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Amoebas, convexity and the volume of integer polytopes

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Abstract.

To any given Laurent polynomial f on $\mathbf{C}_*^n$ we associate two natural convex functions M_f and N_f on $\mathbf{R}^n$. We compute the Hessian of M_f and obtain an explicit formula for the volume of the Newton polytope Δ_f. We also establish asymptotic formulas relating our convex functions to coherent triangulations of Δ_f and to the secondary polytope.

§1.

Let $A \subset \mathbf{Z}^n$ be a finite set and consider a general Laurent polynomial $f(z) = \sum_{\alpha \in A} a_\alpha \, z^\alpha$, with complex coefficients and $z \in \mathbf{C}_*^n$. The Newton polytope Δ_f is defined as the convex hull of A (in $\mathbf{R}^n \supset \mathbf{Z}^n$), or more accurately, as the convex hull of those α for which $a_\alpha \neq 0$. The amoeba $\mathbf{A}_f$ is defined to be the image of the zero set of f under the mapping $\mathrm{Log} : \mathbf{C}_*^n \to \mathbf{R}^n$ given by $(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$. In the sequel we use the notation $|z_j| = t_j$ and $\log |z_j| = x_j$.

We are going to deal with the two functions

$$M_f(x) = \log\Big(\sum_{\alpha \in A} |a_\alpha| \, e^{\langle \alpha, x \rangle}\Big)$$

and

$$N_f(x) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log \big| f(e^{x+i\theta}) \big| \, d\theta_1 \wedge \cdots \wedge d\theta_n.$$

They are both convex functions in $\mathbf{R}^n$ with the property that their gradient mappings map $\mathbf{R}^n$ to the Newton polytope Δ_f. More precisely, the mapping $\operatorname{grad} M_f$ is a diffeomorphism $\mathbf{R}^n \to \operatorname{int} \Delta_f$, whereas

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$\operatorname{grad} N_f$ maps $\mathbf{R}^n$ onto the closed polytope Δ_f with each connected component of $\mathbf{R}^n \setminus \mathbf{A}_f$ being sent to one of the integer vectors $\Delta_f \cap \mathbf{Z}^n$, called the order of that connected component. (See [5] for more on this.)

Introducing the corresponding Monge–Ampère measures

$$\operatorname{Hess} M_f = \operatorname{Jac} \operatorname{grad} M_f \quad \text{and} \quad \operatorname{Hess} N_f = \operatorname{Jac} \operatorname{grad} N_f,$$

we conclude from general facts on convex functions, see [6], that these are both positive measures with total masses equal to $\operatorname{Vol} \Delta_f$.

Let us order the set A as $\{\alpha^0, \alpha^1, \ldots, \alpha^N\}$, and consider, for any increasing multi-index $J = \{j_0, \ldots, j_n\} \in \{0, 1, \ldots, N\}^{1+n}$, the square matrix A_J having the $(1+n)$-vectors $(1, \alpha^{j_k})$ as its columns. Observe that $|\det(A_J)|$ equals $n!$ times the volume of the simplex σ_J with vertices in $\alpha^{j_0}, \ldots, \alpha^{j_n}$. We begin with an explicit computation.

Proposition 1.1 *The push-forward of the measure* $\operatorname{Hess} M_f$ *under the mapping* $\operatorname{Exp}\colon \mathbf{R}^n \to \mathbf{R}^n_+$ *defined by* $(x_1, \ldots, x_n) \mapsto (e^{x_1}, \ldots, e^{x_n})$*, is given by Lebesgue measure times a rational function* h_f / F^{1+n}*, with the polynomial* h_f *explicitly given by*

$$h_f(t) = \sum\nolimits'_{|J|=1+n} \det{}^2(A_J)\, |a_{\alpha^{j_0}}|\, t^{\alpha^{j_0}} \cdots |a_{\alpha^{j_n}}|\, t^{\alpha^{j_n}}.$$

Here the summation is over all increasing multi-indices J, and F is obtained from f by replacing each coefficient a_α by $|a_\alpha|$.

Proof: The gradient of M_f equals the moment map (cf. [4], p.198)

$$\operatorname{grad} M_f(x) = \frac{\sum_{\alpha \in A} \alpha\, |a_\alpha|\, e^{\langle \alpha, x \rangle}}{\sum_{\alpha \in A} |a_\alpha|\, e^{\langle \alpha, x \rangle}} = \frac{\sum_{\alpha \in A} \alpha\, |a_\alpha|\, t^\alpha}{\sum_{\alpha \in A} |a_\alpha|\, t^\alpha},$$

which means that $\operatorname{Hess} M_f(x) = \det\bigl(\partial^2 M_f(x) / \partial x_j\, \partial x_k\bigr)$ is equal to

$$\left| \frac{\sum_{\alpha \in A} \alpha_j \alpha_k\, |a_\alpha|\, t^\alpha}{\sum_{\alpha \in A} |a_\alpha|\, t^\alpha} - \frac{\left(\sum_{\alpha \in A} \alpha_j\, |a_\alpha|\, t^\alpha\right)\left(\sum_{\alpha \in A} \alpha_k\, |a_\alpha|\, t^\alpha\right)}{\left(\sum_{\alpha \in A} |a_\alpha|\, t^\alpha\right)^2} \right|,$$

and if we introduce the abbreviation $c_\alpha = |a_\alpha|\, t^\alpha$ we may re-write the above $n \times n$-determinant as the following $(1+n) \times (1+n)$-determinant:

$$\frac{1}{\left(\sum c_\alpha\right)^{1+n}} \begin{vmatrix} \sum c_\alpha & \sum \alpha_1\, c_\alpha & \cdots & \sum \alpha_n\, c_\alpha \\ \sum \alpha_1\, c_\alpha & \sum \alpha_1 \alpha_1\, c_\alpha & \cdots & \sum \alpha_1 \alpha_n\, c_\alpha \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sum \alpha_n\, c_\alpha & \sum \alpha_n \alpha_1\, c_\alpha & \cdots & \sum \alpha_n \alpha_n\, c_\alpha \end{vmatrix}. \qquad (*)$$

Now we consider the $(1+n)\times(1+N)$-matrix

$$B = \begin{pmatrix} \sqrt{c_{\alpha^0}} & \sqrt{c_{\alpha^1}} & \dots & \sqrt{c_{\alpha^N}} \\ \alpha_1^0\sqrt{c_{\alpha^0}} & \alpha_1^1\sqrt{c_{\alpha^1}} & \dots & \alpha_1^N\sqrt{c_{\alpha^N}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_n^0\sqrt{c_{\alpha^0}} & \alpha_n^1\sqrt{c_{\alpha^1}} & \dots & \alpha_n^N\sqrt{c_{\alpha^N}} \end{pmatrix},$$

and make two observations. First, the determinant $(*)$ is equal to $\det(B\,B^{\mathrm{tr}})/F(t)^{1+n}$. Second, the polynomial h_f is equal to the sum of the squares of all the maximal minors of B. The desired identity $\operatorname{Hess} M_f = h_f/F^{1+n}$ therefore follows from the Cauchy–Binet formula, see [3], which says that the determinant of the product $B\,B^{\mathrm{tr}}$ is indeed equal to the sum of the squares of the minors of B.

We remark that h_f is the non-homogeneous toric Jacobian of the extended gradient $(F, t_1\partial_1 F, \dots, t_n\partial_n F)$, see [2] and Proposition 1.2 in [1], where a similar computation was carried out. Combining our Proposition 1.1 with the fact that the total mass of $\operatorname{Hess} M_f$ is equal to $\operatorname{Vol}\Delta_f$, we obtain the following explicit, elementary, and apparently new formula for the volume of the Newton polytope.

Theorem 1.2 *The volume of the Newton polytope* Δ_f *can be computed by means of the closed formula*

$$\operatorname{Vol}\Delta_f = \int_{\mathbf{R}^n_+} \frac{h_f(t)}{(F(t))^{1+n}}\,\frac{dt_1\wedge\cdots\wedge dt_n}{t_1\cdots t_n}. \qquad (**)$$

We knew a priori that this integral should converge, since the measure $\operatorname{Hess} M_f$ has a finite mass, but the convergence now also follows from the obvious fact that the Newton polytope of h_f is contained in the interior of $(1+n)\,\Delta_f$.

Regarding the function N_f, we recall the following result from [5]. Remember that a polyhedral subdivision is a generalized triangulation whose elements are polyhedra (but not necessarily simplices).

Theorem 1.3 *The piecewise linear convex function* $\max_\alpha(c_\alpha + \langle\alpha, x\rangle)$, *where* $c_\alpha + \langle\alpha, x\rangle = N_f(x)$ *in the component of* $\mathbf{R}^n\setminus\mathbf{A}_f$ *of order* α, *defines a polyhedral subdivision of* $\mathbf{R}^n$ *whose* $(n-1)$*-skeleton is contained in* $\mathbf{A}_f$, *while its Legendre transform similarly defines a dual polyhedral subdivision* $\mathbf{T}_f$ *of* Δ_f. *A vector* α *is a vertex in* $\mathbf{T}_f$ *if and only if* $\mathbf{R}^n\setminus\mathbf{A}_f$ *has a component of order* α.

§2.

In this section we shall study the asymptotic behaviour of Theorems 1.2 and 1.3 as the coefficients a_α tend to infinity. More precisely, we will set $a_\alpha = \lambda^{s_\alpha}$ for some fixed vector $(s_\alpha) \in \mathbf{R}^A$ and $\mathbf{R} \ni \lambda \to \infty$. We recall from [4] that the so-called secondary polytope $\Sigma_A \subset \mathbf{Z}^A$ has the property that its vertices are in bijective correspondence with the coherent triangulations of Δ_f, and that a triangulation is coherent if it can be defined by a convex (or concave) piecewise linear function (as in Theorem 1.3).

For any vertex v of Σ_A, the normal cone N_v, which consists of all vectors $(s_\alpha) \in \mathbf{R}^A$ such that $(s, v) = \max_{w \in \Sigma_A}(s, w)$, has a non-empty interior. Any vector (s_α) from int N_v, that is, such that $(s, v) > (s, w)$ for all $w \in \Sigma_A$ with $v \neq w$, can be used to produce the associated coherent triangulation $\mathbf{T}_v$ of Δ_f in the following way. Let g_s be the piecewise linear concave function on Δ_f whose graph equals the upper boundary of the convex hull of the union of half lines $\{(\alpha, y);\ \alpha \in A, y \leq s_\alpha\}$. Then $\mathbf{T}_v$ is obtained by projecting the linear pieces of the graph of g_s down to Δ_f. Notice that $-g_s$ is the Legendre transform of the piecwise linear convex function $\max_\alpha(s_\alpha + \langle \alpha, x\rangle)$ on $\mathbf{R}^n$.

The polynomial h_f, and hence the whole volume formula in Theorem 1.2, contains one term for each subsimplex σ_J with vertices in A. Asymptotically, it is only the terms corresponding to the disjoint simplices of a coherent triangulation that survive, as shown by the following theorem.

Theorem 2.1 *Let v be a vertex of the secondary polytope Σ_A, and take a vector $(s_\alpha) \in \mathbf{R}^A$ in the interior of the normal cone N_v. Set the coefficients a_α of f equal to λ^{s_α}. Then the term $I_J(\lambda)$ in (**) corresponding to the multi-index J satisfies*

$$\lim_{\lambda \to \infty} I_J(\lambda) = \begin{cases} \operatorname{Vol} \sigma_J, & \text{if } \sigma_J \in \mathbf{T}_v\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Proof: Recalling the formula for h_f, we see that

$$I_J(\lambda) = \int_{\mathbf{R}^n_+} \frac{\det^2(A_J)\, \lambda^{s_{\alpha^{j_0}}}\, t^{\alpha^{j_0}} \cdots \lambda^{s_{\alpha^{j_n}}}\, t^{\alpha^{j_n}}}{(\lambda^{s_{\alpha^0}}\, t^{\alpha^0} + \lambda^{s_{\alpha^1}}\, t^{\alpha^1} + \ldots \lambda^{s_{\alpha^N}}\, t^{\alpha^N})^{1+n}} \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n}.$$

If we perform the monomial substituion $u_k = \lambda^{s_{\alpha^{j_k}}} t^{\alpha^{j_k}} / \lambda^{s_{\alpha^{j_0}}} t^{\alpha^{j_0}}$, for $k = 1, \ldots, n$, we arrive at

$$I_J(\lambda) = \int_{\mathbf{R}_+^n} \frac{|\det(A_J)|\, du_1 \wedge \cdots \wedge du_n}{(1 + u_1 + \cdots + u_n + \delta(\lambda))^{1+n}},$$

where $\delta(\lambda)$ is a finite sum of fractional monomials $\lambda^{r_0} u_1^{r_1} \cdots u_n^{r_n}$, with $r \in \mathbf{Q}^{1+n}$ and $r_0 \neq 0$. Now, it is not hard to verify that the simplex σ_J belongs to the triangulation $\mathbf{T}_v$ precisely if all the exponents r_0 are negative. In this case the term $\delta(\lambda)$ tends to zero, and since the integral of $du_1 \wedge \cdots \wedge du_n/(1+u_1+\cdots+u_n)^{1+n}$ over the positive orthant is equal to $1/n!$, we conclude that $I_J(\lambda) \to |\det(A_J)|/n!$ as claimed. Otherwise, the denominator in the integrand goes to infinity, and the integral $I_J(\lambda)$ tends to zero.

The proof of the next result is essentially parallel to that of Theorem 9 in [7] and will be omitted.

Theorem 2.2 *Let v be a vertex of the secondary polytope Σ_A, and take a vector (s_α) as in Theorem 2.1. Set the coefficients a_α of f equal to λ^{s_α} and denote the new polynomial by f^λ. For large values of the parameter λ the polyhedral subdivision $\mathbf{T}_{f^\lambda}$ from Theorem 1.3 will then coincide with the coherent triangulation $\mathbf{T}_v$.*

We end with a closer look at a one-dimensional case.

Example 2.3 Consider a one-variable polynomial of the form $f(t) = 1 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$. For each $m = 0, 1, \ldots, 2n-2$ the so-called Ostrogradski method for finding the rational part of a primitive function can be realized with the explicit formula

$$\int \frac{t^m\, dt}{f(t)^2} = -\frac{P_m(t)}{f(t)} + \int \frac{Q_m(t)\, dt}{f(t)},$$

where the P_m and Q_m are polynomials of degrees $n-1$ and $n-2$ respectively. To be specific, one has $P_m(t) = \sum_{k=0}^{n-1} A_{m,k} t^k$ and $Q_m(t) = P'_m(t) + \sum_{\ell=0}^{n-2} B_{m,\ell} t^\ell$, with the $(2n-1)\times(2n-1)$-matrix $(B_{m,\ell}, A_{m,k})$ being the inverse of the standard Sylvester matrix (see [4], p.405) whose determinant equals the discriminant D_n of f. Now, if we collect terms in h_f and write $t^{-1} h_f(t) = \sum_{m=0}^{2n-2} C_m t^m$, then it holds that $\sum_m A_{m,k} C_m = (n-k) a_k$ and $\sum_m B_{m,\ell} C_m = -(\ell+1)(n-\ell-1) a_{\ell+1}$. (Here $a_0 = a_n = 1$.) This implies in particular that if we replace the individual terms

$$\int_0^\infty \frac{(j_1 - j_0)^2\, a_{j_0} a_{j_1}\, t^{j_0+j_1-1}\, dt}{f(t)^2}$$

in formula $(**)$ by their principal parts

$$-\left.\frac{(j_1-j_0)^2\,a_{j_0}a_{j_1}\,P_{j_0+j_1-1}(t)}{f(t)}\right|_0^{\infty} = (j_1-j_0)^2\,a_{j_0}a_{j_1}\,A_{j_0+j_1-1,0}$$

then they still sum to $\operatorname{Vol}\Delta_f = n$. In other words, the individual terms of $(**)$, which are not themselves rational functions of the coefficients a_j, can be replaced by rational expressions so that the volume formula still holds true. Since these expressions all have the discriminant D_n as their denominator, this means we have in a canonical way associated polynomials (the numerators) with all subsimplices $[j_0, j_1]$ so that their sum is equal to nD_n. In fact, the linear form on the vector space $\langle 1, t, \ldots, t^{2n-2}\rangle$ given by

$$t^m \mapsto P_m(0) \quad (= A_{m,0})$$

coincides with the toric residue associated to the mapping (f, tf').

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