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Equivariant Diffusions on Principal Bundles

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Let $\pi : P \longrightarrow M$ be a smooth principal bundle with structure group G. This means that there is a C^∞ right multiplication $P \times G \longrightarrow P$, $u \mapsto u \cdot g$ say, of the Lie group G such that π identifies the space of orbits of G with the manifold M and π is locally trivial in the sense that each point of M has an open neighbourhood U with a diffeomorphism

$$\begin{tikzcd} \tau_U : \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

over U, which is equivariant with respect to the right action of G, i.e. if $\tau_u(b) = (\pi(b), k)$ then $\tau_u(b \cdot g) = (\pi(b), kg)$. Assume for simplicity that M is compact. Set $n = dimM$. The fibres, $\pi^{-1}(x)$, $x \in M$ are diffeomorphic to G and their tangent spaces $VT_uP(= kerT_u\pi)$, $u \in P$, are the 'vertical' tangent spaces to P. A *connection* on P, (or on π) assigns a complementary 'horizontal' subspace HT_uP to VT_uP in T_uP for each u, giving a smooth horizontal subbundle HTP of the tangent bundle TP to P. Given such a connection it is a classical result that for any C^1 curve: $\sigma : [0, T] \longrightarrow M$ and $u_0 \in \pi^{-1}(\sigma(0))$ there is a unique horizontal $\tilde{\sigma} : [0, T] \longrightarrow P$ which is a lift of σ, i.e. $\pi(\tilde{\sigma}(t)) = \sigma(t)$ and has $\tilde{\sigma}(0) = u_0$.

In his startling ICM article [8] Itô showed how this construction could be extended to give horizontal lifts of the sample paths of diffusion processes. In fact he was particularly concerned with the case when M is given a Riemannian metric $\langle, \rangle_x$, $x \in M$, the diffusion is Brownian motion on M, and P is the orthonormal frame bundle $\pi : OM \longrightarrow M$. Recall that each $u \in OM$ with $u \in \pi^{-1}(x)$ can be considered as an isometry $u : \mathbb{R}^n \longrightarrow T_xM$, $\langle, \rangle_x$ and a

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horizontal lift $\tilde{\sigma}$ determines parallel translation of tangent vectors along σ

$$
\begin{aligned}
/\!/_t \equiv /\!/(\sigma)_t : \qquad & T_{\sigma(\cdot)}M \to T_{\sigma(t)}M \\
& v \mapsto \tilde{\sigma}(t)(\tilde{\sigma}(0))^{-1}v.
\end{aligned}
$$

The resulting parallel translation along Brownian paths extends also to parallel translation of forms and elements of $\wedge^p TM$. This enabled Itô to use his construction to obtain a semi-group acting on differential forms

$$P_t\phi = \mathbb{E}(/\!/_t^{-1})_*(\phi) = \mathbb{E}\phi(/\!/_t -).$$

As he pointed out this is not the semi-group generated by the Hodge-Kodaira Laplacian, Δ. To obtain that generated by the Hodge-Kodaira Laplacian, Δ, some modification had to be made since the latter contains zero order terms, the so called Weitzenbock curvature terms. The resulting probabilistic expression for the heat semi-groups on forms has played a major role in subsequent development.

In [5] we go in the opposite direction starting with a diffusion with smooth generator $\mathcal{B}$ on P, which is G-invariant and so projects to a diffusion generator $\mathcal{A}$ on M. We assume the symbol $\sigma_{\mathcal{A}}$ has constant rank so determining a sub-bundle E of TM, (so $E = TM$ if $\mathcal{A}$ is elliptic). We show that this set-up induces a 'semi-connection' on P over E (a connection if $E = TM$) with respect to which $\mathcal{B}$ can be decomposed into a horizontal component $\mathcal{A}^H$ and a vertical part $\mathcal{B}^V$. Moreover any vertical diffusion operator such as $\mathcal{B}^V$ induces only zero order operators on sections of associated vector bundles.

There are two particularly interesting examples. The first when $\pi\colon GLM \to M$ is the full linear frame bundle and we are given a stochastic flow $\{\xi_t : 0 \le t < \infty\}$ on M, generator $\mathcal{A}$, inducing the diffusion $\{u_t : 0 \le t < \infty\}$ on GLM by

$$u_t = T\xi_t(u_0).$$

Here we can determine the connection on GLM in terms of the LeJan-Watanabe connection of the flow [12], [1], as defined in [6], [7], in particular giving conditions when it is a Levi-Civita connection. The zero order operators arising from the vertical components can be identified with generalized Weitzenbock curvature terms.

The second example slightly extends the above framework by letting $\pi : P \to M$ be the evaluation map on the diffeomorphism group Diff$M$ of M given by $\pi(h) := h(x_0)$ for a fixed point x_0 in M. The group G corresponds to the group of diffeomorphisms fixing x_0. Again we take a flow $\{\xi_t(x) : x \in M, t \ge 0\}$ on M, but now the process on Diff$M$ is just the right invariant process determined by $\{\xi_t : 0 \le t < \infty\}$. In this case the horizontal lift to the diffeomorphism group of the diffusion $\{\xi_t(x_0) : 0 \le t < \infty\}$ on M is obtained by 'removal of redundant noise', c.f. [7] while the vertical process is a flow of diffeomorphisms preserving x_0, driven by the redundant noise.

Here we report briefly on some of the main results to appear in [5] and give details of a more probabilistic version Theorem 2.5 below: a skew product decomposition which, although it has a statement not explicitly mentioning connections, relates to Itô's pioneering work on the existence of horizontal lifts. The derivative flow example and a simplified version of the stochastic flow example are described in § 3.

The decomposition and lifting apply in much more generality than with the full structure of a principal bundle, for example to certain skew products and invariant processes on foliated manifolds. This will be reported on later. Earlier work on such decompositions includes [4] [13].

§1. Construction

A. If $\mathcal{A}$ is a second order differential operator on a manifold X, denote by $\sigma^{\mathcal{A}} : T^*X \to TX$ its symbol determined by

$$df\left(\sigma^{\mathcal{A}}(dg)\right) = \frac{1}{2}\mathcal{A}\,(fg) - \frac{1}{2}\mathcal{A}\,(f)\,g - \frac{1}{2}f\mathcal{A}\,(g),$$

for C^2 functions f, g. The operator is said to be *semi-elliptic* if $df\left(\sigma^{\mathcal{A}}(df)\right) \geq 0$ for each $f \in C^2(X)$, and *elliptic* if the inequality holds strictly. Ellipticity is equivalent to $\sigma^{\mathcal{A}}$ being onto. It is called a *diffusion operator* if it is semi-elliptic and annihilates constants, and is *smooth* if it sends smooth functions to smooth functions.

Consider a smooth map $p : N \to M$ between smooth manifolds M and N. By a *lift* of a diffusion operator $\mathcal{A}$ on M over p we mean a diffusion operator $\mathcal{B}$ on N such that

$$(1) \qquad \mathcal{B}(f \circ p) = (\mathcal{A}f) \circ p$$

for all C^2 functions f on M. Suppose $\mathcal{A}$ is a smooth diffusion operator on M and $\mathcal{B}$ is a lift of $\mathcal{A}$.

Lemma 1.1. *Let $\sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{A}}$ be respectively the symbols for $\mathcal{B}$ and $\mathcal{A}$. The following diagram is commutative for all $u \in p^{-1}(x)$, $x \in M$:*

$$\begin{tikzcd}
T_u^*N \arrow[r, "\sigma_u^{\mathcal{B}}"] & T_uN \arrow[d, "Tp"] \\
T_x^*M \arrow[u, "(Tp)^*"] \arrow[r, "\sigma_x^{\mathcal{A}}"'] & T_xM.
\end{tikzcd}$$

B. Semi-connections on principal bundles. Let M be a smooth finite dimensional manifold and $P(M, G)$ a principal fibre bundle over M with structure group G a Lie group. Denote by $\pi : P \to M$ the projection and R_a the right translation by a.

Definition 1.2. *Let E be a sub-bundle of TM and $\pi : P \to M$ a principal G-bundle. An E semi-connection on $\pi : P \to M$ is a smooth sub-bundle $H^E TP$ of TP such that*

(i) *$T_u\pi$ maps the fibres $H^E T_u P$ bijectively onto $E_{\pi(u)}$ for all $u \in P$.*
(ii) *$H^E TP$ is G-invariant.*

Notes.
(1) Such a semi-connection determines and is determined by, a smooth horizontal lift:

$$h_u : E_{\pi(u)} \to T_u P, \qquad u \in P$$

such that

(i) $T_u \pi \circ h_u(v) = v$, for all $v \in E_x \subset T_x M$;
(ii) $h_{u \cdot a} = T_u R_a \circ h_u$.

The *horizontal subspace* $H^E T_u P$ at u is then the image at u of h_u, and the composition $h_u \circ T_u P$ is a projection onto $H^E T_u P$.
(2) Let $F = P \times V/\sim$ be an associated vector bundle to P with fibre V. An element of F is an equivalence class $[(u, e)]$ such that $(ug, g^{-1}e) \sim (u, e)$. Set $\tilde{u}(e) = [(u, e)]$. An E semi-connection on P gives a covariant derivative on F. Let Z be a section of F and $w \in E_x \subset T_x M$, the covariant derivative $\nabla_w Z \in F_x$ is defined, as usual for connections, by

$$\nabla_w Z = u(d\tilde{Z}(h_u(w))), \qquad u \in \pi^{-1}(x) = F_x.$$

Here $\tilde{Z} : P \to V$ is $\tilde{Z}(u) = \tilde{u}^{-1} Z(\pi(u))$ considering $\tilde{u}$ as an isomorphism $\tilde{u} : V \to F_{\pi(u)}$. This agrees with the 'semi-connections on E' defined in Elworthy-LeJan-Li [7] when P is taken to be the linear frame bundle of TM and $F = TM$. As described there, any semi-connection can be completed to a genuine connection, but not canonically.

Consider on P a diffusion generator $\mathcal{B}$, which is *equivariant*, i.e.

$$\mathcal{B}f \circ R_a = \mathcal{B}(f \circ R_a), \qquad \forall f, g \in C^2(P, R),\ a \in G.$$

The operator $\mathcal{B}$ induces an operator $\mathcal{A}$ on the base manifold M by setting

$$\mathcal{A}f(x) = \mathcal{B}(f \circ \pi)(u), \qquad u \in \pi^{-1}(x), f \in C^2(M), \tag{2}$$

which is well defined since

$$\mathcal{B}(f \circ \pi)(u \cdot a) = \mathcal{B}((f \circ \pi))(u).$$

Let $E_x := Image(\sigma_x^{\mathcal{A}}) \subset T_x M$, the image of $\sigma_x^{\mathcal{A}}$. Assume the dimension of $E_x = p$, independent of x. Set $E = \cup_x E_x$. Then $\pi : E \to M$ is a subbundle of TM.

Theorem 1.3. *Assume $\sigma^{\mathcal{A}}$ has constant rank. Then $\sigma^{\mathcal{B}}$ gives rise to a semi-connection on the principal bundle P whose horizontal map is given by*

$$h_u(v) = \sigma^{\mathcal{B}}\left((T_u\pi)^*\alpha\right) \tag{3}$$

where $\alpha \in T^*_{\pi(u)}M$ satisfies $\sigma_x^{\mathcal{A}}(\alpha) = v$.

Proof. To prove h_u is well defined we only need to show $\psi(\sigma^{\mathcal{B}}(T_u\pi^*(\alpha))) = 0$ for every 1-form ψ on P and for every α in ker $\sigma_x^{\mathcal{A}}$. Now $\sigma^{\mathcal{A}}\alpha = 0$ implies by Lemma 1.1 that

$$0 = \alpha\sigma^{\mathcal{A}}(\alpha) = (T\pi)^*(\alpha)\sigma^{\mathcal{B}}((T\pi)^*(\alpha)).$$

Thus $T\pi^*(\alpha)\sigma^{\mathcal{B}}(T\pi^*(\alpha)) = 0$. On the other hand we may consider $\sigma^{\mathcal{B}}$ as a bilinear form on T^*P and then for all $\beta \in T_u^*P$,

$$\begin{aligned}
&\sigma^{\mathcal{B}}(\beta + t(T\pi)^*(\alpha), \beta + t(T\pi)^*(\alpha)) \\
&= \sigma^{\mathcal{B}}(\beta,\beta) + 2t\sigma^{\mathcal{B}}(\beta, (T\pi)^*(\alpha)) + t^2\sigma^{\mathcal{B}}((T\pi)^*\alpha, (T\pi)^*\alpha) \\
&= \sigma^{\mathcal{B}}(\beta,\beta) + 2t\sigma^{\mathcal{B}}(\beta, (T\pi)^*(\alpha)).
\end{aligned}$$

Suppose $\sigma^{\mathcal{B}}(\beta, (T\pi)^*(\alpha)) \neq 0$. We can then choose t such that

$$\sigma^{\mathcal{B}}(\beta + t(T\pi)^*(\alpha), \beta + t(T\pi)^*(\alpha)) < 0,$$

which contradicts the semi-ellipticity of $\mathcal{B}$.

We must verify (i) $T_u\pi \circ h_u(v) = v$, $v \in E_x \subset T_xM$ and (ii) $h_{u\cdot a} = T_uR_a \circ h_u$. The first is immediate by Lemma 1.1 and for the second use the fact that $T\pi \circ TR_a = T\pi$ for all $a \in G$ and the equivariance of $\sigma^{\mathcal{B}}$. ■

§2. Horizontal lifts of diffusion operators and decompositions of equivariant operators

A. Denote by $C^\infty\Omega^p$ the space of smooth differential p-forms on a manifold M. To each diffusion operator $\mathcal{A}$ we shall associate a unique operator $\delta^{\mathcal{A}}$. The horizontal lift of $\mathcal{A}$ can be defined to be the unique operator such that the associated operator $\bar{\delta}$ vanishes on vertical 1-forms and such that $\bar{\delta}$ and $\delta^{\mathcal{A}}$ are intertwined by the lift map π^* acting on 1-forms.

Proposition 2.1. *For each smooth diffusion operator $\mathcal{A}$ there is a unique smooth differential operator $\delta^{\mathcal{A}} : C^\infty(\Omega^1) \to C^\infty\Omega^0$ such that*

(1) $\delta^{\mathcal{A}}(f\phi) = df\sigma^{\mathcal{A}}(\phi)_x + f \cdot \delta^{\mathcal{A}}(\phi)$

(2) $\delta^{\mathcal{A}}(df) = \mathcal{A}(f)$.

For example if $\mathcal{A}$ has Hörmander representation

$$\mathcal{A} = \frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{X^j}\mathcal{L}_{X^j} + \mathcal{L}_A$$

for some C^1 vector fields X^i, A then

$$\delta^{\mathcal{A}} = \frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{X^j}\iota_{X^j} + \iota_A$$

where ι_A denotes the interior product of the vector field A acting on differential forms.

Definition 2.2. *Let S be a C^∞ sub-bundle of TN for some smooth manifold N. A diffusion operator $\mathcal{B}$ on N is said to be along S if $\delta^{\mathcal{B}}\phi = 0$ for all 1-forms ϕ which vanish on S; it is said to be strongly cohesive if $\sigma^{\mathcal{B}}$ has constant rank and $\mathcal{B}$ is along the image of $\sigma^{\mathcal{B}}$.*

To be along S implies that any Hörmander form representation of $\mathcal{B}$ uses only vector fields which are sections of S.

Definition 2.3. *When a diffusion operator $\mathcal{B}$ on P is along the vertical foliation VTP of the $\pi : P \to M$ we say $\mathcal{B}$ is vertical, and when the bundle has a semi-connection and $\mathcal{B}$ is along the horizontal distribution we say $\mathcal{B}$ is horizontal.*

If $\pi : P \to M$ has an E semi-connection and $\mathcal{A}$ is a smooth diffusion operator along E it is easy to see that $\mathcal{A}$ has a unique horizontal lift $\mathcal{A}^H$, i.e. a smooth diffusion operator $\mathcal{A}^H$ on P which is horizontal and is a lift of $\mathcal{A}$ in the sense of (1). By uniqueness it is equivariant.

B. The action of G on P induces a homomorphism of the Lie algebra $\mathfrak{g}$ of G with the algebra of right invariant vector fields on P: if $\alpha \in \mathfrak{g}$,

$$A^\alpha(u) = \frac{d}{dt}\bigg|_{t=0} u\exp(t\alpha),$$

and A^α is called the fundamental vector field corresponding to α. Take a basis $A_1, \ldots, A_k$ of $\mathfrak{g}$ and denote the corresponding fundamental vector fields by $\{A_i^*\}$.

We can now give one of the main results from [5]:

Theorem 2.4. *Let $\mathcal{B}$ be an equivariant operator on P with $\mathcal{A}$ the induced operator on the base manifold. Assume $\mathcal{A}$ is strongly cohesive. Then there is a unique semi-connection on P over E for which $\mathcal{B}$ has a decomposition*

$$\mathcal{B} = \mathcal{A}^H + \mathcal{B}^V,$$

where $\mathcal{A}^H$ is horizontal and $\mathcal{B}^V$ is vertical. Furthermore $\mathcal{B}^V$ has the expression $\sum \alpha^{ij} \mathcal{L}_{A_i^*} \mathcal{L}_{A_j^*} + \sum \beta^k \mathcal{L}_{A_k^*}$, where α^{ij} and β^k are smooth functions on P, given by $\alpha^{k\ell} = \tilde{\omega}^k\left(\sigma^{\mathcal{B}}(\tilde{\omega}^\ell)\right)$, and $\beta^\ell = \delta^{\mathcal{B}}(\tilde{\omega}^\ell)$ for $\tilde{\omega}$ any connection 1-form on P which vanishes on the horizontal subspaces of this semi-connection.

We shall only prove the first part of Theorem 2.4 here. The semi-connection is the one given by Theorem 1.3, and we define $\mathcal{A}^H$ to be the horizontal lift of $\mathcal{A}$. The proof that $\mathcal{B}^V := \mathcal{B} - \mathcal{A}^H$ is vertical is simplified by using the fact that a diffusion operator $\mathcal{D}$ on P is vertical if and only if for all C^2 functions f_1 on P and f_2 on M

$$\mathcal{D}(f_1(f_2 \circ \pi)) = (f_2 \circ \pi)\mathcal{D}(f_1). \tag{4}$$

Set $\tilde{f_2} = f_2 \circ \pi$. Note

$$\left(\mathcal{B} - \mathcal{A}^H\right)(f_1\tilde{f_2}) = \tilde{f_2}(\mathcal{B} - \mathcal{A}^H)f_1 + f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f_2} + 2(df_1)\sigma^{\mathcal{B}-\mathcal{A}^H}(d\tilde{f_2}).$$

Therefore to show $(\mathcal{B} - \mathcal{A}^H)$ is vertical we only need to prove

$$f_1(\mathcal{B} - \mathcal{A}^H)\tilde{f_2} + 2(df_1)\sigma^{\mathcal{B}-\mathcal{A}^H}(d\tilde{f_2}) = 0.$$

Recall Lemma 1.1 and use the natural extension of $\sigma^{\mathcal{A}}$ to $\sigma^{\mathcal{A}} : E^* \to E$ and the fact that by (3) $h \circ \sigma_x^{\mathcal{A}} = \sigma^{\mathcal{B}}(T_u\pi)^*$ to see

$$\begin{aligned}\sigma^{\mathcal{A}^H}(d\tilde{f_2}) &= \left(h \circ \sigma^{\mathcal{A}} h^*\right)(df_2 \circ T\pi) = h \circ \sigma^{\mathcal{A}} df_2 \\ &= \sigma^{\mathcal{B}}(df_2 \circ T\pi) = \sigma^{\mathcal{B}}(d\tilde{f_2}),\end{aligned}$$

and so $\sigma^{(\mathcal{B}-\mathcal{A}^H)}(d\tilde{f_2}) = 0$. Also by equation (1)

$$(\mathcal{B} - \mathcal{A}^H)\tilde{f_2} = \mathcal{A}f_2 - \mathcal{A}^H\tilde{f_2} = 0.$$

This shows that $\mathcal{B} - \mathcal{A}^H$ is vertical. ■

Define $\alpha : P \to \mathfrak{g} \otimes \mathfrak{g}$ and $\beta : P \to \mathfrak{g}$ by

$$\alpha(u) = \sum \alpha^{ij}(u) A_i \otimes A_j$$

$$\beta(u) = \sum \beta^k(u) A_k.$$

It is easy to see that $\mathcal{B}^V$ depends only on α, β and the expression is independent of the choice of basis of $\mathfrak{g}$. From the invariance of $\mathcal{B}$ we obtain

$$\begin{aligned} \alpha(ug) &= (ad(g) \otimes ad(g))\, \alpha(u), \\ \beta(ug) &= ad(g)\beta(u) \end{aligned}$$

for all $u \in P$ and $g \in G$.

C. Theorem 2.4 has a more directly probabilistic version. For this let $\pi : P \to M$ be as before and for $0 \leq l < r < \infty$ let $C(l, r; P)$ be the space of continuous paths $y : [l, r] \to P$ with its usual Borel σ-algebra. For such write $l_y = l$ and $r_y = r$. Let $C(*, *; P)$ be the union of such spaces. It has the standard additive structure under concatenation: if y and y' are two paths with $r_y = l_{y'}$ and $y(r_y) = y'(l_{y'})$ let $y + y'$ be the corresponding element in $C(l_y, r_{y'}; P)$. The *basic* σ-algebra of $C(*, *, P)$ is defined to be the pull back by π of the usual Borel σ-algebra on $C(*, *; M)$.

Consider the laws $\{\mathbb{P}_a^{l,r} : 0 \leq l < r, a \in P\}$ of the process running from a between times l and r, associated to a smooth diffusion operator $\mathcal{B}$ on P. Assume for simplicity that the diffusion has no explosion. Thus $\{\mathbb{P}_a^{l,r}, a \in P\}$ is a kernel from P to $C(l, r; P)$. The right action R_g by g in G extends to give a right action, also written R_g, of G on $C(*, *, P)$. Equivariance of $\mathcal{B}$ is equivalent to

$$\mathbb{P}_{ag}^{l,r} = (R_g)_* \mathbb{P}_a^{l,r}$$

for all $0 \leq l \leq r$ and $a \in P$. If so $\pi_*(\mathbb{P}_a^{l,r})$ depends only on $\pi(a)$, l, r and gives the law of the induced diffusion $\mathcal{A}$ on M. We say that such a diffusion $\mathcal{B}$ is *basic* if for all $a \in P$ and $0 \leq l < r < \infty$ the basic σ-algebra on $C(l, r; P)$ contains all Borel sets up to $\mathbb{P}_a^{l,r}$ negligible sets, i.e. for all $a \in P$ and Borel subsets B of $C(l, r; P)$ there exists a Borel subset A of $C(l, r, M)$ s.t. $\mathbb{P}_a(\pi^{-1}(A)\Delta B) = 0$.

For paths in G it is more convenient to consider the space $C_{id}(l, r; G)$ of continuous $\sigma : [l, r] \to G$ with $\sigma(l) = id$ for 'id' the identity element. The corresponding space $C_{id}(*, *, G)$ has a multiplication

$$C_{id}(s, t; G) \times C_{id}(t, u; G) \longrightarrow C_{id}(s, u; G)$$

$$(g, g') \mapsto g \times g'$$

where $(g \times g')(r) = g(r)$ for $r \in [s, t]$ and $(g \times g')(r) = g(t)g'(r)$ for $r \in [t, u]$.

Given probability measures $\mathbb{Q}$, $\mathbb{Q}'$ on $C_{id}(s, t; G)$ and $C_{id}(t, u; G)$ respectively this determines a convolution $\mathbb{Q} * \mathbb{Q}'$ of $\mathbb{Q}$ with $\mathbb{Q}'$ which is a probability measure on $C_{id}(s, u; G)$.

Theorem 2.5. *Given the laws $\{\mathbb{P}_a^{l,r} : a \in P, 0 \le l < r < \infty\}$ of an equivariant diffusion $\mathcal{B}$ as above with $\mathcal{A}$ strongly cohesive there exist probability kernels $\{\mathbb{P}_a^{H,l,r} : a \in P\}$ from P to $C(l,r;P)$, $0 \le l < r < \infty$ and $\mathbb{Q}_y^{l,r}$, defined $\mathbb{P}^{l,r}$ a.s. from $C(l,r,P)$ to $C_{id}(l,r;G)$ such that*

(i) *$\{\mathbb{P}_a^{H,l,r} : a \in P\}$ is equivariant, basic and determining a strongly cohesive generator.*

(ii) *$y \mapsto \mathbb{Q}_y^{l,r}$ satisfies*

$$\mathbb{Q}_{y+y'}^{l_y,r_{y'}} = \mathbb{Q}_y^{l_y,r_y} * \mathbb{Q}_{y'}^{l_{y'},r_{y'}}$$

for $\mathbb{P}^{l_y,r_y} \otimes \mathbb{P}^{l_{y'},r_{y'}}$ almost all y, y' with $r_y = l_{y'}$.

(iii) *For U a Borel subset of $C(l,r,P)$,*

$$\mathbb{P}_a^{l,r}(U) = \int\int \chi_U(y_\cdot \cdot g_\cdot)\mathbb{Q}_y^{l,r}(dg)\mathbb{P}_a^{H,l,r}(dy).$$

The kernels $\mathbb{P}_a^{H,l,r}$ are uniquely determined as are the $\{\mathbb{Q}_y^{l,r} : y \in \mathbb{R}\}$, $\mathbb{P}_a^{H,l,r}$ a.s. in y for all a in P. Furthermore $\mathbb{Q}_y^{l,r}$ depends on y only through its projection $\pi(y)$ and its initial point y_l.

Proof. Fix a in P and let $\{b_t : l \le r \le t\}$ be a process with law $\mathbb{P}_a^{l,r}$. By Theorem 2.4 we can assume that $b_\cdot$ is given by an s.d.e. of the form

$$db_t = \tilde{X}(b_t) \circ dB_t + \tilde{X}^0(b_t)dt + A(b_t) \circ d\beta_t + V(b_t)dt \tag{5}$$

where $\tilde{X} : P \times \mathbb{R}^p \to TP$ is the horizontal lift of some $X : M \times \mathbb{R}^p \to E$, $\tilde{X}^0$ is the horizontal lift of a vector field X^0 on M, while $A : P \times \mathbb{R}^1 \to TP$ and the vector field V are vertical and determine $\mathcal{B}^V$. Here $B_\cdot$ and $\beta_\cdot$ are independent Brownian motions on $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, some q, and we are using the semi-connection on P induced by $\mathcal{B}$ as in Theorem 1.3.

Let $\{\tilde{x}_t : l \le t \le r\}$ satisfy

$$\begin{aligned} d\tilde{x}_t &= \tilde{X}(\tilde{x}_t) \circ dB_t + \tilde{X}^0(\tilde{x}_t)dt \\ \tilde{x}_l &= a \end{aligned} \tag{6}$$

so $\tilde{x}_\cdot$ is the horizontal lift of $\{\pi(b_t) : l \le t \le r\}$. Then there is a unique continuous process $\{g_t : l \le t \le r\}$ in G with $g_l = id$ such that

$$\tilde{x}_t g_t = b_t.$$

We have to analyse $\{g_t : l \le t \le r\}$. Using local trivialisations of $\pi : P \to M$ we see it is a semi-martingale. As in [9], Proposition 3.1 on page 69,

$$db_t = TR_{g_t}(\circ d\tilde{x}_t) + A^{g_t^{-1} \circ dg_t}(b_t)$$

giving

$$\tilde{\omega}(\circ db_t) = \tilde{\omega}\left(A^{g_t^{-1}\circ dg_t}(b_t)\right) = g_t^{-1} \circ dg_t$$

for any smooth connection form $\tilde{\omega} : P \to \mathfrak{g}$ on P which vanishes on $H^E TP$. Thus

$$\begin{aligned} dg_t &= TL_{g_t}\tilde{\omega}\big(A(\tilde{x}_t g_t) \circ d\beta_t + V(\tilde{x}_t g_t)dt\big) \\ g_l &= id, \qquad l \le t \le r. \end{aligned} \tag{7}$$

For $y \in C(l, r : P)$ let $\{g_t^y : l \le t \le r\}$ be the solution of

$$\begin{aligned} dg_t^y &= TL_{g_t^y}\tilde{\omega}\big(A(y_t g_t^y) \circ d\beta_t + V(y_t g_t^y)dt\big) \\ g_l^y &= id \end{aligned} \tag{8}$$

(where the Stratonovich equation is interpreted with '$dy_t d\beta_t = 0$'). Since $\beta.$ and $B.$ and hence $\beta.$ and $\tilde{x}.$ are independent we see $g = g^{\tilde{x}}$ almost surely. For a discussion of some technicalities concerning skew products, see [16].

For $y.$ in $C(*, *; P)$ let $\{h(y)_t : l_y \le t \le r_y\}$ be the horizontal lift of $\pi(y).$, starting at y_{l_y}. This exists for almost all y as can be seen either by the extension of Itô's result to general principal bundles, e.g. using (6), or by the existence of measurable sections using the fact that $\mathcal{A}^H$ is basic. Define $\mathbb{P}_a^{H,l,r}$ to be the law of $\tilde{x}.$ above and $Q_y^{l,r}$ to be that of $g^{h(y)}$. Clearly conditions (i) is satisfied.

To check (ii) take y and y' with $r_y = l_{y'}$. Then

$$h(y + y') = h(y) + h(y')\left(g_{r_y}^{h(y)}\right)^{-1},$$

writing $y = h(y)g^{h(y)}$ and $y' = h(y')g^{h(y')}$. For $r_y \le t \le r_{y'}$ this shows

$$(y + y')_t = h(y')_t\left(g_{r_y}^{h(y)}\right)^{-1} g_t^{h(y+y')}.$$

But $(y + y')_t = y'_t = h(y')_t g_t^{h(y')}$ and so we have $g_t^{h(y+y')} = g_{r_y}^{h(y)} g_t^{h(y')}$ for $t \ge r_y$, giving $g^{h(y+y')} = g^{h(y)} \times g^{h(y')}$ almost surely. This proves (ii).

For uniqueness suppose we have another set of probability measures denoted $\tilde{\mathbb{Q}}_y^{l,r}$ and $\tilde{P}_a^{H,l,r}$ which satisfy (i), (ii), (iii). Since $\{\tilde{\mathbb{P}}_a^{H,l,r}\}_a$ is equivariant and induces $\mathcal{A}$ on M we can apply the preceding argument to it in place of $\{\mathbb{P}_a^{l,r}\}_a$. However since it is basic the term involving β in the stochastic differential equation (6) must vanish. Since it is also strongly cohesive the vertical part V must vanish also and we have $\tilde{\mathbb{P}}_a^{H,l,r} = \mathbb{P}_a^{H,l,r}$. On the other hand in the decomposition $b_t = \tilde{x}_t g_t^{\tilde{x}_t}$ the law of $g_\cdot^{\tilde{x}}$ is determined by those of $b.$ and $\tilde{x}.$ but $\mathbb{Q}_y^{l,r}$ is the conditional law of $g_\cdot^{\tilde{x}\cdot}$ given $\tilde{x} = y.$ and so is uniquely determined as described. ∎

In fact $\mathbb{Q}_y^{l,r}$ is associated to the time dependent generator which at $g \in G$ and $t \in [l,r]$ is $\sum \alpha^{ij}(h(y)_t g)\mathcal{L}_{A_i}\mathcal{L}_{A_j} + \sum \beta^k(h(y)_t g)\mathcal{L}_{A_k}$ for α^{ij} and β^k as defined in Theorem 2.4 while $\mathbb{P}^{H,l,r}$ is associated to $\mathcal{A}^H$.

§3. Stochastic flows and derivative flows

A. Derivative flows. Let $\mathcal{A}$ on M be given in Hörmander form

$$\mathcal{A} = \frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{X^j}\mathcal{L}_{X^j} + \mathcal{L}_A$$

for some vector fields $X^1, \dots X^m$, A. As before let $E_x = span\{X^1(x), \dots, X^m(x)\}$ and assume $dim E_x$ is constant, p, say, giving a sub-bundle $E \subset TM$. The $X^1(x), \dots, X^m(x)$ determine a vector bundle map of the trivial bundle $\underline{\mathbb{R}}^m$

$$X : \underline{\mathbb{R}}^m \longrightarrow TM$$

with $\sigma^{\mathcal{A}} = X(x)X(x)^*$. We can, and will, consider X as a map $X : \underline{\mathbb{R}}^m \to E$.

As such it determines (a) a Riemannian metric $\{\langle,\rangle_x : x \in M\}$ on E (the same as that determined by $\sigma^{\mathcal{A}}$) and (b) a metric connection $\breve{\nabla}$ on E uniquely defined by the requirement that for each x in M,

$$\breve{\nabla}_v X(e) = 0$$

for all $v \in T_xM$ whenever e is orthogonal to the kernel of T_xM. Then for any differentiable section U of E,

$$\breve{\nabla}_v U = Y(x) d\left(Y_.(U(\cdot))\right)(v), \qquad v \in T_xM, \tag{9}$$

where Y is the $\mathbb{R}^m$ valued 1-form on M given by

$$\langle Y_x(v), e\rangle_{\mathbb{R}^m} = \langle X(x)(e), v\rangle_x, \qquad e \in \mathbb{R}^m, v \in E_x, x \in M$$

e.g. [7] where it is referred to as the LeJan-Watanabe connection in this context. By a theorem of Narasimhan and Ramanan [14] all metric connections on E arise this way, see [15], [7].

For $\{B_t : 0 \le t < \infty\}$ a Brownian motion on $\mathbb{R}^m$, the stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{10}$$

determines a Markov process with differential generator $\mathcal{A}$. Over each solution $\{x_t : 0 \le t < \rho\}$, where ρ is the explosion time, there is a 'derivative' process $\{v_t : 0 \le t < \rho\}$ in TM which we can write as $\{T\xi_t(v_0) : 0 \le t < \rho\}$

with $T\xi_t : T_{x_0}M \to T_{x_t}M$ linear. This would be the derivative of the flow $\{\xi_t : 0 \leq t < \rho\}$ of the stochastic differential equation when the stochastic differential equation is strongly complete. In general it is given by a stochastic differential equation on the tangent bundle TM, or equivalently by a covariant equation along $\{x_t : 0 \leq t < \rho\}$:

$$Dv_t = \nabla X(v_t) \circ dB_t + \nabla A(v_t)dt$$

with respect to any torsion free connection. Take P to be the linear frame bundle $GL(M)$ of M, treating $u \in GL(M)$ as an isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$. For $u_0 \in GLM$ we obtain a process $\{u_t : 0 \leq t < \rho\}$ on GLM by

$$u_t = T\xi_t \circ u_0.$$

Let $\mathcal{B}$ be its differential generator. Clearly it is equivariant and a lift of $\mathcal{A}$.

A proof of the following in the context of stochastic flows, is given later. For $w \in E_x$, set

$$Z^w(y) = X(y)Y(x)(w). \tag{11}$$

Theorem 3.1. *The semi-connection ∇ induced by $\mathcal{B}$ is the adjoint connection of the LeJan-Watanabe connection $\breve{\nabla}$ determined by X, as defined by (9),* [7]. *Consequently $\nabla_w V = L_{Z^w} V$ for any vector field V and $w \in E$ also $\nabla_{V(x)} Z^w$ vanishes if $w \in E_x$.*

In the case of the derivative flow the α, β of Theorem 2.4 have an explicit expression: for $u \in GLM$,

$$\begin{cases} \alpha(u) = \dfrac{1}{2}\sum \left(u^{-1}(-)\breve{\nabla}_{u(-)}X^p\right) \otimes \left(u^{-1}(-)\breve{\nabla}_{u(-)}X^p\right) \\ \beta(u) = -\dfrac{1}{2}\sum u^{-1}\breve{\nabla}_{\breve{\nabla}_{u(-)}X^p}X^p - \dfrac{1}{2}u^{-1}\text{Ric}^{\#}u(-). \end{cases} \tag{12}$$

Here $\breve{R}$ is the curvature tensor for $\breve{\nabla}$ and $\breve{Ric}^{\#} : TM \to E$ the Ricci curvature defined by $\breve{Ric}^{\#}(v) = \sum_{j=1}^{p} \breve{R}(v, e^j)e^j$, $v \in T_xM$.

Equivariant operators on GLM determine operators on associated bundles, such as $\wedge^q TM$. If the original operator was vertical this turns out to be a zero order operator (as is shown in [5] for general principal bundles) and in the case of $\wedge^q TM$ these operators are the generalized Weitzenbock curvature operators described in [7]. In particular for differential 1-forms the operator is $\phi \mapsto \phi(Ric^{\#}-)$. To see this, as an illustrative example, given a 1-form ϕ

define $\tilde{\phi} : GLM \to L(\mathbb{R}^n; \mathbb{R})$ by $\tilde{\phi}(u) = \phi_{\pi u} u$ so $\tilde{\phi}(ug) = \phi_{\pi u}(ug-)$. Then

$$\begin{aligned} L_{A_j^*}(\tilde{\phi})(u) &= \frac{d}{dt}\tilde{\phi}(u \cdot e^{A_j t})|_{t=0} \\ &= \frac{d}{dt}\phi_{\pi u}(u \cdot e^{A_j t})|_{t=0} \\ &= \phi_{\pi u}(uA_j-) = \tilde{\phi}(u)(A_j-). \end{aligned}$$

Iterating we have

$$\begin{aligned} \mathcal{B}^V(\tilde{\phi})(u) &= \sum_{i,j} \alpha^{i,j}(u)\phi_{\pi u}(uA_jA_i-) + \sum_k \beta^k(u)\phi_{\pi u}(uA_k-) \\ &= -\frac{1}{2}\tilde{\phi}(u)(u^{-1}Ric^{\#}(u-)) \end{aligned}$$

as required, by using the map $gl(n) \otimes gl(n) \to gl(n)$, $S \otimes T \mapsto S \circ T$, and equation (12).

B. Stochastic flows. In fact Theorem 3.1 can be understood in the more general context of stochastic flows as diffusions on the diffeomorphism groups. For this assume that M is compact and for $r \in \{1, 2, \dots\}$ and $s > r + \dim(M)/2$ let $\mathcal{D}^s = \mathcal{D}^s M$ be the C^∞ manifold of diffeomorphisms of M of Sobolev class H^s, (for example see Ebin-Marsden [2] or Elworthy [3].) Alternatively we could take the space $\mathcal{D}^\infty$ of C^∞ diffeomorphisms with differentiable structure as in [11]. Fix a base point x_0 in M and let $\pi : \mathcal{D}^s \to M$ be evaluation at x_0. This makes $\mathcal{D}^s$ into a principal bundle over M with group the manifold $\mathcal{D}^s_{x_0}$ of H^s- diffeomorphisms θ with $\theta(x_0) = x_0$, acting on the right by composition (although the action of $\mathcal{D}^{s+r}$ is only C^r, for $r = 0, 1, 2, \dots$).

Let $\{\xi_t^s : 0 \le s \le t < \infty\}$ be the flow of (10) starting at time s. Write ξ_t for ξ_t^0. The more general case allowing for infinite dimensional noise is given in [5]. We define probability measures $\{\mathbb{P}_\theta^{s,t} : \theta \in \mathcal{D}^s\}$ on $C([s.t]; M)$ be letting $\mathbb{P}_\theta^{s,t}$ be the law of $\{\xi_r^s \circ \theta : s \le r \le t\}$ (These correspond to the diffusion process on $\mathcal{D}^s$ associated to the right-invariant stochastic differential equation on $\mathcal{D}^s$ satisfied by $\{\xi_t : 0 \le t < \infty\}$ as in [3].) These are equivariant and project by π to the laws given by the stochastic differential equation on M. Assuming that these give a strongly cohesive diffusion on M we are essentially in the situation of Theorem 2.5.

Let $K(x) : \mathbb{R}^m \to \mathbb{R}^m$ be the orthogonal projection onto the kernel of $X(x)$, each $x \in M$. set $K^\perp(x) = id - K(x)$. Consider the $\mathcal{D}^\infty$-valued process $\{\theta_t : 0 \le t < \infty\}$ given by (or as the flow of)

(13) $d\theta_t(x) = X(\theta_t(x))K^\perp(\theta_t(x_0)) \circ dB_t + X(\theta_t(x))Y(\theta_t(x_0))A(\theta_t(x_0))$

for given θ_0 in $\mathcal{D}^\infty$ and, define a $\mathcal{D}^\infty_{x_0}$-valued process $\{g_t : 0 \le t < \infty\}$ by

$$\begin{aligned} (14) \qquad dg_t &= T\theta_t^{-1}\{X(\theta_t g_t-)K(\theta_t x_0)\circ dB_t \\ &\quad + A(\theta_t g_t-)dt - X(\theta_t g_t-)Y(\theta_t x_0)A(\theta_t x_0)dt\} \\ g_0 &= id. \end{aligned}$$

Set $x_t^\theta = \xi_t(\theta_0(x_0))$. Note that $\pi(\theta_t) = \theta_t(x_0) = x_t^\theta$ since

$$X(\theta_t(x_0))K^\perp(\theta_t(x_0)) = X(\theta_t(x_0))$$

and

$$X(\theta_t(x_0))Y(\theta_t(x_0))A(\theta_t(x_0)) = A(\theta_t(x_0)).$$

Thus $\{\theta_t : 0 \le t < \infty\}$ is a lift of $\{x_t^\theta, 0 \le t < \infty\}$. It can be considered to be driven by the 'relevant noise', (from the point of view of $\xi_\cdot(\theta_0(x_0))$, i.e. by the Brownian motion $\tilde{B}_\cdot$ given by

$$\tilde{B}_t = \int_0^t \tilde{/\!/}(x_\cdot^\theta)_s^{-1} K^\perp(x_s^\theta)dB_s$$

where $\{\tilde{/\!/}(x_\cdot^\theta), 0 \le s < \infty\}$ is parallel translation along $x_\cdot^\theta$ with respect to the connection on the trivial bundle $M \times \mathbb{R}^m \to M$ determined by K and $K^\perp$, so that

$$\tilde{/\!/}(x_\cdot^\theta)_s : \mathbb{R}^m \to \mathbb{R}^m$$

is orthogonal and maps the kernel of $X(\theta_\cdot(x_0))$ onto the kernel of $X(x_s^\theta)$ for $0 \le s < \infty$, see [7](chapter 3).

Correspondingly there is the 'redundant noise', the Brownian motion $\{\beta_t : 0 \le t < \infty\}$ given by

$$\beta_t = \int_0^t \tilde{/\!/}(x_\cdot^\theta)_s^{-1} K(x_s^\theta)dB_s.$$

Then, as shown in [7](chapter 3),

(i) $\tilde{B}_\cdot$ has the same filtration as $\{x_s^\theta : 0 \le s < \infty\}$
(ii) $\beta_\cdot$ and $\tilde{B}_\cdot$ are independent
(iii) $dB_t = \tilde{/\!/}_t d\beta_t + \tilde{/\!/}_t d\tilde{B}_t$.

We wish to see how $g_\cdot$ is driven by $\beta_\cdot$. For this observe

$$\int_0^t K(x_s^\theta)\circ dB_s = \int_0^t K(x_s^\theta)dB_s + \int_0^t \Lambda(x_s^\theta)ds$$

for $\Lambda : M \to \mathbb{R}$ given by the Stratonovich correction term. By (iii)

$$\int_0^t K(x_s^\theta)dB_s = \int_0^t \tilde{/\!/}_s d\beta_s = \int_0^t \tilde{/\!/}_s \circ d\beta_s$$

since $\tilde{/\!/}.$ is independent of β by (i) and (ii). Thus equation (14) for $g.$ can be written as

$$\begin{aligned} dg_t &= T\theta_t^{-1}\left\{X(\theta_t g_t-)\tilde{/\!/}(\theta.(x_0))_t \circ d\beta_t + X(\theta_t g_t-)\Lambda(\theta_t(x_0))dt \right. \\ &\quad \left. +A(\theta_t g_t-)dt - X(\theta_t g_t-)Y(\theta_t x_0)A(\theta_t x_0)dt\right\} \end{aligned}$$

and if we define

$$\begin{aligned} dg_t^y &= Ty_t^{-1}\left\{X(y_t g_t-)\tilde{/\!/}(y.(x_0))_t \circ d\beta_t + X(y_t g_t-)\Lambda(y_t(x_0))dt \right. \\ &\quad \left. +A(y_t g_t-)dt - X(y_t g_t-)Y(y_t x_0)A(y_t x_0)dt\right\} \\ g_0 &= id \end{aligned}$$

for any continuous $y : [0,\infty) \to \mathcal{D}^\infty$, we see, by the independence of β and θ that $g. = g^\theta_.$.

By Itô's formula on $\mathcal{D}^s$, for $x \in M$,

$$d(\theta_t g_t(x_0)) = (\circ d\theta_t)(g_t(x)) + T\theta_t(\circ dg_t^\theta(x)).$$

Now

$$\begin{aligned} T\theta_t(\circ dg_t^\theta(x)) &= \{X(\theta_t g_t(x))K(\theta_t x_0) \circ dB_t \\ &\quad +A(\theta_t g_t(x))dt - X(\theta_t g_t(x))Y(\theta_t x_0)A(\theta_t x_0)dt\} \end{aligned}$$

and so by (13) we see that $\theta_t g_t = \xi_t \circ \theta_0$, a.s.

Taking $\theta_0 = id$ we have

Proposition 3.2. *The flow $\xi.$ has the decomposition*

$$\xi_t = \theta_t g_t^{\theta.}, \qquad 0 \le t < \infty$$

for θ and $g^\theta_. \equiv g.$ given by (13) and (14) above. For almost all $\sigma : [0,\infty) \to M$ with $\sigma(0) = x_0$ and bounded measurable $F : C(0,\infty;\mathcal{D}^\infty) \to \mathbb{R}$

$$\mathbb{E}\{F(\xi.)\,|\xi.(x_0) = \sigma\} = \mathbb{E}\left\{F(\tilde{\sigma} g^{\tilde{\sigma}}_.)\right\}$$

where $\tilde{\sigma} : [0,\infty) \to \mathcal{D}^\infty$ is the horizontal lift of σ with $\tilde{\sigma}(0) = id$.

To define the 'horizontal lift' above we can use the fact, from (i) above, that $\theta.$ has the same filtration as $\xi.(x_0)$ and so furnishes a lifting map.

In terms of the semi-connection induced on $\pi : \mathcal{D}^s \to M$ over E, from above, by uniqueness or directly, we see the horizontal lift

$$\begin{aligned} h_\theta &: E_{\theta(x_0)} \longrightarrow T_\theta\mathcal{D}^s \\ h_\theta(v) &: M \longrightarrow TM \end{aligned}$$

is given by $h_\theta(v) = X(\theta(x))Y(\theta(x_0))v$ and the horizontal lift $\tilde{\sigma}.$ from $\tilde{\sigma}_0$ of a C^1 curve σ on M with $\tilde{\sigma}_0(x_0) = \sigma_0$ and $\dot{\sigma}(t) \in E_{\sigma(t)}$, all t, is given by

$$\frac{d}{dt}\tilde{\sigma}_t = X(\tilde{\sigma}_t-)Y(\sigma_t)\dot{\sigma}_t$$

for $\tilde{\sigma}_0 = id$. The lift is the solution flow of the differential equation

$$\dot{y}_t = Z^{\dot{\sigma}}(y_t)$$

on M.

For each frame $u : \mathbb{R}^n \to T_{x_0}M$ there is a homomorphism of principal bundles

$$\begin{array}{ll} \mathcal{D}^s & \to GL(M) \\ \theta & \mapsto T_{x_0}\theta \circ u. \end{array} \tag{15}$$

This sends $\{\xi_t : t \geq 0\}$ to the derivative process $T_x\xi_t \circ u$. (If the latter satisfies the strongly cohesive condition we could apply our analysis to this submersion $\mathcal{D}^s \to GLM$ and get another decomposition of $\xi.$.)

Results in Kobayashi-Nomizu [9] (Proposition 6.1 on page 79) apply to the homomorphism $\mathcal{D}^s \to GL(M)$ of (15). This gives a relationship between the curvature and holonomy groups of the semi-connection $\hat{\nabla}$ on GLM determined by the derivative flow and those of the connection induced by the diffusion on $\mathcal{D}^s \xrightarrow{\pi} M$. It also shows that the horizontal lift $\{\tilde{x}_t : t \geq 0\}$ through u of $\{x_t : t \geq 0\}$ to $GL(M)$ is just $T_{x_0}\theta_t \circ u$ for $\{\theta_t : t \geq 0\}$ the flow given by (13) with $\theta_0 = id$, i.e. the solution flow of the stochastic differential equation

$$dy_t = Z^{\circ dx_t}(y_t).$$

From this and Lemma 1.3.4 of [7] we see that $\hat{\nabla}$ is the adjoint of the LeJan-Watanabe connection determined by the flow, so proving Theorem 3.1 above. However the present construction applies with GLM replaced by any natural bundle over M (e.g. jet bundles, see Kolar-Michor-Slovak [10]), to give semi-connections on these bundles.

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