + (t_m - 1)x_{i,k} = 0. \tag{9}$$

For any $k \in \mathbb{N}$, there is a natural morphism $\varphi_k : G' \twoheadrightarrow M_L^k$. We will sometimes refer to $\varphi_k(g)$ as $g \pmod{\mathfrak{m}^k}$ and equalities in M_L^k will be denoted by $p_1 \overset{k}{\equiv} p_2$.

Remark 2.13. A Zariski presentation on G induces a (set-theoretical) section

$$s : M_L \to G',$$

$$s\left(\varepsilon(t_1 - 1)^{k_1} ... (t_r - 1)^{k_r} x_{ij}\right) := [x_1^{[k_1]}, x_2^{[k_2]}, ..., x_r^{[k_r]}, [x_i, x_j]]^{\varepsilon},$$

defined inductively, where

$$[w_1, w_2, ..., w_n] := [w_1, [w_2, ..., w_n]], \quad [w_1^{[n]}, w_2] := [w_1^{[n-1]}, w_1, w_2].$$

This, accordingly, induces a section of φ_k on each M_L^k denoted by s_k.

Remark 2.14. From Property 2.2(1), we deduce that the kernel of φ_2 equals $\gamma_4(G)$. Moreover $\ker(G' \to M_L^1)$ equals $\gamma_3(G)$.

The previous construction can be summarized in the following.

Proposition 2.15. *Let $\psi(p_1, ..., p_m)$ be a word on the letters $\bar{p} := \{p_1, ..., p_m\}$. If $p_i, q_i \in G'$ and $p_i \overset{k}{\equiv} q_i$ $(i = 1, ..., m)$, then $[g, \psi(\bar{p})] \overset{k+1}{\equiv} [g, \psi(\bar{q})]$, $\forall g \in G$. In particular, if $p \in M_L^k$ then $[g, s_k(p)]$ is a well-defined element of M_L^{k+1}; if $g = x_i$ this element can be written $(t_i - 1)p \in M_L^{k+1}$.*

Remark 2.16. The ring Λ_k is not local, but note that an element $\lambda \in \Lambda_k$ is a unit if an only if $\varepsilon(\lambda) = \pm 1$. To see this note that $\Lambda_k = \mathbb{Z} \oplus \mathfrak{m}/\mathfrak{m}^k$ and the kernel of the evaluation map $\varepsilon : \mathbb{Z} \oplus \mathfrak{m}/\mathfrak{m}^k \to \mathbb{Z}$ is exactly $\mathfrak{m}/\mathfrak{m}^k$.

Notation 2.17. Note that everything in this section can also be reproduced by using the free group $\mathbb{F}$ associated with a Zariski presentation of G and will be denoted by adding a tilde. For instance, $\tilde{M}_L = \Lambda^{\binom{r}{2}}/\mathcal{J}$ is the Alexander Invariant associated with $\mathbb{F}_G$, and $F^i\tilde{M}_L$ is the filtration associated with $\mathfrak{m} \subset \Lambda$.

Note that any automorphism of G that sends x_i to $x_i\alpha_i$, (with $\alpha_i \in G'$) induces a filtered automorphism of M_L^k:

$$(10) \qquad [x_i, x_j] \mapsto [x_i\alpha_i, x_j\alpha_j] \overset{2}{=} [x_i, x_j] + t_j(t_i - 1)\alpha_j - t_i(t_j - 1)\alpha_i.$$

Note that this automorphism induces the identity on $\operatorname{gr} M_L^k$.

The following result is an immediate consequence of Proposition 2.15 and it explains why M_L^k is a more manageable object.

Corollary 2.18. *Under the above conditions the following formula holds in M_L^k:*

$$[x_i\alpha_i, x_j\alpha_j] \overset{k}{\equiv}$$

$$[x_i, x_j]+(t_i-1)\varphi_{k-1}(\alpha_j)-(t_j-1)\varphi_{k-1}(\alpha_i)+(t_j-1)(t_i-1)\varphi_{k-2}(\alpha_j-\alpha_i),$$

and hence the formula (10) *only depends on $\varphi_{k-1}(\alpha_i)$.*

Lemma 2.19. *Under the above conditions*

(1) *The Λ_1-module $\operatorname{gr}^0 M_L = \Lambda_1^{\binom{r}{2}}/\mathcal{W}$ is free of rank $g = \binom{r}{2} - v$, where v is the multiplicity of the combinatorial type of L (Notation* 1.2*).*

(2) *The Λ_2-module $\operatorname{gr}^1 M_L$ is combinatorial.*

The groups $\operatorname{gr}^k M_L \otimes \mathbb{Q}$ are combinatorial, see [23].

Proof. By Proposition 2.3 and the discussion previous to it, for any singular point $P = \ell_{i_1} \cap \ldots \cap \ell_{i_p}$ (of multiplicity $p \geq 2$) there are relations in G of type

(11)

$$R_P \equiv \begin{cases} [x_{i_1}^{\alpha_1^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{=} x_{i_1,i_2} + x_{i_1,i_3} + \ldots + x_{i_1,i_p} + \ldots \\ \hfill \text{terms in } F^1M_L \\ [x_{i_2}^{\alpha_2^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{=} x_{i_2,i_1} + x_{i_2,i_3} + \ldots + x_{i_2,i_p} + \ldots \\ \hfill \text{terms in } F^1M_L \\ \ldots \\ [x_{i_{p-1}}^{\alpha_{i_p}^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{=} x_{i_p,i_1} + x_{i_p,i_2} + \ldots + x_{i_{p-1},i_p} + \ldots \\ \hfill \text{terms in } F^1M_L \end{cases}$$

Since $\mathcal{J} \subset F^1\tilde{M}_L$, one has that $\operatorname{gr}^0 \tilde{M}_L = \operatorname{gr}^0 \tilde{\Gamma} = \Lambda_1^{\binom{r}{2}}$ and hence, by Proposition 2.8, one has $\operatorname{gr}^0 M_L = \Lambda_1^{\binom{r}{2}}/\mathcal{W}^1$. Therefore (1) follows from the fact that the equations of R_P in (11) are independent, since each generator $x_{i_\bullet,i_p}$ appears only once.

To prove the second part, it is enough to see that $\operatorname{gr}^1 M_L$ must be generated (as an Abelian group) by the elements $(t_i - 1)x_{j,k}$ and a generating system of relations is given by $J(i,j,k) \otimes \Lambda_2$ and $(\mathfrak{m}R_P) \otimes \Lambda_2$, for any $i,j,k \in \{1,...,r\}$ and $P \in \operatorname{Sing}(L) \setminus \ell_0$, which is a purely combinatorial system of relators. Q.E.D.

Notation 2.20. Since $\operatorname{gr}^0 M_L$ and $\operatorname{gr}^1 M_L$ only depend on the combinatorics, we will often use the notation $\operatorname{gr}^0 M_{\mathscr{C}}$ and $\operatorname{gr}^1 M_{\mathscr{C}}$ respectively to refer to such groups.

§3. Truncated Alexander Invariant and Homeomorphisms of Ordered Pairs

Let L_1 and L_2 be two ordered line arrangements sharing the ordered combinatorics $\mathscr{C}$. Consider two Zariski presentations $G_1 = \langle \bar{x}; \bar{W}^1(\bar{x})\rangle$ and $G_2 = \langle \bar{x}; \bar{W}^2(\bar{x})\rangle$ of the fundamental groups of X_{L_1} and X_{L_2}, where the subscripts of the generators $\bar{x} := \{x_1, ..., x_r\}$ respect the ordering of the irreducible components. The Abelian groups G_1/G_1' and G_2/G_2' can be canonically identified with $\operatorname{gr}^0 M_{\mathscr{C}}$ so that $x_i \pmod{G_1'} \equiv x_i \pmod{G_2'}$. Hence $\Lambda := \Lambda_{L_1} = \Lambda_{L_2}$ We will study the existence of isomorphisms $h : G_1 \to G_2$ such that $h_* : \operatorname{gr}^0 M_{\mathscr{C}} \to \operatorname{gr}^0 M_{\mathscr{C}}$ is the identity.

Definition 3.1. Let $\mathbb{F}_i$ be the free group associated with the Zariski presentation of G_i, $i = 1, 2$. A morphism $\tilde{h} : \mathbb{F}_1 \to \mathbb{F}_2$ is called a *homologically trivial morphism* if $\tilde{h}_* : \mathbb{F}_1/\mathbb{F}_1' \to \mathbb{F}_2/\mathbb{F}_2'$ satisfies $\tilde{h}_*(x_i) = x_i$. A morphism $h : G_1 \to G_2$ is called a *homologically trivial isomorphism* if it is induced by a homologically trivial morphism $\tilde{h}$, i.e., if $h_* : \operatorname{gr}^0 M_{\mathscr{C}} \to \operatorname{gr}^0 M_{\mathscr{C}}$ is the identity. Note that $\tilde{h}$ might not be unique.

Remarks 3.2.

(1) The above definition is mainly used for isomorphisms; in this setting, we consider that the trivial map is the identity and not the constant morphism. Other authors use *IA-automorphisms* [4] or homologically marked groups and homologically marked morphisms [22].

(2) In other words, a morphism $h : G_1 \to G_2$ is homologically trivial if there exists $(\alpha_1, ..., \alpha_r) \in (G_2')^r$ such that $h(x_i) = x_i\alpha_i$.

(3) Any homologically trivial isomorphism h induces a Λ-module morphism $h : M_1 := M_{L_1} \to M_{L_2} =: M_2$.
(4) Any homologically trivial isomorphism h respects the filtrations F and produces isomorphisms $\operatorname{gr}^i h : \operatorname{gr}^i M_1 \to \operatorname{gr}^i M_2$. By identifying $\operatorname{gr}^1 M_1 \equiv \operatorname{gr}^1 M_{\mathscr{C}} \equiv \operatorname{gr}^1 M_2$, $\operatorname{gr}^1 h$ is the *identity*.

In order to state some properties of homologically trivial isomorphisms, we need to introduce some notation. Note that the homologically trivial morphism $\tilde{h}$ also induces morphisms on the Alexander Invariants of the associated free groups $\tilde{M}_i$ $(i = 1, 2)$ and on their truncations $\tilde{M}_i^j$. Let us denote by $\tilde{h}^i : \tilde{M}_1^i \to \tilde{M}_2^i$ the induced homologically trivial morphisms of the truncated modules. A straightforward computation proves that

$$\tilde{h}(J(x_i, x_j, x_k)) = J(x_i, x_j, x_k) \in \mathbb{F}_2'/\mathbb{F}_2''. \tag{12}$$

Homologically trivial isomorphisms induce a particular kind of isomorphisms of the Λ-modules M_1, M_2 which are worth studying.

Remark 3.3. A direct attempt to prove that two modules are homologically trivial isomorphic is almost intractable. One would have to check if, for some choice $(\alpha_1, \ldots, \alpha_r) \in (G_2')^r \mod G_2''$, such a Λ-module isomorphism exists. The lack of linearity in this approach is the reason why we consider the truncated modules M_1^k, M_2^k.

Applying Corollary 2.18, we are faced with simply solving a linear system as follows. Let $h : G_1 \to G_2$ be a homologically trivial morphism, then there exists $(\alpha_1, \ldots, \alpha_r) \in (G_2')^r \mod G_2''$ such that $h(x_i) = x_i\alpha_i$. Therefore there exist Λ_k-morphisms $h^k : M_1^k \to M_2^k$ induced by h for any $k \in \mathbb{N}$. Note that

$$h^2(x_{i,j}) = x_{i,j} + \sum_{u,v} \alpha_{u,v}^j x_{i,u,v} - \sum_{u,v} \alpha_{u,v}^i x_{j,u,v}$$

where $x_{i,j} \stackrel{2}{\equiv} [x_i, x_j]$, $x_{i,u,v} \stackrel{2}{\equiv} (t_i - 1)x_{u,v}$, and

$$\alpha_w \stackrel{1}{\equiv} \sum_{u<v} \alpha_{u,v}^w x_{u,v}, \tag{13}$$

since

$$h^2(x_{i,j}) \stackrel{2}{\equiv} [h(x_i), h(x_j)] \stackrel{2}{\equiv} [x_i\alpha_i, x_j\alpha_j] \tag{14}$$

only depends on $\varphi_1(\alpha_i)$, the class of $\alpha_i \mod \mathfrak{m}$, by Proposition 2.18. In order to prove that h^2 is well defined, one must solve a linear system of equations on the variables $\alpha_{u,v}^i$ in the Abelian group M_2^k. If an integer solution exists, one can repeat the procedure on M_i^3, obtaining again a linear system of equations in the Abelian group M_2^3, and so on. In this work, we only need to consider h^2.

Let us consider an ordered line arrangement L with a fixed Zariski presentation $G = \langle \bar{x}; \bar{W} \rangle$. Let us denote $\mathscr{C}$ its ordered combinatorics.

In our particular case we can effectively compute the 2-nd truncated Alexander Invariant. The following result is an easy computation.

Lemma 3.4. *For any MacLane arrangement L, the Abelian group M_L^2 is free of rank 29, and its subgroup $\operatorname{gr}^1 M_L^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$ is free of rank 21.*

Theorem 3.5. *There is no homologically trivial isomorphism between $G_\omega := \pi_1(\mathbb{P}^2 \setminus \bigcup L_\omega)$ and $G_{\bar\omega} := \pi_1(\mathbb{P}^2 \setminus \bigcup L_{\bar\omega})$.*

Proof. Fix suitable Zariski presentations $G_\omega = \langle x_1, ..., x_7; W_1^\omega(\bar{x}), ..., W_{13}^\omega(\bar{x}) \rangle$ and $G_{\bar\omega} = \langle x_1, ..., x_7; W_1^{\bar\omega}(\bar{x}), ..., W_{13}^{\bar\omega}(\bar{x}) \rangle$ of the ordered arrangements L_ω and $L_{\bar\omega}$ (for instance, we have used the suitable Zariski presentations provided in [26] and other presentations obtained using the software in [5]). We identify the corresponding free groups $\mathbb{F}_\omega$ and $\mathbb{F}_{\bar\omega}$ with a free group $\mathbb{F}_7$. Recall that their combinatorial type has multiplicity 13, see Notation 1.2. We assume the relations to be ordered upon the following condition:

$$W_i^\omega(\bar{x}) \stackrel{1}{\equiv} W_i^{\bar\omega}(\bar{x}),$$

(in particular $W_i^\omega(\bar{x}) - W_i^{\bar\omega}(\bar{x}) \in F^1 \tilde{M}_{L_{\bar\omega}}$). Let us suppose that a homologically trivial homomorphism $h : G_\omega \to G_{\bar\omega}$ exists. Consider the corresponding elements $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_7$ that induce such a morphism (Remark 3.2(2)). Consider $M_\omega = M_{L_\omega}$ and $M_{\bar\omega} = M_{L_{\bar\omega}}$, the Alexander Invariants of L_ω and $L_{\bar\omega}$. Let $\Lambda := \mathbb{Z}[t_1^{\pm 1}, ..., t_7^{\pm 1}]$ be the ground ring of both Alexander Invariants, where $t_i \stackrel{1}{=} x_i$ as usual. This mapping induces a Λ_2-isomorphism $h^2 : M_\omega^2 \to M_{\bar\omega}^2$. By Corollary 2.18, h^2 only depends on the class $\alpha_i \mod \mathfrak{m}$. As in (13), one has

$$\alpha_k \stackrel{1}{\equiv} \sum_{1 \leq i < j \leq 7} \alpha_{ij}^k x_{ij}, \quad \alpha_{ij}^k \in \mathbb{Z}.$$

By (12) Jacobi relations play no role here.

Let us fix $i = 1, \ldots, 13$. Since $W_i^\omega(\bar{x}) \in \tilde{M}_\omega^2$ vanishes in M_ω^2, one deduces that $\tilde{h}^2(W_i^\omega(\bar{x})) \in \tilde{M}_{L_{\bar\omega}}^2$ should vanish in $M_{\bar\omega}^2$. Equivalently $\tilde{h}^2(W_i^\omega(\bar{x})) - W_i^{\bar\omega}(\bar{x}) \in F^1 \tilde{M}_{\bar\omega}^2$ should also vanish in $F^1 M_{\bar\omega}^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$. The vanishing of these terms, considered in the free abelian group $\operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$, produces a system of linear equations in the variables

α_{ij}^k (actually, even though there are 147 variables, only 126 appear in the equations).

Solving a system with 137 equations and 126 variables is not an easy task, but any computer will help. Using Maple8, it takes 85 seconds of CPU time running on an Athlon at 1.4MHz and 256Kb RAM Memory to obtain the linear set of solutions. It is an affine variety of dimension 98 of the form $(\lambda_1, ..., \lambda_{98}, \kappa_1, ..., \kappa_{28})$ where

$$\kappa_i = q_i + \sum_{j=1}^{98} \varepsilon_j^i \lambda_j,$$

$\varepsilon_j^i \in \{0, \pm 1\}$, and $q_i \in \mathbb{Q}$. Since $\{q_i \mid i = 1, \ldots, 28\} = \{0, \pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}\}$ one concludes that there is no integer solution[1]. Q.E.D.

Corollary 3.6. *There is no orientation preserving homeomorphism between the pairs of ordered arrangements* $(\mathbb{P}^2, \bigcup L_\omega)$ *and* $(\mathbb{P}^2, \bigcup L_{\bar\omega})$.

For Rybnikov's arrangements, one obtains similar results.

Lemma 3.7. *For any Rybnikov's arrangement R, the Abelian group M_R^2 is free of rank* 55, *and its subgroup* $\operatorname{gr}^1 M_R^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\text{RYB}}}$ *is free of rank* 40.

Theorem 3.8. *There is no homologically trivial isomorphism between* $G_+ := \pi_1(\mathbb{P}^2 \setminus \bigcup R_{\omega,\omega})$ *and* $G_- := \pi_1(\mathbb{P}^2 \setminus \bigcup R_{\bar\omega,\omega})$.

Proof. One way to prove this statement is to follow the computational strategy proposed for MacLane arrangements. First one needs Zariski presentations of $G_\pm$. This was done by means of the software in [5]. In this case the linear system obtained consists of 531 equations and 420 variables (again, out of the 792 variables α_{ij}^k, only 420 appear in the equations) and it took the same processor a total of 23,853 seconds of CPU time to compute the solutions. The space of solutions has dimension 252, that is, it can be written as $(\lambda_1, ..., \lambda_{252}, \kappa_1, ..., \kappa_{168})$, where

$$\kappa_i = q_i + \sum_{j=1}^{168} \varepsilon_j^i \lambda_j,$$

$\varepsilon_j^i \in \{0, \pm 1\}$, and $q_i \in \mathbb{Q}$. Since $\{q_i \mid i = 1, \ldots, 168\} = \{0, \pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}\}$, one again concludes that there is no integer solution.

[1]The software is written for Maple8 and can be visited at the following public site `http://riemann.unizar.es/geotop/pub/`.

Another proof that doesn't depend as strongly on computations can be obtained from Theorem 3.5 as follows. Let us assume that a homologically trivial isomorphism exists between G_+ and G_-. Such an isomorphism induces an Λ_2-isomorphism between M^2_+ and M^2_-. Let $\widehat{\Lambda}_2 := \Lambda_2/\mathfrak{m}'$, where $\mathfrak{m}'$ is the ideal generated by $(t_8-1), \ldots, (t_{12}-1)$, and let $\widehat{M^2_\pm}$ denote $M^2_\pm \otimes \widehat{\Lambda}_2$. Note that M^2_ω (resp. $M^2_{\bar\omega}$) can be considered as the $\widehat{\Lambda}_2$-module obtained from the inclusion of the complements $\mathbb{P}^2 \setminus \bigcup R_{\omega,\omega} \hookrightarrow \mathbb{P}^2 \setminus \bigcup L_\omega$ (resp. $\mathbb{P}^2 \setminus \bigcup R_{\bar\omega,\omega} \hookrightarrow \mathbb{P}^2 \setminus \bigcup L_{\bar\omega}$). Moreover, these inclusions define epimorphisms of $\widehat{\Lambda}_2$-modules $\pi : \widehat{M^2_+} \twoheadrightarrow M^2_\omega$ and $\bar\pi : \widehat{M^2_+} \twoheadrightarrow M^2_{\bar\omega}$. Proving the existence of a homologically trivial isomorphism $\tilde h^2$ that matches in the commutative diagram (15), and using Theorem 3.5 one obtains a contradiction.

$$
\begin{array}{ccc}
\widehat{M^2_+} & \xrightarrow{\hat h^2} & \widehat{M^2_-} \\
\pi \downarrow & & \downarrow \bar\pi \\
M^2_\omega & \xrightarrow{\tilde h^2} & M^2_{\bar\omega}
\end{array}
\tag{15}
$$

Consider S_+ the $\widehat{\Lambda}_2$-submodule of $\widehat{M^2_+}$ generated by the elements $x_{i,j}$, $i,j \in \{1, \ldots, 7\}$ and consider the commutative diagram (16). Since $\pi(x_{i,j}) = x_{i,j}$ and $\tilde h^2(x_{i,j}) \equiv x_{i,j} \mod \operatorname{gr}^1 M^2_{\bar\omega}$.

$$
\begin{array}{ccc}
\widehat{M^2_+} & \xrightarrow{\hat h^2} & \widehat{M^2_-} \\
\pi \uparrow & & \downarrow \bar\pi \\
S_+ & \xrightarrow{\tilde h^2} & M^2_{\bar\omega} \\
\pi_| \downarrow & & \\
M^2_\omega & &
\end{array}
\tag{16}
$$

Since M^2_ω and $M^2_{\bar\omega}$ are free Abelian of the same rank, (15) can be obtained from (16), by proving that $\pi_|$ is an isomorphism, which is the statement of Lemma 3.9. Q.E.D.

Lemma 3.9. *The epimorphism* $\pi_|$ *in* (16) *is injective*

Proof. We break the proof in several steps.

(O1) $(t_k - 1)x_{i,j} = 0$ in $\widehat{M^2_+}$ if $\{i,j\} \cap \{8, \ldots, 12\} \neq \emptyset$.

This can be proved case by case (all the equalities are considered in $\widehat{M^2_+}$):

- (a) If $i \in \{3, \ldots, 7\}$ and $j \in \{8, \ldots, 12\}$ (or vice versa, since $x_{i,j} = -x_{j,i}$): this is a consequence of Example 2.12(1) since the lines ℓ_i and ℓ_j intersect transversally.
- (b) If $i,j \in \{8, \ldots, 12\}$: this is a consequence of (a) and the Jacobi relations (Property 2.2(6)).

(c) If $i \in \{1,2\}, j \in \{8, ..., 12\}$: using the Jacobi relations (Property 2.2(6)) and (b) it is enough to check that $(t_k - 1)x_{i,j} = 0$, $i,k \in \{1,2\}$, $j \in \{8, ..., 12\}$. If ℓ_i and ℓ_j intersect at a double point, then (a) proves the result. Otherwise, there exists a line ℓ_m ($m \in \{7, ..., 12\}$) such that ℓ_i, ℓ_j and ℓ_m intersect at a triple point. By Example 2.12(2) one has $(t_k - 1)x_{i,j} + (t_k - 1)x_{m,j} = 0$, but $(t_k - 1)x_{m,j} = 0$ by (b), thus we are done.

(O2) $\operatorname{gr}^1(\widehat{M_+^2}) \subset S_+$. It is a direct consequence of (O1).

(O3) $\operatorname{gr}^0(\widehat{M_+^2}) = \dfrac{S_+}{\operatorname{gr}^1 \widehat{M_+^2}} \oplus \dfrac{\ker\pi + \operatorname{gr}^1 \widehat{M_+^2}}{\operatorname{gr}^1 \widehat{M_+^2}}$, i.e., $\dfrac{S_+}{\operatorname{gr}^1 \widehat{M_+^2}} \cong \operatorname{gr}^0 M_\omega^2$.

Since $\ker\pi$ is generated by $\langle x_{i,j}\rangle_{1\leq i<j\leq 12, j>7}$, it is clear that $\operatorname{gr}^0(\widehat{M_+^2})$ decomposes in the required sum. It remains to prove that it is a direct sum.

One can consider $\operatorname{gr}^0(\widehat{M_+^2})$ as a quotient $\dfrac{\langle x_{i,j}\rangle_{1\leq i<j\leq 12}}{\mathcal{W}}$, hence it is enough to check that there is a system of generators $r_1, ..., r_n$ of $\mathcal{W}$ such that:

$$(*)\qquad \text{either}\quad r_i \in \frac{\ker\pi + \operatorname{gr}^1 \widehat{M_+^2}}{\operatorname{gr}^1 \widehat{M_+^2}}, \quad \text{or} \quad r_i \in \frac{S_+}{\operatorname{gr}^1 \widehat{M_+^2}}.$$

Note that a system of relators can be obtained combinatorially as $x_{i,j} = 0$ (if $\{\ell_i, \ell_j\}$ is a double point) or $x_{i,j} + x_{i,k} = 0$ (if $\{\ell_i, \ell_j, \ell_k\}$ is a triple point). Relations coming from double points satisfy (*). For the triple point relations note that any triple point $\{\ell_i, \ell_j, \ell_k\}$ such that $\{i,j\} \subset \{8, ..., 12\}$, verifies that $k \in \{8, ..., 12\}$; therefore, condition (*) is also satisfied.

(O4) $\operatorname{gr}^1(\widehat{M_+^2}) \cong \operatorname{gr}^1 M_\omega^2$.

By (O1), the Abelian group $\widehat{M_+^2}$ is generated by $x_{i,j}$, $i,j \in \{1, \ldots, 12\}$, and $(t_k - 1)x_{i,j}$, $i,j,k \in \{1, \ldots, 7\}$; the relators are obtained from the singular points (see Example 2.12) and the Jacobi relations $\mathcal{J}$.

By (O1) and the proof of Lemma 2.19(2), we find that $\operatorname{gr}^1(\widehat{M_+^2})$ is generated by the elements $(t_k - 1)x_{i,j}$, $i,j,k \in \{1, \ldots, 7\}$ and the relations are exactly those in $\mathcal{J}$ and the relations (7) and (9) in Example 2.12. The arguments used in (O3) also show that only double and triple points in $\mathscr{C}_{\mathrm{ML}}$ provide non-trivial relations and thus one obtains the same system of generators and relations of $\operatorname{gr}^1 M_\omega^2$.

Q.E.D.

Remark 3.10. The proof of Lemma 3.9 is combinatorial and depends strongly on the properties of $\mathscr{C}_{\mathrm{RYB}}$. This lemma corresponds to a key statement of the proof of [26, Lemma 4.3] which is worth mentioning.

§4. Homologically Rigid Fundamental Groups

This last section will be devoted to proving that the fundamental groups of $R_{\omega,\omega}$ and $R_{\omega,\bar{\omega}}$ are not isomorphic.

Remark 4.1. Associated with a combinatorial type $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, there is a family of groups, where

$$H_{\mathscr{C}} := \frac{\bigoplus_{\ell\in\mathcal{L}} \langle x_\ell\rangle\mathbb{Z}}{\langle\sum_{\ell\in\mathcal{L}} x_\ell\rangle\mathbb{Z}}$$

and $\operatorname{gr}^i M_{\mathscr{C}}$ is given by generators and relations as a quotient of $H_{\mathscr{C}}^{\otimes(i+1)}$, $i = 0, 1$ as described in Lemma 2.19. Note that, if $\mathscr{C}$ has a realization L, then one has identifications $H_{\mathscr{C}} \equiv H_1(\mathbb{P}^2 \setminus \bigcup L; \mathbb{Z})$ and $\operatorname{gr}^i M_{\mathscr{C}} \equiv \operatorname{gr}^i M_L$.

Notation 4.2. There is a natural injective map $\Gamma(\mathscr{C}) \hookrightarrow \operatorname{Aut}(H_{\mathscr{C}})$ given by the permutation of the generators of $H_{\mathscr{C}}$; we identify $\Gamma(\mathscr{C})$ with its image in $\operatorname{Aut}(H_{\mathscr{C}})$. Another subgroup of $\operatorname{Aut}(H_{\mathscr{C}})$, denoted by $\operatorname{Aut}^1(H_{\mathscr{C}})$, is defined as those automorphisms of $H_{\mathscr{C}}$ that induce an automorphism of $\operatorname{gr}^1 M_{\mathscr{C}}$. It is easily seen that $\{\pm 1_{H_{\mathscr{C}}}\} \times \operatorname{Aut}(H_{\mathscr{C}}) \subset \operatorname{Aut}^1(H_{\mathscr{C}})$.

Definition 4.3. A line combinatorics $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ is called *homologically rigid* if $\operatorname{Aut}^1(H_{\mathscr{C}}) = \{\pm 1\} \times \Gamma(\mathscr{C})$.

The first goal of this section is to prove that Rybnikov's combinatorial type $\mathscr{C}_{\mathrm{RYB}} := (\mathcal{R}, \mathcal{P})$ (described in (2)) is homologically rigid; we will follow the ordering (1). Results of this sort have been studied by M.Falk in [10, Corollary 3.24].

In order to do so, we are going to study $\operatorname{Aut}^1(H_{\mathscr{C}})$ for an ordered combinatorics $\mathscr{C} = (\mathcal{L}, \mathcal{P})$ having at most triple points. We will denote $\mathcal{P}_j := \{P \in \mathcal{P} \mid \#P = j\}$, $j = 2, 3$, and $\mathcal{L} := \{\ell_0, \ell_1, \dots, \ell_r\}$. Let us first describe the groups $H_{\mathscr{C}}$ and $\operatorname{gr}^1 M_{\mathscr{C}}$:

$$H_{\mathscr{C}} := \frac{\langle x_0\rangle\mathbb{Z} \oplus \langle x_1\rangle\mathbb{Z} \oplus \cdots \oplus \langle x_r\rangle\mathbb{Z}}{\langle x_0 + x_1 + \ldots + x_r\rangle\mathbb{Z}}, \quad \operatorname{gr}^1 M_{\mathscr{C}} := \frac{\bigwedge^2 H_{\mathscr{C}}}{R_2 \oplus R_3}$$

where R_2 is the subgroup generated by $x_{i,j}$ ($\{\ell_i, \ell_j\} \in \mathcal{P}_2$) and R_3 is the subgroup generated by $x_{i,k} + x_{j,k}$ and $x_{i,j} + x_{i,k}$ ($\{\ell_i, \ell_j, \ell_k\} \in \mathcal{P}_3$).

Any isomorphism $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ induces a map $\wedge^2\psi : \bigwedge^2 H_{\mathscr{C}} \to \bigwedge^2 H_{\mathscr{C}}$. Let us represent $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ by means of a matrix $A^\psi :=$

$(a_i^j) \in \mathrm{Mat}(r+1,\mathbb{Z})$ such that $\psi(x_i) := \sum_{j=0}^r a_i^j x_j$ (note that such a matrix is not uniquely determined: each column is only well defined modulo the vector $1\!\!1_{r+1} := (1,\ldots,1)$). The conditions required for this map to define a morphism on the quotient $\mathrm{gr}^1 M_{\mathscr{C}}$ are called *admissibility conditions* and can be expressed as follows:

(17)

$$\begin{vmatrix} a_u^i & a_v^i & 1 \\ a_u^j & a_v^j & 1 \\ a_u^k & a_v^k & 1 \end{vmatrix} = 0, \qquad \text{if } \{\ell_i,\ell_j,\ell_k\} \in \mathcal{P}_3, \{\ell_u,\ell_v\} \in \mathcal{P}_2$$

$$\begin{vmatrix} a_\bullet^i & a_u^i + a_v^i + a_w^i & 1 \\ a_\bullet^j & a_u^j + a_v^j + a_w^j & 1 \\ a_\bullet^k & a_u^k + a_v^k + a_w^k & 1 \end{vmatrix} = 0 \quad \text{if } \{\ell_i,\ell_j,\ell_k\}, \{\ell_u,\ell_v,\ell_w\} \in \mathcal{P}_3$$

$$(\bullet = u, v, w)$$

(also note that such conditions are invariant on the coefficient vectors $(a_\bullet^0, a_\bullet^1, \ldots, a_\bullet^{12})$ modulo $1\!\!1_{13}$). We summarize these facts.

Proposition 4.4. *Any morphism $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ whose associated matrix A^ψ satisfies the admissibility conditions* (17) *produces a well-defined morphism $\wedge^2\psi : M^1_{\mathscr{C}} \to M^1_{\mathscr{C}}$.*

We are going to express the admissibility conditions (17) in a more useful way. Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ and let A^ψ be a matrix representing ψ. Fix $P \in \mathcal{P}_3$ and consider the submatrix $A_P^\psi \in \mathrm{Mat}(3 \times 12, \mathbb{Z})$ of A^ψ which contains the rows associated with P. Let $\Sigma_k := \mathbb{Z}^{k+1}/1\!\!1_{k+1}$, $k \in \mathbb{N}$. We denote by $v_0(P), v_1(P) \ldots, v_r(P) \in \Sigma_2$, the column vectors (mod $1\!\!1_3$) of A_P^ψ.

Lemma 4.5.

(1) *The vectors $v_0(P), v_1(P) \ldots, v_r(P)$ span Σ_2.*
(2) $\sum_{j=0}^r v_j(P) = 0 \in \Sigma_2$.
(3) *For any $Q \in \mathcal{P}$ and for any $\ell_u \in Q$, the vectors $v_u(P)$ and $\sum_{\ell_i \in Q} v_i(P)$ are linearly dependent (i.e., span a sublattice of Σ_2 of rank less than two). In particular, if $\sum_{\ell_i \in Q} v_i(P) \neq 0$, then $\{v_i(P) \mid \ell_i \in Q\}$ spans a rank-one sublattice of Σ_2.*
(4) *There exists $Q \in \mathcal{P}_3$ such that $\{v_i(P) \mid \ell_i \in Q\}$ spans a rank-two sublattice of Σ_2 and $\sum_{\ell_i \in Q} v_i(P) = 0$.*

Proof.

(1) ψ is an automorphism.
(2) The sum of the columns of A^ψ is a multiple of $1\!\!1_{r+1}$.

(3) It is an immediate consequence of the admissibility conditions (17).
(4) If no such Q exists, then all the vectors $V_i(P)$ are linearly dependent, which contradicts (1). The last part is a consequence of (3).

Q.E.D.

Definition 4.6. Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorics; we say $\mathscr{C}' := (\mathcal{L}', \mathcal{P}')$ is a *subcombinatorics of* $\mathscr{C}$ if $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{P}' := \{P \cap \mathcal{L} \mid P \in \mathcal{P}, \#(P \cap \mathcal{L}) \geq 2\}$.

We define a subcombinatorics $\operatorname{Adm}_\psi(P) \subset \mathscr{C}$ as follows:

$$\mathcal{L}(\operatorname{Adm}_\psi(P)) := \{\ell_i \in \mathcal{R} \mid v_i(P) \neq 0\}.$$

Note that,

$$(18) \quad \ell_i \notin \mathcal{L}(\operatorname{Adm}_\psi(P)) \iff \text{the } i^{\text{th}} \text{ column of } A_P^\psi \text{ is a multiple of } \mathbb{1}_3.$$

This motivates the following definition.

Definition 4.7. A line combinatorics $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ with only double and triple points is called *3-admissible* if it is possible to assign a non-zero vector $v_i \in \mathbb{Z}^2$ to each $\ell_i \in \mathcal{L}$ such that:

(1) There exists $P \in \mathcal{P}_3$, such that $\{v_j \mid \ell_j \in P\}$ spans a rank-two sublattice.
(2) For every $P \in \mathcal{P}$ and for every $\ell_i \in P$, v_i and $\sum_{\ell_j \in P} v_j$ are linearly dependent.
(3) $\sum_{\ell_i \in \mathcal{L}} v_i = (0,0)$.

Remarks 4.8. The conditions of Definition 4.7 can be made more precise.

(1) If $P = \{\ell_i, \ell_j\} \in \mathcal{P}_2$, then v_i and v_j are proportional, in notation, $v_i \| v_j$.
(2) If $P \in \mathcal{P}_3$ verifies condition (1) then $\sum_{\ell_i \in P} v_i = (0,0)$.

Examples 4.9.

(1) With the above notation, $\operatorname{Adm}_\psi(P)$ is 3-admissible by Lemma 4.5.
(2) The combinatorics $\mathcal{M}_3$ of a triple point (that is, $\mathcal{L}_{\mathcal{M}_3} := \{0,1,2\}, \mathcal{P}_{\mathcal{M}_3} := \{\{0,1,2\}\}$) is 3-admissible, simply using $v_0 := (1,0)$, $v_1 := (0,1)$, $v_2 := (-1,-1)$.
(3) Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorics such that
   - $\mathcal{L} := \mathcal{L}_0 \coprod \mathcal{L}_1 \coprod \mathcal{L}_2$;

- $\mathcal{L}_1$ and $\mathcal{L}_2$ define non-empty subcombinatorics in general position w.r.t. $\mathcal{L}_0$ (that is, $\ell_i \in \mathcal{L}_1$ and $\ell_j \in \mathcal{L}_2$ implies that $\{\ell_i, \ell_j\} \in \mathcal{P}$);
- at most one line of $\mathcal{L}_0$ *intersects* $\mathcal{L}_1 \cup \mathcal{L}_2$ in a non-multiple point of $\mathcal{L}_0$.

Then $\mathscr{C}$ is not 3-admissible. It is enough to see that if $\{v_i\}_{\ell_i \in \mathcal{L}}$ is a set of non-zero vectors satisfying conditions (2) and (3) of Definition 4.7, one has that $v_i \| v_j$.

- $\ell_i \in \mathcal{L}_1$ and $\ell_2 \in \mathcal{L}_2$; since $\{\ell_i, \ell_j\} \in \mathcal{P}$ then, by Remark 4.8(1), $v_i \| v_j$.
- $\ell_i, \ell_j \in \mathcal{L}_1$; considering any $\ell_k \in \mathcal{L}_2$ and using the previous case, one has $v_i \| v_k \| v_j$. The same argument works for $\ell_i, \ell_j \in \mathcal{L}_2$. In particular, $v_i \| v_j$ if $\ell_i, \ell_j \in \mathcal{L}_1 \cup \mathcal{L}_2$.
- $\ell_i \in \mathcal{L}_0$ and $P \in \mathcal{P}$ such that $P \cap \mathcal{L}_0 = \{\ell_i\}$. Since all the vectors associated with P but one are proportional, then this must also be the case for v_i.

All the vectors (but at most one) are proportional. To conclude we apply condition (3) of Definition 4.7.

(4) Ceva's line combinatorics is 3-admissible.

Proof. Ceva's line combinatorics is given by the following realization:

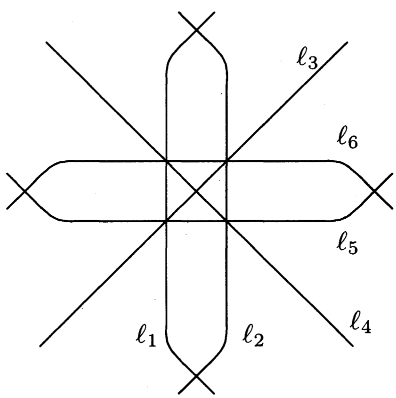

Fig. 2. Ceva's line combinatorics

$$\mathcal{L}_{\text{CEVA}} := \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{P}_{\text{CEVA}} := \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}.$$

For example, the following is a 3-admissible set of vectors for Ceva:

$$\{v_1 = v_2 = (1,0), v_3 = v_4 = (0,1), v_5 = v_6 = (-1,-1)\}$$

Q.E.D.

(5) MacLane's line combinatorics $\mathscr{C}_{\mathrm{ML}}$ is not 3-admissible.

Proof. We will use the combinatorics given in Figure 1. Let us assume that MacLane is 3-admissible, then one has a list of non-zero vectors $v_0, v_1, ..., v_7$ associated with each line. We will first see that v_0 and v_1 cannot be proportional. If they were ($v_0 \| v_1$), using $\{0,1,2\}$, $\{1,6\}$ and $\{2,3\}$ one would have that $v_0 \| v_2 \| v_6 \| v_3$ and finally, using $\{3,6,7\}$, one obtains $v_0 \| v_7$, and hence, by $\{1,4,7\}$ and $\{4,5\}$, all vectors are proportional, which contradicts condition (1) of Definition 4.7.

Therefore, v_0 and v_1 are linearly independent. After a change of basis, one can assume that $v_0 = (\alpha, 0)$, $v_1 = (0, \beta)$, $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, and therefore $v_2 = (-\alpha, -\beta)$. We will briefly describe the conditions and how they affect the vectors until a contradiction is reached:

$$\xrightarrow{\substack{\{0,7\} \\ \{1,6\}}} \begin{cases} v_7 = (\gamma, 0) \\ v_6 = (0, \delta) \end{cases} \quad \xrightarrow{\substack{\{3,6,7\} \\ \{0,5,6\}}} \begin{cases} v_3 = (-\gamma, -\delta) \\ v_5 = (-\alpha, -\delta) \end{cases} \quad \xrightarrow{\{1,3,5\}} \begin{cases} \alpha = -\gamma \\ \beta = 2\delta. \end{cases}$$

Hence v_3 does not satisfy the condition (2) of Definition 4.7 with $v_2 = (\gamma, -2\delta)$ on the double point $\{2,3\}$. Q.E.D.

Definition 4.10. A combinatorics $\mathscr{C}$ with only double and triple points is *pointwise 3-admissible* if the only 3-admissible subcombinatorics of $\mathscr{C}$ are isomorphic to $\mathcal{M}_3$.

Remark 4.11. In fact, it can be proved that a combinatorics $\mathscr{C}$ is pointwise 3-admissible if and only if its first resonance variety does not contain non-local components ([18]).

Proposition 4.12. *If $\mathscr{C}$ is a pointwise 3-admissible combinatorics then any $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a permutation ψ_3 of $\mathcal{P}_3$.*

Proof. Let $\mathcal{L} := \{\ell_0, ..., \ell_r\}$ denote the set of lines of $\mathscr{C}$ and let $\mathcal{P}_3 \subset \mathcal{P}$ denote the set of triple points of $\mathscr{C}$. We will first prove that any isomorphism $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a map $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$. Consider a triple point $P := \{\ell_i, \ell_j, \ell_k\} \in \mathcal{P}_3$; then $\mathrm{Adm}_\psi(P)$ is an admissible subcombinatorics of $\mathscr{C}$ (Example 4.9(1)) and defines a triple point. The map $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ given by $\psi_3(P) := \mathrm{Adm}_\psi(P)$ is defined.

We will next prove that such a map is indeed injective, and hence a permutation (in order for this to make sense, one has to assume that $\#\mathcal{P}_3 > 1$ and hence $r \geq 4$). Assume ψ_3 is not injective, and let $\psi_3(P_1) = \psi_3(P_2) = Q = \{\ell_u, \ell_v, \ell_w\}$. One has to consider two different cases depending on whether or not P_1 and P_2 share a line:

(1) If P_1, P_2 do not share a line, i.e, $P_1 := \{\ell_{i_1}, \ell_{j_1}, \ell_{k_1}\}$ and $P_2 := \{\ell_{i_2}, \ell_{j_2}, \ell_{k_2}\}$, where all the subscripts are pairwise different. By reordering the columns, let us write $Q = \{\ell_0, \ell_1, \ell_2\}$. Let $A^{\psi}_{P_1,P_2}$ be the submatrix of A^{ψ} corresponding to the rows $\{i_1, j_1, k_1, i_2, j_2, k_2\}$. Using (18):

$$A^{\psi}_{P_1,P_2} := \begin{pmatrix} a_0^{i_1} & a_1^{i_1} & a_2^{i_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{j_1} & a_1^{j_1} & a_2^{j_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{k_1} & a_1^{k_1} & a_2^{k_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{i_2} & a_1^{i_2} & a_2^{i_2} & a_3^{i_2} & \dots & a_r^{i_2} \\ a_0^{j_2} & a_1^{j_2} & a_2^{j_2} & a_3^{i_2} & \dots & a_r^{i_2} \\ a_0^{k_2} & a_1^{k_2} & a_2^{k_2} & a_3^{i_2} & \dots & a_r^{i_2} \end{pmatrix}, \tag{19}$$

where $a_0^{i_\bullet} + a_1^{i_\bullet} + a_2^{i_\bullet} = a_0^{j_\bullet} + a_1^{j_\bullet} + a_2^{j_\bullet} = a_0^{k_\bullet} + a_1^{k_\bullet} + a_2^{k_\bullet}$, $\bullet = 1, 2$. The sublattice K of Σ_5 generated by the columns (mod $\mathbb{1}_6$) should have maximal rank equal to 5. Note that rank (K) equals the rank of the matrix $\overline{A}^{\psi}_{P_1,P_2}$ obtained by subtracting the last row from the first ones, forgetting the last row and replacing the first column by the sum of the first three:

$$\overline{A}^{\psi}_{P_1,P_2} = \begin{pmatrix} b_0^{i_1} & b_1^{i_1} & b_2 & b_3 & \dots & b_r \\ b_0^{j_1} & b_1^{j_1} & b_2 & b_3 & \dots & b_r \\ b_0^{k_1} & b_1^{k_1} & b_2 & b_3 & \dots & b_r \\ b_0^{i_2} & b_1^{i_2} & 0 & 0 & \dots & 0 \\ b_0^{j_2} & b_1^{j_2} & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{20}$$

which does not have rank 5.

(2) If P_1, P_2 share a line, say $P_1 := \{i, j_1, k_1\}$ and $P_1 := \{i, j_2, k_2\}$. Then, analogously to the previous case, one obtains a similar matrix to (19) but where the rows i_1 and i_2 are identified, and we proceed in a similar way.

Q.E.D.

Definition 4.13. Three triple points $P, Q, R \in \mathcal{P}$ of a line combinatorics $(\mathcal{L}, \mathcal{P})$ are said to be *in a triangle* if $P \cap Q = \{\ell_1\}$, $P \cap R = \{\ell_2\}$ and $Q \cap R = \{\ell_3\}$ are pairwise different.

Proposition 4.14. *For any $\psi \in \operatorname{Aut}^1(H_{\mathcal{C}})$, $\mathcal{C}$ pointwise 3-admissible, ψ_3 satisfies the following Triangle Property: $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ preserves triangles, that is, if $P_1, P_2, P_3 \in \mathcal{P}_3$ are in a triangle, then $\psi_3(P_1), \psi_3(P_2)$, $\psi_3(P_3)$ are also in a triangle.*

Proof. Let $P_1, P_2, P_3 \in \mathcal{P}_3$ be three triple points in a triangle, $P_1 := \{\ell_i, \ell_j, \ell_k\}$, $P_2 := \{\ell_k, \ell_l, \ell_m\}$, $P_3 := \{\ell_m, \ell_n, \ell_i\}$. Let us assume that $\psi_3(P_1), \psi_3(P_2), \psi_3(P_3)$ are not in a triangle. One has two possibilities, either two of them do not share a line or three of them share a line.

(1) Two of them, say $\psi_3(P_1), \psi_3(P_2)$ do not share a line. After reordering, we can suppose that $\psi_3(P_1) = \{\ell_0, \ell_1, \ell_2\}$ and $\psi_3(P_2) = \{\ell_3, \ell_4, \ell_5\}$. For $\psi_3(P_3)$ there are several possibilities but we may assume $\psi_3(P_3) \subset \{\ell_0, \ell_3, \ell_6, \ell_7, \ell_8\}$. Consider the submatrix $A^{\psi}_{P_1,P_2,P_3}$ of A^{ψ} given by the rows of P_1, P_2, P_3. Applying (18) successively to the rows defined by P_1, P_2, P_3 one has:

(21)
$$A^{\psi}_{P_1,P_2,P_3} := \begin{pmatrix} a_0^i & a_1^i & a_2^i & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0^j & a_1^j & a_2^j & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3^l & a_4^l & a_5^l & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3^m & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0^n & a_1 & a_2 & a_3^n & a_4 & a_5 & a_6^n & a_7^n & a_8^n & a_9 & \dots & a_r \end{pmatrix};$$

moreover, $a_1^i = a_1$, $a_2^i = a_2$,

$$a_0^i + a_1 + a_2 = a_0^j + a_1^j + a_2^j = a_0 + a_1 + a_2 (\Rightarrow a_0^i = a_0),$$
$$a_3 + a_4 + a_5 = a_3^l + a_4^l + a_5^l = a_3^m + a_4 + a_5 (\Rightarrow a_3^m = a_3)$$

and

$$a_0 + a_3 + a_6 + a_7 + a_8 = a_0^n + a_3^n + a_6^n + a_7^n + a_8^n.$$

As in the proof of Proposition 4.12 we need rank $(\overline{A}^{\psi}_{P_1,P_2,P_3}) = 5$, where $\overline{A}^{\psi}_{P_1,P_2,P_3}$ is the matrix obtained from $A^{\psi}_{P_1,P_2,P_3}$ by subtracting the last row from the first ones and forgetting the last row. We obtain:

(22)
$$\bar{A}^{\psi}_{P_1,P_2,P_3} := \begin{pmatrix} b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0^j & b_1^j & b_2^j & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3^l & b_4^l & b_5^l & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \end{pmatrix}$$

which cannot have rank 5.

(2) If $\psi_3(P_1), \psi_3(P_2)$ and $\psi_3(P_3)$ have a common line, say ℓ_0, we follow the same strategy and obtain the desired result.

Q.E.D.

Our main goal is to check if ψ_3 is induced by an element of $\mathrm{Aut}(\mathscr{C})$. The next example shows that we need *enough* triangles.

Example 4.15. Note that Proposition 4.12 does not automatically ensure that in general an automorphism of the combinatorics is produced. For instance, consider the combinatorics $\mathscr{C}$ given by the lines $\{0,1,\ldots,6\}$, and the following triple points $\{0,1,2\}$, $\{2,3,4\}$, and $\{4,5,6\}$ (the remaining intersections are double points). It is easy to see that such a combinatorics is pointwise 3-admissible. Let $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ be given by the following matrix:

$$A^{\psi} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix},$$

satisfying the admissibility relations. This induces the following maps:

$$\begin{array}{ccc} \mathcal{P}_3 & \overset{\psi_3}{\to} & \mathcal{P}_3 \\ \{0,1,2\} & \mapsto & \{0,1,2\} \\ \{2,3,4\} & \mapsto & \{4,5,6\} \\ \{4,5,6\} & \mapsto & \{2,3,4\} \end{array} \qquad \begin{array}{ccc} \mathrm{gr}^1 M_{\mathscr{C}} & \overset{\wedge^2\psi}{\to} & \mathrm{gr}^1 M_{\mathscr{C}} \\ x_{0,1} & \mapsto & x_{0,1} \\ x_{2,3} & \mapsto & x_{4,5} \\ x_{4,5} & \mapsto & x_{2,3}, \end{array}$$

where $\mathrm{gr}^1 M_{\mathscr{C}} \cong \langle x_{0,1}\rangle\mathbb{Z} \oplus \langle x_{2,3}\rangle\mathbb{Z} \oplus \langle x_{4,5}\rangle\mathbb{Z}$.

However, the given permutation is not induced by an automorphism of the combinatorics, because the point $\{2,3,4\}$ (which is the only one that shares a line with the other two) is not fixed.

We want to apply the previous results to $\mathscr{C}_{\mathrm{RYB}}$. First we will check that $\mathscr{C}_{\mathrm{RYB}}$ is pointwise 3-admissible.

Lemma 4.16. *An admissible subcombinatorics of $\mathscr{C}_{\mathrm{RYB}}$ cannot have lines in both $\mathcal{R}_1 := \{3,4,5,6,7\}$ and $\mathcal{R}_2 := \{8,9,10,11,12\}$.*

Proof. Any subcombinatorics of $\mathscr{C}_{\mathrm{RYB}}$ having lines in both $\mathcal{R}_1$ and $\mathcal{R}_2$ verifies the conditions of Example 4.9(3). Q.E.D.

Lemma 4.17. *$\mathscr{C}_{\mathrm{ML}}$ is pointwise 3-admissible.*

Proof. It is not difficult to prove that any combinatorics (with only double and triple points) of less than 6 lines (other than $\mathcal{M}_3$) is not 3-admissible. In Example 4.9(5) it is shown for the whole combinatorics of eight lines. Let us check the remaining cases, that is, six and seven lines. Note that, up to a combinatorics automorphism, there is a unique way to remove one line. There are, however, two possible ways to remove two lines, depending on whether they intersect or not at a triple point. Therefore we only have to check the following cases:

(1) For 7 lines
$\{0,1,2,3,4,5,6\}$:
$\overset{\{3,6\}}{\to} v_3\|v_6 \overset{\{1,6\},\{2,3\}}{\to} v_3\|v_1\|v_2 \overset{\{1,4\}}{\to} v_3\|v_4 \overset{\{4,5\}}{\to} v_3\|v_5.$

(2) For 6 lines, removing two lines intersecting at a triple point
$\{0,2,3,4,5,6\}$: $\overset{\{3,5\}}{\to} v_5\|v_3 \overset{\{2,3\},\{3,6\},\{4,5\}}{\to} v_5\|v_2\|v_6\|v_4.$

(3) For 6 lines, removing two lines intersecting at a double point
$\{0,2,3,4,5,7\}$: $\overset{\{2,3\},\{2,4\}}{\to} v_2\|v_3\|v_4 \overset{\{3,7\},\{4,5\}}{\to} v_2\|v_5\|v_7.$

Q.E.D.

Proposition 4.18. $\mathscr{C}_{\text{RYB}}$ *is pointwise 3-admissible.*

Proof. An immediate consequence of Lemmas 4.16 and 4.17.

Q.E.D.

Remarks 4.19. In order to prove that for any $\psi \in \operatorname{Aut}^1(H_{\mathscr{C}_{\text{RYB}}})$, ψ_3 comes from an element of $\operatorname{Aut}(\mathscr{C}_{\text{RYB}})$ we need to know more combinatorial properties of $\mathscr{C}_{\text{RYB}}$ and $\mathscr{C}_{\text{ML}}$.

(1) The triple point $\{0,1,2\} \in \mathcal{P}_3$ in $\mathscr{C}_{\text{RYB}}$ is the only one that belongs to 36 triangles.
(2) Any triple point $P \in \mathcal{P}_3$ in $\mathscr{C}_{\text{RYB}}$ except for $\{3,6,7\}$ and $\{8,11,12\}$ satisfies that $\{0,1,2\}$ and P are in a triangle.
(3) Any two triple points in $\mathscr{C}_{\text{ML}}$ sharing a line belong to a triangle.
(4) For any three triple points P_1, P_2, P_3 in $\mathscr{C}_{\text{ML}}$, there exists another triple point Q such that Q, P_i, P_j belong to a triangle $(i,j \in \{1,2,3\})$.

Proposition 4.20. *Let* $\psi \in \operatorname{Aut}^1(\mathscr{C}_{\text{RYB}})$.

(1) $\psi_3(\{0,1,2\}) = \{0,1,2\}$
(2) ψ_3 *either preserves (resp. exchanges) the triple points of* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *in* $\mathscr{C}_{\text{RYB}}$ *inducing an automorphism*
(3) *The action of* ψ_3 *on* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *comes from an automorphism (resp. isomorphism) of their combinatorics.*

Proof. Part (1) is true by Propositions 4.12-4.14 and Remark 4.19(1). By Remark 4.19(2), the points $\{3,6,7\}$ and $\{8,11,12\}$ are either preserved or exchanged. In order to prove (2) and (3), we may suppose that $\psi_3(\{3,6,7\}) = \{3,6,7\}$. Recall that the subcombinatorics defined by $\mathcal{R}_0 \cup \mathcal{R}_1$ is isomorphic to $\mathscr{C}_{\mathrm{ML}}$. Since triangles are preserved by ψ_3 (Proposition 4.14), according to Remark 4.19(3) the images of any two triple points in $\mathcal{R}_0 \cup \mathcal{R}_1$ sharing a line also share a line. This implies, using Remark 4.19(4) again, that the image of any three triple points on $\mathcal{R}_0 \cup \mathcal{R}_1$ sharing a line are also three points sharing a line. Since any line in $\mathcal{R}_1$ or $\mathcal{R}_2$ has at least three triple points, we conclude (2) and (3). Q.E.D.

Proposition 4.21. *Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$; then $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ is induced by an automorphism of $\mathscr{C}_{\mathrm{RYB}}$.*

Proof. We will use Proposition 4.20 repeatedly. We can compose ψ with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$ in order to have ψ_3 preserve the triple points in $\mathcal{R}_i$. Recall that $\psi_3|_{\mathcal{R}_0\cup\mathcal{R}_i}$ comes from an automorphism φ_i of $\mathscr{C}_{\mathrm{ML}}$ which respects $\{0,1,2\}$. Composing again with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$ we may suppose that φ_1 is the identity on $\{0,1,2\}$. It is enough to prove that it is also the case for φ_2. If it is not the case, we may assume (by conjugation with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$) that $\varphi_2(0) = 1$. There are two possibilities to be checked, depending on whether 9 and 10 are fixed or permuted. In both cases, the triple points $\{0,1,2\}$, $\{3,6,7\}$ and $\{8,11,12\}$ are fixed. Using the arguments in the proofs of Propositions 4.12 and 4.14, one can obtain the induced matrices A^ψ with all the admissibility relations, which, modulo $\mathbb{1}_{13}$ do not have a maximal rank in either case. Q.E.D.

Proposition 4.22. *$\mathscr{C}_{\mathrm{RYB}}$ is homologically rigid.*

Proof. Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$; by Propositions 4.21 and 4.14, ψ_3 comes from an automorphism of the combinatorics. Composing with the inverse of such an automorphism, we may suppose that $\psi_3 = 1_{\mathcal{P}_3}$. It is hence enough to prove that any isomorphism $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$ that induces the identity on ψ_3 is just $\pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. From the definition of $\mathrm{Adm}_\psi(P)$, $P \in \mathcal{P}_3$, we deduce the following. Let us fix the j^{th}-column; all the entries in this column corresponding to $P \in \mathcal{P}_3$ such that $j \notin P$ are equal. We deduce from this that we can choose A^ψ such that all the elements outside the diagonal are constant in their column. Adding multiples of $\mathbb{1}_{13}$, we obtain that A^ψ can be chosen to be diagonal. We also know that for each $P \in \mathcal{P}_3$, the diagonal terms corresponding to $\mathcal{P}$ are equal and since any two elements can be joined by a *chain* of triple points, we deduce that all the diagonal terms are equal. Since ψ is an automorphism, they are equal to ± 1 and then $\psi = \pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. Q.E.D.

Therefore we can prove the main result.

Theorem 4.23. *The fundamental groups of the two complex realizations of Rybnikov's combinatorics are not isomorphic.*

Proof. Let G_+ and G_- be the fundamental groups of $R_{\omega,\omega}$ and $R_{\bar{\omega},\omega}$ respectively. Any isomorphism $\tilde{\psi} : G_+ \to G_-$ will produce an automorphism $\psi : H_{\mathscr{C}_{\mathrm{RYB}}} \equiv G_+/G'_+ \to G_-/G'_- \equiv H_{\mathscr{C}_{\mathrm{RYB}}}$, that is, we can consider $\psi \in \mathrm{Aut}^1(\mathscr{C}_{\mathrm{RYB}})$. By Theorem 4.22, ψ induces an automorphism of $\mathscr{C}_{\mathrm{RYB}}$. Since the identifications $H_{\mathscr{C}_{\mathrm{RYB}}} \equiv G_\pm/G'_\pm$ are made up to the action of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$, we may assume that ψ induces $\pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. Moreover, eventually exchanging $R_{\omega,\omega}$ (resp. $R_{\bar{\omega},\omega}$) for $R_{\bar{\omega},\bar{\omega}}$ (resp. $R_{\omega,\bar{\omega}}$), see Example 1.10 by means of the automorphism given by complex conjugation, we may assume that $\psi = 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$ (Proposition 4.22). Therefore $\tilde{\psi}$ is a homologically trivial isomorphism between G_+ and G_-, something which is ruled out by Theorem 3.8. Q.E.D.

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