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Subadjunction theorem

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Abstract.

We give a subadjunction theorem which relates the multi-adjoint linear system of the ambient space and the linear system of the restricted bundle on a subvariety.

§1. Introduction

Let M be a complex manifold and L be a line bundle on M and S be a submanifold of M. It is a basic question whether the restriction map

$$H^0(M, \mathcal{O}_M(L)) \to H^0(S, \mathcal{O}_S(L))$$

is surjective.

In this paper we shall consider this question for multi-adjoint type line bundles under certain geometric conditions.

Let us state our result precisely. Let M be a complex manifold of dimension n and let S be a closed complex submanifold of M. Then we consider a class of continuous function $\Psi : M \longrightarrow [-\infty, 0)$ such that

1. $\Psi^{-1}(-\infty) \supset S$,
2. if S is k-dimensional around a point x, there exists a local coorinate $(z_1, \ldots, z_n)$ on a neighbourhood of x such that $z_{k+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \mid \Psi(z) - (n-k) \log \sum_{j=k+1}^{n} \mid z_j \mid^2 \mid < \infty.$$

The set of such functions Ψ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_M[\Psi]$ on S as the minimum element of the partially ordered set of positive

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measueres $d\mu$ satisfying

$$\int_{S_k} f d\mu \geq \limsup_{t\to\infty} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} dV_M$$

for any nonnegative continuous function f with $\operatorname{supp} f \subset\subset M$. Here S_k denotes the k-dimensional component of S, σ_m denotes the volume of the unit sphere in $\mathbf{R}^{m+1}$, and $\chi_{R(\Psi,t)}$ denotes the characteristic funciton of the set

$$R(\Psi, t) = \{x \in M \mid -t-1 < \Psi(x) < -t\}.$$

Theorem 1.1. *Let M be a projective manifold with a continuous volume form dV_M, let L be a holomorphic line bundle over M with a C^∞-hermitian metric h_L, let S be a compact complex submanifold of M, let $\Psi : M \longrightarrow [-\infty, 0)$ be a continuous function and let K_M be the canonical bundle of M.*

1. $\Psi \in \sharp(S) \cap C^\infty(M\backslash S)$,
2. $\Theta_{h\cdot e^{-(1+\epsilon)\Psi}} \geq 0$ *for every* $\epsilon \in [0, \delta]$ *for some* $\delta > 0$,
3. *there is a positive line bundle on M.*

Then every element of $H^0(S, \mathcal{O}_S(m(K_M + L)))$ extends to an element of $H^0(M, \mathcal{O}_M(m(K_M + L)))$.

One may think that the assumption on the existence of the function Ψ is somewhat technical or restrictive. But as one see in the last section, this is not the case. In fact one may construct such a function by using an effective $\mathbf{Q}$-divisor on M.

The results in this paper may be considered as a generalization of [6] to the case of nontrivial normal bundles. We also note that there exists another type of subadjunction theorem due to Y. Kawamata ([2]). This is a reserch announcement. The detailed proof will be published elsewhere.

§2. Setch of the proof of Theorem 1.1

Here we shall give a sketch of the proof of Theorem 1.1. Let M, S, L be as in Theorem 1.1. Let h_S be a canonical AZD ([8]) of $K_M + L\mid_S$. Let A be a sufficiently ample line bundle on M. Let us define the singular hemitian metric on $m(K_M + L)\mid_S$ by

$$h_{m,S} := K(A + m(K_M + L)\mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)^{-1}$$

Then as in [8], we see that

$$h_S := \liminf_{m\to\infty} \sqrt[m]{h_{m,S}}$$

holds. Hence $\{\sqrt[m]{h_{m,S}}\}$ is considered to be an algebraic approximation of h_S. We consider the Bergman kernel

$$K(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S) = \sum_i \mid \sigma_i^{(m)} \mid^2,$$

where $\{\sigma_i^{(m)}\}$ is a complete orthonormal basis of $A^2(S, A+m(K_M+L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)$. We note that (cf. [3, p.46, Proposition 1.4.16])

$$K(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)(x)$$

$$= \sup\{\mid \sigma \mid^2 (x) \mid \sigma \in A^2(S, A+m(K_M+L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S), \| \sigma \| = 1\}$$

holds for every $x \in S$. We note that there exists a positive constant C_0 independent of m such that

$$h_{m,S} \leq C_0 \cdot h_A \cdot h_S^m$$

holds for every $m \geq 1$ as in [8]. Let h_M be a canoncal AZD of $K_X + L$ and let ν denote the numerical Kodaira dimension of $(K_M + L, h_M)$, i.e.,

$$\nu := \lim_{m \to \infty} \frac{\log \dim H^0(M, \mathcal{O}_M(A + m(K_M + L)) \otimes \mathcal{I}(h_M^m))}{\log m}.$$

For simplicity we shall consider the case that ν is equal to the numerical Kodaira dimension of $K_M + L$. Otherwise the proof should be modified a little bit.

Inductively on m, we extend each

$$\sigma \in A^2(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)$$

to a section

$$\tilde{\sigma} \in A^2(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)$$

with the estimate

$$\| \tilde{\sigma} \| \leq C \cdot m^{-\nu} \| \sigma \|$$

where $\| \quad \|$'s denote the L^2-norms respectively, C is a positive constant indpendent of m and we have defined

$$\tilde{K}_m(x) := \sup\{\mid \tilde{\sigma} \mid^2 (x) \mid \| \tilde{\sigma} \mid_S \| = 1, \| \tilde{\sigma} \| \leq C \cdot m^{-\nu}\}$$

and set

$$\tilde{h}_m = \frac{1}{\tilde{K}_m}.$$

If we take C sufficiently large, then $\tilde{h}_m$ is well defined for every $m \geq 0$. By easy inductive estimates, we see that

$$\tilde{h}_\infty := \liminf_{m \to \infty} \sqrt[m]{\tilde{h}_m}$$

exists and gives an extension of h_S. Then by [4] for every $m \geq 1$, we may extend every element of $A^2(m(K_M + L) \mid_S, dV_M^{-1} \cdot h_L \cdot h_S^{m-1}, dV_M[\Psi])$ to $A^2(m(K_M + L), dV_M^{-1} \cdot h_L \cdot \tilde{h}_\infty, dV_M)$. This completes the proof of Theorem 1.1.

§3. Generalization of Theorem 1.1

Let M be a smooth projective variety and let (L, h_L) be a singlar hermitian line bundle on M such that $\Theta_{h_L} \geq 0$ on M. Let dV be a C^∞-volume form on M. Let $\sigma \in \Gamma(\bar{M}, \mathcal{O}_{\bar{M}}(m_0 L) \otimes \mathcal{I}(h))$ be a global section. Let α be a positive rational number ≤ 1 and let S be an irreducible subvariety of M such that $(M, \alpha(\sigma))$ is logcanonical but not KLT(Kawamata log-terminal) on the generic point of S and $(M, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of S for every $0 < \epsilon << 1$. We set

$$\Psi = \alpha \log h_L(\sigma, \sigma).$$

We shall assume that S is not contained in the singular locus of h, where the singular locus of h means the set of points where h is $+\infty$.

For the moment we shall consider the case that S is smooth (when S is not smooth, we just need to take an embedded resolution of S). In this case Ψ may not belong to $\sharp(S)$, since Ψ may not have the prescribed singularity along S as in the definition of $\sharp(S)$. Then as in Section 2.1, we may define a (possibly singular measure) $dV[\Psi]$ on S. This can be viewed as follows. Let $f : N \longrightarrow M$ be a logresolution of $(X, \alpha(\sigma))$. Then as before we may define the singular volume form $f^* dV[f^*\Psi]$ on the divisorial component of $f^{-1}(S)$. The singular volume form $dV[\Psi]$ is defined as the fibre integral of $f^* dV[f^*\Psi]$.

The proof [4] and hence proof of Theorem 1.1 also works in this case except a minor difference. The difference is that $dV_M[\Psi]$ (which is defined similarly as above) may have singularities along some Zariski closed subset of S. Let $d\mu_S$ be a C^∞-volume form on S and let φ be the function on S defined by

$$\varphi := \log \frac{dV[\Psi]}{d\mu_S}.$$

Theorem 3.1. *Let M,S,Ψ be as above. Suppose that S is smooth. Then every element of $A^2(S, \mathcal{O}_S(m(K_M + dL)), e^{-(m-1)\varphi} \cdot dV^{-m} \cdot h_L^m, dV[\Psi])$ extends to an element of*

$$H^0(M, \mathcal{O}_M(m(K_M + dL))).$$

As we mentioned as above the smoothness assumption on S is just to make the statement simpler.

As an example of an application, we have :

Corollary 3.1 ([6]). *Let $\pi : X \longrightarrow \Delta$ be a semistable degeneration of projective variety over the unit disk. Let $X_0 = \pi^{-1}(0) = \sum_i D_i$ be the irreducible decomposition. Then we have that*

$$\sum_i P_m(D_i) \leq P_m(X_t)$$

holds where t is any regular value of π and P_m denotes the m-th plurigenus.

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