

## A link between the asymptotic expansions of the Bergman kernel and the Szegö kernel

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### Introduction

Let  $\Omega$  be a strictly pseudoconvex domain in  $\mathbb{C}^n$ . Then the Bergman kernel  $K^B$  and the Szegö kernel  $K^S$  of  $\Omega$  have singularities at the boundary diagonal. These singularities admit asymptotic expansions in powers and log of the defining function of  $\Omega$  ([3], [2]) and, moreover, the coefficients of which can be expressed in terms of local invariants of the CR structure of the boundary  $\partial\Omega$  as an application of the parabolic invariant theory developed in [4], [5], [1], [8], [6] and others. While these works provide a geometric algorithm of expressing the expansion of each kernel, it is not easy to read relations between them from this construction — for example, we can say very little about the relation between the log term coefficients of  $K^B$  and  $K^S$ , cf. §2.

In this note we present a method of relating these asymptotic expansions. Our strategy is to construct a meromorphic family of kernel functions  $K_s$ ,  $s \in \mathbb{C}$ , such that  $K^B$  and  $K^S$  are realized as special values of  $K_s$ . In the case of the unit ball,  $\{|z| < 1\}$ , such a family is given by

$$K_s(z) = \pi^{-n} \Gamma(n-s) (1 - |z|^2)^{s-n},$$

where  $\Gamma(\alpha)$  is the gamma function, and  $K_{-1}$ ,  $K_0$  give  $K^B$ ,  $K^S$ , respectively. Note that, for  $s < 0$ ,  $K_s$  is characterized as the Bergman kernel for the weighted  $L^2$  norm defined by the measure  $(1 - |z|^2)^{-s-1} / \Gamma(-s) dV$ , see §1. For general strictly pseudoconvex domains, we begin by defining  $K_s$  for  $s < 0$  as the weighted Bergman kernel, and then extend to  $s \in \mathbb{C}$  by analytic continuation. Here we only consider the asymptotic expansion of  $K_s$  and define the analytic continuation as a meromorphic family of formal series, see §2. We then apply the invariant theory to express  $K_s$  in terms of geometric invariants of the boundary (Theorem 2). In these expansions, all  $K_s$  contain the same invariants up to universal

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constants depending polynomially on  $s$ . These formulae, in particular, give a relation between  $K^B = K_{-1}$  and  $K^S = K_0$ .

Note that the kernel functions  $K_s$  for  $s \in \mathbb{Z}$  have been introduced in Hirachi–Komatsu [7] and the present note is a continuation of that work. In [7],  $K_s$  are defined as the solutions of simple holonomic systems, which naturally arise from Kashiwara’s microlocal analysis of the Bergman kernel [9]. While this point of view is not given explicitly in this note, this is also the main tool of the proofs of Theorems 1 and 2; the details will be given in my forthcoming paper.

### §1. Weighted Bergman kernels

Let  $\Omega \subset \mathbb{C}^n$  be a domain with  $C^\infty$  smooth boundary. Then there is a function  $r \in C^\infty(\overline{\Omega})$ , called a *defining function*, such that  $\Omega = \{r > 0\}$  and  $dr \neq 0$  on  $\partial\Omega$ . Fixing such an  $r$ , we define for  $s < 0$  a weighted  $L^2$  norm on  $\Omega$  by

$$(1) \quad \|f\|_s^2 = \int_{\Omega} |f(z)|^2 \frac{r(z)^{-s-1}}{\Gamma(-s)} dV(z),$$

where  $dV(z)$  is the standard Lebesgue measure on  $\mathbb{C}^n$ . Let

$$H_s(\Omega, r) := \{f \in \mathcal{O}(\Omega) : \|f\|_s < \infty\},$$

the Hilbert space of weighted  $L^2$  holomorphic functions on  $\Omega$ . If we take a complete orthonormal system  $\{h_j\}_{j=0}^\infty$  of  $H_s(\Omega, r)$ , then the series

$$K_s[r](z, \bar{w}) := \sum_j h_j(z) \overline{h_j(w)}$$

converges for  $(z, w) \in \Omega \times \Omega$  and define a function, which is shown to be independent of the choice of  $\{h_j\}$ . We call  $K_s[r]$  the *weighted Bergman kernel*. Note that the Bergman kernel  $K^B$  is given by  $K_{-1}[r]$ , which is clearly independent of the choice of  $r$ .

In case  $s = 0$ , the right-hand side of (1) does not make sense because  $\Gamma(-s)$  has simple poles at  $s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ . However, we may define  $\|\cdot\|_0$  by taking the limit

$$\lim_{s \rightarrow -0} \|f\|_s^2 = \int_{\partial\Omega} |f|^2 d\sigma(z), \quad f \in C^0(\overline{\Omega}),$$

where  $d\sigma$  is the volume element on  $\partial\Omega$  normalized by the condition

$$d\sigma \wedge dr = dV \quad \text{on } \partial\Omega.$$

Thus it is natural to define  $H_0(\Omega, r) := \ker \bar{\partial}_b \subset L^2(\partial\Omega, d\sigma)$ , where  $\bar{\partial}_b$  is the tangential Cauchy–Riemann operator of  $\partial\Omega$ . Since each  $f \in H_0(\Omega, r)$  admits an extension to  $f \in \mathcal{O}(\Omega)$ , we may also regard  $H_0(\Omega, r) \subset \mathcal{O}(\Omega)$ . The Szegö kernel is then defined by  $K^S[r](z, \bar{w}) := \sum_j h_j(z) \overline{h_j(w)}$ , where  $\{h_j\}_j$  is a complete orthonormal system of  $H_0(\Omega, r)$ .

**Model case.** In the case of the unit ball  $\Omega_0$ , we may take  $r(z) = 1 - |z|^2$ . Then the monomials of  $z$  form a complete orthogonal system of  $H_s(\Omega_0, r)$  (cf. [7]) and thus

$$K_s[r](z, \bar{w}) = \sum_{\alpha} \frac{z^{\alpha} \bar{w}^{\alpha}}{\|z^{\alpha}\|_s^2} = \frac{\Gamma(n-s)}{\pi^n} (1 - z \cdot \bar{w})^{s-n}.$$

The right-hand side is a meromorphic function of  $s \in \mathbb{C}$  (with parameters  $z, w \in \Omega$ ) and, thus  $K_s[r]$  ( $s < 0$ ) can be analytically continued to a meromorphic function of  $s \in \mathbb{C}$ , which we also denote by  $K_s[r]$ . Then, in particular,  $K_0[r]$  gives the Szegö kernel  $K^S[r]$ .

## §2. Asymptotic expansions of the weighted Bergman kernels

In what follows, we assume that  $\Omega$  is strictly pseudoconvex, and mainly consider the restriction to the diagonal of the kernel functions  $K_s[r](z) := K_s[r](z, \bar{z})$ .

It is known from the work of Fefferman [3] that the boundary singularity of the Bergman kernel  $K^B(z)$  takes the form  $\varphi r^{-n-1} + \psi \log r$ , where  $\varphi, \psi \in C^{\infty}(\bar{\Omega})$ . Based on his analysis, G. Komatsu has shown that the weighted Bergman kernels  $K_s[r]$  admit similar expansions.

**Theorem ([10]).** *For  $s < 0$ , the weighted Bergman kernel  $K_s[r]$  admits the following asymptotic expansion at the boundary:*

$$(2) \quad K_s[r] = \begin{cases} \varphi^{(s)}[r] r^{s-n} + \psi^{(s)}[r] \log r & \text{if } s \in \mathbb{Z}, \\ \varphi^{(s)}[r] r^{s-n} & \text{if } s \notin \mathbb{Z}, \end{cases}$$

where  $\varphi^{(s)}[r], \psi^{(s)}[r] \in C^{\infty}(\bar{\Omega})$ .

If we introduce the functions

$$\Phi_s[r] = \begin{cases} \Gamma(-s) r^s & \text{if } s \in \mathbb{C} \setminus \mathbb{N}_0, \\ \frac{(-1)^{s+1}}{s!} r^s \log r & \text{if } s \in \mathbb{N}_0, \end{cases}$$

then we may rewrite the expansions (2) in a unified form:

$$(3) \quad K_s[r](z) = \sum_{j=0}^{\infty} \varphi_j^{(s)}[r](z) \Phi_{s-n+j}[r](z), \quad \varphi_j^{(s)}[r] \in C^{\infty}(\bar{\Omega}).$$

Here the coefficients  $\varphi_j^{(s)}[r]$  are not uniquely determined because  $r$  and  $z$  are not independent.

Our basic result that enables us to define the meromorphic family  $K_s[r]$ ,  $s \in \mathbb{C}$ , is the following

**Theorem 1.** *The coefficients  $\varphi_j^{(s)}[r]$  of (3) can be chosen so that  $\varphi_j^{(s)}[r] = \sum_{k=0}^{2j} a_{j,k}[r] s^k$  holds for functions  $a_{j,k}[r] \in C^\infty(\bar{\Omega})$  that are independent of  $s$ .*

Taking  $\varphi_j^{(s)}[r]$  as in the theorem above and then using the relation  $s \Phi_{s+j}[r] = -r \Phi_{s+j-1}[r] - j \Phi_{s+j}[r]$ , we may rewrite (3) in the form

$$(4) \quad K_s[r] = \sum_{j=-\infty}^{\infty} a_j[r] \Phi_{s-n+j}[r],$$

where  $a_j[r] \in C^\infty(\bar{\Omega})$  are independent of  $s$  and satisfies  $a_j[r] = O(r^{-2j})$  for  $j < 0$  (hence the boundary singularity of  $a_j[r] \Phi_{s-n+j}[r]$  gets weaker as  $|j| \rightarrow \infty$ ). Note that  $a_j[r]$  modulo  $O(r^\infty)$  is now uniquely determined by  $r$ , and moreover it is shown that map  $r \mapsto a_j[r]$  is given by a partial differential operator.

Now we define  $K_s[r]$  for  $s \in \mathbb{C} \setminus (-\infty, 0)$  by the formula (4), which is regarded as formal series. Then we can show, in particular, that  $K_0[r]$  gives the asymptotic expansion of the Szegő kernel  $K^S[r]$ .

### §3. Transformation law and an invariant expansion of $K_s[r]$

We next examine the transformation law of  $a_j[r]$  under biholomorphic maps  $F: \tilde{\Omega} \rightarrow \Omega$ . Recall [3] that  $F$  can be extended to a diffeomorphism up to the boundary. So, for a defining function  $r$  of  $\Omega$ , we may give a defining function of  $\tilde{\Omega}$  by

$$(5) \quad \tilde{r} := |\det F'|^{-2/(n+1)} r \circ F,$$

where  $\det F'$  is the holomorphic Jacobian of  $F$ . Now from the definition of the norm  $\|\cdot\|_s$ , we see that the weighted Bergman kernel transforms according to

$$(6) \quad K_s[\tilde{r}] = |\det F'|^{2(n-s)/(n+1)} K_s[r] \circ F.$$

Thus, substituting these transformation laws into (4), we get

$$(7) \quad a_j[\tilde{r}] = |\det F'|^{2j/(n+1)} a_j[r] \circ F$$

by the uniqueness of the expansion (4).

Our next task is to construct functionals of  $r$  that transform like this under biholomorphic maps — and hopefully express  $a_j[r]$  in terms of these functionals. Here we utilize the ambient metric construction of [4]. Associated to each  $r$ , we first define a Lorentz-Kähler metric  $g = g[r]$  on a neighborhood of  $\mathbb{C}^* \times \partial\Omega \subset \mathbb{C}^* \times \mathbb{C}^n$  by  $g[r] = \sum_{j,k=0}^n g_{j\bar{k}} dz_j d\bar{z}_k$ , where  $g_{j\bar{k}} = \partial^2 r_{\#} / \partial z_j \partial \bar{z}_k$ . Let  $R = R[r]$  be the curvature of  $g$  and  $R^{(p,q)} = \bar{\nabla}^{q-2} \nabla^{p-2} R$  be its iterated covariant derivatives. Then consider complete contractions of the form

$$W_{\#} = \text{contr} \left( R^{(p_1, q_1)} \otimes \dots \otimes R^{(p_m, q_m)} \right),$$

with  $\sum p_l = \sum q_l = m + w$ . Such a contraction  $W_{\#}$  assigns to each  $r$  a smooth function  $W[r] := W_{\#}[r]|_{z_0=0}$  on  $\bar{\Omega}$  near  $\partial\Omega$ . We call the functional  $r \mapsto W[r]$  a *Weyl functional of weight  $w$* . If  $W$  has weight  $w$ , then under (5), we have the desired transformation law

$$W[\tilde{r}] = |\det F'|^{2w/(n+1)} W[r] \circ F.$$

It is a natural hope that all  $a_j$  can be expressed in terms of these Weyl functionals. However, at this stage, it is hard to deal with the case of arbitrary  $r$ . So we here choose a good class of defining functions in such a way that we can apply the invariant theory of [4], [1], [6]. To specify a class of defining functions, following [6], we consider the following complex Monge-Ampère equation

$$(-1)^n \det \left( \partial^2 U / \partial z^j \partial \bar{z}^k \right)_{0 \leq j, k \leq n} = |z_0|^{2n}$$

for a function  $U(z_0, z)$  on  $\mathbb{C}^* \times \bar{\Omega}$ . This equation admits asymptotic solutions along  $\mathbb{C}^* \times \partial\Omega$  of the form

$$U = r_{\#} + r_{\#} \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_{\#})^k,$$

where  $r$  is a  $C^\infty$  defining function of  $\Omega$ ,  $r_{\#}(z_0, z) = |z_0|^2 r(z)$  and  $\eta_k \in C^\infty(\bar{\Omega})$ . For such a solution  $U$ , the smooth part  $r_{\#} = |z_0|^2 r$  is uniquely determined. So, for each  $\Omega$ , we may define  $\mathcal{F}_\Omega$  to be the totality of  $r$  that arises as the smooth part of an asymptotic solution  $U$ . This class  $\mathcal{F}_\Omega$  is shown to be preserved under the pull-back (5).

Now we use Weyl functionals to express  $K_s[r]$  for  $r \in \mathcal{F}_\Omega$ . The invariant theory of [6] implies that each  $a_j[r]$  admits an asymptotic expansion

$$(8) \quad a_j[r] = \sum_{k=0}^{\infty} W_{j,k}[r] r^k, \quad r \in \mathcal{F}_\Omega,$$

where  $W_{j,k}$  is a linear combination of Weyl functionals of weight  $j+k$ . Hence, using  $r\Phi_{s-m}[r] = (m-s)\Phi_{s-m+1}[r]$  to absorb all explicit  $r$  in (8) into other  $\Phi_{s-l}[r]$ , we get

**Theorem 2.** *If  $r \in \mathcal{F}_\Omega$ , then  $K_s[r]$  admits an expansion*

$$(9) \quad K_s[r] = \sum_{j=0}^{\infty} W_j^{(s)}[r] \Phi_{s-n+j}[r],$$

where each  $W_j^{(s)}$  is a linear combination of Weyl functionals of weight  $j$  whose coefficients are polynomials in  $s$  of degree  $\leq 2j$ .

The first three terms of the expansion are given by

$$\pi^n K_s[r] = \Phi_{s-n}[r] + \frac{1}{24} \|R\|_{z_0=1}^2 \Phi_{s-n+2}[r] + O(r^{s-n-3}).$$

Here the second term  $W_{s-n+1}^{(s)}$  vanishes. Thus we see in particular that the Bergman and the Szegő kernels have the same expansion in  $\Phi_s[r]$  up to this order.

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