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A Frontier of White Noise Analysis, in line with Itô Calculus

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Dedicated to Professor K. Itô

Abstract.

This note discusses two topics; one is the notion of the multiple Wiener integral and the other is the Lévy-Itô decomposition of a Lévy process.

Both have been taken up by Professor K. Itô showing the significance in stochastic analysis. The extensive development of the stochastic analysis at present largely depends on these discoveries by him.

It seems to be a good time to remind his profound ideas and to discuss some of future directions in probability theory.

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§1. Introduction

The White Noise Analysis has extensively developed in the last quarter of a century and now it is the time to be in search of further directions of research which will be still in line with the Itô's original contribution toward stochastic analysis.

Two directions are proposed in this line.

(1) Analysis of white noise functionals parameterized by a contour or a surface, namely some kinds of random fields.
Those fields are assumed to live in the Hilbert space (L^2) of white noise functionals (or of Brownian functionals). In order to analyze those functionals we shall appeal to the variational calculus for the random fields, and there is requested to introduce a suitable class of generalized white noise functionals. A good tool from the stochastic analysis for this purpose is the direct sum decomposition of the Hilbert space (L^2) in terms of

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the subspaces of *multiple Wiener integrals* (see [2]) or of those involving homogeneous chaos.

We then come to introduce classes of generalized white noise functionals, where we should note that our generalization can be done for each subspace given by the decomposition mentioned above. Representations of those functionals can be given by the so-called S-transform which looks like an infinite dimensional analogue of the Laplace transform.

To discuss a variational calculus for random fields in question some more probabilistic interpretation is necessary; however the decomposition of the basic Hilbert space is the milestone of the further study of random fields formed by white noise functionals.

(2) Reduction of random complex systems.

Given a random, evolutional complex system, we first try to obtain the *innovation* of the system. Under some reasonable conditions, we may assume that the innovation comes from a Lévy process. Some concrete results on this topic can be seen in [5].

Then, we are naturally led to the decomposition of the Lévy process established by Itô (see [1]). The decomposition itself was obtained earlier, by taking a relaxed view of rigor. When nonlinear functionals of innovation are considered, finer and rigorous results on the decomposition are necessary, so that the results in [1] are quite helpful. This fact can be seen as soon as we come to the analysis of nonlinear functionals of Poisson process (which is an elemental process) or of compound Poisson noise. It is noted that interesting results are obtained by viewing dissimilarity to the Gaussian case.

§2. Topic (1)

2.1. Decomposition of the white noise space

Let μ be the standard white noise measure on a space E^* of generalized functions on $\mathbb{R}$, and let $L^2(E^*, \mu) = (L^2)$ be the space of white noise functionals. One of the basic tools for the analysis on (L^2) is the direct sum decomposition due to Itô and Wiener:

$$(L^2) = \oplus_{n \geq 0} H_n,$$

where H_n is the space of multiple Wiener integrals or homogeneous chaos of degree n.

Another tool is the so-called S-transform (see e.g. I. Kubo and S. Takenaka [8] or [4, Chapt. 2]) defined by

$$(S\varphi)(\xi) = \exp[-\frac{1}{2}\|\xi\|^2] \int_{E^*} \exp[\langle x, \xi \rangle]\varphi(x)d\mu(x),$$

where φ is an (L^2)-functional. The S-transform gives a nice representation of white noise functionals. Indeed, the transform plays fundamental roles in two ways; one is that a visualized representation of φ is given, and the other is that it helps to introduce generalized white noise functionals which are very important in advanced white noise analysis.

2.2. The isomorphism

$$H_n \cong L^2(\mathbb{R}^n)^\wedge$$

(up to $\sqrt{n!}$), where $L^2(\mathbb{R}^n)^\wedge$ denotes the subspace of $L^2(\mathbb{R}^n)$ containing only symmetric functions. This isomorphism extends to

$$H_n^- \cong H^{-\frac{n+1}{2}}(\mathbb{R}^n)^\wedge,$$

where $H^m(\mathbb{R}^n)^\wedge$ is the subspace of the Sobolev space of order m involving symmetric functions. Define a space $(L^2)^-$ of generalized functionals by

$$(L^2)^- = \oplus c_n H_n^-,$$

where c_n is a certain sequence of positive numbers such that $c_n \to 0$. For details, we refer to the literature [4, Chapt.3.A]. It is noted that the choice of the $\{c_n\}$ depends on the singularity of functionals to be discussed.

Another space $(S)^*$ of generalized functional is defined by a Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*,$$

which is an infinite dimensional analogue of the triple to define the Schwartz distributions. The Rigorous definition and the properties of $(S)^*$ are found in [4, Chapt.4]. Also see [9, Chapt.4].

The spaces $(L^2)^-$ and $(S)^*$ of generalized white noise functionals are basic concept of the white noise analysis; indeed, the choice of these spaces is one of the advantages of our analysis. By using these spaces $(L^2)^-$ and $(S)^*$, we can carry on the analysis of white noise functionals in sufficiently large class and find significant applications in other fields like quantum dynamics. Actually, to be surprising enough, we can find an interesting application to the non-commutative geometry (e.g. [10]).

§3. Topic (2)

3.1. Elemental processes involved in innovation

We shall discuss the analysis of stochastic processes and random fields. The approach is done in line with

$$\textit{Reduction} \rightarrow \textit{Synthesis} \rightarrow \textit{Analysis}.$$

The step of the reduction is realized by taking the innovation for the random complex phenomena in question. Then, we are given a (Gaussian) white noise and/or a Poisson noise, both of which are *elemental* generalized stochastic processes with independent values at every time point, as is illustrated just below. The two cases have much similarity in the analysis on L^2-spaces; however dissimilarity is also interesting. Hence, they are discussed separately, except those properties in common.

In the multi-dimensional parameter case, say $\mathbb{R}^d$-parameter in general, innovations should still be generalized stochastic processes having independent values at every point in $\mathbb{R}^d$.

To fix the idea, we first take the one-dimensional parameter case. Then, under mild, and in fact reasonable assumptions, we may assume that innovation is the time derivative of some Lévy process with stationary independent increments. Let it be denoted by $L(t)$. We may assume it has no non-random part. Then, the Lévy-Itô decomposition gives us an expression of the form

$$L(t) = cB(t) + X(t),$$

where c is a constant, $B(t)$ is a Brownian motion, and $X(t)$ is a compound Poisson process which consists of independent Poisson processes with different heights of jumps (see [1]). Thus, we can conclude that a Brownian motion and each Poisson process being a component of the $X(t)$ are all *elemental* Lévy processes.

After the reduction follows the step of synthesis which means the construction of the functionals of (Gaussian) white noise and Poisson noises of different heights of jumps. It is, formally speaking, easy to form a general space of functionals of the innovation, however we need profound background. Actually rigorous interpretation can be seen e.g. in [6].

The collection of S-transforms of (L^2)-functionals $\{\varphi\}$ forms a Reproducing Kernel Hilbert space $\mathbf{F}$ which is isomorphic to (L^2). In short, a φ has a representation in terms of a functional (indeed, non-random functional) of a smooth function ξ. We can therefore appeal to the classical theory of functional analysis in order to carry on the calculus on (L^2).

3.2. Random fields

Coming to random fields, a certain class of them is introduced. Let $\mathbf{C}$ be a class of manifolds C in $\mathbb{R}^d$:

$$\mathbf{C} = \{C;\ \text{convex},\ C^\infty\text{-manifold},\ \approx\ S^{d-1}\},$$

where $\approx$ means a diffeomorphism.

The topology is introduced to $\mathbf{C}$ by using the Euclidean distance.

Given a random field $X(C) = X(C,x), x \in E^*(\mu)$, indexed by $C \in \mathbf{C}$. The S-transform is now of the form

$$(SX(C))(\xi) = \exp[-\frac{1}{2}\|\xi\|^2]\int_{E^*} \exp[\langle x,\xi\rangle]X(C,x)d\mu(x),$$

and it defines a U-functional:

$$(SX(C))(\xi) = U(C,\xi),\ \xi \in E.$$

Suppose that $X(C)$ is a linear functional of a multiple Wiener integral of degree n. Then, the associated U-functional is expressed in the form

$$U(C,\xi) = \int F(C,\mathbf{u})\xi(u)^{n\otimes}du^n.$$

Thus, we are ready to apply the classical theory of calculus of variations to the functional $U(C,\xi)$ of variable C (see [5, Chapt.6]).

We then come to the case of a Poisson noise.

The case of functionals of a single Poisson process $P(t)$, we introduce the U-transform following the paper by K. Saitô and A. Tsoi [14]. It is given by the formula

$$(U\varphi)(\xi) = C_P(\xi)^{-1}\int \exp[i < x,\xi >]\varphi(x)d\mu_P(x),$$

where μ_P is the probability distribution of Poisson noise $\dot{P}$, and where $C_P(\xi)$ is the characteristic functional of $\dot{P}$. Namely,

$$C_P(\xi) = \exp[\lambda\int(e^{i\xi(u)} - 1)du].$$

For simplicity, the intensity λ is taken to be 1 in this subsection.

Fact. Under the U-transform, a discrete chaos (in Wiener's sense) φ of degree n has a representation of the form

$$\int \cdots \int F(u)\Pi_1^n(e^{i\xi(u_j)} - 1)du^n,$$

where $u = (u_1, \cdots, u_n)$. (For proof see [14].)

Generalized Poisson noise functionals can be introduced by using this representation. We can further play a similar game for the analysis of them to the Gaussian case.

Generalization to the multi-dimensional parameter case is just straightforward. Also, a random field $X(C)$ of functional of Poisson noise is defined, and its variation can be discussed in a similar manner to the Gaussian case.

3.3. Stochastic variational equation

First note that, as a generalization of the infinitesimal equation, there is given a stochastic variational equation for a random field $X(C)$:

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C),$$

where $\{Y(s), s \in C\}$ is the innovation, and where $C' < C$ denotes that C' lies inside of C.

As was explained before, we apply S-transform in the Gaussian case, and we are given a variational equation for U-functional. When we come back to a random function by using S^{-1}, a difficulty arises. This can be illustrated as follows.

To fix the idea, we consider a Gaussian random field $X(C)$ which has a *causal* representation in terms of white noise. Namely, it is a linear functional of x expressed in the form:

$$X(C) = \int_{(C)} F(C, u)x(u)du^d,$$

where $F(C, u)$ is smooth in (C, u), and where (C) denotes the domain enclosed by the ovaloid C. Then, U-functional can be expressed as

$$U(C, \xi) = \int_{(C)} F(C, u)\xi(u)du^d.$$

The variation of U is easily computed to have

$$\delta U(C,\xi) = \int_C F(C,s)\xi(s)\delta n(s)ds + \int_{(C)} \delta F(C,u)\xi(u)du^d,$$

where the δF is the variation of F in the variable C which is defined in the classical sense, and where ds is the surface element over C.

In what follows, the representation of $X(C)$ in the above form is assumed to be *canonical.* Namely, the sigma-field $\mathbf{B}_C(X)$ generated by $X(C')$ with C' inside of C is equal to the sigma-field $\mathbf{B}_C(x)$ generated by $x(u)$, u being inside of C for every C.

Theorem. *The variation $\delta X(C)$ is given by*

$$\delta X(C) = \int_C F(C,s)x(s)\delta n(s)ds + \int_{(C)} \delta F(C,u)x(u)du^d.$$

The innovation is obtained from the first term of the right side.

Proof. This result may be thought of as an easy consequence of the definition of $\delta X(C,x)$, but not quite. A rigorous proof needs, in addition to the functional analysis, the following considerations on the restriction of the parameter and the choice of C. Hence, what follow in Subsection 3.4 are to be included in the proof. For details, see [5].

3.4. Other variations

Restriction of the parameter.

i) Gaussian case.

Given an $\mathbb{R}^d$ parameter white noise (E^*,μ). For $f \in L^2(\mathbb{R}^d)$ the stochastic bilinear form $\langle x,f\rangle, x \in E^*$, is defined which is subject to a Gaussian distribution $N(0,\|f\|^2)$. Take f to be an indicator function such that

$$f_t(u) = \chi_{I(t)}(u),\ t = (t_1,t_2,\cdots,t_d), I(t) = \Pi_j[0,t_j].$$

Then, $X(I(t)) = \langle x, f_t\rangle$ is a Brownian sheet, that is,

$$E[X(I(t))] = 0, E[X(I(t))\cdot X(I(s))] = \Pi_j(t_j \wedge s_j).$$

Set $t_d = 1$. Then, we are given an $\mathbb{R}^{d-1}$-dimensional parameter Brownian sheet. It is now ready to have an $\mathbb{R}^{d-1}$-dimensional parameter white noise by applying partial derivatives. Similarly, much lower dimensional parameter Brownian sheet and white noise can be derived.

The idea is the same for the restriction of the parameter to a hypersurface C which is a smooth ovaloid.

Proposition. *The restriction of the parameter of white noise can be done with the help of Brownian sheet.*

ii) Poisson case

For the Poisson noise the same trick can be applied as is easily shown. Just remind that the characteristic functional $C_P(\xi)$ of a Poisson noise with $\mathbb{R}^d$-parameter is

$$C_P(\xi) = \exp[\lambda \int (e^{i\xi(t)} - 1) dt^d],$$

where $\lambda > 0$ is the intensity. The associated measure on E^* is denoted by μ_p.

Take ξ to be $I(t)$ as above, and form a stochastic bilinear form $\langle x, f_t \rangle, x \in E^*(\mu_p)$, which is to be a Poisson sheet. A restriction of the parameter to a hyperplane defines a lower dimensional parameter Poisson noise. Also a restriction to a hypersurface is given.

As in the Gaussian case we can state a proposition. Since it is similar, so that we omit.

It is noted that observation of on a Poisson sheet, we can see invariance of Poisson noise under some transformations of the parameter space. This will be reported later.

Choice of the C.

One may ask why the parameter C for a random field should be taken to be a smooth ovaloid (convex, closed, without boundary). For one thing, the deformation of C will be done by members of a known transformation group acting on $\mathbb{R}^d$. For another reason, a white noise parameterized by a point in C should be defined. Indeed, as soon as the variational calculus is applied we are naturally led to define an innovation or white noise with parameter set C, as we have seen in Subsection 3.3. For this purpose a Gel'fand triple of functions spaces on C should be defined. Our assumptions for C guarantees the existence of a Gel'fand triple to be requested.

With the considerations mentioned above, we can deal with variational equations, not only for a Gaussian type field reviewed in this section, but also more general non-Gaussian fields and even in the case where $X(C)$ is a nonlinear functional of Poisson noise.

The above considerations are more significant if functionals of Poisson noise are discussed. For example we take a causal and linear functional of a homogeneous chaos, sample function of which is denoted by

x. It is expressed in the form

$$X(C) = \int_{(C)} F(C,u)x(u)du^d.$$

Note that, the *canonical property* (which is an easy generalization of the notion on canonical property discussed in [3, II [8]]) of the above representation follows easily under the assumption that the kernel $F(C,u)$ never vanishes. Hence, the innovation can immediately be obtained.

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