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Extremal weight modules of quantum affine algebras

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Abstract.

Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra, and let $\mathbf{U}_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra introduced by Drinfeld and Jimbo. In [11] Kashiwara introduced a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $V(\lambda)$, having a global crystal base for an integrable weight λ of level 0. We call it an *extremal weight module.* It is isomorphic to the Weyl module introduced by Chari-Pressley [6]. In [12, §13] Kashiwara gave a conjecture on the structure of extremal weight modules. We prove his conjecture when $\widehat{\mathfrak{g}}$ is an untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type ADE, using a result of Beck-Chari-Pressley [5]. As a by-product, we also show that the extremal weight module is isomorphic to a universal standard module, defined via quiver varieties by the author [16, 18]. This result was conjectured by Varagnolo-Vasserot [19] and Chari-Pressley [6] in a less precise form. Furthermore, we give a characterization of global crystal bases by an almost orthogonality propery, as in the case of global crystal base of highest weight modules.

§1. Introduction

In the conference, I gave a survey on quiver varieties and finite dimensional representations of quantum affine algebras. Since I already wrote a survey article [17] on this subject, I will discuss a different one in this paper. But it is related to my talks since I will study extremal weight modules which turn out to be isomorphic to universal standard modules, which was one of the main objects in my talk.

Let us describe Kashiwara's conjecture [12, §13] on extremal weight modules when $\widehat{\mathfrak{g}}$ is the untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type ADE. The notation will be explained in the next section.

Let λ be a dominant integral weight of $\mathfrak{g}$. We write $\lambda = \sum_{i\in I} m_i \varpi_i$, where ϖ_i is the i-th fundamental weight of $\mathfrak{g}$. We consider λ, ϖ_i as level 0 weights of $\widehat{\mathfrak{g}}$ by identifying them with $\lambda - \sum_i m_i a_i^\vee \Lambda_0$, $\Lambda_i - a_i^\vee \Lambda_0$,

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where $c = \sum_i a_i^\vee h_i$ is the central element, and Λ_i is the ith fundamental weight of $\widehat{\mathfrak{g}}$. Let $V(\lambda)$ be the extremal weight module of extremal weight λ with a global crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda), V^{\mathbb{Z}}(\lambda))$ (see §2.5 for definition). Let us define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module

$$\widetilde{V}(\lambda) \overset{\text{def.}}{=} \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i},$$

where we take and fix any ordering of I to define the tensor product. It has $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module automorphisms $z_{i,\nu}$ ($i \in I$, $\nu = 1, \dots, m_i$) (see §3.2).

Set $\widetilde{\mathcal{L}}(\lambda) \overset{\text{def.}}{=} \bigotimes_{i\in I} \mathcal{L}(\varpi_i)^{\otimes m_i}$, $\widetilde{u}_\lambda \overset{\text{def.}}{=} \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i}$. Let $\widetilde{\mathcal{B}}_0(\lambda)$ be the connected component of the crystal $\bigotimes_{i\in I} \mathcal{B}(\varpi_i)^{\otimes m_i}$ containing $\widetilde{u}_\lambda$ mod $q\widetilde{\mathcal{L}}(\lambda)$. There is a (subset of) global base $\{G(b) \mid b \in \mathcal{B}_0(\lambda)\}$ (see §3.2). Let $\widetilde{\mathcal{B}}(\lambda) \overset{\text{def.}}{=} \{s(z)b \mid b \in \widetilde{\mathcal{B}}_0(\lambda),\ s \in (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)\}$ where $s(z) = \prod_{i\in I} s_{\lambda^{(i)}}(z_{i,1}, \dots, z_{i,m_i})$ runs over the set $(\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)$ of products of Schur functions.

There exists a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism

$$\Phi_\lambda \colon V(\lambda) \to \widetilde{V}(\lambda)$$

sending u_λ to $\widetilde{u}_\lambda$ (see §3.2).

Theorem 1. (1) *Φ_λ is injective.*

(2) *$\Phi_\lambda(\mathcal{L}(\lambda)) \subset \widetilde{\mathcal{L}}(\lambda)$.*

Let Φ^0_λ be the induced map $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \widetilde{\mathcal{L}}(\lambda)/q\widetilde{\mathcal{L}}(\lambda)$.

(3) *Φ^0_λ gives a bijection between $\mathcal{B}(\lambda)$ and $\widetilde{\mathcal{B}}(\lambda)$.*

(4) *Φ_λ maps the global crystal base $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$ to $\{s(z)G(b) \mid b \in \widetilde{\mathcal{B}}_0(\lambda),\ s \in (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)\}$.*

While the author was preparing this article, he learned that Kashiwara also noticed that his conjecture follows from [5] when $\mathfrak{g}$ is of type ADE. In fact, some arguments (the proof of the injectivity of Φ_λ, the definition of $(\ ,\ )$, etc.) has been improved from the original form after the discussion with him. After the author posted the first version of this paper to the network archive, he was informed that Jonathan Beck also proved a part of Kashiwara's conjecture [4].

§2. Preliminaries

2.1. Affine Lie algebra

Let us fix notations for the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. (For a moment we do not assume that $\mathfrak{g}$ is of type ADE.)

(1) $\widehat{I}$: the index set of simple roots,
(2) $\{\alpha_i\}_{i\in\widehat{I}}$: the set of simple roots; $\{h_i\}_{i\in\widehat{I}}$: the set of simple coroots,
(3) $\widehat{P}^* \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}h_i\oplus\mathbb{Z}d$: the dual weight lattice; $\widehat{P} = \mathrm{Hom}_{\mathbb{Z}}(\widehat{P}^*, \mathbb{Z})$: the weight lattice,
(4) $\widehat{\mathfrak{h}} \overset{\text{def.}}{=} \widehat{P}^* \otimes_{\mathbb{Z}} \mathbb{Q}$: the Cartan subalgebra,
(5) the simple root $\alpha_i \in \widehat{P}$ defined by $\langle h_i, \alpha_j\rangle = a_{ij}, \langle d, \alpha_j\rangle = \delta_{0j}$, where a_{ij} is the Cartan matrix of $\widehat{\mathfrak{g}}$,
(6) the fundamental weight $\Lambda_i \in \widehat{P}$ defined by $\langle h_i, \Lambda_j\rangle = \delta_{ij}, \langle d, \Lambda_j\rangle = 0$.
(7) $\widehat{Q} \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}\alpha_i$: the root lattice; $\widehat{Q}^\vee \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}h_i$: the coroot lattice,
(8) $\widehat{Q}_+ \overset{\text{def.}}{=} \sum_{i\in\widehat{I}}\mathbb{Z}_{\geq 0}\alpha_i$; $\widehat{P}_+ \overset{\text{def.}}{=} \{\lambda \in \widehat{P} \mid \langle h_i, \lambda\rangle \geq 0 \text{ for all } i \in \widehat{I}\}$: the set of integral dominant weights,
(9) the unique element $c = \sum_{i\in\widehat{I}} a_i^\vee h_i$ ($a_i^\vee \in \mathbb{Z}_{\geq 0}$) satisfying
$\left\{h \in \widehat{Q}^\vee \,\middle|\, \langle h, \alpha_j\rangle = 0 \text{ for all } j \in \widehat{I}\right\} = \mathbb{Z}c,$
(10) the unique element $\delta = \sum_{i\in\widehat{I}} a_i\alpha_i$ ($a_i \in \mathbb{Z}_{\geq 0}$) satisfying
$\left\{\lambda \in \widehat{Q} \,\middle|\, \langle h_i, \lambda\rangle = 0 \text{ for all } i \in \widehat{I}\right\} = \mathbb{Z}\delta,$
(11) the symmetric bilinear form (,) on $\widehat{\mathfrak{h}}^*$, uniquely characterized by $\langle h_i, \lambda\rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$, $\langle c, \lambda\rangle = (\delta, \lambda)$, for $\lambda \in \widehat{\mathfrak{h}}^*$,
(12) $h \overset{\text{def.}}{=} \sum_{i\in\widehat{I}} a_i$: the Coxeter number; $h^\vee \overset{\text{def.}}{=} \sum_{i\in\widehat{I}} a_i^\vee$: the dual Coxeter number.

The symmetric bilinear form (,) is known to be nondegenerate, and defines an isomorphism $\nu\colon \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ by $\langle h, \lambda\rangle = (\nu(h), \lambda)$ for $\lambda \in \widehat{\mathfrak{h}}^*$. For example, $\nu(c) = \delta$. This coincides with one in [9, §6].

For $\beta \in \widehat{\mathfrak{h}}^*$ with $(\beta, \beta) \neq 0$, we set $\beta^\vee \overset{\text{def.}}{=} \frac{2\beta}{(\beta,\beta)}$. We have $\nu(h_i) = \alpha_i^\vee$.

We have an element $0 \in \widehat{I}$ such that $\{\alpha_i \mid i \neq 0\}$ is the set of simple roots of $\mathfrak{g}$. It is known $a_0^\vee = a_0 = 1$ for the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. We denote $\widehat{I} \setminus \{0\}$ by I.

Let $\mathrm{cl}\colon \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*/\mathbb{Q}\delta$ be the natural projection. Let $\widehat{\mathfrak{h}}^{*0} \overset{\text{def.}}{=} \{\lambda \in \widehat{\mathfrak{h}}^{*0} \mid \langle c, \lambda\rangle = 0\}$, $\widehat{P}^0 \overset{\text{def.}}{=} \widehat{P}\cap\widehat{\mathfrak{h}}^{*0}$ (level 0 weights). We identify $\mathrm{cl}(\widehat{\mathfrak{h}}^{*0}) \subset \widehat{\mathfrak{h}}^*/\mathbb{Q}\delta$ with the dual of the Cartan subalgebra $\mathfrak{h}$ of the finite dimensional Lie algebra $\mathfrak{g}$, which is $\bigoplus_{i\in I}\mathbb{Q}h_i$. Similarly we identify $\mathrm{cl}(\widehat{P}^0)$ with the weight lattice P of $\mathfrak{g}$. We define the root lattice of $\mathfrak{g}$ by $Q \overset{\text{def.}}{=} \bigoplus_{i\in I}\mathbb{Z}\alpha_i$. For $i \in I$, we set $\varpi_i \overset{\text{def.}}{=} \Lambda_i - a_i^\vee\Lambda_0 \in \widehat{P}^0$. Then $\mathrm{cl}(\varpi_i)$ is identified with the ith fundamental weight of $\mathfrak{g}$. Let $\widehat{P}^{0,+} \overset{\text{def.}}{=} \left\{\lambda \in \widehat{P}^0 \,\middle|\, \langle h_i, \lambda\rangle \geq 0 \text{ for } i \in I\right\}$.

Its projection $\mathrm{cl}(\widehat{P}^{0,+})$ is the set of dominant integrable weights of $\mathfrak{g}$. Let $P^\vee \stackrel{\text{def.}}{=} \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$. The fundamental coweights $\varpi_i^\vee$ are defined by $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$ for $i, j \in I$. We extend $\varpi_i^\vee$ to a homomorphism $\widehat{Q} \to \mathbb{Z}$ by setting $\langle \varpi_i^\vee, \delta \rangle = 0$.

Let Δ (resp. Δ_+) be the set of roots (resp. positive roots) of $\mathfrak{g}$. The set of roots $\widehat{\mathcal{R}}$ of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_+ \sqcup \widehat{\mathcal{R}}_-$, where

$$\widehat{\mathcal{R}}_+ = \begin{array}{c} \{k\delta + \alpha \mid k \geq 0, \alpha \in \Delta_+\} \sqcup \{k\delta \mid k > 0\} \\ \sqcup \{k\delta - \alpha \mid k > 0, \alpha \in \Delta_+\}, \end{array}, \qquad \widehat{\mathcal{R}}_- = -\widehat{\mathcal{R}}_+.$$

The roots of the form $k\delta \pm \alpha$ ($k \in \mathbb{Z}$, $\alpha \in \Delta$) are called *real* roots, while roots $k\delta$ are called *imaginary* roots. The multiplicities of real roots are 1, and those of imaginary roots are equal to the rank of $\mathfrak{g}$, i.e., $\#I$.

Set

$$\mathcal{R}_> \stackrel{\text{def.}}{=} \{k\delta + \alpha \mid k \geq 0, \alpha \in \Delta^+\}, \quad \mathcal{R}_< \stackrel{\text{def.}}{=} \{k\delta - \alpha \mid k > 0, \alpha \in \Delta^+\},$$
$$\mathcal{R}_0 \stackrel{\text{def.}}{=} \{k\delta \mid k > 0\} \times I, \quad \mathcal{R} \stackrel{\text{def.}}{=} \mathcal{R}_> \sqcup \mathcal{R}_0 \sqcup \mathcal{R}_<.$$

These are sets of roots, counted with multiplicities.

For $i \in \widehat{I}$, we define the reflection s_i acting on $\widehat{\mathfrak{h}}^*$ by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. Moreover, s_i acts also on $\widehat{\mathfrak{h}}$ by $s_i(h) = h - \langle h, \alpha_i \rangle h_i$. The actions of s_i preserve $\widehat{P}$, $\widehat{Q}$, $\widehat{Q}^\vee$ and $\widehat{\mathfrak{h}}^{*0}$. We have $s_i\delta = \delta$, $s_i c = c$. If $i \in I$, the corresponding reflection s_i preserves $\mathfrak{h}$, P, $P^\vee$ and Q. The *Weyl group* W (resp. *affine Weyl group* $\widehat{W}$) of $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) is the subgroups of $\mathrm{GL}(\mathfrak{h}^*)$ (resp. $\mathrm{GL}(\widehat{\mathfrak{h}}^*)$) generated by s_i for $i \in I$ (resp. $i \in \widehat{I}$). We define the *extended Weyl group* $\widetilde{W}$ as the semidirect product $\widetilde{W} \stackrel{\text{def.}}{=} W \ltimes P^\vee$, using the W-action on $P^\vee$. It is known that $\widehat{W}$ is a normal subgroup of $\widetilde{W}$, and the quotient $\mathcal{T} \stackrel{\text{def.}}{=} \widetilde{W}/\widehat{W}$ is a finite group isomorphic to a subgroup of the group of the diagram automorphisms of $\widehat{\mathfrak{g}}$, i.e., bijections $\tau\colon I \to I$. Moreover, $\widetilde{W}$ is isomorphic to $\mathcal{T} \ltimes \widehat{W}$.

When we consider $\xi \in P^\vee$ as an element of $\widetilde{W}$, we denote it by t_ξ. We have $t_\xi(\lambda) = \lambda - \langle \xi, \lambda \rangle \delta$ for $\xi \in P^\vee$, $\lambda \in \widehat{\mathfrak{h}}^{*0}$.

Lemma 2.1. *We have*

$$\sum_{\alpha \in \widehat{\mathcal{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathcal{R}}_-)} (\alpha, \xi) = h^\vee \langle \varpi_i^\vee, \xi \rangle, \qquad \sum_{\alpha \in \widehat{\mathcal{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathcal{R}}_-)} (\alpha^\vee, \xi) = h \langle \varpi_i^\vee, \xi \rangle.$$

Proof. From the above description of the root system $\widehat{\mathcal{R}}_+$, we have

$$\widehat{\mathcal{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathcal{R}}_-) = \{\beta + n\delta \mid \beta \in \Delta_+, 0 \le n < \langle \varpi_i^\vee, \beta\rangle\}.$$

Therefore

$$\sum_{\alpha \in \widehat{\mathcal{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathcal{R}}_-)} (\alpha, \xi) = \sum_{\beta \in \Delta_+} (\beta, \xi)\langle \varpi_i^\vee, \beta\rangle = \sum_{\beta \in \Delta_+} \frac{a_i}{a_i^\vee}(\beta, \xi)(\beta, \varpi_i).$$

We consider the bilinear form on $\mathfrak{h}^*$ defined by

$$\Phi(\xi, \eta) \overset{\text{def.}}{=} \sum_{\beta \in \Delta_+} (\beta, \xi)(\beta, \eta).$$

By [9, Corollary 8.7] it is equal to $h^\vee(\xi, \eta)$ and we get the assertion. We give a proof since the corresponding equality for the second equation cannot be found there.

From the definition, it is invariant under the Weyl group W. So there is a constant c such that $\Phi(\xi, \eta) = c(\xi, \eta)$. Let $\theta = \delta - \alpha_0$ be the highest root of $\mathfrak{g}$. Then we have

$$(\theta, \theta) = (\alpha_0, \alpha_0) = 2.$$

On the other hand, we have

$$\Phi(\theta, \theta) = \sum_{\beta \in \Delta_+} (\beta, \theta)(\beta, \theta).$$

If $\beta = \sum_i n_i \alpha_i \in \Delta_+$, we have $0 \le n_i \le a_i$. So we have

$$(\beta, \theta) = -\sum_i n_i(\alpha_i, \alpha_0) > 0,$$
$$(\beta, \theta) = (\theta, \theta) - \sum_i (n_i - a_i)(\alpha_i, \alpha_0) \le 2,$$

where the equality holds when $\beta = \theta$. (Note that $(\alpha_i, \alpha_0) = a_{0i}$ is a negative integer.) Therefore

$$\Phi(\theta, \theta) = \sum_{\beta \in \Delta_+} (\beta, \theta) + 2 = 2(\rho, \theta) + 2$$
$$= 2\sum_{i \in I} (\varpi_i, \theta) + 2 = 2\sum_{i \in I} a_i^\vee + 2 = 2h^\vee,$$

where ρ is the half sum of the positive roots of $\mathfrak{h}$, which is known to be equal to $\sum_{i \in I} \varpi_i$. Therefore we have $c = h^\vee$ and get the first equation. A similar calculation shows the second equation. □

2.2. Quantum affine algebra

Let $\mathbf{U}_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra. We follow the notation in [1, 12]. We choose a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in \widehat{I}$. We set $q_s = q^{1/d}$. (Later we assume $\mathfrak{g}$ is of type ADE. Then $d = 1$ and $q_s = q$.) Then the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is the associative algebra over $\mathbb{Q}(q_s)$ with 1 generated by elements e_i, f_i $(i \in \widehat{I})$, q^h $(h \in d^{-1}\widehat{P}^*)$, $q^{\pm c/2}$ with certain defining relations. As customary, we set $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q^{(\alpha_i,\alpha_i)h_i/2}$, $e_i^{(p)} = e_i^p/[p]_{q_i}!$, $f_i^{(p)} = f_i^p/[p]_{q_i}!$.

Let $\mathbf{U}'_q(\widehat{\mathfrak{g}})$ be the quantized enveloping algebra with $\mathrm{cl}(\widehat{P})$ as a weight lattice. It is the subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by e_i, f_i $(i \in \widehat{I})$, q^h $(h \in d^{-1}\bigoplus_i \mathbb{Z}h_i)$, $q^{\pm c/2}$. The quotient $\mathbf{U}'_q(\widehat{\mathfrak{g}})/(q^{\pm c/2} - 1)$ is denoted by $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ and called a *quantum loop algebra* in [16, 18].

Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$ (resp. $\mathbf{U}_q(\widehat{\mathfrak{g}})^-$) be the $\mathbb{Q}(q_s)$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by elements e_i's (resp. f_i's). Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^0$ be the $\mathbb{Q}(q_s)$-subalgebra generated by elements q^h $(h \in d^{-1}\widehat{P}^*)$. We have the triangular decomposition $\mathbf{U}_q(\widehat{\mathfrak{g}}) \cong \mathbf{U}_q(\widehat{\mathfrak{g}})^+ \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})^0 \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})^-$.

For $\xi \in \widehat{Q}$, we define the *root space* $\mathbf{U}_q(\widehat{\mathfrak{g}})_\xi$ by

$$\mathbf{U}_q(\widehat{\mathfrak{g}})_\xi \overset{\text{def.}}{=} \{x \in \mathbf{U}_q(\widehat{\mathfrak{g}}) \mid q^h x q^{-h} = q^{\langle h,\xi\rangle} x \text{ for all } h \in \widehat{P}^*\}.$$

Let $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ be the $\mathbb{Z}[q_s, q_s^{-1}]$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by elements $e_i^{(n)}$, $f_i^{(n)}$, q^h for $i \in I$, $n \in \mathbb{Z}_{>0}$, $h \in d^{-1}\widehat{P}^*$.

Let us introduce a $\mathbb{Q}(q_s)$-algebra involutive automorphism $\vee$ and $\mathbb{Q}(q_s)$-algebra involutive anti-automorphisms $*$ and ψ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ by

$$\begin{gathered} e_i^\vee = f_i, \quad f_i^\vee = e_i, \quad (q^h)^\vee = q^{-h}, \\ e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}, \\ \psi(e_i) = q_i^{-1}t_i^{-1}f_i, \quad \psi(f_i) = q_i^{-1}t_i e_i, \quad \psi(q^h) = q^h. \end{gathered}$$

We define a $\mathbb{Q}$-algebra involutive automorphism $\overline{}$ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ by

$$\begin{gathered} \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h}, \\ \overline{a(q_s)u} = a(q_s^{-1})\overline{u} \quad \text{for } a(q_s) \in \mathbb{Q}(q_s) \text{ and } u \in \mathbf{U}_q(\widehat{\mathfrak{g}}). \end{gathered}$$

In this article, we take the coproduct Δ on $\mathbf{U}_q(\widehat{\mathfrak{g}})$ given by

$$\begin{gathered} \Delta q^h = q^h \otimes q^h, \quad \Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i, \\ \Delta f_i = f_i \otimes 1 + t_i \otimes f_i. \end{gathered} \tag{2.2}$$

Let us denote by Ω the $\mathbb{Q}$-algebra anti-automorphism $* \circ \overline{} \circ \vee$ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$. We have

$$\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(q^h) = q^{-h}, \quad \Omega(q_s) = q_s^{-1}.$$

A $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M is called *integrable* if

(1) all e_i, f_i $(i \in I)$ are locally nilpotent, and

(2) it admits a *weight space decomposition*:

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad \text{where } M_\lambda = \{u \in M \mid q^h u = q^{\langle h,\lambda\rangle} u \text{ for all } h \in \widehat{P}^*\}.$$

Let $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ be the modified enveloping algebra [13, Part IV]. It is defined as

$$\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}}) \overset{\text{def.}}{=} \bigoplus_{\lambda \in \widehat{P}} \mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda, \quad \mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda \overset{\text{def.}}{=} \mathbf{U}_q(\widehat{\mathfrak{g}}) \Big/ \sum_{h \in \widehat{P}^*} \mathbf{U}_q(\widehat{\mathfrak{g}})(q^h - q^{\langle h,\lambda\rangle}) .$$

Here the multiplication is given by

$$a_\lambda x = x a_{\lambda-\xi} \quad \text{for } \xi \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi, \qquad a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda,$$

where a_λ is considered as the image of 1 in the above definition of $\mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda$.

Let λ, $\mu \in \widehat{P}_+$. Let $V(\lambda)$ (resp. $V(-\mu)$) be the irreducible highest (resp. lowest) weight module of weight λ (resp. $-\mu$) [13, §3.5]. Then there is a surjective homomorphism

$$\mathbf{U}_q(\widehat{\mathfrak{g}}) a_{\lambda-\mu} \ni u \longmapsto u(u_\lambda \otimes u_{-\mu}) \in V(\lambda) \otimes V(-\mu), \tag{2.3}$$

where u_λ (resp. $u_{-\mu}$) is a highest (resp. lowest) weight vector of $V(\lambda)$ (resp. $V(-\mu)$).

2.3. Braid group action

For each $w \in \widehat{W}$, there exists an $\mathbb{Q}(q)$-algebra automorphism T_w [13, §39] (denoted there by $T''_{w,1}$). Also, for any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M, there exists $\mathbb{Q}(q)$-linear map $T_w \colon M \to M$ satisfying $T_w(xu) = T_w(x)T_w(u)$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})$, $u \in M$ [13, §5]. We denote T_{s_i} by T_i hereafter. By [13, 39.4.5] we have

$$\Omega \circ T_w \circ \Omega = T_w. \tag{2.4}$$

Lemma 2.5. *We have*

$$(\psi \circ T_w \circ \psi)(x) = (-1)^{N^\vee} q^{-N} T_{w^{-1}}^{-1}(x) \quad \textit{for all } w \in \widehat{W},\ x \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi,$$

where

$$N = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha, \xi), \qquad N^\vee = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_0)} (\alpha^\vee, \xi).$$

Proof. Let $T''_{i,-1}$ be the automorphism defined in [13, §37]. A direct calculation shows $\psi \circ T_i \circ \psi = T''_{i,-1}$. By [loc. cit., 37.2.4] we have $T''_{i,-1}(x) = (-1)^{\langle h_i, \xi\rangle} q^{-(\alpha_i,\xi)} T_i^{-1}(x)$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi$. Let $w = s_{i_m} \dots s_{i_1}$ be a reduced expression of w. Then

$$(\psi \circ T_w \circ \psi)(x) = (-1)^{N^\vee} q^{-N} \left(T_{i_m}^{-1} \dots T_{i_1}^{-1}\right)(x),$$

where

$$\begin{aligned} N^\vee &= \langle h_{i_1} + s_{i_1} h_{i_2} + \dots + s_{i_1} \dots s_{i_{m-1}} h_{i_m}, \xi\rangle, \\ N &= (\alpha_{i_1} + s_{i_1}\alpha_{i_2} + \dots + s_{i_1} \dots s_{i_{m-1}} \alpha_{i_m}, \xi). \end{aligned}$$

Since we have $\widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-) = \left\{ s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid k = 1, \dots, m \right\}$, we are done. □

As in [2, 5], the definition of the automorphism T_w of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ can be extended to the case $w \in \widetilde{W}$ by setting

$$\tau e_i = e_{\tau(i)}, \quad \tau f_i = f_{\tau(i)}, \quad \tau q^{h_i} = q^{h_{\tau(i)}}, \quad \tau q^d = q^d.$$

2.4. Crystal base

We shall briefly recall the notion of crystal bases. For the notion of (abstract) crystals, we refer to [11, 1].

For $n \in \mathbb{Z}$ and $i \in \widehat{I}$, let us define an operator acting on any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M by

$$\widetilde{F}_i^{(n)} \overset{\text{def.}}{=} \sum_{k \geq \max(0,-n)} f_i^{(n+k)} e_i^{(k)} a_k^n(t_i),$$

$$\text{where} \quad a_k^n(t_i) \overset{\text{def.}}{=} (-1)^k q_i^{k(1-n)} t_i^k \prod_{\nu=1}^{k-1} (1 - q_i^{n+2\nu}).$$

And we set $\widetilde{e}_i \overset{\text{def.}}{=} \widetilde{F}_i^{(-1)}$, $\widetilde{f}_i \overset{\text{def.}}{=} \widetilde{F}_i^{(1)}$.

These operators are different from those used for the definition of crystal bases in [10], but it gives us the same crystal bases by [12, Proposition 6.1].

A direct calculation shows

$$\psi(\widetilde{e}_i) = (1 - q_i)\widetilde{f}_i. \tag{2.6}$$

Let $\mathbf{A}_0 \overset{\text{def.}}{=} \{f(q_s) \in \mathbb{Q}(q_s) \mid f \text{ is regular at } q_s = 0\}$.

Definition 2.7. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module. A pair $(\mathcal{L}, \mathcal{B})$ is called a *crystal base* of M if it satisfies

(1) $\mathcal{L}$ is a free $\mathbf{A}_0$-submodule of M such that $M \cong \mathbb{Q}(q_s) \otimes_{\mathbf{A}_0} \mathcal{L}$,
(2) $\mathcal{L} = \bigoplus_{\lambda \in \widehat{P}} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for $\lambda \in \widehat{P}$,
(3) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{Q} \otimes_{\mathbf{A}_0} \mathcal{L}$,
(4) $\widetilde{e}_i\mathcal{L} \subset \mathcal{L}$, $\widetilde{f}_i\mathcal{L} \subset \mathcal{L}$ for all $i \in \widehat{I}$,
(5) if we denote operators on $\mathcal{L}/q\mathcal{L}$ induced by $\widetilde{e}_i$, $\widetilde{f}_i$ by the same symbols, we have $\widetilde{e}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}$, $\widetilde{f}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}$,
(6) for any $b, b' \in \mathcal{B}$ and $i \in \widehat{I}$, we have $b' = \widetilde{f}_i b$ if and only if $b = \widetilde{e}_i b'$.

We define functions $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{Z}_{\geq 0}$ by $\varepsilon_i(b) \overset{\text{def.}}{=} \max\{n \geq 0 \mid \widetilde{e}_i^n b \neq 0\}$, $\varphi_i(b) \overset{\text{def.}}{=} \max\{n \geq 0 \mid \widetilde{f}_i^n b \neq 0\}$. We set $\widetilde{e}_i^{\max} b \overset{\text{def.}}{=} \widetilde{e}_i^{\varepsilon_i(b)} b$, $\widetilde{f}_i^{\max} b \overset{\text{def.}}{=} \widetilde{f}_i^{\varphi_i(b)} b$.

Let $\overline{}$ be an automorphism of $\mathbb{Q}(q_s)$ sending q_s to q_s^{-1}. Let $\overline{\mathbf{A}_0}$ be the image of $\mathbf{A}_0$ under $\overline{}$, that is, the subring of $\mathbb{Q}(q_s)$ consisting of rational functions regular at $q_s = \infty$.

Definition 2.8. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module with a crystale base $(\mathcal{L}, \mathcal{B})$. Let $\overline{}$ be an involution of an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M satisfying $\overline{xu} = \overline{x}\,\overline{u}$ for any $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})$, $u \in M$. Let $M^{\mathbb{Z}}$ be a $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule of M such that $\overline{M^{\mathbb{Z}}} = M^{\mathbb{Z}}$, $u - \overline{u} \in (q_s - 1)M^{\mathbb{Z}}$ for $u \in M^{\mathbb{Z}}$. We say that M *has a global base* $(\mathcal{L}, \mathcal{B}, M^{\mathbb{Z}})$ if the following conditions are satisfied

(1) $M \cong \mathbb{Q}(q_s) \otimes_{\mathbb{Z}[q_s, q_s^{-1}]} M^{\mathbb{Z}} \cong \mathbb{Q}(q_s) \otimes_{\mathbf{A}_0} \mathcal{L} \cong \mathbb{Q}(q_s) \otimes_{\overline{\mathbf{A}_0}} \overline{\mathcal{L}}$,
(2) $\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}} \to \mathcal{L}/q_s\mathcal{L}$ is an isomorphism.

As a consequence of the definition, natural homomorphisms

$$\mathbf{A}_0 \otimes_{\mathbb{Z}} \left(\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}\right) \to \mathcal{L}, \quad \overline{\mathbf{A}_0} \otimes_{\mathbb{Z}} \left(\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}\right) \to \overline{\mathcal{L}},$$
$$\mathbb{Z}[q_s, q_s^{-1}] \otimes_{\mathbb{Z}} \left(\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}\right) \to M^{\mathbb{Z}},$$

are isomorphisms.

Let G be the inverse isomorphism $\mathcal{L}/q_s\mathcal{L} \to \mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}$. Then $\{G(b) \mid b \in \mathcal{B}\}$ is a base of M. It is called a *global crystal base* of M. The above conditions imply $\overline{G(b)} = G(b)$.

For a dominant weight $\lambda \in \widehat{P}_+$, the irreducible highest weight module $V(\lambda)$ has a global crystal base [10]. If $\lambda, \mu \in \widehat{P}_+$, then the tensor product $V(\lambda) \otimes V(-\mu)$ also has a global crystal base. Moreover, $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ has a global crystal base $\left(\mathcal{L}(\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})), \mathcal{B}(\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})), \tilde{\mathbf{U}}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})\right)$ such that the homomorphism (2.3) maps a global base of $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ to the union of that of $V(\lambda) \otimes V(-\mu)$ and 0 [13, Part IV]. Furthermore, the global base is invariant under $*$ [11, 4.3.2].

2.5. Extremal vectors

A crystal $\mathcal{B}$ over $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is called *regular* if, for any $J \subsetneq \widehat{I}$, $\mathcal{B}$ is isomorphic (as a crystal over $\mathbf{U}_q(\mathfrak{g}_J)$) to the crystal associated with an integrable $\mathbf{U}_q(\mathfrak{g}_J)$-module. (It was called *normal* in [11].) Here $\mathbf{U}_q(\mathfrak{g}_J)$ is the subalgebra generated by e_j, f_j $(j \in J)$, q^h $(h \in d^{-1}\widehat{P}^*)$. By [11], the affine Weyl group $\widehat{W}$ acts on any regular crystal. The action S is given by

$$S_{s_i} b = \begin{cases} \widetilde{f}_i^{\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \geq 0, \\ \widetilde{e}_i^{-\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \leq 0 \end{cases}$$

for the simple reflection s_i. We denote S_{s_i} by S_i hereafter.

Definition 2.9. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module. A vector $u \in M$ with weight $\lambda \in \widehat{P}$ is called *extremal*, if the following holds for all $w \in \widehat{W}$:

$$\begin{cases} e_i T_w u = 0 & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ f_i T_w u = 0 & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases} \tag{2.10}$$

In this case, we define $S_w u$ so that

$$S_i S_w u = \begin{cases} f_i^{(\langle h_i, w\lambda\rangle)} S_w u & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ e_i^{(-\langle h_i, w\lambda\rangle)} S_w u & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases}$$

This is well-defined, i.e., $S_w u$ depends only on w.

Similarly, for a vector b of a regular crystal B with weight λ, we say that b is *extremal* if it satisfies

$$\begin{cases} \widetilde{e}_i S_w b = 0 & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ \widetilde{f}_i S_w b = 0 & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases}$$

Lemma 2.11. *Suppose that an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M has a crystal base $(\mathcal{L}, \mathcal{B})$. If $u \in \mathcal{L} \subset M$ is an extremal vector of weight λ*

satisfying $b \overset{\text{def.}}{=} u \bmod q\mathcal{L} \in \mathcal{B}$, *then* b *is an extremal vector, and we have*

$$S_w u = (-1)^{N_+^\vee} q^{-N_+} T_w u, \quad S_w b = S_w u \bmod q\mathcal{L} \quad \textit{for all } w \in \widehat{W},$$

where $N_+ = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} \max\left((\alpha, \lambda), 0\right)$, *and* $N_+^\vee$ *is given by replacing* α *by* $\alpha^\vee$.

Proof. The equation $S_w b = S_w u \bmod q\mathcal{L}$ follows from the definition of S_w.

If $v \in M_\xi$ satisfies $e_i v = 0$ (resp. $f_i v = 0$), we have

$$T_i v = (-q_i)^{\xi_i} f_i^{(\xi_i)} v \qquad \left(\text{resp. } T_i v = e_i^{(\xi_i)} v\right),$$

where $\xi_i = \langle h_i, \xi \rangle$. The rest of the proof is the same as that of Lemma 2.5. □

The following follows from a formula for the crystal $\mathcal{B}(\widetilde{\mathbf{U}}_q(\widehat{\mathfrak{g}}))$ (see [12, App. B]):

Lemma 2.12. *Let* $\lambda \in P^0$. *The followings hold for* $b = b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) = \mathcal{B}(\infty) \otimes T_\lambda \otimes \mathcal{B}(-\infty)$ *with* $\operatorname{wt} b_1 \in \mathbb{Z}\delta$:
$\widetilde{e}_i b = 0$ *or* $\widetilde{f}_i b = 0$ *if and only if* $\varepsilon_i(b_1) \le \max(-\langle h_i, \lambda \rangle, 0)$.

For $\lambda \in P$, Kashiwara defined the $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $V(\lambda)$ generated by u_λ with the defining relation that u_λ is an extremal vector of weight λ [11][1]. It is written as

$$V(\lambda) = \mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda / I_\lambda, \qquad I_\lambda \overset{\text{def.}}{=} \bigoplus_{b \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q) G(b),$$

where $\mathcal{B}(\lambda) \overset{\text{def.}}{=} \{b \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) \mid b^* \text{ is extremal}\}$. Thus $V(\lambda)$ has a crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ together with a $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule $V^{\mathbb{Z}}(\lambda)$ with a global crystal base, naturally induced from that of $\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda$. If λ is dominant or anti-dominant, then $V(\lambda)$ is isomorphic to the highest weight module or the lowest weight module. So there is no fear of the confusion of the notation.

[1]He denoted it by $V^{\max}(\lambda)$.

2.6. Drinfeld realization

The quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ has another realization, due to [8, 2]. It is isomorphic to an associative algebra over $\mathbb{Q}(q_s)$ with generators $x_{i,r}^{\pm}$ $(i \in I,\ r \in \mathbb{Z})$, q^h $(h \in d^{-1}\widehat{P}^*)$, $h_{i,m}^{\pm}$ $(i \in I,\ m \in \mathbb{Z}\setminus\{0\})$ with certain defining relations (see [2, §4]). The isomorphism depends on the choice of $o\colon I \to \{\pm 1\}$, and is given by

$$x_{i,r}^{+} = o(i)^r T_{\varpi_i^\vee}^{-r}(e_i), \quad x_{i,r}^{-} = o(i)^r T_{\varpi_i^\vee}^{r}(f_i),$$

$$\left[x_{i,r}^{+}, x_{j,s}^{-}\right] = \delta_{ij}\frac{q^{(r-s)c/2}\psi_{i,r+s}^{+} - q^{-(r-s)c/2}\psi_{i,r+s}^{-}}{q-q^{-1}},$$

$$\text{where}\quad \psi_i^{\pm}(u) \equiv \sum_{r=0}^{\infty}\psi_{i,\pm r}^{\pm}u^{\pm r} \overset{\text{def.}}{=} t_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{m=1}^{\infty} h_{i,\pm m}u^{\pm m}\right).$$

By (2.4) we have

$$\Omega(x_{i,r}^{\pm}) = x_{i,-r}^{\mp}, \quad \Omega(h_{i,m}) = h_{i,-m} \quad \text{for } i \in I,\ r \in \mathbb{Z},\ m \in \mathbb{Z}\setminus\{0\}.$$

2.7. The crystal base of $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$

Let us recall results in [5]. We assume $\mathfrak{g}$ is of type ADE hereafter. We choose a reduced expression $s_{i_1}\cdots s_{i_N}$ of $2\rho = 2\sum_{i\in I}\varpi_i$ in a suitable way (see [loc. cit.] for detail), and consider a periodic doubly infinite sequence $(\dots, i_{-1}, i_0, i_1, \dots)$ of $\widehat{I}$ by setting $i_k = i_{k \bmod N}$. Let

$$\beta_k \overset{\text{def.}}{=} \begin{cases} s_{i_0}s_{i_{-1}}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0. \end{cases}$$

We have

$$\mathcal{R}_{>} = \{\beta_k \mid k \le 0\}, \quad \mathcal{R}_{<} = \{\beta_k \mid k > 0\}. \tag{2.13}$$

We define

$$E_{\beta_k}^{(n)} \overset{\text{def.}}{=} \begin{cases} T_{i_0}^{-1}T_{i_{-1}}^{-1}\dots T_{i_{k+1}}^{-1}(e_{i_k}^{(n)}) & \text{if } k \le 0, \\ T_{i_1}T_{i_2}\dots T_{i_{k-1}}(e_{i_k}^{(n)}) & \text{if } k > 0. \end{cases}$$

We denote $E_{\beta_k}^{(1)}$ by E_{β_k}. These are *root vectors* for $\mathcal{R}_{>}$ and $\mathcal{R}_{<}$. By [13, 40.1.3] we have $E_{\beta_k}^{(n)} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$. Explicit relations among $E_{\beta_k}^{(n)}$ and $x_{i,r}^{\pm}$ can be found in [5, Lemma 1.5].

We define $P_{m,i}$ $(m > 0,\ i \in I)$ by

$$1 + \sum_{m>0} P_{m,i} u^m = \exp\left(- \sum_{m>0} \frac{(o(i) q^{c/2} u)^r h_{i,r}}{[r]_{q_i}} \right).$$

We also define $\widetilde{P}_{m,i} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$ by replacing $h_{i,r}$ by $-h_{i,r}$. These are root vectors for $\mathcal{R}_0$. We also set $P_{-m,i} = \Omega(P_{m,i})$ $(m > 0, i \in I)$.

Let $\mathbf{c}\colon \mathcal{R} \to \mathbb{Z}_{\geq 0}$ be a map such that $\mathbf{c}(\alpha) = 0$ except for finitely many α. We denote its restrictions to $\mathcal{R}_>$, $\mathcal{R}_>$, $\mathcal{R}_0$ by $\mathbf{c}_>$, $\mathbf{c}_<$, $\mathbf{c}_0$ respectively. We define $E_{\mathbf{c}_>}, E_{\mathbf{c}_<} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$ by

$$E_{\mathbf{c}_>} \overset{\text{def.}}{=} E_{\beta_0}^{(\mathbf{c}(\beta_0))} E_{\beta_{-1}}^{(\mathbf{c}(\beta_{-1}))} \cdots, \qquad E_{\mathbf{c}_<} \overset{\text{def.}}{=} \cdots E_{\beta_2}^{(\mathbf{c}(\beta_2))} E_{\beta_1}^{(\mathbf{c}(\beta_1))}.$$

Next, given $\mathbf{c}_0$, we associate an I-tuple of partitons $(\lambda^{(i)})_{i\in I}$ as

$$\lambda^{(i)} \overset{\text{def.}}{=} (1^{\mathbf{c}_0(\delta,i)} 2^{\mathbf{c}_0(2\delta,i)} \cdots k^{\mathbf{c}_0(k\delta,i)} \cdots).$$

As in [15] we denote it also in another notation:

$$\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots).$$

We define the corresponding Schur function

$$S_{\mathbf{c}_0} \overset{\text{def.}}{=} \prod_{i\in I} \det\left(\widetilde{P}_{{}^t\lambda_k^{(i)} - k + l, i} \right)_{1\leq k,l\leq t},$$

where $t \geq l(\lambda^{(i)})$ and ${}^t\lambda^{(i)}$ means the transpose partition of $\lambda^{(i)}$. Note that $\widetilde{P}_{m,i}$ corresponds to an elementary symmetric function, while $P_{m,i}$ corresponds to a complete symmetric function, up to sign.

Now a main result of [5] says that

(1) $B_{\mathbf{c}} \overset{\text{def.}}{=} \overline{E_{\mathbf{c}_>} \cdot S_{\mathbf{c}_0} \cdot E_{\mathbf{c}_<}}$ is contained in $\mathcal{L}(\infty) \cap \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})^+$,
(2) $\{B_{\mathbf{c}} \bmod q\mathcal{L}(\infty) \mid \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\mathcal{R}}\}$ is the crystal base of $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$.

Set $(\mathbb{Z}_{\geq 0}^{\mathcal{R}_0})(\lambda) \overset{\text{def.}}{=} \left\{ \mathbf{c}_0 \in \mathbb{Z}_{\geq 0}^{\mathcal{R}_0} \,\middle|\, l(\lambda^{(i)}) \leq \langle h_i, \lambda \rangle \text{ for all } i \in I \right\}$, where $(\lambda^{(i)})_{i\in I}$ is the I-tuple of partition corresponding to $\mathbf{c}_0$ as above.

We apply $\vee$ to the above crystal base to get

$$F_{\mathbf{c}_>} \overset{\text{def.}}{=} (E_{\mathbf{c}_>})^\vee, \quad F_{\mathbf{c}_<} \overset{\text{def.}}{=} (E_{\mathbf{c}_<})^\vee, \quad S_{\mathbf{c}_0}^- \overset{\text{def.}}{=} (S_{\mathbf{c}_0})^\vee.$$

2.8. Extremal weight modules and the Drinfeld realization

Extremal weight modules are defined in terms of Chevalley generators. We shall rewrite the definition in terms of Drinfeld generators, and derive several easy consequences in this subsection.

The following is a consequence of [12, Theorem 5.3].

Lemma 2.14. *Let u be a vector of an integrable $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module M with weight $\lambda \in \widehat{P}^{0,+}$. Then the following conditions are equivalent:*

(1) u is an extremal vector.

(2) $x^+_{i,r} u = 0$ for all $i \in I$, $r \in \mathbb{Z}$.

Remark 2.15. The extremal weight module $V(\lambda)$ is isomorphic to the Weyl module $W_q(\lambda)$ introduced by Chari-Pressley [6]. This result was refered as 'an unpublished work' of Kashiwara in [loc. cit., Proposition 4.5]. Let us give Kashiwara's proof here. Let $\lambda = \sum_{i\in I} m_i \varpi_i \in \widehat{P}^{0,+}$. Then $W_q(\lambda)$ is integrable and contains a vector w_λ of weight λ which satisfies the above condition (2). Therefore, there is a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism $V(\lambda) \to W_q(\lambda)$, sending v_λ to w_λ. (The integrablity of $W_q(\lambda)$ was proved via the isomorphism $V(\lambda) \cong W_q(\lambda)$ in [loc. cit.]. So one must give another proof of the integrablity as sketched in [loc. cit.].) Since $W_q(\lambda)$ is generated by w_λ by definition, the homomorphism is surjective. On the other hand, any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module generated by a vector u of weight λ satisfying the above condition (2) is a quotient of $W_q(\lambda)$ [loc. cit., Proposition 4.6]. Therefore $V(\lambda)$ and $W_q(\lambda)$ are isomorphic.

Corollary 2.16. *Let u be an extremal vector with weight $\lambda \in \widehat{P}^{0,+}$. Then $S^-_{\mathbf{c}_0} u = \overline{S^*_{\mathbf{c}_0}} u = 0$ for $\mathbf{c}_0 \notin (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)$.*

Proof. It is enough to show the assertion for $u = u_\lambda \in V(\lambda)$. We have a $\mathbb{Q}(q)$-vector space isomorphism

$$V(\lambda) \ni x u_\lambda \mapsto x^\vee u_{-\lambda} \in V(-\lambda).$$

Therefore it is enough to show $S_{\mathbf{c}_0} u_{-\lambda} = \Omega(S_{\mathbf{c}_0}) u_{-\lambda} = 0$. By [6, Proposition 4.3], which is applicable thanks to Lemma 2.14, we have

$$\widetilde{P}_{m,i} u_{-\lambda} = 0 \qquad \text{for } |m| > \langle h_i, \lambda \rangle.$$

(More precisely, we apply [loc. cit.] after composing an automorphism $x^\pm_{i,r} \mapsto -x^\mp_{i,-r}$, $h_{i,m} \mapsto -h_{i,-m}$.) Now the assertion follows from a standard result in the theory of symmetric polynomials. $\square$

§3. A study of extremal weight modules

3.1. Fundamental representations

By [12, §5.2] there is a unique $\mathbf{U}_q'(\widehat{\mathfrak{g}})$-linear automorphism z_i of $V(\varpi_i)$ with weight δ, which sends u_{ϖ_i} to $u_{\varpi_i+\delta}$. (Note that d_i in [12, §5.2] is equal to 1 for untwisted $\widehat{\mathfrak{g}}$.)

Proposition 3.1. $h_{i,1}u_{\varpi_i} = o(i)(-1)^{1-h}q^{-h^\vee}z_i u_{\varpi_i}$.

Proof. We have

$$h_{i,1}u_{\varpi_i} = t_i^{-1}\left[x_{i,1}^+, x_{i,0}^-\right]u_{\varpi_i} = o(i)t_i^{-1}T_{\varpi_i^\vee}^{-1}(e_i)f_i u_{\varpi_i}.$$

Let us write $T_{\varpi_i^\vee} = \tau T_w$ with $w \in \widehat{W}$. Then Lemma 2.11 implies

$$T_{\varpi_i^\vee}^{-1}(e_i)f_i u_{\varpi_i} = (-1)^{N_+^{\prime\vee}}q^{N_+'}S_w^{-1}\left(e_{\tau^{-1}(i)}S_w(f_i u_{\varpi_i})\right), \tag{3.2}$$

where $N_+' = \sum_{\alpha\in\widehat{\mathfrak{R}}_+\cap w^{-1}(\widehat{\mathfrak{R}}_-)}\max((\alpha, s_i\varpi_i), 0) - \max((\alpha,\varpi_i),0)$, and $N_+^{\prime\vee}$ is given by replacing α by $\alpha^\vee$. Since $\widehat{\mathfrak{R}}_+\cap w^{-1}(\widehat{\mathfrak{R}}_-) = \widehat{\mathfrak{R}}_+\cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathfrak{R}}_-) = \{\beta + n\delta \mid \beta\in\Delta_+, 0\le n < \langle\varpi_i,\alpha\rangle\}$, we have

$$\max((\alpha,\varpi_i),0) = (\alpha,\varpi_i), \quad \max((\alpha, s_i\varpi_i),0) = \begin{cases} 0 & \text{if } \alpha = \alpha_i, \\ (\alpha, s_i\varpi_i) & \text{otherwise.}\end{cases}$$

Therefore

$$N_+' = (\alpha_i,\varpi_i) - \sum_{\alpha\in\widehat{\mathfrak{R}}_+\cap w^{-1}(\widehat{\mathfrak{R}}_-)}(\alpha,\alpha_i) = (\alpha_i,\varpi_i) - h^\vee,$$

where we have used Lemma 2.1. Similarly we have $N_+^{\prime\vee} = 1 - h$. Now the assertion follows from the definition of the Weyl group action S. □

Remark 3.3. Let $W(\varpi_i) \overset{\text{def.}}{=} V(\varpi_i)/(z_i - 1)V(\varpi_i)$. This is a finite dimensional irreducible $\mathbf{U}_q'(\widehat{\mathfrak{g}})$-module [12, §5.2]. The above proposition says that $W(\varpi_i)$ has the Drinfeld polynomial

$$P_j(u) = \begin{cases} 1 & \text{if } j \ne i, \\ 1 + o(i)(-1)^h q^{-h^\vee}u & \text{if } j = i.\end{cases}$$

Proposition 3.4. $(\tilde{P}_{\pm1,i})^\vee u_{\varpi_i} = z_i^\pm u_{\varpi_i}$.

Proof. Let us endow a new $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure on $V(-\varpi_i)$ by

$$x \cdot u \stackrel{\text{def.}}{=} x^\vee \cdot u, \quad (x \in \mathbf{U}_q(\widehat{\mathfrak{g}}), u \in V(-\varpi_i)).$$

We denote it by $V(-\varpi_i)^\vee$. Then there is a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module isomorphism $V(\varpi_i) \cong V(-\varpi_i)^\vee$ sending u_{ϖ_i} to $u_{-\varpi_i}$. Using this isomorphism, we can calculate $(\tilde{P}_{\pm 1,i})^\vee u_{\varpi_i}$ exactly as in the above proposition (in fact, more easily) to get the assertion. □

3.2. Tensor product modules

Let $\lambda = \sum_{i\in I} m_i\varpi_i \in \widehat{P}^{0,+}$. We define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $\widetilde{V}(\lambda)$, $\widetilde{\mathcal{L}}(\lambda)$, $\widetilde{\mathcal{B}}(\lambda)$, $\widetilde{u}_\lambda$ as in the introduction. Let $z_{i,\nu}$ ($i \in I$, $\nu = 1, \dots, m_i$) be the $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-linear automorphism of $\widetilde{V}(\lambda)$ obtained by the action of $z_i \colon V(\varpi_i) \to V(\varpi_i)$ on the ν-th factor. Obviously they are commuting: $z_{i,\nu}z_{j,\mu} = z_{j,\mu}z_{i,\nu}$. Let

$$\breve{V}(\lambda) \stackrel{\text{def.}}{=} \mathbf{U}_q(\widehat{\mathfrak{g}})[z_{i,\nu}^{\pm}]_{i\in I,\nu=1,\dots,m_i} \cdot \widetilde{u}_\lambda, \quad \breve{\mathcal{L}}(\lambda) \stackrel{\text{def.}}{=} \widetilde{\mathcal{L}}(\lambda) \cap \breve{V}(\lambda),$$
$$\breve{\mathcal{B}}(\lambda) \stackrel{\text{def.}}{=} \bigotimes_{i\in I} \mathcal{B}(\varpi_i)^{\otimes m_i}, \quad \breve{V}^{\mathbb{Z}}(\lambda) \stackrel{\text{def.}}{=} \bigotimes_{i\in I} \left(V(\varpi_i)^{\mathbb{Z}}\right)^{\otimes m_i} \cap \breve{V}(\lambda).$$

By [12, §8], the submodule $\breve{V}(\lambda)$ has

(1) the unique bar involution $\overline{}$ satisfying
$\overline{xu} = \overline{x}\,\overline{u}$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})[z_{i,\nu}^{\pm}]_{i\in I,\nu=1,\dots,m_i}$, $u \in \breve{V}(\lambda)$,

(2) the crystal base $(\breve{\mathcal{L}}(\lambda), \breve{\mathcal{B}}(\lambda))$, and

(3) the $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule $\breve{V}^{\mathbb{Z}}(\lambda)$ and the global crystal base $\{G(b) \mid b \in \breve{\mathcal{B}}(\lambda)\}$.

The module $\widetilde{V}(\lambda)$ contains the extremal vector $\widetilde{u}_\lambda$ of weight λ. Therefore there exists a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism $\Phi_\lambda \colon V(\lambda) \to \widetilde{V}(\lambda)$ sending u_λ to $\widetilde{u}_\lambda$. The image is contained in $\breve{V}(\lambda)$.

Recall that a function $\mathbf{c}_0 \in \mathcal{R}_0 \to \mathbb{Z}_{\geq 0}$ defines an I-tuple of partitions $(\lambda^{(i)})_{i\in I}$ as §2.7. We define an endomorphism of $\widetilde{V}(\lambda)$ by

$$s_{\mathbf{c}_0}(z^{\pm}) \stackrel{\text{def.}}{=} \prod_{i\in I} s_{\lambda^{(i)}}(z_{i,1}^{\pm}, \dots, z_{i,m_i}^{\pm}),$$

where $s_{\lambda^{(i)}}$ is the Schur polynomial corresponding to the partition $\lambda^{(i)}$. If $l(\lambda^{(i)}) > m_i$, it is understood as 0.

Proposition 3.5. $\Phi_\lambda(S_{\mathbf{c}_0}^- u_\lambda) = s_{\mathbf{c}_0}(z) \cdot \widetilde{u}_\lambda$, $\Phi_\lambda(\overline{S_{\mathbf{c}_0}^* u_\lambda}) = s_{\mathbf{c}_0}(z^{-1}) \cdot \widetilde{u}_\lambda$.

Proof. On level 0 modules, we have

$$\Delta h_{i,\pm m} = h_{i,\pm m} \otimes 1 + 1 \otimes h_{i,\pm m} + \text{a nilpotent term}$$

by [7]. Up to sign, the transition between $h_{i,m}$'s and $P_{k,i}$'s is the same as that between power sums and elementary symmetric functions. The above equation means that Δ coincides with the standard coproduct on symmetric polynomials modulo nilpotent terms [15, Chap. I, §5, Ex. 25]. Therefore we have

$$\Delta P_{k,i} = \sum_{s=0}^{k} P_{s,i} \otimes P_{k-s,i} + \text{a nilpotent term}.$$

Using Corollary 2.16 and Proposition 3.4, we have the assertion. $\square$

3.3. Detemination of extremal vectors

Proposition 3.6. *Suppose* $\lambda \in \widehat{P}^{0,+}$. *Consider* $B_{\mathbf{c}} = \overline{F_{\mathbf{c}_>} \cdot S^-_{\mathbf{c}_0} \cdot F_{\mathbf{c}_<}}$ *with* $\mathrm{wt}\, B_{\mathbf{c}} \in \mathbb{Z}\delta$, *and set* $b_1 \overset{\text{def.}}{=} B_{\mathbf{c}} \bmod q\mathcal{L}(\infty) \in \mathcal{B}(\infty)$ *and* $b \overset{\text{def.}}{=} b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\widetilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})a_\lambda)$. *If* b *and* b^* *are extremal, then we have* $\mathbf{c}_> \equiv 0 \equiv \mathbf{c}_<$ *and* $\mathbf{c}_0 \in (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)$.

Proof. Assume $\mathbf{c}_> \not\equiv 0$ and take the largest number $k \leq 0$ satisfying $\mathbf{c}(\beta_k) \neq 0$. Let $w = s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}$.

Since b^* is extremal, we can consider b as an element of $\mathcal{B}(\lambda)$. We have

$$b = B_{\mathbf{c}} u_\lambda \bmod q\mathcal{L}(\lambda).$$

By Lemma 2.11, we have

$$S_w^{-1} b = (-1)^{N^\vee} q^N T_w^{-1}(B_{\mathbf{c}}) \cdot S_w^{-1}(u_\lambda) \bmod q\mathcal{L}(\lambda)$$

for some integers $N^\vee$, N. By [11, 8.2.2] there exists a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear isomorphism

$$V(\lambda) \to V(w^{-1}\lambda); \qquad S_w^{-1}(u_\lambda) \mapsto u_{w^{-1}\lambda},$$

respecting the crystal bases. Therefore we have

$$(-1)^{N^\vee} q^N T_w^{-1}(B_{\mathbf{c}}) u_{w^{-1}\lambda} \bmod q\mathcal{L}(w^{-1}\lambda) \in \mathcal{B}(w^{-1}\lambda).$$

(In fact, this is equal to $S^*_{w^{-1}} S_{w^{-1}} b$.) Let us denote this by $b'_1 \otimes t_{w^{-1}\lambda} \otimes b'_2$.

We have

$$T_w^{-1}(B_{\mathbf{c}}) = T_w^{-1}(\overline{F_{\mathbf{c}_>}}) \cdot T_w^{-1}(\overline{S^-_{\mathbf{c}_0}}) \cdot T_w^{-1}(\overline{F_{\mathbf{c}_<}}).$$

It is clear that $T_w^{-1}(\overline{F_{\mathbf{c}_<}}) \in \mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-$. We also have $T_w^{-1}(\overline{S_{\mathbf{c}_0}^-}) \in \mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-$ by [3, Lemma 2]. (More precisely, we apply [loc. cit.] after composing $\bar{\ } \circ \vee$. Note that $T_w^{-1} = \bar{\ } \circ \vee \circ T_w \circ \bar{\ } \circ \vee$ by [13, 39.4.5].) Moreover, by our choice of k, we have

$$T_w(\overline{F_{\mathbf{c}_>}}) = f_{i_k}^{(\mathbf{c}(\beta_k))} T_{i_k}(f_{i_{k-1}}^{(\mathbf{c}(\beta_{k-1}))}) \cdots \in f_{i_k}^{(\mathbf{c}(\beta_k))} \left(\mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-\right).$$

Therefore we have

$$b_2' = u_{-\infty}, \qquad b_1' = T_w^{-1}(B_{\mathbf{c}}) \bmod q\mathcal{L}(\infty), \qquad \varepsilon_{i_k}(b_1') = \mathbf{c}(\beta_k),$$

where the last equality follows from [13, 38.1.6]. Since $b_1' \otimes t_{w^{-1}\lambda} \otimes u_{-\infty}$ is extremal, Lemma 2.12 implies

$$\mathbf{c}(\beta_k) \le \max(-\langle h_{i_k}, w^{-1}\lambda\rangle, 0). \tag{3.7}$$

However, we have $\langle h_{i_k}, w^{-1}\lambda\rangle = (w\alpha_{i_k}^\vee, \lambda) \ge 0$ for $\lambda \in \widehat{P}^{0,+}$, because $w\alpha_{i_k} \in \widehat{\mathfrak{R}}_>$ by (2.13). So the right hand side of (3.7) is 0, and this contradicts with the choice of k. Therefore $\mathbf{c}_> \equiv 0$. Applying $*$, we similarly get $\mathbf{c}_< \equiv 0$. Now the last assertion is a consequence of Corollary 2.16. □

Proof of Theorem 1. We first prove (2), (3), (4) and then (1).

(2) Recall that any vector $b \in \mathcal{B}(\lambda)$ is connected to an extremal vector [11, 9.3.3]. Moreover, an extremal vector can be mapped by $\widetilde{f}_i^{\max}$ to an extremal vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$. (See [12, Proof of Theorem 5.1]). Therefore

$$\mathcal{B}(\lambda) = \left\{X_l \cdots X_1 \overline{S_{\mathbf{c}_0}^-} \bmod q\mathcal{L}(\Lambda) \,\middle|\, \mathbf{c}_0 \in (\mathbb{Z}_{\ge 0}^{\mathfrak{R}_0})(\lambda),\ X_\mu \text{ is } \widetilde{e}_i \text{ or } \widetilde{f}_i\right\} \setminus \{0\}$$

by Proposition 3.6. Then $\mathcal{L}(\lambda)$ is spanned by $\{X_l \cdots X_1 \overline{S_{\mathbf{c}_0}^-}\}$ over $\mathbf{A}_0$, by Nakayama's lemma. Note that Φ_λ commutes with the operators $\widetilde{e}_i$, $\widetilde{f}_i$ and $\widetilde{\mathcal{L}}(\lambda)$ is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$. Therefore it is enough to show that $\Phi_\lambda(\overline{S_{\mathbf{c}_0}^-}) \in \widetilde{\mathcal{L}}(\lambda)$. But this follows from Proposition 3.5.

(3) By Proposition 3.5, we have

$$\Phi_\lambda^0(\overline{S_{\mathbf{c}_0}^-} \bmod q\mathcal{L}(\lambda)) \in \widetilde{\mathcal{B}}(\lambda) \qquad \text{for } \mathbf{c}_0 \in (\mathbb{Z}_{\ge 0}^{\mathfrak{R}_0})(\lambda).$$

As in the proof of (1), we conclude that $\Phi_\lambda^0(\mathcal{B}(\lambda)) \subset \widetilde{\mathcal{B}}(\lambda) \sqcup \{0\}$. From the definition, it is obvious that the image contains $\widetilde{\mathcal{B}}(\lambda)$. Consider $\operatorname{Ker}\Phi_\lambda^0 \cap \mathcal{B}(\lambda)$. It is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$. Since any vector is connected to an extremal vector, $\operatorname{Ker}\Phi^0_\lambda \cap \mathcal{B}(\lambda)$ contains an extremal vector if it is nonempty. But we already checked that every extremal vector is mapped to a nonzero vector. Hence $\operatorname{Ker}\Phi^0_\lambda \cap \mathcal{B}(\lambda) = \emptyset$. Now suppose b_1, $b_2 \in \mathcal{B}(\lambda)$ satisfy $\Phi^0_\lambda(b_1) = \Phi^0_\lambda(b_2)$. We want to show $b_1 = b_2$. Applying $\widetilde{e}_i$, $\widetilde{f}_i$'s, we may assume $b_1 = \overline{S^-_{\mathbf{c}_0}} \bmod q\mathcal{L}(\lambda)$. By [12, §5.1] b_2 is also extremal. Applying $\widetilde{f}_i^{\max}$'s if necessarily, we may assume b_2 is of form $b_2^- \otimes t_\lambda \otimes u_{-\infty}$, and hence $b_2 = \overline{S^-_{\mathbf{c}'_0}} \bmod q\mathcal{L}(\lambda)$. By this process, b_1 may be changed, but still is of form $b_1^- \otimes t_\lambda \otimes u_{-\infty}$, so we may assume $b_1 = \overline{S^-_{\mathbf{c}_0}} \bmod q\mathcal{L}(\lambda)$ after we change $\mathbf{c}_0$. By Proposition 3.5, we have $s_{\mathbf{c}_0}(z) \cdot \widetilde{u}_\lambda = \Phi^0_\lambda(b_1) = \Phi^0_\lambda(b_2) = s_{\mathbf{c}'_0}(z) \cdot \widetilde{u}_\lambda$. This implies $\mathbf{c}_0 = \mathbf{c}'_0$ and hence $b_1 = b_2$.

(4) By the uniqueness, Φ_λ respects the bar involutions on $V(\lambda)$ and $\widetilde{V}(\lambda)$. Since $V^{\mathbb{Z}}(\lambda) = \mathbf{U}^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})u_\lambda$, we have $\Phi_\lambda(V^{\mathbb{Z}}(\lambda)) \subset \check{V}^{\mathbb{Z}}(\lambda)$. Therefore we have

$$\Phi_\lambda\left(\mathcal{L}(\lambda) \cap \overline{\mathcal{L}(\lambda)} \cap V^{\mathbb{Z}}(\lambda)\right) \subset \check{\mathcal{L}}(\lambda) \cap \overline{\check{\mathcal{L}}(\lambda)} \cap \check{V}^{\mathbb{Z}}(\lambda).$$

Now the assertion follows from (3).

(1) It is easy to see that $\widetilde{\mathcal{B}}(\lambda)$ is linearly independent. Therefore $\Phi^0_\lambda \colon \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \widetilde{\mathcal{L}}(\lambda)/q\widetilde{\mathcal{L}}(\lambda)$ is injective.

Let $\{G(b)\}$ be the global crystal base of $V(\lambda)$. Let $0 \neq \sum f_b(q)G(b) \in \operatorname{Ker}\Phi_\lambda$. Multiplying a power of q, we may assume $f_b(q) \in \mathbf{A}_0$ for all b and $f_{b_0}(0) \neq 0$ for some b_0. Then $\sum f_b(0)b \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ is mapped to 0 by Φ^0_λ. The injectivity of Φ^0_λ implies that $f_b(0) = 0$ for all b. This is a contradiction. □

Remark 3.8. Theorem 1 together with Proposition 3.5 implies that $S^-_{\mathbf{c}_0}u_\lambda$ and $\overline{S^*_{\mathbf{c}_0}}u_\lambda$ are elements of the global base.

3.4. Standard modules

Let us briefly recall the properties of the universal standard module $M(\lambda)$ with a weight $\lambda = \sum m_i\varpi_i \in \widehat{P}^{0,+}$ introduced in [16, 18]. (We do not review its definition, which is based on quiver varieties.) Let $G_\lambda \overset{\text{def.}}{=} \prod_i \mathrm{GL}_{m_i}(\mathbb{C})$. Its maximal torus consisting of diagonal matrices is denoted by H_λ. Their representation rings are denoted by $R(G_\lambda)$, $R(H_\lambda)$ respectively. They are isomrphic to $\bigotimes_i \mathbb{Z}[x^\pm_{i,1}, \dots, x^\pm_{i,m_i}]^{\mathfrak{S}_{m_i}}$ and $\bigotimes_i \mathbb{Z}[x^\pm_{i,1}, \dots, x^\pm_{i,m_i}]$ respectively. The universal standard module $M(\lambda)$ is a $\mathbf{U}'^{\mathbb{Z}}_q(\widehat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda)$-module which is integrable (in fact, it satisfies a stronger condition 'l–integrability') and contains a vector $[0]_\lambda$ with

$$x^+_{i,r}[0]_\lambda = 0 \quad \text{for any } i \in I,\ r \in \mathbb{Z}, \quad q^h[0]_\lambda = q^{\langle h,\lambda\rangle}[0]_\lambda,$$
$$M(\lambda) = \left(\mathbf{U}'^{\mathbb{Z}}_q(\widehat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda)\right)[0]_\lambda,$$
$$\psi^\pm_i(u)[0]_\lambda = q^{m_i}\left(\prod_{\nu=1}^{m_i} \frac{1-q^{-1}x_{i,\nu}u}{1-qx_{i,\nu}u}\right)^{\pm}[0]_\lambda,$$

where $(\ )^\pm$ denotes the expansion at $u=0$ and ∞ respectively. (In fact, we have $M(\lambda) = \mathbf{U}'^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})[0]_\lambda$ by the proof of Theorem 1.) Moreover, $M(\lambda)$ is free of finite rank as an $R(G_\lambda)$-module. And $M(\lambda)$ is simple if we tensor the quotient field of $\mathbb{Z}[q,q^{-1}] \otimes R(G_\lambda)$.

On the other hand, we have a $\bigotimes_{i\in I}\mathbb{Z}[z^\pm_{i,1},\dots,z^\pm_{i,m_i}]^{\mathfrak{S}_{m_i}}$ -module structure on $V(\lambda)$ given by $s_{\mathbf{c}_0}(z)u_\lambda = S^-_{\mathbf{c}_0}u_\lambda$ and $s_{\mathbf{c}_0}(z^{-1})u_\lambda = \overline{S^*_{\mathbf{c}_0}}u_\lambda$ by the above discussion. We make it a $R(G_\lambda) = \bigotimes_{i\in I}\mathbb{Z}[x^\pm_{i,1},\dots,x^\pm_{i,m_i}]^{\mathfrak{S}_{m_i}}$ -module structure by setting $x_{i,\nu} = o(i)(-1)^{1-h}q^{-h^\vee}z_{i,\nu}$.

Theorem 2. *There exists a unique* $\mathbf{U}'^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})\otimes_{\mathbb{Z}}R(G_\lambda)$ *-isomorphism* $V^{\mathbb{Z}}(\lambda) \to M(\lambda)$ *sending* u_λ *to* $[0]_\lambda$.

This result follows from Theorem 1 as explained in [18, 1.23]. The calculation of Drinfeld polynomial, which was not given there, is done in Proposition 3.1.

Correction to [18]:

Delete $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ in Theorem 1.22.

Replace $R(G_\lambda)$ in page 411, line 5 by $R(H_\lambda)$.

Delete 'and forgetting the symmetric group invariance' in Remark 1.23.

Replace 'the submodule above' in line 8, 'the submodule $\mathbf{U}^{\mathbb{Z}}_q(\mathbf{L}\mathfrak{g})[x_{k,\nu}]_{k\in I,\nu=1,\dots,\lambda_k}\bigotimes_{k\in I}[0]^{\otimes\lambda_k}_{\Lambda_k}$.

§4. A bilinear form

Kashiwara proved that the crystal base $\mathcal{B}(\lambda)$ is an orthonomal base with respect to a natural bilinear form when λ is dominant [10, 5.1.1]. We prove a similar result for $\lambda \in \widehat{P}^{0,+}$ in this section. This generalizes a result of Varagnolo-Vasserot [20, Theorem A] from fundamental representations to arbitray λ.

Proposition 4.1 (Kashiwara). *The extremal weight module $V(\lambda)$ has a unique bilinear form $(\ ,\ )$ satisfying*

$$(u_\lambda, G(b)) = \begin{cases} 1 & \text{if } G(b) = u_\lambda, \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$$(xu, v) = (u, \psi(x)v) \quad \text{for } x \in \mathbf{U}_q(\widehat{\mathfrak{g}}),\ u, v \in V(\lambda). \tag{4.3}$$

Proof. We define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure on $\operatorname{Hom}(V(\lambda), \mathbb{Q}(q))$ by

$$\langle xf, u\rangle \overset{\text{def.}}{=} \langle f, \psi(x)u\rangle, \quad x \in \mathbf{U}_q(\widehat{\mathfrak{g}}), f \in \operatorname{Hom}(V(\lambda), \mathbb{Q}(q)), u \in V(\lambda).$$

This defines a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure since $\psi\colon \mathbf{U}_q(\widehat{\mathfrak{g}}) \to \mathbf{U}_q(\widehat{\mathfrak{g}})^{\mathrm{opp}}$ is an algebra homomorphism. Let u^λ be the unique linear form such that

$$\langle u^\lambda, G(b)\rangle = \begin{cases} 1 & \text{if } G(b) = u_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then u^λ has a weight λ. We claim that u^λ is an extremal vector. From the definition all elements in a weight space $\operatorname{Hom}(V(\lambda), \mathbb{Q}(q))_\xi$ vanish on $V(\lambda)_\xi$. Since weights of $V(\lambda)$ are contained in the convex hull of $W\lambda$ [12, Theorem 5.3], the weights of $V'(\lambda)$ also have the same property. Therefore u^λ is an extremal vector. Now we have a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-algebra homomorphism $V(\lambda) \to V'(\lambda) \subset \operatorname{Hom}(V(\lambda), \mathbb{Q}(q))$ sending u_λ to u^λ. This defines a bilinear form satisfying the desired properties. The uniqueness follows from the uniqueness of the above homomorphism. □

Remark 4.4. The uniqueness holds even if (4.3) holds only for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. In fact, this condition together with (4.2) automatically implies (4.3) for $x = q^d$ as follows. When $u = u_\lambda$, (4.2) implies (4.3) for $x = q^d$. For a general case, we write $u = xu_\lambda$ with $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})_\xi$. Then

$$\begin{aligned}(q^d u, v) &= q^{\langle d,\xi\rangle}(xq^d u_\lambda, v) = q^{\langle d,\xi\rangle}(q^d u_\lambda, \psi(x)v) = q^{\langle d,\xi\rangle}(u_\lambda, q^d\psi(x)v) \\ &= (u_\lambda, \psi(x)q^d v) = (xu_\lambda, q^d v) = (u, q^d v),\end{aligned}$$

where we have used $\psi(x) \in \mathbf{U}'_q(\widehat{\mathfrak{g}})_{-\xi}$.

Lemma 4.5. *Let M be an integrable $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module with a bilinear form $(\ ,\ )$ satisfying (4.3) for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. Then*

$$(T_w u, v) = (-1)^{N^\vee} q^N (u, T_{w^{-1}} v) \quad \text{for all } w \in \widehat{W},\ u \in M_\xi,\ v \in M,$$

where N and $N^\vee$ are as in Lemma 2.5.

Proof. Let $T'_{i,1}$ be the operator defined in [13, 5.2.1]. A direct calculation shows $(T_i u, v) = (u, T'_{i,1} v)$ for $u \in M_\xi$, $v \in M$. (We may assume that v is contained in a weight space. Thanks to (4.3) for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, both hand sides are 0 unless the weight of v is $s_i\xi + m\delta$ for some $m \in \mathbb{Z}$.) By [loc. cit., 5.2.3], we have $T'_{i,1} v = (-1)^{\langle h_i, \xi\rangle} q^{(\alpha_i,\xi)} T_i v$. The rest of the proof is the same as that of Lemma 2.5. □

Lemma 4.6. *Let M and $(\ ,\ )$ be as above. Let u, $v \in M$ be extremal vectors. Then*

$$(S_w u, v) = (u, S_{w^{-1}} v).$$

Proof. Let ξ be the weight of u. Using Lemmas 2.11, 4.5, we have

$$(S_w u, v) = (-1)^{N_+^\vee + N_+^{\vee\prime} + N^\vee} q^{-N_+ - N'_+ + N} (u, S_{w^{-1}} v),$$

where

$$N = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha, \xi), \qquad N_+ = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} \max((\alpha,\xi), 0),$$
$$N'_+ = \sum_{\alpha' \in \widehat{\mathfrak{R}}_+ \cap w(\widehat{\mathfrak{R}}_-)} \max((\alpha', w\xi), 0),$$

and $N^\vee$, $N_+^\vee$, $N_+^{\vee\prime}$ are defined in similar ways. Noticing $\alpha' \in \widehat{\mathfrak{R}}_+ \cap w(\widehat{\mathfrak{R}}_-) \Leftrightarrow -w^{-1}\alpha' \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)$, we have $N = N_+ + N'_+$. Similarly we have $N^\vee = N_+^\vee + N_+^{\vee\prime}$. Therefore we have the assertion. □

In order to study $(\ ,\ )$ on $V(\lambda)$ we need to relate it to a bilinear form on the tensor product module $\widetilde{V}(\lambda)$.

Lemma 4.7. *We have $(z_i u, z_i v) = (u, v)$ for $u, v \in V(\varpi_i)$.*

Proof. By the uniqueness, it is enough to show that $(z_i u, z_i v)$ satisfies (4.2, 4.3). The property (4.3) is clear. If $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, then it holds since z_i is $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-linear. It also holds for $x = q^d$ thanks to $z_i q^d z_i^{-1} = q^{-a_0 d_i} q^d$.

Let us check (4.2). Since $\dim V(\varpi_i)_{\varpi_i} = 1$ by [12, Proposition 5.10], it is enough to show that $(z_i u_{\varpi_i}, z_i u_{\varpi_i}) = 1$. But this follows from the previous lemma. □

We define a $\mathbb{Q}(q)[z_i^{\pm}]$-valued bilinear form $(\!(\ , \ )\!)$ on $V(\varpi_i)$ by

$$(\!(u,v)\!) = \begin{cases} z_i^m(z_i^{-m}u,v) & \text{if } \mathrm{wt}(u) = \mathrm{wt}(v) + md_i\delta \text{ for } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Since z_i is $\mathbf{U}_q'(\widehat{\mathfrak{g}})$-linear, we have

$$(\!(xu,v)\!) = (\!(u,\psi(x)v)\!) \quad \text{for } x \in \mathbf{U}_q'(\widehat{\mathfrak{g}}),\ u,v \in V(\varpi_i).$$

By Lemma 4.7 we have

$$(\!(z_i^m u_{\varpi_i}, z_i^n u_{\varpi_i})\!) = z_i^{m-n}. \tag{4.8}$$

We define a $\mathbb{Q}(q)[z_{i,\nu}^{\pm}]_{i\in I,\nu=1,\dots,m_i}$-valued bilinear form $(\!(\ , \ )\!)$ on $\widetilde{V}(\lambda)$ by

$$(\!(u,v)\!) \stackrel{\text{def.}}{=} \prod_{i,\nu} (\!(u_{i,\nu}, v_{i,\nu})\!),$$

where $u_{i,\nu}$, $v_{i,\nu}$ is the ν-th $V(\varpi_i)$-factor of $u,v \in \widetilde{V}(\lambda)$. We define a bilinear form $(\ , \ )^{\sim}$ on $\widetilde{V}(\lambda)$ by

$$(u,v)^{\sim} \stackrel{\text{def.}}{=} \prod_{i\in I} \frac{1}{m_i!} \left[(\!(u,v)\!) \prod_{\mu\neq\nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right]_1,$$

where $[f]_1$ denote the constant term in f.

Lemma 4.9. *Let* $\mathbf{c}_0$, $\mathbf{c}_0' \in (\mathbb{Z}_{\geq 0}^{\mathfrak{R}_0})(\lambda)$. *Then* $(s_{\mathbf{c}_0}(z)\widetilde{u}_\lambda, s_{\mathbf{c}_0'}(z)\widetilde{u}_\lambda)^{\sim} = \delta_{\mathbf{c}_0,\mathbf{c}_0'}$.

Proof. Let $f = f(z)$ and $g = g(z)$ be polynomials in $z_{i,\nu}$'s ($i \in I$, $\nu = 1,\dots,m_i$). By (4.8) we have

$$(f(z)\widetilde{u}_\lambda, g(z)\widetilde{u}_\lambda)^{\sim} = \prod_{i\in I} \frac{1}{m_i!} \left[f\overline{g} \prod_{\mu\neq\nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right]_1,$$

where $\overline{g} = g(\dots, z_{i,\nu}^{-1}, \dots)$. Considered as a bilinear form on the Laurent polynomial ring, it coincides with one in [15, Chap.VI, §9] with $q = t$. The Schur functions give an orthogonal base with respect to that bilinear form. Therefore we have the assertion. □

Proposition 4.10. *Let* $u,v \in V(\lambda)$. *Then* $(u,v) = (\Phi_\lambda(u), \Phi_\lambda(v))^{\sim}$.

Proof. It is enough to show that $(\Phi_\lambda(u), \Phi_\lambda(v))^\sim$ satisfies conditions in Proposition 4.1. It is clear that the condition (4.3) holds for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. By Remark 4.4, it is enough to check (4.2). From (4.3) for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, it is enough to check (4.2) when $\mathrm{cl}(\mathrm{wt}(b)) = \mathrm{cl}(\lambda)$, i.e., $\mathrm{wt}(b) = \lambda + m\delta$ for some $m \in \mathbb{Z}$. Since weights of $V(\lambda)$ is contained in the convex hull of $W\lambda$, b is an extremal vector. We have

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (\Phi_\lambda(S_w u_\lambda), \Phi_\lambda(S_w G(b)))^\sim$$

by Lemma 4.6. We take S_w as sufficiently many compositions of $\widetilde{f}_i^{\max}$, we may assume $S_w u_\lambda = S^-_{\mathbf{c}_0} u_\lambda$, $S_w G(b) = S^-_{\mathbf{c}'_0} u_\lambda$. (Recall that $S^-_{\mathbf{c}_0} u_\lambda$ is an element of the global basis as we explained in Remark 3.8.) Then

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (s_{\mathbf{c}_0}(z)\widetilde{u}_\lambda, s_{\mathbf{c}'_0}(z)\widetilde{u}_\lambda)^\sim = \delta_{\mathbf{c}_0,\mathbf{c}'_0} = \delta_{u_\lambda, G(b)},$$

where we have used Proposition 3.5 and Lemma 4.9. □

From the proof of Proposition 4.10 the bilinear form $(\ ,\ )$ on $V(\lambda)$ defined in Proposition 4.1 also has the following characterization: it satisfies (4.3) and $(S_{\mathbf{c}_0} u_\lambda, S_{\mathbf{c}'_0} u_\lambda) = \delta_{\mathbf{c}_0,\mathbf{c}'_0}$. Since these conditions are symmetric, we have the following:

Corollary 4.11. *The bilinear form $(\ ,\ )$ on $V(\lambda)$ is symmetric, i.e., $(u,v) = (v,u)$.*

Proposition 4.12. (1) $(\mathcal{L}(\lambda), \mathcal{L}(\lambda)) \subset \mathbf{A}_0$.
Let $(\ ,\ )_0$ be the $\mathbb{Q}$-valued bilinear form on $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ induced by $(\ ,\ )|_{q=0}$ on $\mathcal{L}(\lambda)$.

(2) $(\widetilde{e}_i u, v)_0 = (u, \widetilde{f}_i v)_0$ *for $u, v \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$.*

(3) $\mathcal{B}(\lambda)$ *is an orthonormal base with respect to $(\ ,\ )_0$. In particular, $(\ ,\ )_0$ is positive definite.*

(4) $\mathcal{L}(\lambda) = \{u \in V \mid (u,u) \in \mathbf{A}_0\}$.

Proof. We shall prove

- there exist representatives $\widetilde{b}$ for all $b \in \mathcal{B}(\lambda)_\xi \subset \mathcal{L}(\lambda)_\xi / q\mathcal{L}(\lambda)_\xi$ such that $(\widetilde{b}, \widetilde{b'}) \equiv \delta_{bb'} \mod q\mathbf{A}_0$ for $b, b' \in \mathcal{B}(\lambda)_\xi$

by the induction on (ξ, ξ). Since $\mathcal{L}(\lambda)_\xi$ is spanned by $\widetilde{b}$'s over $\mathbf{A}_0$, this implies the above equations for *any* representatives $\widetilde{b}$. It also implies (1) and (3). Recall $(\widetilde{e_i b}, \widetilde{b'}) = (1 - q_i)(\widetilde{b}, \widetilde{f_i b'})$ by (2.6). Therefore the above assertion also implies (2).

First suppose that b is extremal. Since we may assume that $\mathrm{wt}(b) = \mathrm{wt}(b')$ by (4.3), we may assume b' is also extremal by [12, 5.3]. Then we may assume $\widetilde{b} = S_{\mathbf{c}_0} u_\lambda$, $\widetilde{b'} = S_{\mathbf{c}'_0} u_\lambda$ by applying S_w for some $w \in \widehat{W}$. But, in this case, the assertion has been already shown in Lemma 4.9 and Proposition 4.10.

Now we start the induction. Recall that (ξ, ξ) is bounded from above and $b \in \mathcal{B}(\lambda)$ is extremal if $(\mathrm{wt}\, b, \mathrm{wt}\, b)$ is maximal ([11, §9.3]). Therefore when (ξ, ξ) is maximal, both b and b' are extremal. We have already proved the assertion this case.

Now assuming the above for ξ such that $(\xi, \xi) > a$, let us prove it for ξ with $(\xi, \xi) = a$. For $i \in I$, suppose that $\langle h_i, \xi \rangle \geq 0$. We consider $\widetilde{e}_i b$. If $\widetilde{e}_i b \neq 0$, then we have

$$(\mathrm{wt}(\widetilde{e}_i b), \mathrm{wt}(\widetilde{e}_i b)) = (\xi + \alpha_i, \xi + \alpha_i) > (\xi, \xi).$$

Therefore we have

$$\left(\widetilde{f}_i \widetilde{e}_i \widetilde{b}, \widetilde{b'}\right) = \frac{1}{1-q} \left(\widetilde{e}_i \widetilde{b}, \widetilde{e}_i \widetilde{b'}\right) \equiv \delta_{\widetilde{e}_i b, \widetilde{e}_i b'} \equiv \delta_{bb'} \mod q\mathbf{A}_0$$

by the induction hypotheis. Hence the assertion holds if we replace the representative $\widetilde{b}$ by another representative $\widetilde{f}_i \widetilde{e}_i \widetilde{b}$. Similarly, if $\langle h_i, \xi \rangle \leq 0$ and $\widetilde{f}_i b \neq 0$, we replace $\widetilde{b}$ by $\widetilde{e}_i \widetilde{f}_i \widetilde{b}$ to get the assertion.

Since we may suppose that b is not extremal, there exists $w \in \widehat{W}$ such that $S_w b$ satisfies $\widetilde{e}_i S_w b \neq 0$ if $\langle h_i, w\xi \rangle \geq 0$ and $\widetilde{f}_i S_w b \neq 0$ if $\langle h_i, w\xi \rangle \leq 0$. Then we have $(\widetilde{f}_i \widetilde{e}_i S_w \widetilde{b}, S_w \widetilde{b'})$ or $(\widetilde{e}_i \widetilde{f}_i S_w \widetilde{b}, S_w \widetilde{b'})$ is in $\delta_{bb'} + q\mathbf{A}_0$. Therefore we are done.

The statement (4) follows from [13, 14.2.2]. □

The following result generalizes [20, Theorem A] from fundamental representations to arbitrary λ:

Theorem 3. (1) $\{G(b)\}_{b \in \mathcal{B}(\lambda)}$ *is almost orthonormal for* $(\ ,\ )$, *that is,* $(G(b), G(b')) \equiv \delta_{bb'} \mod q\mathbb{Z}[q]$.

(2) $\{\pm G(b) \mid b \in \mathcal{B}(\lambda)\} = \left\{u \in V^{\mathbb{Z}}(\lambda) \mid \overline{u} = u,\ (u, u) \equiv 1 \bmod q\mathbb{Z}[q]\right\}$.

Proof. We claim

$$(u, v) \in \mathbb{Z}[q, q^{-1}] \quad \text{for } u, v \in V^{\mathbb{Z}}(\lambda).$$

The assertion is obvious for the special case $u = u_\lambda$ by (4.2). For general case, we may assume $u = x u_\lambda$ for $x \in \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$. Then $(x u_\lambda, v) = (u_\lambda, \psi(x) v)$. Since $\psi(x) \in \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ and $V^{\mathbb{Z}}(\lambda)$ is stable under the action of $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$, the assertion follows from the special case.

Combining with Proposition 4.12, we have

$$(G(b), G(b')) - \delta_{bb'} \in \mathbb{Z}[q, q^{-1}] \cap q\mathbf{A}_0 = q\mathbb{Z}[q].$$

This is the statement (1). The statement (2) follows from the argument of [13, 14.2.3]. □

Remark 4.13. Lusztig conjectures that the universal standard module $M(\lambda)$, more precisely its tensor product of $\otimes_{R(G_\lambda)} R(H_\lambda)$, which is isomorphic to $\breve{V}^{\mathbb{Z}}(\lambda)$, has a signed base characterized by the almost orthogonality property Theorem 3(2), with respect to geometrically defined bilinear form and bar involution [14]. (See §3.4 for notations.) Recently Varagnolo-Vasserot [20] give a proof of the conjecture by showing that $\{G(b) \mid b \in \breve{\mathcal{B}}(\lambda)\}$ satisfies the property. They also conjecture that the global base $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$ of $V(\lambda)$ satisfies the almost orthogonality property with respect to the geometric bilinear form and bar involution. Their conjecture follows from Theorem 3(2) since the geometric bilinear form and bar involution coincide with ones used in this paper, as Varangnolo and Vasserot proved that the formers satisfy the conditions in Proposition 4.1 (more precisely (4.3) and $(S_{\mathbf{c}_0} u_\lambda, S_{\mathbf{c}_0'} u_\lambda) = \delta_{\mathbf{c}_0, \mathbf{c}_0'}$) and the equality $\overline{xu} = \overline{x}\,\overline{u}$. Remark that these hold only after an *appropriate* normalization of universal standard modules so that we have $x_{i,\nu} = \pm z_{i,\nu}$. This is the normalization in [20] different from ours. This point is clarified during discussion with Varagnolo-Vasserot in Februrary 2002.

Added in Proof. Results of this paper are generalized to the case of arbitrary affine algebras in the paper "Crystal bases and two-sided cells of quantum affine algebras" by J. Beck and H. Nakajima, to appear in Duke Math. J.

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