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Seshadri constants and a criterion for bigness of pseudo-effective line bundles

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Abstract.

We state some fundamental properties of the intersection theory for pseudo-effective line bundles defined by Tsuji, and some applications of that theory. As one of applications, we generalize Seshadri's criterion for ampleness of nef line bundles as a criterion for bigness of pseudo-effective line bundles.

§1. Intersection theory for pseudo-effective line bundles

We consider a smooth complex projective variety X of dimension n and a line bundle L on X. A line bundle L is *pseudo-effective* if it has a singular Hermitian metric h with positive curvature: $\Theta_h \geq 0$ in the sense of current, here $\Theta_h := (2\pi)^{-1}\sqrt{-1}\ \overline{\partial}\partial \log h$. This is equivalent to the original algebraic definition of the pseudo-effectivity, namely the first Chern class $c_1(L) \in \overline{N}_{eff}$, here $\overline{N}_{eff}$ is the closure of the convex cone in the (real) Néron-Severi group $NS_{\mathbf{R}}(X) \subset H^2(X, \mathbf{R})$ generated by first Chern classes of effective $\mathbf{R}$-line bundles. By definition, a singular Hermitian metric h on L is written as $h = e^{-\varphi}h_0$ for a smooth Hermitian metric h_0 on L and $\varphi \in L^1_{loc}(X)$. The *multiplier ideal sheaf*

$$\mathcal{I}(h) = \mathcal{I}(X, h)$$

of $h = e^{-\varphi}h_0$ is defined by the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is integrable. By a theorem of Nadel (cf. [D1, §5]), $\mathcal{I}(h)$ is coherent provided φ is quasi-plurisubharmonic, i.e., the current $\sqrt{-1}\partial\overline{\partial}\varphi$ is bounded from below by a smooth real $(1,1)$-form on X (in particular case $\Theta_h \geq 0$). We sometimes use a variant $\mathcal{I}_+(h) := \mathcal{I}_+(e^{-\varphi})$ $:= \lim_{\varepsilon \downarrow 0} \mathcal{I}(e^{-(1+\varepsilon)\varphi})$. In case φ is quasi-plurisubharmonic, we regard the (complete pluri-)polar set of φ as the singular locus of h:

$$\operatorname{Sing} h := \{x \in X;\ \varphi(x) = -\infty\}.$$

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For a subvariety Y in X (i.e., a closed integral subscheme of X), we say that the restriction $h|_Y$ is *well-defined*, if $Y \not\subset \operatorname{Sing} h$.

Definition 1.1 ([T, Definition 2.9]). Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. Let C be an integral (i.e., reduced and irreducible) curve in X such that $h|_C$ is well-defined. The *intersection number* $(L, h) \cdot C$ is the real number

$$(L,h) \cdot C := \limsup_{m \longrightarrow \infty} m^{-1} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) \cdot \mathcal{O}_C).$$

The original definition [T, Definition 2.9] is given for a slightly wider class of curves. The following proposition gives alternative definitions.

Proposition 1.2. *Let $C \subset X$ be an integral curve such that $h|_C$ is well-defined. Then*

$$\begin{aligned}(L,h) \cdot C &= \lim_{m \longrightarrow \infty} m^{-1} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) \cdot \mathcal{O}_C) \\ &= \lim_{m \longrightarrow \infty} m^{-1} \deg_C \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m).\end{aligned}$$

More precisely we assert that limits exist and they coincide. As for the last term, we define $\deg_C \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) := mL \cdot C + \deg_C \mathcal{I}(h^m)$, and "$\deg_C$" as follows: Let $\nu : C' \longrightarrow C \subset X$ be the normalization of C, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf such that $C \not\subset \operatorname{supp} \mathcal{O}_X/\mathcal{I}$. Then we set $\deg_C \mathcal{I} := \deg_{C'} \nu^{-1}\mathcal{I} \cdot \mathcal{O}_{C'}$ as the degree of the invertible sheaf $\nu^{-1}\mathcal{I} \cdot \mathcal{O}_{C'}$ on C'.

By Demailly [D2, 4.1.1], every pseudo-effective line bundle L has a unique (up to certain equivalence of singularities) class of singular Hermitian metrics $h_{\min}$ with minimal singularities such that $\Theta_{h_{\min}} \geq 0$. In case L is semi-ample, we have $\mathcal{I}(h_{\min}^m) = \mathcal{O}_X$ for every $m \in \mathbf{N}$, and hence $(L, h_{\min}) \cdot C = L \cdot C$. In case L is big and it admits the so-called Zariski decomposition $L = P + N$ (i.e., P and N are $\mathbf{Q}$-divisors such that P nef, N effective, and that the natural injection $H^0(X, [mP]) \longrightarrow H^0(X, mL)$ is bijective for all $m \in \mathbf{N}$), $(L, h_{\min}) \cdot C = P \cdot C$ holds for $C \not\subset \operatorname{SBs}|L| := \bigcap_{m\in\mathbf{N}} \operatorname{Bs}|mL|$ the stable base locus (cf. [Tk2, 2.11]).

In [Tk2], we introduce a variant of the above intersection numbers. It is defined by using a variant of multiplier ideals due to Ein and Kawamata (refer [DEL, 1.7], [L] for a systematic treatment). These are defined algebraically, and therefore they fit into algebraic methods. In the rest of this section, let us assume the Kodaira-Iitaka dimension $\kappa(X, L) \geq 0$, and denote $h_{\min}$ the singular Hermitian metric with minimal singularities on L. Let k be a positive integer such that $H^0(X, \mathcal{O}_X(kL)) \neq 0$. We take a log-resolution $\mu : X' \longrightarrow X$ of the linear system $|kL|$ such that $\mu^*|kL| = |V| + E$, where $|V|$ is a free linear system, E the fixed part, and $E + \mathrm{Exc}(\mu)$ has simple normal crossing support. Given such a log-resolution plus a rational number $c > 0$, we define $\mathcal{J}(c \cdot |kL|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cE])$, this is independent of the choice of μ. In case L is big, there exists a large $k_0 \in \mathbf{N}$ such that the family of ideals $\{\mathcal{J}(\frac{c}{k} \cdot |kL|)\}_{k>k_0}$ has a unique maximal element. We denote the maximal element by

$$\mathcal{J}(c \cdot ||L||) = \mathcal{J}(X, c \cdot ||L||)$$

and call it the *asymptotic multiplier ideal sheaf* associated to c and $|L|$. Even in case L is not big, but still $\kappa(X, L) \geq 0$, one can modify the above definition and can define $\mathcal{J}(c \cdot ||L||)$. Let us recall the following fundamental properties.

Lemma 1.3. (1) $\mathcal{J}(||L||) \subset \mathcal{I}(h_{\min})$.
(2) $\mathcal{I}_+(h_{\min}) \subset \mathcal{J}(||L||) \subset \mathcal{I}(h_{\min})$, *provided L is big.*
(3) *The following natural inclusions are isomorphisms for every m:*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{J}(||mL||)) \longrightarrow H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_{\min}^m)) \longrightarrow H^0(X, \mathcal{O}_X(mL)).$$

Those above mentioned multiplier ideals $\mathcal{J}(||mL||), \mathcal{I}(h_{\min}^m)$ are used as an important tool in their proofs of the invariance of plurigenera due to Siu [S] and Kawamata [K]. Using asymptotic multiplier ideals, we introduce a variant of intersection numbers.

Definition-Proposition 1.4 ([Tk2]). *Let $C \subset X$ be an integral curve such that $C \not\subset \mathrm{SBs}\,|L|$. Then*
(1) *the following limit exists:*

$$||L; C|| := \lim_{m \to \infty} m^{-1} \deg_C \mathcal{O}_C(mL) \otimes \mathcal{J}(||mL||).$$

(2) $0 \leq ||L; C|| \leq (L, h_{\min}) \cdot C \leq L \cdot C$.
(3) $||L; C|| = (L, h_{\min}) \cdot C$, *provided L is big.*

The middle and the last inequalities in (2) can be strict.

§2. Seshadri-type criterion

We generalize the following Seshadri's criterion.

Definition-Theorem 2.1 (cf. [H, I §7]). *Assume L is nef.* Seshadri's constant *of L at $x \in X$ is defined by*

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all integral curve C passing through x, and $\mathrm{mult}_x C$ *is the multiplicity of C at x.*

(1) Seshadri's criterion: *L is ample if and only if the global Seshadri's constant* $\varepsilon(L) := \inf_{x\in X} \varepsilon(L,x)$ *is positive.*

(2) (cf. [D1, 7.9]) *For every positive d-dimensional subvariety Y passing through x, $L^d \cdot Y \geq \varepsilon(L,x)^d \mathrm{mult}_x Y$ holds.*

It is suggestive to think of the Seshadri's constant $\varepsilon(L,x)$ as measuring how positive L is locally near x. By means of our intersection theory, we introduce an invariant to measure a local positivity of pseudo-effective line bundles.

Definition 2.2. Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. *Seshadri's constant of* (L,h) at $x \in X$ is defined by

$$\varepsilon((L,h),x) := \inf_{C \ni x} \frac{(L,h)\cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all integral curve C passing through x and $h|_C$ is well-defined.

One can also define this type of invariant in terms of $||L;C||$. On the other hand, we have a global invariant.

Definition 2.3. The *volume* of L is the real number

$$v(L) = v(X,L) := \limsup_{m\to\infty} \frac{n!}{m^n} h^0(X, \mathcal{O}_X(mL)).$$

By definition, a line bundle is *big* if its volume is positive. In case L is nef, $v(L) = L^n$ holds. In case L is big, it has a singular Hermitian metric h with $\Theta_h \geq 0$ such that $\varepsilon((L,h),x) > 0$ for every point x outside some divisor (this is an easy consequence of Kodaira's lemma). Conversely, as in Seshadri's criterion, our criterion guarantees a positivity of the global volume by means of local positivities.

Theorem 2.4. *A line bundle L is big if and only if it has a singular Hermitian metric h with $\Theta_h \geq 0$ such that $\varepsilon((L,h),x) > 0$ for some point $x \in X - \mathrm{Sing}\, h$. Moreover in that case*

$$v(L) \geq \varepsilon((L,h),x)^n.$$

We also have an analogous statement of Theorem 2.1(2). Theorem 2.4 is proved by using the following approximation result of Seshadri's constants, which is an analogue of the approximation theorem of volumes due to Fujita. In this sense Theorem 2.5 is a local version of Fujita [F] (see also [DEL, 3.2]): Theorem 2.6 below.

Theorem 2.5. *Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. Assume L is big, and write $L \equiv A + E$ for an ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor E (Kodaira's lemma). Let δ be a positive number. Then there exist a birational modification $\mu_\delta : X_\delta \longrightarrow X$ and a decomposition $\mu_\delta^* L \equiv A_\delta + E_\delta$ with an ample $\mathbf{Q}$-divisor A_δ and an effective $\mathbf{Q}$-divisor E_δ such that μ_δ is isomophic over $X - (\operatorname{Sing} h \cup E)$, $E_\delta \subset \mu_\delta^{-1}(\operatorname{Sing} h \cup E)$, and such that*

$$\varepsilon(A_\delta, \mu_\delta^{-1}(x)) \geq (1 - \delta)\varepsilon((L, h), x)$$

for every $x \in X - (\operatorname{Sing} h \cup E)$. Moreover $A_\delta^n \geq v(L) - \delta$ holds as below.

Theorem 2.6 ([F]). *Let L be a big line bundle on X. Given any $\varepsilon > 0$, there exists a birational modification $\mu_\varepsilon : X'_\varepsilon \longrightarrow X$ and a decomposition $\mu_\varepsilon^* L \equiv A_\varepsilon + E_\varepsilon$, where E_ε is en effective $\mathbf{Q}$-divisor and A_ε is an ample $\mathbf{Q}$-divisor with $A_\varepsilon^n > v(L) - \varepsilon$.*

References

[D1] Demailly J.-P., *L^2 vanishing theorem for positive line bundles and adjunction theory*, Transcendental Methods in Algebraic Geometry, Lecture Notes in Math., vol. 1646, Springer-Verlag, 1996, pp. 1–97.

[D2] Demailly J.-P., *Méthodes L^2 et résultats effectifs en géométrie algébrique*, Astérisque **266** (2000) 59–90.

[DEL] Demailly J.-P., Ein L.-Lazarsfeld R., *A subadditivity property of multiplier ideals*, Michigan J. of Math. **48** (2000) 137–156.

[F] Fujita T., *Approximating Zariski decomposition of big line bundles*, Kodai. Math. J. **17** (1994) 1–3.

[H] Hartshorne R., *Ample subvarities of algebraic varieties*, Lecture Notes in Math., vol. 156, Springer-Verlag, Berlin, 1970.

[K] Kawamata Y., *Deformations of canonical singularities*, J. Amer. Math. Soc. **12** (1999) 85–92.

[L] Lazarsfeld R., *Multiplier Ideals for Algebraic Geometers*, August, 2000.

[S] Siu Y.-T., *Invariance of plurigenera*, Invent. math. **134** (1998) 661–673.

[Tk1] Takayama S., *Seshadri constants and a criterion for bigness of pseudo-effective line bundles*, preprint, 2001.

[Tk2] Takayama S., *Iitaka's fibrations via multiplier ideals*, preprint, 2001.

[T] Tsuji H., *Numerically trivial fibrations*, math.AG/0001023.

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