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Meromorphic mappings and deficiencies

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Abstract.

In this note, we shall discuss elimination theorems of defects of hypersurfaces or rational moving targets for a meromorphic mapping or a holomorphic curve into $\mathbf{P}^n(\mathbf{C})$ by its small deformation.

§1. Introduction.

Value distribution theory is to study how intersects the image of a mapping to divisors in a target space. Liouville theorem asserts that the image of a meromorphic function is dense in the projective space $\mathbf{P}^1(\mathbf{C})$, and also Picard theorem asserts that the image covers all points on $\mathbf{P}^1(\mathbf{C})$ except for at most two points. Nevanlinna theory is a quantitative refinement of Picard theorem. Nevanlinna deficiency $\delta_f(a)$ express that $\delta_f(a) = 1$ if the image $f(\mathbf{C})$ omits a-point and $\delta_f(a) > 0$ if f covers a point a relatively few times. For a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, Nevanlinna's defect relations or Crofton's formulae assert that Nevanlinna defects or Valiron defects of a mapping are very few.

We shall now discuss on defects for a family of mappings, that is, elimination theorems of defects of hyperplanes, hypersurfaces or rational moving targets for a meromorphic mapping or a holomorphic curve into $\mathbf{P}^n(\mathbf{C})$ by its small deformation. Here a samll deformation $\tilde{f}$ of f means that the difference of order functions of $\tilde{f}$ and f is relatively small.

§2. Preliminaries.

Let $z = (z_1, ..., z_m)$ be the natural coordinate system in $\mathbf{C}^m$. Set

$$\langle z, \xi \rangle = \sum_{j=1}^{m} z_j \xi_j \text{ for } \xi = (\xi_1, ..., \xi_m), \|z\|^2 = \langle z, \overline{z} \rangle, \; B(r) = \left\{ z \middle| \|z\| < r \right\},$$

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$$\partial B(r) = \left\{z \middle| \|z\| = r\right\},\ \psi = dd^c \log \|z\|^2 \text{ and } \sigma = d^c \log \|z\|^2 \wedge \psi^{m-1},$$

where $d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$ and $\psi^k = \psi \wedge \cdots \wedge \psi$ (k-times).

Let f be a nonconstant meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $\mathcal{L} = [\mathbf{H}^d]$ be the line bundle over $\mathbf{P}^n(\mathbf{C})$ which is determined by d-th tensor power of the hyperplane bundle $[\mathbf{H}]$. A hypersurface D of degree d in $\mathbf{P}^n(\mathbf{C})$ is given by the divisor of a holomorphic section $s \in H^0(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(\mathcal{L}))$ which is determined by a homogeneous polynomial $P(w)$ of degree d. A metric $a = \{a_\alpha\}$ on the line bundle $\mathcal{L}$ is given by $a_\alpha = (\sum_{j=0}^n |w_j/w_\alpha|^2)^d$ in a neighborhood $U_\alpha = \{w \in \mathbf{P}^n(\mathbf{C}) | w_\alpha \neq 0\}$.

The Nevanlinna's order function $T_f(r, \mathcal{L})$ of f for the line bundle $\mathcal{L}$ is given by:

$$T_f(r, \mathcal{L}) := \int_{r_0}^{r} \frac{dt}{t} \int_{B(t)} f^*\omega \wedge \psi^{m-1},$$

where $\omega = \{\omega_\alpha\} = dd^c \log(\sum_{j=0}^n |w_j/w_\alpha|^2)^d$ in U_α. We say that f is transcendental if $\displaystyle\lim_{r\to+\infty} \frac{T_f(r,\mathcal{L})}{\log r} = +\infty$. The norm of a section s is given by

$$\|s\|^2 := \frac{|s_\alpha|^2}{a_\alpha} = \frac{|P(w)|^2}{(\sum_{j=0}^n |w_j|^2)^d}.$$

The proximity function $m_f(r, D)$ of D is defined by

$$m_f(r, D) := \int_{\partial B} \log \frac{1}{\|s_f\|} \sigma = \int_{\partial B} \log \frac{\|f\|^d}{|P(f)|} \sigma.$$

The Nevanlinna deficiency $\delta_f(D)$ and the Valiron deficiency $\Delta_f(D)$ of D for f is defined by

$$\delta_f(D) := \liminf_{r\to\infty} \frac{m_f(r, D)}{T_f(r, \mathcal{L})} \text{ and } \Delta_f(D) := \limsup_{r\to\infty} \frac{m_f(r, D)}{T_f(r, \mathcal{L})}.$$

Using Stok's theorem, the Nevanlinna's order function $T_f(r) := T_f(r, [\mathbf{H}])$ of f for the hyperplane bundle $[\mathbf{H}]$ is written as:

$$T_f(r) = \int_{\partial B(r)} \log\Big(\sum_{j=0}^n |f_j|^2\Big)^{1/2} \sigma + O(1) = \int_{\partial B(r)} \log \sum_{j=0}^n |f_j| \sigma + O(1).$$

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, and ϕ be a meromorphic mapping of $\mathbf{C}^m$ into the dual projective space $\mathbf{P}^n(\mathbf{C})^*$ which is called a moving target for f. Then the proximity function $m_f(r, \phi)$ of a moving target ϕ into $\mathbf{P}^n(\mathbf{C})^*$ is given by:

$$m_f(r, \phi) := \int_{\partial B} \log \frac{\|f\| \|\phi\|}{|\langle f, \phi \rangle|} \sigma.$$

The Nevanlinna deficiency $\delta_f(\phi)$ and the Valiron deficiency $\Delta_f(\phi)$ of a moving target ϕ for f are defined similarly. (See [5])

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then f has a reduced representation $(f_0 : \ldots : f_n)$, and we write $f = (f_0, \ldots, f_n)$ the same letter as the mapping f. Denote $D^\alpha f = (D^\alpha f_0, \ldots, D^\alpha f_n)$ for a multi-index α, where $D^\alpha f_j = \partial^{|\alpha|} f_j / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}, \alpha = (\alpha_1, \ldots, \alpha_m)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Fujimoto [2] defined the generalized Wronskian of f by

$$W_{\alpha^0,\ldots,\alpha^n}(f) = \det(D^{\alpha^k} f : 0 \leq k \leq n),$$

for $n+1$ multi-indices $\alpha^k = (\alpha_1^k, \ldots, \alpha_m^k)$, $(0 \leq k \leq n)$.

§ 2-2. Some Results

Molzon-Shiffman-Sibony [6] defined the projective logarithmic capacity $C(E)$ of a set E on $\mathbf{P}^n(\mathbf{C})$, and they gave a criterion of positivity of projective logarithmic capacity for a subset of $\mathbf{P}^n(\mathbf{C})$

Proposition 1 ([3]). *Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then, for $H \in \mathbf{P}^n(\mathbf{C})^*$,*

$$\lim_{r \to +\infty} \frac{m_f(r, H)}{T_f(r)} = 0,$$

outside a set $E \subset \mathbf{P}^n(\mathbf{C})^*$ of projective logarithmic capacity zero.

Proposition 2 ([3]).

$$\mathcal{A} := \left\{ (1, a_1, \ldots, a_n, a_1^2, a_1 a_2, \ldots, a_1^{i_1} \cdots a_n^{i_n}, \ldots, \prod_{k=1}^{n} a_k^d) \mid a_j \in \mathbf{C} \right\}$$

is of positive projective logarithmic capacity.

§3. Elimination of defects of meromorphic mappings.

For a meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all defects by a small deformation of f.

Theorem 1. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental meromorphic mapping, and d is a positive integer. Then there exists a regular matrix $L = (l_{ij})_{0 \leq i,j \leq n}$ of the form $l_{i,j} = c_{ij} g_j + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbf{C} : 0 \leq i, j \leq n)$ such that $\det L \neq 0$ and $\tilde{f} = L \cdot f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is a meromorphic mapping without Nevanlinna defects of hypersurfaces of degree at most d, and satisfies $|T_f(r) - T_{\tilde{f}}(r)| = O(\log r)$ $(r \to \infty)$, where g_j $(j = 1, \ldots, n)$ are some monomials on $\mathbf{C}^m$.*

Theorem 2. *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental holomorphic curve. Then there exists a regular matrix $L = (l_{ij})_{0 \le i,j \le n}$ of the form $l_{i,j} = c_{ij} g_j + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbf{C} : 0 \le i, j \le n)$ such that* $\det L \neq 0$ *and $\tilde{f} = L \cdot f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is a holomorphic curve without Nevanlinna defects of rational moving targets and satisfies $|T_f(r) - T_{\tilde{f}}(r)| = o(T_f(r))(r \to \infty)$, where g_j $(j = 1, ..., n)$ are some transcendental entire functions on $\mathbf{C}$ satisfying $T_{g_j}(r) = o(T_{g_{j+1}}(r))$, $(j = 1, ..., n-1)$ and $T_{g_n}(r) = o(T_f(r))$ $(r \to \infty)$ which are constructed by using Edrei-Fuchs' theorem* [1].

Note that we cannot replace all transcendental entire functions g_j by rational functions.

Remark 1. In Theorem 1 and 2, mappings f may be linearly degenarate or of infinite order, and also if f is of finite order we can replace "Nevanlinna deficiency" by "Valiron deficiency" in the conclusion.

Remark 2. I first proved Theorem 1 for a meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ and hyperplanes [3], and also for a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and hypersurfaces [4]. The case where $m > 1$ in Theorem 1 is not yet published. Theorem 2 is found in [5].

We now give a very short sketch of the proof of Theorem 1 for $m \ge 1$. We need following lemmas.

Lemma 1. *There are monomials $g_1, ..., g_n$ in $\mathbf{C}^m$ such that any n derivatives in $\{D^\alpha g := (D^\alpha g_1, ..., D^\alpha g_n) | |\alpha| \le n+1\}$ are linearly independent over the field $\mathcal{M}$ of meromorphic functions on $\mathbf{C}^m$, where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{Z}_{\ge 0}$ is a multi-index and $D^\alpha g_k = \partial^{|\alpha|} g_k / \partial {z_1}^{\alpha_1} \cdots \partial {z_m}^{\alpha_m}$.*

Lemma 2. *Let $h = (h_0 : h_1 : \cdots : h_n)$ be a reduced representation of a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $g_1, ..., g_n$ linearly independent monomials as in Lemma 1. Then there exists $(\tilde{a}_1, ..., \tilde{a}_n)$ such that*

$$f := (h_0 : h_1 + \tilde{a}_1 g_1 h_0 : h_2 + \tilde{a}_2 g_2 h_0 : \cdots : h_n + \tilde{a}_n g_n h_0)$$

is a reduced representation of a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

Sketch of the proof of Theorem 1:

There is a regular linear change L_1 of $\mathbf{P}^n(\mathbf{C})$ such that $h := L_1 \cdot f \equiv (h_0 : \cdots : h_n) : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is a reduced representation of the meromorphic mapping h which satisfies

$$m_h(r, H_j) = o(T_h(r)) \quad (r \to +\infty), \quad (j = 0, 1, .., n),$$

where $H_j = \{(w_0 : \cdots : w_n) | w_j = 0\}$.

Consider the Veronese mapping v_d given by monomials of degree d. We first deform a meromorphic mapping h to $\tilde{h} := (h_0 : h_1 + \tilde{a}_1 g_1 h_0 : h_2 + \tilde{a}_2 g_2 h_0 : \cdots : h_n + \tilde{a}_n g_n h_0)$ by using $g_1, ..., g_n$ as in Lemma 1, and compose it to the Veronese mapping v_d. We write the composed mapping as $\tilde{f} = v_d \circ \tilde{h} = (\tilde{f}_0, ..., \tilde{f}_s)$.

We next choose a sequence of integers $\{m_{j,i}\}$ with large gaps such that $m_{j,i}^{(s+1)^2} < m_{j,i+1}$ for (j=1,...,n; i=1,...,m). We consider monomials $g_j = g_{j,1}(z_1) \cdots g_{j,m}(z_m)$, where $g_{j,i}(z_i) = z_i^{m_{j,i}}$ (j=1,...,n; i=1,...,m). Then we can prove Lemma 1 and Lemma 2. In the proof of Theorem 1, the key point is an auxiliary mapping F which is constructed by using the generalized Wronskian of $\tilde{f}_0, ..., \tilde{f}_s$. By using Proposition 1 and 2, we can choose complex numbers $\tilde{a}_1, ..., \tilde{a}_n$ in Lemma 2 such that F is nonconstant and $\Delta_F(H_{\mathbf{a}}) = 0$ for some suitable vector $\mathbf{a} \in \mathbf{C}^{s+1} \setminus \{0\}$ constructed by using $\tilde{a}_1, ..., \tilde{a}_n$. Another part of the proof is essentially similar to the method of [3]. Detail is omitted here.

§4. A space of meromorphic mappings.

We shall introduce a distance on the space $\mathcal{F}$ of meromorphic mappings into $\mathbf{P}^n(\mathbf{C})$. Let $f = (f_0 : ... : f_n)$ and $g = (g_0 : ... : g_n)$ be reduced representations of meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then we define the distance $d(f, g) := d_1(f, g) + d_2(f, g)$, where

$$d_1(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \int_n^{n+1} dt \int_{\partial B(t)} \inf_{\theta} \left\| \frac{f(z)}{\|f(z)\|} - e^{i\theta} \frac{g(z)}{\|g(z)\|} \right\| \sigma \leq 1,$$

which is a distance and it can not distinguish mappings which are rational or transcendental, and

$$d_2(f, g) := \liminf_{\alpha \to +1} \limsup_{r \to \infty} \left\{ \left| \frac{T_f(r)}{(\log r)^\alpha + T_f(r)} - \frac{T_g(r)}{(\log r)^\alpha + T_g(r)} \right| \right\},$$

which is a pseudodistance and it distinguishs mappings which are rational or transcendental.

In our case, a small deformation $\tilde{f}$ is represented as a form $\tilde{f} = (h_0, h_1 + a_1 g_1 h_0 :, ..., : h_n + a_n g_n h_0)$. Also, we can choose $(a_1, ..., a_n)$ such that $\|\mathbf{a}\| := |a_1| + \cdots + |a_n|$ is as small as possible. So, we can choose $\hat{f} := L_1^{-1} \cdot \tilde{f}$ which is also a small deformation without Nevanlinna defects such that $d(\hat{f}, f)$ is as small as possible. Hence we see meromorphic mappings without Nevanlinna defects are dense in the subset $\mathcal{F}_T \subset \mathcal{F}$ of transcendental meromorphic mappings on this distance.

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