Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 127–133

Recent development on Grauert domains

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§1. Introduction

The purpose of this article is to give a short survey on the recent development of a canonical complex structure, the so called *adapted complex structure*, on the tangent bundle of a real-analytic Riemannian manifold.

It was observed by Grauert [G] that a real-analytic manifold X could be embedded in a complex manifold as a maximal totally real submanifold. One way to see this is to complexify the transition functions defining X. However, this complexification is not unique. In [G-S] and [L-S], Guillemin-Stenzel and independently Lempert-Szöke encompass certain conditions on the ambient complex structure to make the complexification canonical for a given real-analytic Riemannian manifold. In short, they were looking for a complex structure, on part of the cotangent bundle T^*X, compatible with the canonical symplectic structure on T^*X. Equivalently, it is to say that there is a unique complex structure, the *adapted complex structure*, on part of the tangent bundle of X making the leaves of the Riemannian foliation on TX into holomorphic curves. The set of tangent vectors of length less than r equipped with the adapted complex structure is called a *Grauert tube* T^rX. For each X, there corresponds a $r_{max}(X) \geq 0$ which is the maximal real number such that the adapted complex structure is defined on T^rX for all $r \leq r_{max}(X)$. Though each Grauert tube over the same Riemannian manifold are diffeomorphic to each other, it was proved in [K1] and [Sz1] that T^rX and T^sX are biholomorphically nonequivalent when $r \neq s$. A domain D in which the adapted complex structure is defined and $X \subset D \subset TX$, is called a *Grauert domain.* The largest one of such Grauert domains is called the *maximal Grauert domain* in TX. In general, the maximal Grauert domain is strictly larger than $T^{r_{max}}X$. They are the same when X is a symmetric space of rank-one. The domain of definition depends on the geometry of X. Lempert and Szöke have the following estimate on the existence of domain of definition.

Received March 25, 2002

Theorem (Lempert-Szöke). *If the sectional curvatures of X are $\geq \lambda, \lambda < 0$ and the adapted complex structure exists on T^rX then $r < \frac{\pi}{2\sqrt{-\lambda}}$.*

§2. Rigidity of Grauert tubes

Since the adapted complex structure is constructed canonically associated to the Riemannian metric g of X, the differentials of the isometries of X are automorphisms of T^rX. Conversely, it is interesting to see whether all automorphisms of T^rX come from the differentials of the isometries of X or not. When the answer is affirmative, we say the Grauert tube is *rigid.*

With respect to the adapted complex structure, the length square function $\rho(x, v) = |v|^2, v \in T_xX$, is strictly plurisubharmonic. When the center X is compact, the Grauert tube T^rX is exhausted by ρ, hence is a Stein manifold with smooth strictly pseudoconvex boundary when the radius is less than the critical one. Applying the existence theorem of Cheng-Yau, there exists an invariant complete Kähler-Einstein metric g_{KE} of negative scalar curvature -1. Let ω_{KE}, which is a symplectic form on T^rX, denote the imaginary part of g_{KE}. Burns and Hind proved that (T^rX, ω_{KE}) is symplectomorphic to $(T^*X, d(pdq))$ via a symplectomorphism fixing X where pdq is the canonical Liouville 1-form on the cotangent bundle. Together with the fact that the automorphism group of T^rX is a compact Lie group, they (cf. [B], [B-H]) were able to prove the following rigidity result for Grauert tubes over compact real-analytic Riemannian manifolds.

Theorem (Burns-Hind). *Any Grauert tube of finite radius over a compact real-analytic Riemannian manifold is rigid.*

When X is non-compact nothing particular is known, not even to the general existence of a Grauert tube over X, i.e., the r_{max} could very well shrink to zero. When X is non-compact, most of the good properties in the compact cases were lacking since the length square function ρ is no longer an exhaustion. By now, the only two non-compact cases we are sure about the existence of Grauert tubes are those over co-compact real-analytic Riemannian manifolds, the Grauert tubes are simply the lifting of the Grauert tubes over their compact quotients, and Grauert tubes over real-analytic homogeneous Riemannian manifolds. In [K2], the author proved the following characterization on Grauert tubes.

Theorem (Kan 1). *If a Grauert tube T^rX is covered by the ball, then X is the real hyperbolic space.*

Using this and an extended version of Wong-Rosay theorem on the characterization of the unit ball, Kan and Ma (cf. [K-M 1,2] and [K3]) proved the rigidity for Grauert tubes over compact or non-compact locally symmetric spaces.

Later on, the author generalized the Wong-Rosay characterization to a general setting in any complex manifold and hence obtained:

Theorem (Kan 2). *Let $T^r X$ be a Grauert tube over homogeneous Riemannian manifold of $r < r_{max}$. Then $T^r X$ is either rigid or the ball.*

Here we need the condition $r < r_{max}$ since the proof heavily relies on the strictly pseudoconvexity of some good boundary points. We don't know whether it is possible to have more general rigidity other than this since the homogeneous spaces seem to be the best we could expect for Grauert tubes' construction to exist.

§3. Maximal Grauert domains

It is interesting to see whether the rigidity holds for $T^{r_{max}} X$ when X is not compact. As mentioned in the introduction, the maximal Grauert domain coincide with $T^{r_{max}} X$ when X is a symmetric space of rank-one. In [BHH], the authors considered the maximal Grauert domains over non-compact symmetric spaces. They showed that such maximal Grauert domains could be described algebraically which are correspondent to domains defined and studied by Akhiezer and Gindikin in [A-G]. They proved that

Theorem (Burns-Halverscheid-Hind 1).

(1) *The maximal Grauert domain over a non-compact symmetric space is either rigid or Hermitian symmetric.*

(2) *When X is a non-compact symmetric space of rank-one, $T^{r_{max}} X$ is never rigid.*

They also verified a conjecture of Akhiezer and Gindikin on the Steinness of such domains.

Theorem (Burns-Halverscheid-Hind 2). *The maximal Grauert domain over a non-compact symmetric space is Stein.*

By now, all examples we know are Stein. It is natural to ask whether all Grauert tubes or maximal Grauert domains are Stein. Recently Halverscheid and Iannuzzi [H-I] answer this question negatively. The example they consider is the 3-dimensional Heisenberg group. Their calculation works for generalized Heisenberg groups as well.

Theorem (Halverscheid-Iannuzzi). *The maximal Grauert domain over a generalized Heisenberg group is neither holomorphically separable nor holomorphically convex.*

§4. On the Kähler potential and CR invariants

Another characteristic feature of a Grauert tube over a compact Riemannian manifold is that it is exhausted by a non-negative strictly plurisubharmonic function whose square root satisfies the complex homogeneous Monge-Ampère equation away form the zero section. Emphasizing on this Monge-Ampère equation, some very nice results were obtained by Aguilar and by Stenzel.

In this section, we ask X to be compact. It is clear from the construction that a Grauert tube $T^rX, r < r_{max}$ is a Stein manifold with smooth strictly pseudoconvex boundary points. The existence of an invariant complete Kähler-Einstein metric of negative scalar curvature -1 was guaranteed. Since the construction of a Grauert tube is decided by the Monge-Ampère equation, it was expected that there might be a chance that this Kähler-Einstein metric is completely determined by the length square function ρ. R. Aguilar established a connection between potentials for Kähler-Einstein metrics in a neighborhood of X and the Riemannian density function of X. He proved that this occurs only when the density function of X depends solely on the geodesic distance function(such kind of manifold is called a *harmonic manifold*).

Theorem (Aguilar). *Suppose the Grauert tube T^rX admits a Kähler-Einstein metric with a Kähler potential that solely depends on ρ. Then X is a harmonic manifold.*

It is clear that the $(2n-1)$-dimensional strictly pseudoconvex boundary $\partial(T^rX)$ of the Grauert tube T^rX is a CR manifold when X is compact. The one-form $\theta = -Im\ \partial\rho$ has provided a pseudohermitian structure on it. There are two natural families of curves on $\partial(T^rX)$: the orbits of the geodesic flows coming from the Riemannian metric of X and chains, which are CR-invariants used to characterize CR manifolds.

In [St], Stenzel asked the question that whether the above two kinds of curves are related. He studied this pseudohermitian structure via the Fefferman metric and then related the pseudohermitian invariants of $\partial(T^rX)$ to the invariants of the ambient Kähler metric and eventually to the original metric of X.

Theorem (Stenzel).

(1) *Suppose there exists a $\delta > 0$ such that the orbits of the geodesic*

flows are chains on $\partial(T^r X)$ for all $r < \delta$. Then X is an Riemannian Einstein manifold.

(2) *If X is a harmonic manifold, then the orbits of the geodesic flows are chains on $\partial(T^r X)$, for all $r < r_{max}$.*

§5. Unbounded Grauert tubes

When $r = \infty$, i.e., when the whole tangent bundle TX is a Grauert tube of infinite radius, the situation is completely different from the cases of finite radii. In this case, we call TX an *unbounded Grauert tube.*

One trivial example is by taking $X = S^2$ with the standard metric. The adapted complex structure is defined on the whole tangent bundle, which is biholomorphic to the complex quadric $\mathcal{Q} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^2 + z_2^2 + z_3^2 = 1\}$. The unbounded Grauert tube TS^2 is clearly not rigid.

One interesting question is to ask whether unbounded Grauert tubes over compact Riemannian manifolds have algebraic embeddings in $\mathbb{C}^N$ similar to the above round sphere case. Verifying the existence of a pair of real-valued exhaustion functions with the growth properties related to Demailly's conjecture on the characterization of affine algebraic manifolds. Aguilar and Burns proved the following

Theorem (Aguilar-Burns 1).
Suppose $\Omega = TX$ is an unbounded Grauert tube over a compact manifold X. Then Ω is an affine algebraic manifold.

They also classify all possible unbounded Grauert tubes TX when X is of dimension 2.

Theorem (Aguilar-Burns 2).
Suppose Ω is an unbounded Grauert tube over a compact manifold X^2. Then Ω is biholomorphic to one of $\mathbb{C}^* \times \mathbb{C}^*$, $(\mathbb{C}^* \times \mathbb{C}^*)/Z_2$, $\mathcal{Q}$ or $\mathcal{Q}/Z_2$.

§6. Other applications

There are also some interesting applications to this adapted complex structure done by R. Szöke in [Sz2] and [Sz3]. In [Sz2], Szöke tried to link the adapted complex structure over compact rank-one symmetric spaces to a complex structure J_S defined on the punctured tangent bundle. The latter is preserved by the normalized geodesic flow which makes it possible to quantize the energy function over the symplectic manifold

$\overset{\circ}{T}X$. He showed that the limit of the push forward of the adapted complex structure under an appropriate family of diffeomorphism exists and agrees with J_S.

In [Sz3], Szöke extended the method to treat all compact symmetric spaces. He proved that after appropriate rescalings, the bundle of (1,0) tangent vectors with respect to the adapted complex structure on TX has a specific limit bundle.

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