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Intersection multiplicities of holomorphic and algebraic curves with divisors

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Abstract.

Here we discuss the intersection multiplicities of holomorphic and algebraic curves with divisors on an algebraic variety as an analogue to the abc-Conjecture. We announce some new results.

§1. Introduction

We discuss the truncation of counting functions in the second main theorem (S.M.T.) for holomorphic curves and algebraic ones, that is, the bound of intersection multiplicities of holomorphic curves with divisors in transcendental and algebraic cases; the problem has a strong analogue with the abc-Conjecture of Masser-Oesterlé.

abc-Conjecture. *Let $a, b, c \in \mathbf{Z}$ be coprime and satisfy*

$$a + b + c = 0.$$

Then for an arbitrary $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that

$$\max\{|a|, |b|, |c|\} \leqq C(\epsilon) \left(\prod_{\substack{p>1 \text{ prime} \\ p|abc}} p \right)^{1+\epsilon} . \tag{1.1}$$

It is the key point of the conjecture that the multiplicities of primes of a, b and c in the right hand side of (1.1) are counted only by "$1+\epsilon$". The analogue for holomorphic curves $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is known:

Theorem 1.2. (Nevanlinna-Cartan-Nochka '83) *Let m be the dimension of the linear span of $f(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C})$. Let $H_i, 1 \leqq i \leqq q$, be*

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hyperplanes in general position. Then,

$$(q-2n+m-1)T_f(r) \leqq \sum_{i=1}^{q} N_m(r, f^*H_i) + S_f(r),$$
$$S_f(r) = O(\log r) + \delta \log T_f(r)||_{E(\delta)}, \quad \operatorname{meas} E(\delta) < \infty.$$

Here we use the notation:

$$T_f(r) = T_f(r; O(1)) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^* c_1(O(1)),$$
$$N_k(r, f^*D) = \int_1^r \frac{\sum_{|z|<t} \min\{k, \operatorname{ord}_z f^*D\}}{t} dt,$$

and "$||_{E(\delta)}$" means that the estimate holds for $r \notin E(\delta)$ as $r \to \infty$.

H. Fujimoto '82 obtained similar estimates for associated curves in Ahlfors-Weyl-Stoll theory.

§2. Lemma on logarithmic differential

Let φ be a logarithmic differential on V with C^∞-coefficients. For a meromorphic mapping $g : \mathbf{C}^m \to V$ set

$$g^*\varphi = \sum_{i=1}^{m} \xi_i dz_i.$$

Let γ denote the rotationary invariant probability measure on the sphere $\{\|z\| = r\}$.

Lemma 2.1. (Nog. '77–) *Let the notation be as above. Then we have*

$$m(r, \xi_i) := \int_{\|z\|=r} \log^+ \xi_i(z)\gamma(z) = S_f(r).$$

Vitter '77 proved this for $f : \mathbf{C}^m \to \mathbf{P}^1(\mathbf{C})$ and $\varphi = dw/w$.

Theorem 2.2. (Griffiths, Carlson, King, Sakai, Shiffman, Stoll, Kodaira, ..., Nog.) *Let $f : \mathbf{C}^n \to V$ be a differentiably non-degenerate meromorphic mapping to a complex projective algebraic manifold V. Let $L \to V$ be a line bundle with complete linear system $|L|$. Let $D = \sum D_i \in |L|$ be a divisor only with simple normal crossings. Then,*

$$T_f(r, L) + T_f(r, K_V) \leqq \sum_i N_1(r, f^*D_i) + S_f(r).$$

In stead of the curvature method employed by Griffiths et al, we may apply Lemma 2.1 directly to prove Theorem 2.2, where a positivity assumption for L is not needed.

Fundamental conjecture for holomorphic curves. *Let $L \to V$ be as above and $\dim V = n$. Let $D = \sum D_i \in |L|$. Let $f : \mathbf{C} \to V$ be a Zariski non-degenerate holomorphic curve. Then there exists a number $k = k(D, n)$ such that*

$$T_f(r, L) + T_f(r, K_V) \leqq \sum N_k(r, f^*D_i) + S_f(r).$$

If D has only simple normal crossings, then $k = n$

§3. Holomorphic mappings

We call a complex Lie group M a *semi-torus* if it admits an exact sequence:

$$0 \to (\mathbf{C}^*)^t \to M \to M_0 \to 0,$$

where M_0 is a complex torus. Using the compactification $(\mathbf{P}^1(\mathbf{C}))^t \supset (\mathbf{C})^t$, we take a compactification $\bar{M}$ of M, and set $\partial M = \bar{M} \setminus M$. Let $\bar{D}$ be a reduced divisor on $\bar{M}$ such that no irreducible component of $\bar{D}$ is contained in ∂M. If every l irreducible components of $\bar{D} + \partial M$ has the intersection of pure codimension l, $\bar{D} + \partial M$ is said to be *in general position.*

Theorem 3.1. ([NWY00, NWY02]) *Let M be a semi-torus, and let $L \to \bar{M}$ be a line bundle. Let $\bar{D} \in |L|$ be a divisor without support in ∂M. If M is not compact, we assume that $\bar{D} + \partial M$ is in general position. Set $D = \bar{D} \cap M$. Let $f : \mathbf{C} \to M$ be a holomorphic curve which is Zariski-nondegenerate as a curve in $\bar{M}$. Then we have*

$$T_f(r; L) = N_{k_0}(r; f^*D) + S_f(r).$$

Here, if the order ρ_f of f is finite, $k_0 = k_0(D, \rho_f)$; otherwise, $k_0 = k_0(D, f)$.

Remark. (i) The assumption for $\bar{D} + \partial M$ being in general position is necessary, by examples.

(ii) By examples of singular D on abelian M, k_0 must depend on D. In the case of abelian varieties, Yamanoi lately obtained

$$T_f(r; L) = N_1(r; f^*D) + o(T_f(r; c_1(D))).$$

There is an application of Theorem 3.1 to the Kobayashi hyperbolicity of a covering over abelian varieties, ramified over D.

Using the method of the proof of Theorem 3.1, we have

Theorem 3.2. *Let $f : \mathbf{C}^n \to M$ be a differentiably non-degenerate holomorphic mapping to a semi-torus M of dimension n. Let $L \to \bar{M}$ be a line bundle and $\bar{D} \in |L|$. Set $D = \bar{D} \cap M$. Then,*

$$T_f(r, L) \leqq N_n(r, f^*D) + S_f(r).$$

Taking this opportunity, we would like to make some minor corrections to [N98] and [NWY02].

Remark. (i) The statements given in [N98], Proposition (1.8) (ii), and in [SY96], Theorem (2.2), are too strong to claim; there were gaps in both proofs. The proof given in [N98] implies only that

$$\dim \mathrm{St}(X_k(f)) > 0.$$

But this gives no effect to the other part of the paper, in particular to the proof of the Main Theorem of [M98], for it was not used. If the holomorphic curve is of finite order, then Proposition (1.8) (ii) in [N98] will hold, but in general there is a counter-example (cf. [NW02c]).

(ii) Because of the correction above, in [NWY02] p. 146, 4th–2nd lines from below, the space W_k is to be defined as the π_2-projection of the Zariski closure of $J_k(f)(\mathbf{C})$ as in [NWY00].

§4. Function fields

Nevanlinna theory is an approximation theory of complex numbers by meromorphic functions, as the Diophantine approximation is the approximation of algebraic numbers by rationals or algebraic numbers of a fixed number field (the inverse of Vojta's observation). Over algebraic function fields, one may think the approximation of rational functions by rational functions.

Let R be an algebraic curve of genus g, and let L be a line bundle on R with degree $\deg L$.

Theorem 4.1. *Let $H_j, 1 \leqq j \leqq q$, be linear forms in general position on $\mathbf{P}^n(\mathbf{C})$ with coefficients in $H^0(R, L)$. For an arbitrarily given $\epsilon > 0$, set*

$$k(\epsilon) = (n+1)\left\{\left(\left[\max\left\{g + \frac{2n}{\epsilon}, 2g-2\right\}\right] + 2\right)\deg L - g + 1\right\} - 1.$$

Then for an arbitrary $x : R \to \mathbf{P}^n(\mathbf{C})$, we have

$$(q - 2n - \epsilon)\mathrm{ht}(x) \leqq \sum_{j=1}^{q} N_{k(\epsilon)}(H_j(x)) + C(\epsilon, q).$$

Here $\mathrm{ht}(x) = \deg x^* O(1)$ and $N_{k(\epsilon)}(\cdot)$ is the counting function of zeros of a linear form with truncation level $k(\epsilon)$.

In the proof we use the results of Nochka '83, the method of Steinmetz '85-Shirosaki '91, J. Wang '00, the author '97, and the Riemann-Roch.

Motivated by the similar problem modeled after the "abc-Conjecture" over abelian varieties and here over function fields, A. Buium ([Bu98]) proved

Theorem 4.2. *Let A be an abelian variety, and let D be an effective divisor on A such that D is hyperbolic. Then there is a constant $N(R, A, D)$ such that for an arbitrary morphism $f : R \to A$ with $f(R) \not\subset D$,*

$$\mathrm{ord}_x f^* D \leqq N(R, A, D), \qquad x \in R.$$

A. Buium used a method based on Kolchin's differential algebra. He conjectured there should be a proof by standard algebraic geometry, and the theorem should be valid for ample D. It is indeed true in more general form:

Theorem 4.3. ([NW0x]) *There is a function*

$$N : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$$

such that the following statement holds: Let C be a smooth compact curve of genus g, let A be an abelian variety of dimension n, let D be an ample effective divisor on A with $d = c_1(D)^n$, and let $f : C \to A$ be a morphism with $f(C) \not\subset D$. Then

$$\mathrm{ord}_x f^* D \leqq N(g, n, d), \qquad x \in C.$$

As an application we have a finiteness

Theorem 4.4. ([NW0x]) *Let C' be an affine algebraic curve, let A be an abelian variety and let D be an ample effective divisor on A. Then, either $\exists f : C' \to D$, non-constant, or there are only finitely many non-constant morphisms from C' to $A \setminus D$.*

Remark. If there is a nonconstant $f : C' \to D$, there may be infinitely many non-constant $g : C' \to A \setminus D$, by example.

N.B. The following list of references is not intended to be complete, but sufficient to trace up the necessary papers by referring their references.

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