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Integral Representation of Linear Functionals on Vector Lattices and its Application to BV Functions on Wiener Space

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Abstract.

We consider vector lattices $\mathbb{D}$ generalizing quasi-regular Dirichlet spaces and give a characterization for bounded linear functionals on $\mathbb{D}$ to have a representation by an integral with respect to smooth measures. Applications to BV functions on Wiener space are also discussed.

§1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the Banach space of all continuous functions on X with supremum norm. The Riesz representation theorem says that every bounded linear operator on $C(X)$ is realized by an integral with respect to a certain finite signed measure on X. As a variant of this fact, Fukushima [7] proved that, for any quasi-regular Dirichlet form $(\mathcal{E},\mathcal{F})$ and for $u\in\mathcal{F}$, $\mathcal{E}(u,\cdot)$ is represented as an integral by a smooth signed measure if and only if

$$|\mathcal{E}(u,v)|\le C_k\|v\|_{L^\infty}\qquad\text{for all } v\in\mathcal{F}_{b,F_k},\ k\in\mathbb{N}$$

holds for some nest $\{F_k\}_{k\in\mathbb{N}}$ and some constants C_k, $k\in\mathbb{N}$. As its applications, he gave a characterization for additive functionals of function type for $(\mathcal{E},\mathcal{F})$ to be semimartingales, and also proved the smoothness of the measures associated with BV functions ([7, 8, 9]).

In this paper, we show a corresponding result in the framework of vector lattices generalizing quasi-regular Dirichlet spaces. The proof is similar to that in [7] but based on a purely analytical argument, unlike [7] where probabilistic methods are used together. Typical examples

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which lie in this framework are first order Sobolev spaces derived from a gradient operator and fractional order Sobolev spaces by a real interpolation method with differentiability index between 0 and 1. Using such results, we can improve the smoothness of the measure associated with BV functions on Wiener space, discussed in [8, 9].

We can find several studies closely related to this article regarding the Riesz representation theorem, e.g., in [22, 14]. Their frameworks are based on Markovian semigroups and the function spaces are derived from their generators, which seems to be suitable for complex interpolation spaces. Ours is based on the lattice property instead and fits for real interpolation spaces.

The organization of this paper is as follows. In Section 2, we give a general framework and preparatory lemmas, which are slight modifications of what have been developed already in the case of Dirichlet spaces or in the framework of the nonlinear potential theory. We also give some examples there. In Section 3, the representation theorems are proved. In Section 4, we discuss some applications to BV functions on Wiener space.

§2. Framework and main results

Let X be a topological space and λ a Borel measure on X. Let $L^0(X)$ be the space of all λ-equivalence classes of real-valued Borel measurable functions on X. We will adopt a standard notation to describe function spaces and their norms, such as $L^p(X)$ (or simply L^p) and $\|\cdot\|_{L^p}$.

We suppose that a subspace $\mathbb{D}$ of $L^0(X)$ equipped with norm $\|\cdot\|_{\mathbb{D}}$ satisfies the following.

(A1) $(\mathbb{D}, \|\cdot\|_{\mathbb{D}})$ is a separable and uniformly convex Banach space.

(A2) (Consistency condition) If a sequence in $\mathbb{D}$ converges to 0 in $\mathbb{D}$, then its certain subsequence converges to 0 λ-a.e.

Since $\mathbb{D}$ is assumed to be uniformly convex, it is reflexive and the Banach-Saks property holds: every bounded sequence in $\mathbb{D}$ has a subsequence whose arithmetic means converge strongly in $\mathbb{D}$ (see [16, 19, 13] for the proof). The following lemma is proved by a standard argument.

Lemma 2.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence bounded in $\mathbb{D}$. Then, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that f_{n_k} converges to some f weakly in $\mathbb{D}$ and the arithmetic means $(1/k)\sum_{j=1}^k f_{n_j}$ converge to f strongly in $\mathbb{D}$. Moreover, $\|f\|_{\mathbb{D}} \le \liminf_{k\to\infty} \|f_{n_k}\|_{\mathbb{D}}$. If furthermore f_n converges to some g λ-a.e., then it holds that $g \in \mathbb{D}$ and f_n weakly converges to g in $\mathbb{D}$.*

Proof. By virtue of the Banach-Alaoglu theorem and the reflexivity of $\mathbb{D}$, $\{f_n\}_{n\in\mathbb{N}}$ is weakly relatively compact in $\mathbb{D}$. Using the Banach-Saks property together, we can prove the first claim. The second one follows from the Hahn-Banach theorem. The last one is a consequence of the consistency condition (A2). Q.E.D.

We further assume the following.

(A3) (Normal contraction property) For every $f \in \mathbb{D}$, $\check{f} := 0 \vee f \wedge 1$ belongs to $\mathbb{D}$ and $\|\check{f}\|_{\mathbb{D}} \leq \|f\|_{\mathbb{D}}$.

(A4) For every f in $\mathbb{D}_b := \mathbb{D} \cap L^\infty$, f^2 belongs to $\mathbb{D}$. Moreover, $\sup\{\|f^2\|_{\mathbb{D}} \mid \|f\|_{\mathbb{D}} + \|f\|_{L^\infty} \leq 1\}$ is finite.

The condition (A4) is equivalent to the following:

(A4)' for every f and g in $\mathbb{D}_b$, fg belongs to $\mathbb{D}$. Moreover, $\sup\{\|fg\|_{\mathbb{D}} \mid \|f\|_{\mathbb{D}} + \|f\|_{L^\infty} \leq 1,\ \|g\|_{\mathbb{D}} + \|g\|_{L^\infty} \leq 1\}$ is finite.

Indeed, it is clear that (A4)' implies (A4). To show the converse implication, use the identity $fg = ((f+g)/2)^2 - ((f-g)/2)^2$ and the subadditivity of the norm $\|\cdot\|_{\mathbb{D}}$.

We introduce a sufficient condition for (A3) and (A4).

Lemma 2.2. *Under* (A1) *and* (A2), *the following condition* (C) *implies* (A3) *and* (A4):

(C) *when χ is a bounded and infinitely differentiable function on $\mathbb{R}$ with $\chi(0) = 0$ and $\|\chi'\|_\infty \leq c$ for some $c > 0$, then $\chi \circ v \in \mathbb{D}$ and $\|\chi \circ v\|_{\mathbb{D}} \leq c\|v\|_{\mathbb{D}}$ for every $v \in \mathbb{D}$.*

Proof. To show (A3), apply Lemma 2.1 with a sequence $\{\chi_n \circ v\}_{n\in\mathbb{N}}$ so that $\|\chi_n\|_\infty \leq 1$ and $\|\chi_n'\|_\infty \leq 1$ for every n, and χ_n converges pointwise to $\chi(x) = 0 \vee x \wedge 1$. (A4) is similarly proved. Q.E.D.

For $f \in L^0(X)$, we set $f_+(z) = f(z) \vee 0$ and $f_-(z) = -(f(z) \wedge 0)$.

Lemma 2.3. *Let $f \in \mathbb{D}$. Then $f_+ \in \mathbb{D}$ and $\|f_+\|_{\mathbb{D}} \leq \|f\|_{\mathbb{D}}$.*

Proof. Define $f_n := 0 \vee f \wedge n = n(0 \vee (f/n) \wedge 1)$, $n \in \mathbb{N}$. Then $\|f_n\|_{\mathbb{D}} \leq n\|f/n\|_{\mathbb{D}} = \|f\|_{\mathbb{D}}$ by (A3) and $f_n \to f_+$ pointwise. Lemma 2.1 finishes the proof. Q.E.D.

Lemma 2.4. *For every $f \in \mathbb{D}$, $(-a) \vee f \wedge a \to 0$ weakly in $\mathbb{D}$ as $a \downarrow 0$ and $(-a) \vee f \wedge a \to f$ weakly in $\mathbb{D}$ as $a \to \infty$.*

Proof. It is enough to notice that

$$\|(-a) \vee f \wedge a\|_{\mathbb{D}} = \|0 \vee f \wedge a - 0 \vee (-f) \wedge a\|_{\mathbb{D}} \leq 2\|f\|_{\mathbb{D}},$$

$$(-a) \vee f \wedge a \to 0 \text{ pointwise as } a \downarrow 0,$$

$$(-a) \vee f \wedge a \to f \text{ pointwise as } a \to \infty$$

and to use Lemma 2.1. Q.E.D.

For a measurable set A, we let $\mathbb{D}_A := \{f \in \mathbb{D} \mid f = 0\ \lambda\text{-a.e. on } X\backslash A\}$ and $\mathbb{D}_{b,A} := \mathbb{D}_A \cap L^\infty$. A sequence $\{F_k\}_{k\in\mathbb{N}}$ of increasing sets in X is called a *nest* if each F_k is closed and $\bigcup_{k=1}^\infty \mathbb{D}_{F_k}$ is dense in $\mathbb{D}$. A nest $\{F_k\}_{k\in\mathbb{N}}$ is called (λ-)*regular* if, for all k, any open set O with $\lambda(O\cap F_k) = 0$ satisfies $O \subset X \setminus F_k$. A subset N of X is called *exceptional* if there is a nest $\{F_k\}_{k\in\mathbb{N}}$ such that $N \subset \bigcap_{k=1}^\infty (X\setminus F_k)$. When A is a subset of X, we say that a statement depending on $z \in A$ holds *quasi everywhere* (q.e. in abbreviation) if it does for every $z \in A \setminus N$ for a certain exceptional set N. For a nest $\{F_k\}_{k\in\mathbb{N}}$, we denote by $C(\{F_k\})$ the set of all functions f on X such that f is continuous on each F_k. A function f on X is said to be *quasi-continuous* if there is a nest $\{F_k\}_{k\in\mathbb{N}}$ such that $f \in C(\{F_k\})$. We say that a Borel measure μ on X is *smooth* if it does not charge any exceptional Borel sets and there exists a nest $\{F_k\}_{k\in\mathbb{N}}$ such that $\mu(F_k) < \infty$ for all k. A set function ν on X which is given by $\nu = \nu_1 - \nu_2$ for some smooth measures ν_1 and ν_2 with finite total mass is called a *finite signed smooth measure.* A *signed smooth measure* ν with attached nest $\{F_k\}_{k\in\mathbb{N}}$ is a map from $\mathcal{R} := \{A \subset X \mid A \text{ is a Borel set of some } F_k\}$ to $\mathbb{R}$ such that ν is represented as $\nu(A) = \nu_1(A) - \nu_2(A)$, $A \in \mathcal{R}$, for some smooth Borel measure ν_1 and ν_2 satisfying $\nu_i(F_k) < \infty$ for each $i = 1, 2$ and $k \in \mathbb{N}$. When we want to emphasize the dependency of $\mathbb{D}$, we write $\mathbb{D}$-nest, $\mathbb{D}$-smooth, and so on.

We further assume the following quasi-regularity conditions.

(QR1) There exists a nest consisting of compact sets.

(QR2) There exists a dense subset of $\mathbb{D}$ whose elements have quasi-continuous λ-modifications.

(QR3) There exists a countable subset $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ and an exceptional set N such that each φ_n has a quasi-continuous λ-modification $\tilde{\varphi}_n$ and $\{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$ separates the points of $X \setminus N$.

Every quasi-regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies all conditions (A1)–(A4) and (QR1)–(QR3) (and (C)) when letting $\mathbb{D} = \mathcal{F}$ and $\|f\|_{\mathbb{D}} = (\mathcal{E}(f,f) + \|f\|_{L^2}^2)^{1/2}$. We give other examples in the last part of this section.

Lemma 2.5. *There exist some* $\rho \in \mathbb{D}$ *and some countable subset* $\mathcal{C} = \{h_n\}_{n\in\mathbb{N}}$ *of* $\mathbb{D}_b$ *such that* $0 \le \rho \le 1$ λ-a.e., $h_n \ge 0$ λ-a.e. *for all* n, $\mathcal{C} - \mathcal{C} := \{h - \hat{h} \mid h, \hat{h} \in \mathcal{C}\}$ *is dense in* $\mathbb{D}$, *and for each* $n \in \mathbb{N}$, $\rho \ge c_n h_n$ λ-a.e. *for some* $c_n \in (0, \infty)$.

Proof. Let $\{f_m\}_{m\in\mathbb{N}}$ be a countable dense subset of $\mathbb{D}$. Denote by $\mathcal{C} = \{h_n\}_{n\in\mathbb{N}}$ the set of all arithmetic means of finite number of functions in $\{(f_m)_+ \wedge M,\ (f_m)_- \wedge M\}_{m\in\mathbb{N},\, M\in\mathbb{N}}$. Then $\mathcal{C} - \mathcal{C}$ is dense in $\mathbb{D}$ by

Lemma 2.4 and the Banach-Saks property. Define

$$\rho = \sum_{n=1}^{\infty} \frac{h_n}{2^n(\|h_n\|_{\mathbb{D}} + \|h_n\|_{L^\infty} + 1)}.$$

Then ρ and $\mathcal{C}$ satisfy the conditions in the claim. Q.E.D.

We will fix ρ and $\mathcal{C}$ satisfying the statement in the lemma above. Note that we can always take $\rho \equiv 1$ if $1 \in \mathbb{D}$.

Take a strictly increasing and right-continuous function $\xi : [0, \infty) \to [0, \infty)$ with $\xi(0) = 0$. For an open set $O \subset X$, we define

(1) $$\text{cap}_\xi(O) = \inf\{\xi(\|f\|_{\mathbb{D}}) \mid f \in \mathbb{D} \text{ and } f \geq \rho \ \lambda\text{-a.e. on } O\}.$$

For any subset A of X, we define the capacity of A by

$$\text{cap}_\xi(A) = \inf\{\text{cap}_\xi(O) \mid O \supset A,\ O : \text{open}\}.$$

It should be noted that $\text{cap}_\xi(A) \leq \xi(\|\rho\|_{\mathbb{D}}) < \infty$ for every $A \subset X$.

The following lemma is proved in the same way as in [10].

Lemma 2.6. *For every open set O, there exists a unique function e_O in $\mathbb{D}$ attaining the infimum in* (1). *Moreover,* $0 \leq e_O \leq 1$ λ-a.e.

Proof. The uniqueness follows from the uniform convexity of $\mathbb{D}$. The existence is deduced by the Banach-Saks property and (A2). The last claim is a consequence of (A3). Q.E.D.

We will discuss some basic properties of the capacity.

Lemma 2.7. *Let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of open sets such that $\text{cap}_\xi(O_n) \to 0$. Then, there exists a sequence $\{n_k\} \uparrow \infty$ such that $e_{O_{n_k}} \to 0$* λ-a.e.

Proof. Since $\|e_{O_n}\|_{\mathbb{D}} \to 0$, the claim is clear from (A2). Q.E.D.

Lemma 2.8. *If* $\text{cap}_\xi(A) = 0$, *then $A \cap \{\rho > 0\}$ is a λ-null set.*

Proof. Take a decreasing open sets $\{O_n\}_{n\in\mathbb{N}}$ such that $A \subset O_n$ and $\text{cap}_\xi(O_n) \to 0$. Since $e_{O_n} \geq \rho$ λ-a.e. on $\bigcap_{k=1}^\infty O_k$, we have $\rho = 0$ λ-a.e. on $\bigcap_{k=1}^\infty O_k$ by virtue of Lemma 2.7. This implies the assertion. Q.E.D.

Lemma 2.9. *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of increasing closed sets. Then the following are equivalent.*

(i) *$\{A_k\}_{k\in\mathbb{N}}$ is a nest.*

(ii) $\lim_{k\to\infty} \text{cap}_\xi(X \setminus A_k) = 0$.

Proof. Suppose (i) holds. Take a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathbb{D}$ such that $f_k \in \mathbb{D}_{A_k}$ and $f_k \to \rho$ in $\mathbb{D}$. Since $\rho - f_k = \rho$ λ-a.e. on $X \setminus A_k$, we have $\mathrm{cap}_\xi(X \setminus A_k) \le \xi(\|\rho - f_k\|_{\mathbb{D}}) \to 0$ as $k \to \infty$.

Next, suppose (ii) holds. It suffices to prove that each $h \in \mathcal{C}$ can be approximated in $\mathbb{D}$ by functions in $\bigcup_k \mathbb{D}_{A_k}$. Since $\rho \ge ch$ λ-a.e. for some $c > 0$, it holds that $e_{X\setminus A_k} \ge ch$ λ-a.e. on $X \setminus A_k$ for each k. Let $f_k = (h - c^{-1}e_{X\setminus A_k})_+ \in \mathbb{D}_{A_k}$. By Lemma 2.7, there exists a sequence $\{k'\}$ diverging to infinity such that $e_{X\setminus A_{k'}} \to 0$ λ-a.e. Therefore, $f_{k'} \to h$ λ-a.e. as $k' \to \infty$. On the other hand, $\|f_k\|_{\mathbb{D}} \le \|h\|_{\mathbb{D}} + c^{-1}\|e_{X\setminus A_k}\|_{\mathbb{D}}$, which is bounded in k. From Lemma 2.1, we can take arithmetic means of some subsequence of $\{f_{k'}\}$, which belong to $\bigcup_k \mathbb{D}_{A_k}$, so that they converge to h in $\mathbb{D}$. Q.E.D.

As is seen from this lemma, any choices of $\mathcal{C}$, ρ and ξ are consistent with the notion of nest. From now on, we treat only the case $\xi(t) = t$ and write cap in place of cap_ξ.

Lemma 2.10. *For any sequence of subsets* $\{A_k\}_{k\in\mathbb{N}}$ *in* X, *it follows that* $\mathrm{cap}\big(\bigcup_{k=1}^\infty A_k\big) \le \sum_{k=1}^\infty \mathrm{cap}(A_k)$.

Proof. When $O_1, \dots, O_k$ are open sets, it is easy to see the inequality $\mathrm{cap}\big(\bigcup_{j=1}^k O_j\big) \le \sum_{j=1}^k \mathrm{cap}(O_j)$. Indeed, since $\sum_{j=1}^k e_{O_j} \ge \rho$ λ-a.e. on $\bigcup_{j=1}^k O_j$, we have

$$\mathrm{cap}\left(\bigcup_{j=1}^k O_j\right) \le \left\|\sum_{j=1}^k e_{O_j}\right\|_{\mathbb{D}} \le \sum_{j=1}^k \|e_{O_j}\|_{\mathbb{D}} = \sum_{j=1}^k \mathrm{cap}(O_j).$$

Now, let $\varepsilon > 0$. Take an open set O_k for each $k \in \mathbb{N}$ such that $O_k \supset A_k$ and $\mathrm{cap}(O_k) < \mathrm{cap}(A_k) + \varepsilon 2^{-k}$. Let $U_k = \bigcup_{j=1}^k O_j$. Since $\|e_{U_k}\|_{\mathbb{D}} \le \|\rho\|_{\mathbb{D}} < \infty$, Lemma 2.1 assures the existence of a subsequence $\{e_{U_{k'}}\}$ of $\{e_{U_k}\}$ and $e \in \mathbb{D}$ such that $e_{U_{k'}}$ converges to e weakly in $\mathbb{D}$ and the arithmetic means of $\{e_{U_{k'}}\}$ converge to e in $\mathbb{D}$. Since $e \ge \rho$ λ-a.e. on $\bigcup_{k=1}^\infty O_k$ by using (A2), we have

$$\begin{aligned}\mathrm{cap}\left(\bigcup_{k=1}^\infty A_k\right) &\le \mathrm{cap}\left(\bigcup_{k=1}^\infty O_k\right) \le \|e\|_{\mathbb{D}} \le \liminf_{k'\to\infty} \|e_{U_{k'}}\|_{\mathbb{D}} \\ &= \lim_{k\to\infty} \mathrm{cap}(U_k) \le \lim_{k\to\infty} \sum_{j=1}^k \mathrm{cap}(O_j) \le \varepsilon + \sum_{k=1}^\infty \mathrm{cap}(A_k).\end{aligned}$$

Since ε is arbitrary, we obtain the claim. Q.E.D.

The following series of lemmas are now proved in a standard way as in the case of quasi-regular Dirichlet spaces; see e.g. [11, 15] for the proof.

Lemma 2.11. *Suppose that* $f \in \mathbb{D}$ *has a quasi-continuous* λ*-modification* $\tilde{f}$. *Then, we have* $\operatorname{cap}(\{\tilde{f} > \lambda\}) \le \lambda^{-1}\|f\|_{\mathbb{D}}$ *for each* $\lambda > 0$.

Lemma 2.12. (i) *When* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence of quasi-continuous functions, there exists a nest* $\{F_k\}_{k\in\mathbb{N}}$ *such that* $f_n \in C(\{F_k\})$ *for every* n.

(ii) *If* $f_n \in \mathbb{D}$ *has a quasi-continuous* λ*-modification* $\tilde{f}_n$ *and converges to* f *in* $\mathbb{D}$ *as* $n \to \infty$, *then* f *has a quasi-continuous* λ*-modification* $\tilde{f}$ *and there exists a sequence* $\{n_l\} \uparrow \infty$ *and a nest* $\{F_k\}_{k\in\mathbb{N}}$ *such that every* $\tilde{f}_n$ *belongs to* $C(\{F_k\})$ *and* $\tilde{f}_{n_l}$ *converges to* $\tilde{f}$ *uniformly on each* F_k. *In particular,* $\tilde{f}_{n_l}$ *converges to* $\tilde{f}$ *q.e.*

(iii) *Every* $f \in \mathbb{D}$ *has a quasi-continuous* λ*-modification* $\tilde{f}$.

Lemma 2.13. *There exists a regular nest* $\{K_k\}_{k\in\mathbb{N}}$ *such that* K_k *is a separable and metrizable compact space with respect to the relative topology for any* k.

Lemma 2.14. *Suppose that* $\{F_k\}_{k\in\mathbb{N}}$ *is a regular nest and* $f \in C(\{F_k\})$. *If* $f \ge 0$ λ-a.e. *on an open set* O, *then* $f \ge 0$ *on* $O \cap \bigcup_k F_k$.

Lemma 2.15. *If* u_1 *and* u_2 *are quasi-continuous functions and* $u_1 = u_2$ λ-a.e., *then* $u_1 = u_2$ *q.e.*

In what follows, $\tilde{f}$ always means a quasi-continuous λ-modification of a function f, a particular version of which is sometimes chosen to suit the context.

We can also prove the next two propositions as in [10] (see also [21, Section 2]) by using Lemma 3.1 below together, though they are not used later in this article.

Proposition 2.16. *For any subset* A *of* X, *there exists a unique element* e_A *in the set* $\{f \in \mathbb{D} \mid \tilde{f} \ge \tilde{\rho}$ *q.e. on* $A\}$ *minimizing the norm* $\|f\|_{\mathbb{D}}$. *Moreover,* $0 \le e_A \le 1$ λ-a.e. *and* $\operatorname{cap}(A) = \|e_A\|_{\mathbb{D}}$.

Proposition 2.17. cap *is a Choquet capacity.*

We remark that the assumption (A4) is not necessary so far. The following are our main theorems, which are stated in [7] in the case of quasi-regular Dirichlet spaces.

Theorem 2.18. *Under* (A1)–(A4) *and* (QR1)–(QR3), *for a bounded linear functional* T *on* $\mathbb{D}$, *the next two conditions are equivalent.*

(i) *There exist a nest* $\{F_k\}_{k\in\mathbb{N}}$ *and positive constants* $\{C_k\}_{k\in\mathbb{N}}$ *such that for each* $k \in \mathbb{N}$,

$$|T(v)| \le C_k \|v\|_{L^\infty(X)} \quad \text{for all } v \in \mathbb{D}_{b,F_k}.$$

(ii) *There exists a signed smooth measure* ν *with some attached nest* $\{F'_k\}_{k\in\mathbb{N}}$ *such that*

$$T(v) = \int_X \tilde{v}(z)\,\nu(dz) \quad \textit{for all } v \in \bigcup_{k=1}^{\infty} \mathbb{D}_{b,F'_k}.$$

Moreover, the measure ν *is uniquely determined.*

Theorem 2.19. *Under* (A1)–(A4) *and* (QR1)–(QR3), *for a bounded linear functional* T *on* $\mathbb{D}$ *and a positive constant* C, *the next two conditions are equivalent.*

(i) $|T(v)| \le C\|v\|_{L^\infty(X)}$ *for all* $v \in \mathbb{D}_b$.

(ii) *There exists a finite signed smooth measure* ν *on* X *such that the total variation of* ν *is dominated by* C *and*

$$T(v) = \int_X \tilde{v}(z)\,\nu(dz) \quad \textit{for all } v \in \mathbb{D}_b.$$

In addition, ν *is uniquely determined. Moreover, if* (C) *in Lemma 2.2 holds, we may replace* $\mathbb{D}_b$ *in* (i) *by* $\mathcal{L}$ *that satisfies the following:*

($\mathcal{L}$) $\mathcal{L}$ *is a* $\mathbb{D}$*-dense subspace of* $\mathbb{D}_b$ *such that, for each* $\varepsilon > 0$, *there is a* C^∞ *function* χ *on* $\mathbb{R}$ *with* $|\chi| \le 1+\varepsilon$, $0 \le \chi' \le 1$, $\chi(x) = x$ *on* $[-1, 1]$, *and* $\chi \circ v \in \mathcal{L}$ *for every* $v \in \mathcal{L}$.

Before ending this section, we give a few examples of $\mathbb{D}$ other than quasi-regular Dirichlet spaces. Suppose that X is a separable Banach space and H a separable Hilbert space which is continuously and densely imbedded to X. The inner product and the norm of H will be denoted by $\langle \cdot,\cdot \rangle_H$ and $\|\cdot\|_H$, respectively. The topological dual X^* of X is identified with a subspace of H. Let λ be a finite Borel measure on X. When K is a separable Hilbert space, we denote by $L^p(X \to K)$ the L^p space consisting of K-valued functions on the measure space (X, λ).

Define function spaces $\mathcal{F}C_b^1$ and $(\mathcal{F}C_b^1)_{X^*}$ on X by

$$\mathcal{F}C_b^1 = \left\{ u : X \to \mathbb{R} \;\middle|\; \begin{array}{l} u(z) = f(\ell_1(z), \dots, \ell_m(z)), \\ \ell_1, \dots, \ell_m \in X^*,\ f \in C_b^1(\mathbb{R}^m) \\ \text{for some } m \in \mathbb{N} \end{array} \right\},$$

$$(\mathcal{F}C_b^1)_{X^*} = \left\{ G : X \to X^* \;\middle|\; \begin{array}{l} G(z) = \sum_{j=1}^m g_j(z)\ell_j, \\ g_1, \dots, g_m \in \mathcal{F}C_b^1, \\ \ell_1, \dots, \ell_m \in X^* \text{ for some } m \in \mathbb{N} \end{array} \right\},$$

where $C_b^1(\mathbb{R}^m)$ is the set of all bounded functions f on $\mathbb{R}^m$ that have bounded and continuous first-order derivatives. Let $u \in \mathcal{F}C_b^1$ and $\ell \in X^* \subset H \subset X$. We define $\partial_\ell u$ by $\partial_\ell u(z) = \lim_{\varepsilon\to 0}(u(z+\varepsilon\ell) - u(z))/\varepsilon$.

The H-derivative ∇u is a unique map from X to H that satisfies the relation

$$\langle \nabla u(z), \ell \rangle_H = \partial_\ell u(z), \qquad \ell \in X^* \subset H.$$

We assume that, if $u \in \mathcal{F}C_b^1$ and $v \in \mathcal{F}C_b^1$ coincide on a measurable set A, then $\nabla u = \nabla v$ λ-a.e. on A. Let $p \geq 1$. We also assume that $(\nabla, \mathcal{F}C_b^1)$ is closable as a map from L^p to $L^p(X \to H)$. We denote by $W^{1,p}$ the domain of the closure of $(\nabla, \mathcal{F}C_b^1)$ and extend the domain of ∇ to $W^{1,p}$ naturally. The space $W^{1,p}$ is a separable Banach space with norm $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|\nabla f\|_{L^p(X \to H)}$.

Proposition 2.20. *Suppose also $p > 1$. Then, when we regard $W^{1,p}$ as $\mathbb{D}$, the conditions* (A1)–(A4), (QR1)–(QR3), *and* (C) *are satisfied.*

Proof. From the results of [21], (A1) and (QR1) hold. Since X is separable, X^* is also separable with respect to the weak* topology (see Corollary after Proposition 8 in Chapter IV of [5] for the proof). When $\{\ell_n\}_{n \in \mathbb{N}}$ is a countable dense set of X^*, $\varphi_n(\cdot) = \arctan \ell_n(\cdot)$ and $N = \emptyset$ assure the validity of (QR3). The remaining conditions are easily checked. Q.E.D.

In order to give another example, we introduce real interpolation spaces. Let B_0 and B_1 be separable Banach spaces. We assume that B_0 is continuously imbedded to B_1 for simplicity. For parameters $q \in (1, \infty)$ and $\theta \in (0,1)$, we define the space $(B_0, B_1)_{\theta,q}$ by all elements $f \in B_1$ such that there exist some B_j-valued measurable functions $u_j(t)$ on $[0, \infty)$ $(j = 0, 1)$ satisfying

$$(2) \quad u_0(t) + u_1(t) = f \text{ a.e. } t, \int_0^\infty (t^{j-\theta} \|u_j(t)\|_{B_j})^q \frac{dt}{t} < \infty \ (j = 0,\, 1).$$

We set the norm of $f \in (B_0, B_1)_{\theta,q}$ by

$$\|f\|_{(B_0,B_1)_{\theta,q}} = \inf_{u_0,\, u_1} \left[\max_{j=0,\,1} \left(\int_0^\infty (t^{j-\theta} \|u_j(t)\|_{B_j})^q \frac{dt}{t} \right)^{1/q} \right],$$

where the infimum is taken over all pairs u_0 and u_1 satisfying (2). From the general theory of real interpolation, $(B_0, B_1)_{\theta,q}$ is a Banach space, we have continuous imbeddings $B_0 \hookrightarrow (B_0, B_1)_{\theta,q} \hookrightarrow B_1$, and B_0 is dense in $(B_0, B_1)_{\theta,q}$. Keeping the notation in the previous example, we have the following proposition.

Proposition 2.21. *Let $p \in (1, \infty)$, $q \in (1, \infty)$, and $\theta \in (0,1)$. Then $\mathbb{D} := (W^{1,p}, L^p)_{\theta,q}$ satisfies* (A1)–(A4), (QR1)–(QR3) *and* (C).

Proof. In general, we can prove that $(B_0, B_1)_{\theta,q}$ is uniformly convex if B_0 or B_1 is, in the same way as Proposition V.1 of [3]. Therefore, $\mathbb{D}$ is uniformly convex. The separability, (QR1) and (QR2) come from those of $W^{1,p}$. (QR3) is proved in the same way as the case of $W^{1,p}$. (A2) is clearly true. We will prove (C). Let χ be as in (C) in Lemma 2.2. Let $f \in \mathbb{D}$ and take u_0 and u_1 satisfying (2). Set $v_0(t) = \chi \circ u_0(t)$ and $v_1(t) = \chi \circ f - \chi \circ u_0(t)$. Then $v_0(t) + v_1(t) = \chi \circ f$ and it is easy to see that $\|v_0(t)\|_{W^{1,p}} \le c\|u_0(t)\|_{W^{1,p}}$ and $\|v_1(t)\|_{L^p} \le c\|u_1(t)\|_{L^p}$. This implies that $\chi \circ f \in \mathbb{D}$ and $\|\chi \circ f\|_{\mathbb{D}} \le c\|f\|_{\mathbb{D}}$. Q.E.D.

§3. Proof of Theorems 2.18 and 2.19

First, we will prove that (ii) implies (i) in Theorem 2.18. We take $F_k = F'_k$ and $C_k = |\nu|(F_k) < \infty$. Let $v \in \mathbb{D}_{b,F_k}$ and $M = \|v\|_{L^\infty(X)}$. We can take a quasi-continuous λ-modification $\tilde{v}$ so that $|\tilde{v}| \le M$ everywhere. Then $|T(v)| \le M|\nu|(F_k) = C_k M$. Therefore, (i) holds.

Next, we will prove that (i) implies (ii) in Theorem 2.18. Take a nest $\{E_k^{(1)}\}_{k\in\mathbb{N}}$ so that $\tilde{\rho} \in C(\{E_k^{(1)}\})$. Define $E_k^{(2)} = E_k^{(1)} \cap \{\tilde{\rho} \ge 1/k\}$.

Lemma 3.1. *$\{E_k^{(2)}\}_{k\in\mathbb{N}}$ is a nest.*

Proof. Clearly, $\{E_k^{(2)}\}_{k\in\mathbb{N}}$ is a sequence of increasing closed sets. Define $\rho_k = \rho \wedge (1/k)$, $k \in \mathbb{N}$. Then $\rho_k \to 0$ weakly in $\mathbb{D}$ by Lemma 2.4. Take a sequence $\{k_j\} \uparrow \infty$ so that $\hat{\rho}_m := (1/m)\sum_{j=1}^m \rho_{k_j}$ converges to 0 in $\mathbb{D}$ as $m \to \infty$. Since $\hat{\rho}_m + e_{X\setminus E_{k_m}^{(1)}} \ge \rho$ λ-a.e. on $X \setminus E_{k_m}^{(2)}$, we have $\mathrm{cap}(X \setminus E_{k_m}^{(2)}) \le \|\hat{\rho}_m\|_{\mathbb{D}} + \|e_{X\setminus E_{k_m}^{(1)}}\|_{\mathbb{D}} \to 0$ as $m \to \infty$. Therefore, $\{E_k^{(2)}\}_{k\in\mathbb{N}}$ is a nest. Q.E.D.

Define $E_k^{(3)} = F_k \cap K_k \cap E_k^{(2)}$, $k \in \mathbb{N}$, where K_k is what appeared in Lemma 2.13. Then $\{E_k^{(3)}\}_{k\in\mathbb{N}}$ is a regular nest consisting of separable and metrizable compact sets. Given $k \in \mathbb{N}$, let $\{U_{k,n}\}_{n\in\mathbb{N}}$ be a countable open basis of $E_k^{(3)}$. The totality of every union of finite elements in $\{U_{k,n}\}_{n\in\mathbb{N}}$ will be denoted by $\{V_{k,n}\}_{n\in\mathbb{N}}$. Take a countable family $\mathcal{O}$ of open sets in X such that for every $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, some $O \in \mathcal{O}$ satisfies that $O \supset V_{k,n}$ and $\mathrm{cap}(O) < \mathrm{cap}(V_{k,n}) + \varepsilon$.

Recall the condition (QR3) and set $\mathcal{S} = \{(-M) \vee \tilde{\varphi}_n \wedge M \mid n \in \mathbb{N},\ M \in \mathbb{N}\}$. The functions of $\mathcal{S}$ separate the points of $X \setminus N$. Fix a countable subset $\mathcal{D}$ of $\mathbb{D}$ such that $\{f \in \mathcal{D} \mid \|f\|_{L^\infty(X)} \le M\}$ is a dense set of $\{f \in \mathbb{D} \mid \|f\|_{L^\infty(X)} \le M\}$ for each $M \in \mathbb{N}$. Take a nest $\{\hat{F}_k\}_{k\in\mathbb{N}}$ so that $\hat{F}_k \subset E_k^{(3)}$ for each k, $N \subset \bigcap_{k\in\mathbb{N}}(X \setminus \hat{F}_k)$ and the quasi-continuous

λ-modifications of all elements in $\mathcal{S} \cup \mathcal{D} \cup \{e_O \mid O \in \mathcal{O}\}$ belong to $C(\{\hat{F}_k\})$. Denote by $\mathcal{A}$ the algebra generated by $\mathcal{S} \cup \{1 \wedge M\tilde{\rho} \mid M \in \mathbb{N}\}$. Note that all functions in $\mathcal{A}$ and $\tilde{\rho}$ belong to $C(\{\hat{F}_k\})$. From the Stone-Weierstrass theorem, $\{f|_{\hat{F}_k} \mid f \in \mathcal{A}\}$ is dense in $C(\hat{F}_k)$ with uniform topology for any k.

Lemma 3.2. *There exist a nest $\{F'_k\}_{k\in\mathbb{N}}$ and functions $\{\psi_n\}_{n\in\mathbb{N}}$ in $\bigcup_k \mathbb{D}_{\hat{F}_k}$ satisfying the following:*

(i) $F'_k \subset \hat{F}_k$ *for all* k;
(ii) *the quasi-continuous λ-modification $\tilde{\psi}_n$ belongs to $C(\{F'_k\})$ for all n;*
(iii) $0 \le \psi_n \le 1$ λ-a.e. *on X and $\tilde{\psi}_n = 1$ on F'_n for all n.*

Proof. Take a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \bigcup_k \mathbb{D}_{\hat{F}_k}$ such that $\|\eta_n - \rho\|_{\mathbb{D}} < 1/(n2^{n+1})$, $n \in \mathbb{N}$. By Lemma 2.11, there exists an open set G_n so that $G_n \supset \{|\tilde{\eta}_n - \tilde{\rho}| > 1/(2n)\}$ and $\mathrm{cap}(G_n) < 2^{-n}$ for each n. Take a nest $\{E_k\}_{k\in\mathbb{N}}$ such that $E_k \subset \hat{F}_k$ and $\{\tilde{\eta}_n\}_{n\in\mathbb{N}} \subset C(\{E_k\})$. Then $\tilde{\eta}_n \ge 1/(2n)$ on $E_n \setminus G_n$ since $\tilde{\rho} \ge 1/n$ on E_n. Define $F'_k = E_k \setminus \bigcup_{n=k}^{\infty} G_n$ and $\psi_n = 0 \vee 2n\eta_n \wedge 1$. Then $\psi_n \in \bigcup_k \mathbb{D}_{\hat{F}_k}$, $\tilde{\psi}_n = 1$ on F'_n, $\{F'_k\}_{k\in\mathbb{N}}$ is a sequence of increasing closed sets, and by Lemma 2.10, we have $\mathrm{cap}(X \setminus F'_k) \le \mathrm{cap}(X \setminus E_k) + \sum_{n=k}^{\infty} \mathrm{cap}(G_n) \to 0$ as $k \to \infty$. Q.E.D.

Now, fix $n \in \mathbb{N}$ and take $m \in \mathbb{N}$ so that $\psi_n \in \mathbb{D}_{\hat{F}_m}$. Define $T_n : \mathbb{D}_b \to \mathbb{R}$ by $T_n(f) = T(\psi_n f)$. Since $\psi_n f \in \mathbb{D}_{b,\hat{F}_m} \subset \mathbb{D}_{b,F_m}$, the statement (i) of Theorem 2.18 implies $|T_n(f)| \le C_m \|\psi_n f\|_{L^\infty(X)} \le C_m \|f\|_{L^\infty(X)}$.

For an arbitrary $f \in C(\hat{F}_m)$, we can take $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{j\to\infty} \|f_j - f\|_{C(\hat{F}_m)} = 0$. Then,

$$|T_n(f_i) - T_n(f_j)| \le C_m \|\psi_n(f_i - f_j)\|_{L^\infty(X)} \le C_m \|f_i - f_j\|_{L^\infty(\hat{F}_m)} \to 0$$

as $i \ge j \to \infty$. The limit of $\{T_n(f_j)\}_{j\in\mathbb{N}}$, denoted by $\hat{T}_n(f)$, satisfies $|\hat{T}_n(f)| \le C_m \|f\|_{C(\hat{F}_m)}$. Therefore, $\hat{T}_n$ is a bounded linear functional on $C(\hat{F}_m)$. On account of the Riesz representation theorem, there exists an associated finite signed measure ν_n on $\hat{F}_m$ such that $\hat{T}_n(f) = \int_{\hat{F}_m} f\, d\nu_n$ for every $f \in C(\hat{F}_m)$. We extend ν_n to a measure on X by letting $\nu_n(A) := \nu_n(A \cap \hat{F}_m)$.

Lemma 3.3. *The measure ν_n charges no exceptional sets.*

Proof. Since the measure $|\nu_n|$ restricted on $\hat{F}_m$ is regular, it is enough to prove that $|\nu_n|(K) = 0$ for any compact set $K \subset \hat{F}_m$ of null capacity. Take such K. Then we can take a sequence $\{O_j\}_{j\in\mathbb{N}}$ from $\mathcal{O}$

so that $K \subset O_j$ for all j and $\lim_{j\to\infty} \mathrm{cap}(O_j) = 0$. Indeed, for each j, take an open set O such that $K \subset O$ and $\mathrm{cap}(O) < 1/j$. Since K is compact, there is a set V in $\{V_{k,m}\}_{k\in\mathbb{N}}$ such that $K \subset V \subset O$. Choose $O_j \in \mathcal{O}$ so that $V \subset O_j$ and $\mathrm{cap}(O_j) \le \mathrm{cap}(V) + 1/j$.

Let $f_j := e_{O_j}$. Since $\{\hat{F}_k\}$ is a regular nest, Lemma 2.14 implies that $\tilde{f}_j = 1$ on $K \subset O_j \cap \hat{F}_m$. We may also assume that $0 \le \tilde{f}_j \le 1$ everywhere. Since $\lim_{j\to\infty} \|f_j\|_{\mathbb{D}} = 0$, we can suppose $f_j \to 0$ λ-a.e. as $j \to \infty$ by taking a subsequence if necessary. Since $\{\tilde{f}_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(|\nu_n|)$, the arithmetic means $\{\hat{f}_j\}_{j\in\mathbb{N}}$ of a further subsequence of $\{\tilde{f}_j\}_{j\in\mathbb{N}}$ converge strongly in $L^2(|\nu_n|)$. Take a sequence $\{j_l\} \uparrow \infty$ such that $\hat{f}_{j_l}$ converges $|\nu_n|$-a.e. as $l \to \infty$. Define $f(z) = \liminf_{l\to\infty} \hat{f}_{j_l}(z)$. Then $0 \le f \le 1$ on X, $f = 1$ on K, and $f = 0$ λ-a.e. by the way of construction.

Given $h \in \mathcal{A}$, we have

$$\int_{\hat{F}_m} \hat{f}_{j_l} h \, d\nu_n = \hat{T}_n(\hat{f}_{j_l} h|_{\hat{F}_m}) = T_n(\hat{f}_{j_l} h) = T(\psi_n \hat{f}_{j_l} h). \tag{3}$$

When l tends to ∞, the left-hand side of (3) converges to $\int_{\hat{F}_m} fh\, d\nu_n$ by the dominated convergence theorem. On the other hand, $\{\psi_n \hat{f}_{j_l} h\}_{l\in\mathbb{N}}$ is bounded in $\mathbb{D}$ by (A4)'. Since they converge to 0 λ-a.e., they also converge weakly to 0 in $\mathbb{D}$ by Lemma 2.1. Therefore, the right-hand side of (3) converges to 0 as $l \to \infty$. Namely, $\int_{\hat{F}_m} fh\, d\nu_n = 0$. Since $\{h|_{\hat{F}_m} \mid h \in \mathcal{A}\}$ is dense in $C(\hat{F}_m)$, we conclude that $f\, d\nu_n = 0$, therefore, $|\nu_n|(K) = 0$. Q.E.D.

Lemma 3.4. *For all $f \in \mathbb{D}_b$, $T_n(f) = \int_X \tilde{f}\, d\nu_n$.*

Proof. We can take a sequence $\{f_j\}_{j\in\mathbb{N}}$ from $\mathcal{D}$ so that $\{f_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(X)$, f_j converges to f in $\mathbb{D}$ and $\tilde{f}_j$ converges to $\tilde{f}$ outside some Borel exceptional set N_0. Note that $\tilde{f}_j|_{\hat{F}_m} \in C(\hat{F}_m)$. Then, $T_n(f_j) \to T_n(f)$ as $j \to \infty$, while

$$\begin{aligned} T_n(f_j) &= \hat{T}_n(\tilde{f}_j|_{\hat{F}_m}) = \int_X \tilde{f}_j\, d\nu_n = \int_{X\setminus N_0} \tilde{f}_j\, d\nu_n \\ &\overset{j\to\infty}{\longrightarrow} \int_{X\setminus N_0} \tilde{f}\, d\nu_n = \int_X \tilde{f}\, d\nu_n \end{aligned}$$

by means of the dominated convergence theorem. Q.E.D.

For any k, $l \in \mathbb{N}$, we have $\tilde{\psi}_k\, d\nu_l = \tilde{\psi}_l\, d\nu_k$. Indeed, For $f \in \mathcal{A}$,

$$\int_X f\tilde{\psi}_k\, d\nu_l = T_l(f\psi_k) = T(\psi_l f \psi_k) = T_k(f\psi_l) = \int_X f\tilde{\psi}_l\, d\nu_k.$$

Therefore, we can define a signed smooth measure ν by $\nu = \nu_n$ on F'_n $(n = 1, 2, \ldots)$, which is well-defined by the fact that $\tilde{\psi}_n = 1$ on F'_n. Then for any $f \in \mathbb{D}_{b,F'_k}$, we have $\psi_k f = f$ and

$$T(f) = T(\psi_k f) = T_k(f) = \int_X \tilde{f}\, d\nu_k = \int_X \tilde{f}\, d\nu.$$

Thus, (ii) holds.

In order to prove the uniqueness of ν, it is enough to show that $\nu \equiv 0$ if $\int_X \tilde{v}\, d\nu = 0$ for all $v \in \bigcup_k \mathbb{D}_{b,F_k}$, where $\{F_k\}_{k\in\mathbb{N}}$ is a nest attached with ν. Following the same procedure as in the proof of (i)$\Rightarrow$(ii), take the nests $\{\hat{F}_k\}_{k\in\mathbb{N}}$ and $\{F'_k\}_{k\in\mathbb{N}}$, the function space $\mathcal{A}$, and the sequence of functions $\{\psi_n\}_{n\in\mathbb{N}}$. For any $n \in \mathbb{N}$ and $f \in \mathcal{A}$, we have $f\psi_n \in \mathbb{D}_{b,\hat{F}_n} \subset \mathbb{D}_{b,F_n}$, therefore $\int_X f\tilde{\psi}_n\, d\nu = 0$. Since $\{f|_{\hat{F}_n} \mid f \in \mathcal{A}\}$ is dense in $C(\hat{F}_n)$, we have $\tilde{\psi}_n\, d\nu = 0$. In particular, $\nu = 0$ on F'_n because $\tilde{\psi}_n = 1$ on F'_n. This implies that $\nu \equiv 0$.

The implication (ii)$\Rightarrow$(i) of Theorem 2.19 is proved in the same way as in Theorem 2.18. Because of the result and the proof of (i)$\Rightarrow$(ii) of Theorem 2.18, Theorem 2.19 (i) implies that there exists a finite signed smooth measure ν with some attached nest $\{F'_k\}_{k\in\mathbb{N}}$ such that the total variation is dominated by C and $T(v) = \int_X \tilde{v}(z)\, \nu(dz)$ for all $v \in \bigcup_k \mathbb{D}_{b,F'_k}$. It is easy to show that this identity holds for all $v \in \mathbb{D}_b$ by an approximation argument.

The uniqueness of ν is clear from the corresponding result of Theorem 2.18. The final claim is also deduced by an approximation argument and the use of Lemma 2.1.

This completes the proof of Theorems 2.18 and 2.19.

§4. Application to BV functions on Wiener space

First, we will review some results of [9]. Let E be a separable Banach space and H a separable Hilbert space which is continuously and densely imbedded to E. We use the notations in the end of Section 2 with letting $X = E$. Define a Gaussian measure μ on E by the following identity:

$$\int_E \exp(\sqrt{-1}\,\ell(z))\, \mu(dz) = \exp(-\|\ell\|_H^2/2), \qquad \ell \in E^* \subset H.$$

When Y is a separable Hilbert space and ρ is a nonnegative measurable function on E, we denote by $L^p(E \to Y; \rho)$ in this section the L^p space consisting of Y-valued functions on E with underlying measure $\rho\, d\mu$. We omit $E \to Y$ and ρ from the notation when $Y = \mathbb{R}$ and $\rho \equiv 1$, respectively, and write simply L^p for $L^p(E \to \mathbb{R}; 1)$. We also set

$L^{\infty-} = \bigcap_{p>1} L^p$ and denote by L^p_+ the set of all nonnegative functions in L^p.

If $u \in \mathcal{F}C_b^1$ and $v \in \mathcal{F}C_b^1$ coincide on a measurable set A, then $\nabla u = \nabla v$ μ-a.e. on A. See Proposition I.7.1.4 of [4] for the proof.

For $p \geq 1$, $Cl_p(E)$ denotes the set of all functions ρ in L^1_+ such that $(\nabla, \mathcal{F}C_b^1)$ is closable as a map from $L^p(\rho)$ to $L^p(E \to H; \rho)$. A simple example for such ρ is a function which is uniformly away from 0. Suppose $\rho \in Cl_p(E)$. We write $W^{1,p}(\rho)$ instead of $W^{1,p}$ when regarding $(E, \rho\, d\mu)$ as (X, λ) in Section 2. When $p > 1$, $W^{1,p}(\rho)$ satisfies all the conditions (A1)–(A4), (QR1)–(QR3) and (C).

Let F^ρ be the topological support of the measure $\rho\, d\mu$. Since $L^0(E \to Y; \rho)$ is identified with $L^0(F^\rho \to Y; \rho)$, we abuse the notation and $W^{1,p}(\rho)$ is also regarded as a function space on F^ρ. When $\rho \in Cl_2(E)$, an associated Dirichlet form $(\mathcal{E}^\rho, W^{1,2}(\rho))$ on $L^2(F^\rho; \rho)$ is defined by

$$\mathcal{E}^\rho(f,g) = \int_{F^\rho} \langle \nabla f, \nabla g \rangle_H \, \rho\, d\mu, \qquad f,\, g \in W^{1,2}(\rho).$$

This is a quasi-regular Dirichlet form and a finite signed measure ν on F^ρ is smooth with respect to $\mathcal{E}^\rho$ if and only if ν is $W^{1,2}(\rho)$-smooth.

For each $G \in (\mathcal{F}C_b^1)_{E^*}$, the (formal) adjoint $\nabla^* G$ is defined by the following identity:

$$\int_E (\nabla^* G) u \, d\mu = \int_E \langle G, \nabla u \rangle_H \, d\mu \qquad \text{for all } u \in \mathcal{F}C_b^1.$$

Denote by $L(\log L)^{1/2}$ the space of all functions f on E such that $\Phi \circ |f| \in L^1$, where $\Phi(x) = x((\log x) \vee 0)^{1/2}$. We say that a real measurable function ρ on E is of bounded variation ($\rho \in BV(E)$) if $\rho \in L(\log L)^{1/2}$ and

$$V(\rho) := \sup_G \int_E (\nabla^* G) \rho \, d\mu < \infty,$$

where G is taken over all functions in $(\mathcal{F}C_b^1)_{E^*}$ such that $\|G(z)\|_H \leq 1$ for every $z \in E$.

Let $\{T_t\}_{t>0}$ be the Ornstein-Uhlenbeck semigroup, which is associated with $\mathcal{E}^1$. It is strongly continuous, analytic and contractive on L^p for any $p \in (1, \infty)$.

We recall some results discussed in [9].

Theorem 4.1. (i) *For $\rho \in BV(E)$, $\|\nabla T_t \rho\|_{L^1} \leq V(\rho)$ for every $t > 0$.*

(ii) *$BV(E)$ is a vector lattice. Namely, it is a vector space, and for each $\rho \in BV(E)$, ρ_+ also belongs to $BV(E)$.*

(iii) *A function ρ belongs to $BV(E)$ if and only if $\rho \in L^1$ and there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $W^{1,1}(:= W^{1,1}(1))$ such that $\|\rho_n\|_{W^{1,1}}$ is bounded in n and $\rho_n \to \rho$ in L^1 as $n \to \infty$.*

(iv) *Each $\rho \in BV(E)$ has a unique finite Borel measure ν and a unique H-valued Borel function σ on E such that $\|\sigma\|_H = 1$ ν-a.e. and for every $G \in (\mathcal{F}C_b^1)_{E^*}$,*

$$\int_E (\nabla^* G)\rho \, d\mu = \int_E \langle G, \sigma\rangle_H \, d\nu.$$

The measure ν is $W^{1,2}(|\rho|+1)$-smooth. If moreover $\nu \in Cl_2(E)$, then $\nu|_{E\setminus F^\rho} = 0$ and ν is $W^{1,2}(\rho)$-smooth.

In what follows, we will write ν_ρ for ν in the theorem above. In this section, we improve the result for the smoothness of ν_ρ. In view of the proof of Theorem 4.1 (iv) (Theorem 3.9 of [9]), the smoothness of ν_ρ is derived from the smoothness of ν_ℓ for each $\ell \in E^*$, where ν_ℓ is a unique finite signed measure on E satisfying

$$\int_E \partial_\ell u(z)\rho(z)\,\mu(dz) = -2\int_E u(z)\,\nu_\ell(dz), \qquad u \in \mathcal{F}C_b^1.$$

Therefore, applying Theorem 2.19 with $\mathcal{L} = \mathcal{F}C_b^1$, if we show that the functional

$$I_\ell : \mathcal{F}C_b^1 \ni u \mapsto \int_E \partial_\ell u(z)\rho(z)\,\mu(dz) \in \mathbb{R} \tag{4}$$

extends continuously on $\mathbb{D}$, where $\mathbb{D}$ satisfies (A1)–(A4), (QR1)–(QR3), and (C), and has $\mathcal{F}C_b^1$ as a dense set, then we can say that ν_ℓ, hence ν_ρ, is $\mathbb{D}$-smooth. It is obvious that I_ℓ extends to a continuous functional on $W^{1,p}(|\rho|+1)$ (and $W^{1,p}(\rho)$ if furthermore $\rho \in Cl_p$) for every $p \geq 1$. Also, if $\rho \in L^q$ for some $q \in (1,\infty)$, then I_ℓ extends to a continuous functional on $W^{1,q/(q-1)}(:= W^{1,q/(q-1)}(1))$ by Hölder's inequality. Therefore, we have the following results.

Proposition 4.2. *Let $\rho \in BV(E)$. Then, ν_ρ is $W^{1,p}(|\rho|+1)$-smooth for every $p > 1$. If moreover $\rho \in Cl_p$, then ν is $W^{1,p}(\rho)$-smooth.*

Proposition 4.3. *Let $\rho \in BV(E) \cap L^q$ for some $q \in (1,\infty)$. Then, ν_ρ is $W^{1,q/(q-1)}$-smooth.*

In Proposition 4.2, the smaller p is, the stronger the claim is.

Now, we will give other examples of $\mathbb{D}$ so that ν_ρ is $\mathbb{D}$-smooth. Let us recall the Sobolev spaces in the context of Malliavin calculus. We give several notations in somewhat informal way. We refer to [12] for precise definitions. Let $L = -\nabla^*\nabla$ be the Ornstein-Uhlenbeck operator, which is regarded as a generator of $\{T_t\}_{t>0}$. The Sobolev space $\mathbb{D}^{\alpha,p}$, $\alpha \in \mathbb{R}$, $1 < p < \infty$, is given by $\mathbb{D}^{\alpha,p} = (1-L)^{-\alpha/2}(L^p)$. Each $\mathbb{D}^{\alpha,p}$ is a separable Banach space with norm $\|f\|_{\mathbb{D}^{\alpha,p}} := \|(1-L)^{\alpha/2} f\|_{L^p}$. The topological dual of $\mathbb{D}^{\alpha,p}$ is identified with $\mathbb{D}^{-\alpha,q}$, $q = p/(p-1)$. When $n \in \mathbb{N}$, by Meyer's equivalence, ∇^n is defined as a continuous operator from $\mathbb{D}^{n,p}$ to $L^p(E \to H^{\otimes n})$ and $\|\cdot\|_{L^p} + \|\nabla^n \cdot\|_{L^p(E\to H^{\otimes n})}$ gives a norm on $\mathbb{D}^{n,p}$ which is equivalent to $\|\cdot\|_{\mathbb{D}^{n,p}}$. In particular, $W^{1,p}(:= W^{1,p}(1))$ is identical with $\mathbb{D}^{1,p}$ as a set and their norms are mutually equivalent.

We define another Sobolev space $\mathbb{E}^{\alpha,p}$, $\alpha \in \mathbb{R}$, $1 < p < \infty$, firstly introduced in [24], by

$$\mathbb{E}^{\alpha,p} = \begin{cases} \mathbb{D}^{\alpha,p} & \text{if } \alpha \in \mathbb{Z}, \\ (\mathbb{D}^{k+1,p}, \mathbb{D}^{k,p})_{k+1-\alpha,p} & \text{if } k < \alpha < k+1,\ k \in \mathbb{Z}. \end{cases}$$

The general theory of real interpolation implies that $(\mathbb{E}^{\alpha,p})^*$ is identified with $\mathbb{E}^{-\alpha,q}$, where $q = p/(p-1)$ (see also [24]). When $0 < \alpha < 1$ and $1 < p < \infty$, $\mathbb{E}^{\alpha,p}$ satisfies conditions (A1)–(A4), (QR1)–(QR3), and (C) by virtue of Proposition 2.21, if $\mathbb{E}^{\alpha,p}$ is equipped with a norm deduced by $(W^{1,p}, L^p)_{1-\alpha,p}$. For such indices, $\mathcal{F}C_b^1$ is dense in $\mathbb{E}^{\alpha,p}$ since $W^{1,p}$ is dense in $\mathbb{E}^{\alpha,p}$. For later use, following [1, 2], we introduce another equivalent norm on $\mathbb{E}^{\alpha,p}$ based on the K-method by

$$\|f\|'_{\mathbb{E}^{\alpha,p}} = \left(\int_0^1 (\varepsilon^{-\alpha} K(\varepsilon, f))^p \frac{d\varepsilon}{\varepsilon} \right)^{1/p},$$

where

$$K(\varepsilon, f) = \inf\left\{ \|f_1\|_{L^p} + \varepsilon \|f_2\|_{W^{1,p}} \mid f = f_1 + f_2,\ f_1 \in L^p,\ f_2 \in \mathbb{D}^{1,p} \right\}.$$

The connection between $BV(E)$ and $\mathbb{E}^{\alpha,p}$ is given as follows.

Theorem 4.4. *Let $q > 1$. Then $BV(E) \cap L^q \subset \mathbb{E}^{\alpha,p}$ if $1 < p < q$ and $\alpha < (1/p - 1/q)/(1 - 1/q)$. Also, this inclusion is continuous when $BV(E) \cap L^q$ is equipped with norm $\|f\|_{BV(E)\cap L^q} = V(f) + \|f\|_{L^q}$. In particular, $BV(E) \cap L^{\infty-} \subset \mathbb{E}^{\alpha,p}$ if $p > 1$ and $\alpha p < 1$.*

For the proof, we need the following estimates.

Lemma 4.5. (i) *When $\theta/a + (1-\theta)/b = 1/p$ with $0 < \theta < 1$, a, b, $p \geq 1$, we have $\|f\|_{L^p} \leq \|f\|_{L^a}^{\theta} \|f\|_{L^b}^{1-\theta}$.*

(ii) *For each $r \geq 0$ and $p \in (1, \infty)$, there exists some C such that $\|(1-L)^r T_t f\|_{L^p} \leq C t^{-r} \|f\|_{L^p}$ for every $t \in (0, 1]$ and $f \in L^p$.*

Proof. The claim (i) follows from a simple application of Hölder's inequality. The claim (ii) is a consequence of Theorem 6.13 (c) of Chapter 2 in [18], since $\{T_t\}_{t>0}$ is an analytic semigroup on L^p. Q.E.D.

Proof of Theorem 4.4. Let $f \in BV(E) \cap L^q$ with $V(f)+\|f\|_{L^q} \leq 1$. In the following, c_i denotes a constant depending only on p and q. By Theorem 4.1 (i), $\|\nabla T_t f\|_{L^1} \leq V(f) \leq 1$ for any $t > 0$. By virtue of Meyer's equivalence and Lemma 4.5 (ii), for $t \in (0,1]$,

$$\|\nabla T_t f\|_{L^q} \leq c_1 \|(1-L)^{1/2} T_t f\|_{L^q} \leq c_2 t^{-1/2}.$$

Applying Lemma 4.5 (i) with $a = 1$ and $b = q$, that is, $\theta = (1/p - 1/q)/(1-1/q)$, we have $\|\nabla T_t f\|_{L^p} \leq (c_2 t^{-1/2})^{1-\theta}$ for $t \in (0,1]$, therefore,

$$\|T_t f\|_{W^{1,p}} \leq c_3 t^{-(1-\theta)/2}. \tag{5}$$

From the identity

$$\begin{aligned} f - T_t f &= -\int_0^t \frac{d}{ds} T_s f\, ds = -\int_0^t L T_s f\, ds \\ &= \int_0^t \{((1-L)^{1/2} T_{s/2})^2 f - T_s f\}\, ds, \end{aligned}$$

we obtain, for $t \in (0,1]$,

$$\begin{aligned} \|f - T_t f\|_{L^p} &\leq \int_0^t \|((1-L)^{1/2} T_{s/2})^2 f\|_{L^p}\, ds + t\|f\|_{L^p} \\ &\leq \int_0^t c_4 s^{-1/2} \|(1-L)^{1/2} T_{s/2} f\|_{L^p}\, ds + t \\ &\leq \int_0^t c_5 s^{-1/2} \|T_{s/2} f\|_{W^{1,p}}\, ds + t \\ &\leq \int_0^t c_6 s^{-1/2} s^{-(1-\theta)/2}\, ds + t \leq c_7 t^{\theta/2}. \end{aligned} \tag{6}$$

Here we used Lemma 4.5 (ii) in the second line and (5) in the last line. By combining (5) and (6), for each $\varepsilon \in (0,1]$,

$$K(\varepsilon, f) \leq \|f - T_{\varepsilon^2} f\|_{L^p} + \varepsilon \|T_{\varepsilon^2} f\|_{W^{1,p}} \leq c_8 \varepsilon^{\theta},$$

and, if $\alpha \in (0,\theta)$,

$$\left(\int_0^1 (\varepsilon^{-\alpha} K(\varepsilon, f))^p \frac{d\varepsilon}{\varepsilon} \right)^{1/p} \leq c_8 \{p(\theta - \alpha)\}^{-1/p} < \infty.$$

This proves the claim. Q.E.D.

Using Theorem 4.4, we obtain the $\mathbb{E}^{\alpha,p}$-smoothness of ν_ρ by the following proposition.

Proposition 4.6. *Let $\rho \in BV(E) \cap L^q$, $q > 1$. Then the map I_ℓ in* (4) *extends continuously on $\mathbb{E}^{\alpha,p}$ if $p > q/(q-1)$ and $\alpha p > q/(q-1)$. Therefore, ν_ρ is $\mathbb{E}^{\alpha,p}$-smooth for such α and p with $\alpha \in (0,1)$. In particular, if $\rho \in BV(E) \cap L^{\infty-}$, then ν_ρ is $\mathbb{E}^{\alpha,p}$-smooth for any α, p with $\alpha \in (0,1)$ and $\alpha p > 1$.*

Proof. Due to Meyer's equivalence, the map $u \mapsto \partial_\ell u$ is continuous from $\mathbb{D}^{1,p}$ to L^p and from L^p to $\mathbb{D}^{-1,p}$, respectively. By the real interpolation theorem, it is continuous from $\mathbb{E}^{\alpha,p}$ to $\mathbb{E}^{\alpha-1,p}$ for any $\alpha \in (0,1)$. The claim follows from the fact $(\mathbb{E}^{1-\alpha,p/(p-1)})^* = \mathbb{E}^{\alpha-1,p}$ and $BV(E) \cap L^q \subset \mathbb{E}^{1-\alpha,p/(p-1)}$ by the assumption and Theorem 4.4. Q.E.D.

REMARK 4.7. (i) In [24], it is proved that $\mathbb{D}^{\alpha+\varepsilon,p} \hookrightarrow \mathbb{E}^{\alpha,p} \hookrightarrow \mathbb{D}^{\alpha-\varepsilon,p}$ for every $\alpha \in \mathbb{R}$, $1 < p < \infty$ and $\varepsilon > 0$. Therefore, Theorem 4.4 and Proposition 4.6 remain valid if we replace $\mathbb{E}^{\alpha,p}$ by $\mathbb{D}^{\alpha,p}$.

(ii) When $\rho \in BV(E)$ is an indicator function of some set A, ν_ρ can be regarded as a surface measure of A. The smoothness of ν_ρ that is proved in the proposition above is consistent with Theorem 9 of [6] saying that the Hausdorff measure of codimension n on Wiener space does not charge any set of (α,p)-capacity as long as $p > 1$ and $\alpha p > n$.

Lastly, we give a few nontrivial examples of BV functions, referring to the work [2]. Note that by combining Theorem 4.8 and Theorem 4.4 we recover a part of the results in [2].

Theorem 4.8. (i) *Let F be a function such that $F \in \mathbb{D}^{2,p}$ and $\|\nabla F\|_H^{-1} \in L^q$ for some $p > 1$ and $q > 1$ with $1/p + 1/q < 1$. Let $A = \{F < x\}$ with $x \in \mathbb{R}$. Then $1_A \in BV(E)$.*

(ii) *Suppose that (E, H, μ) is a classical Wiener space on $[0,1]$. For $x > 0$, set $A = \{w \in E \mid \max_{0 \le s \le 1} |w(s)| < x\}$. Then $1_A \in BV(E)$.*

Proof. (i): From the assumptions, we have $\nabla^*(\nabla F/\|\nabla F\|_H) \in L^a$ for some $a > 1$. Indeed, keeping in mind the fact $\|\nabla F\|_H$, $\|\nabla^2 F\|_{H\otimes H} \in L^p$ due to Meyer's equivalence, let ε tend to 0 in the identity

$$\nabla^* \left(\frac{\nabla F}{\sqrt{\|\nabla F\|_H^2 + \varepsilon}} \right) = -\frac{LF}{\sqrt{\|\nabla F\|_H^2 + \varepsilon}} + \frac{\langle \nabla F \otimes \nabla F, \nabla^2 F \rangle_{H\otimes H}}{(\|\nabla F\|_H^2 + \varepsilon)^{3/2}}.$$

Now, set $\psi_n(y) = n 1_{[x-1/n,x]}(y)$, $\varphi_n(y) = \int_y^\infty \psi_n(z)\,dz$, and $\rho_n = \varphi_n(F)$. Then we have

$$\|\nabla \rho_n\|_{L^1(E\to H)} = \int_E \psi_n(F) \|\nabla F\|_H \, d\mu$$

$$
\begin{aligned}
&= -\int_E \left\langle \nabla \rho_n, \frac{\nabla F}{\|\nabla F\|_H} \right\rangle_H d\mu \\
&= -\int_E \rho_n \nabla^* \left(\frac{\nabla F}{\|\nabla F\|_H} \right) d\mu \\
&\leq \left\| \nabla^* \left(\frac{\nabla F}{\|\nabla F\|_H} \right) \right\|_{L^1},
\end{aligned}
$$

which is bounded in n. Since $\{\rho_n\}_{n\in\mathbb{N}}$ is uniformly bounded and converges to 1_A pointwise, Theorem 4.1 (iii) completes the proof.

(ii): Set $\rho_n(w) = 0 \vee n\,(1 - \max_{0\le s\le 1}|w(s)|/x) \wedge 1$, $w \in E$ for each $n \in \mathbb{N}$. By the calculation in the proof of Theorem 3.1 of [2], we have $\rho_n \in W^{1,1}$ and $\|\rho_n\|_{W^{1,1}}$ is bounded in n. Since $\{\rho_n\}_{n\in\mathbb{N}}$ is uniformly bounded and tends to 1_A pointwise, Theorem 4.1 (iii) completes the proof. Q.E.D.

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