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An induction theorem for Springer's representations

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§1. Statement of the result

1.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic p. We fix a prime number l different from p. Let G be a connected reductive algebraic group over $\mathbf{k}$. Let W be the Weyl group of G. Let $\mathcal{B}$ be the variety of Borel subgroups of G. For any $g \in G$ let $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. According to Springer [S], W acts naturally on the l-adic cohomology $H^n(\mathcal{B}_g)$. (Springer's original definition of the W action is valid only when g is unipotent and p is 0 or is large. Here we adopt the definition given in [L1] which is valid without restrictions on g and p.)

1.2. Let L be a Levi subgroup of a parabolic subgroup P of G. Let W' be the Weyl group of L (naturally a subgroup of W). Let $\mathcal{B}'$ be the variety of Borel subgroups of L (naturally a subvariety of $\mathcal{B}$). Let $u \in L$ be unipotent. Let $\mathcal{B}'_u = \{B' \in \mathcal{B}'; u \in B'\}$. Then the W-module $H^n(\mathcal{B}_u)$ and the W'-module $H^n(\mathcal{B}'_u)$ are well defined.

Theorem 1.3. *We have*

$$\sum_n (-1)^n H^n(\mathcal{B}_u) = \mathrm{ind}_{W'}^{W}\Big(\sum_n (-1)^n H^n(\mathcal{B}'_u)\Big)$$

(*equality of virtual W-modules*).

This result was stated without proof in [AL] in the case where $p = 0$. Here we provide a proof valid for any p (answering a question that J. C. Jantzen asked me).

In the remainder of this paper we assume that $p > 1$ and that $\mathbf{k}$ is an algebraic closure of the finite field F_p with p elements. (By standard results, if the theorem holds for such $\mathbf{k}$ then it holds for any $\mathbf{k}$.)

Let $\mathcal{Z}$ be the identity component of the centre of L. Clearly, the theorem is a consequence of Propositions 1.4, 1.5 below (these will be proved in Sections 2 and 3 respectively).

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Proposition 1.4. *There exists $t \in \mathcal{Z}$ such that for any $n \in \mathbf{Z}$, the W-modules $H^n(\mathcal{B}_{tu}), \mathrm{ind}_{W'}^{W}(H^n(\mathcal{B}'_u))$ are isomorphic.*

Proposition 1.5. *For any $t \in \mathcal{Z}$, the virtual W-modules $\sum_n (-1)^n H^n(\mathcal{B}_u)$, $\sum_n (-1)^n H^n(\mathcal{B}_{tu})$ are equal.*

§2. Proof of Proposition 1.4

2.1. For $g \in G$, let g_s be the semisimple part of g and let $Z_G^0(g_s)$ be the identity component of the centralizer of g_s in G. For any torus T in G we denote by $Z_G(T)$ the centralizer of T in G. Let

$$\mathcal{U} = \{g \in G; Z_G^0(g_s) \subset L\}.$$

Since $g \in Z_G^0(g_s)$, we see that $\mathcal{U} \subset L$.

Lemma 2.2. *(a) $\mathcal{U}$ is an open dense subset of L.*
(b) Let $\mathcal{Z}_r = \mathcal{U} \cap \mathcal{Z}$. There exist non-trivial characters $h_a : \mathcal{Z} \to \mathbf{k}^*$, $(a = 1, \ldots, N)$ such that $\mathcal{Z}_r = \mathcal{Z} - \bigcup_{a=1}^{N} \ker(h_a)$.
(c) If $B \in \mathcal{B}$, $g \in \mathcal{U} \cap B$, then $\mathcal{Z} \subset B$.

We prove (a). Let $\mathcal{T}$ be the variety of semisimple classes in L. Then $\mathcal{U}$ is the inverse image under the canonical map $L \to \mathcal{T}$ of a subset $\mathcal{U}'$ of $\mathcal{T}$. It is enough to show that $\mathcal{U}'$ is non-empty, open in $\mathcal{T}$. Let T be a maximal torus of L. Then $\mathcal{T} = T/W'$. Let $\mathcal{T}'$ be the inverse image of $\mathcal{U}'$ under the open map $T \to \mathcal{T}$. It is enough to show that $\mathcal{T}'$ is non-empty, open in T. Now $\mathcal{T}' = \mathcal{U} \cap T = \{t \in T; Z_G^0(t) \subset L\}$. Let R (resp. R') be the set of roots of G (resp. of L) with respect to T. Then $R' \subset R$ and

$$\begin{aligned} \mathcal{T}' &= \{t \in T; \{\alpha \in R; \alpha(t) = 1\} \subset R'\} \\ &= \{t \in T; \alpha(t) \neq 1 \quad \forall \alpha \in R - R'\}. \end{aligned}$$

This is non-empty, open in T and (a) is proved.

We prove (b). In the setup of (a) we have

$$\mathcal{Z}_r = \mathcal{Z} \cap \mathcal{T}' = \{t \in \mathcal{Z}; \alpha(t) \neq 1 \quad \forall \alpha \in R - R'\}.$$

It is enough to observe that, if $\alpha \in R - R'$, then $\alpha|_{\mathcal{Z}}$ is not identically 1.

We prove (c). We have $g_s \in B$. We can find a maximal torus T' of G such that $g_s \in T' \subset B$. Since $g_s \in T'$, we have $T' \subset Z_G^0(g_s)$. Since $Z_G^0(g_s) \subset L$, we have $T' \subset L = Z_G(\mathcal{Z})$. Thus, $\mathcal{Z} \subset Z_G(T') = T'$. Since $T' \subset B$, we have $\mathcal{Z} \subset B$. The lemma is proved.

2.3. Let $\dot{G} = \{(g, B); B \in \mathcal{B}, g \in B\}$, $\tilde{\mathcal{U}} = \{(g, B) \in \dot{G}; g \in \mathcal{U}\}$, $\dot{L} = \{(l, \beta); \beta \in \mathcal{B}', l \in \beta\}$, $Y = \{(l, \beta) \in \dot{L}; l \in \mathcal{U}\}$.

The direct image of $\bar{\mathbf{Q}}_l$ under the first projection $\dot{G} \to G$, $\tilde{\mathcal{U}} \to \mathcal{U}$, $\dot{L} \to L$, $Y \to \mathcal{U}$, is denoted by K, J, K', J' respectively. By [L1], K is an intersection cohomology complex with W-action and K' is an intersection cohomology complex with W'-action. Now $J = K|_{\mathcal{U}}$, (resp. $J' = K'|_{\mathcal{U}}$) inherits a W-action (resp. W'-action) from K (resp. K').

Lemma 2.4. *Let $g \in \mathcal{U}, n \in \mathbf{Z}$. The W-modules $\mathcal{H}^n_g(J)$, $\mathrm{ind}_{W'}^W(\mathcal{H}^n_g(J'))$ are isomorphic. (Here $\mathcal{H}^n_g(J)$, $\mathcal{H}^n_g(J')$ are stalks at g of the cohomology sheaves of J, J'.)*

Let $\mathcal{B}_{\mathcal{Z}} = \{B \in \mathcal{B}; \mathcal{Z} \subset B\}$. Then $\mathcal{B}_{\mathcal{Z}} = \bigsqcup_{d \in D} \mathcal{B}_{\mathcal{Z},d}$ (decomposition into connected components isomorphic to $\mathcal{B}'$ by $B \mapsto B \cap L$), where D is an indexing set. Let $d_0 \in D$ be such that $\mathcal{B}_{\mathcal{Z},d_0} = \{B \in \mathcal{B}; B \subset P\}$. Note that D is a homogeneous W-space and the isotropy group at d_0 is W'. Now

$$\tilde{\mathcal{U}} = \{(g, B) \in \dot{G}; B \in \mathcal{B}_{\mathcal{Z}}, g \in \mathcal{U}\} = \bigsqcup_{d \in D} \tilde{\mathcal{U}}_d$$

where $\tilde{\mathcal{U}}_d = \{(g, B) \in \dot{G}; B \in \mathcal{B}_{\mathcal{Z},d}, g \in \mathcal{U}\}$. We have an isomorphism $\tilde{\mathcal{U}}_d \xrightarrow{\sim} Y$ given by $(g, B) \mapsto (g, B \cap L)$. It follows that $J = \bigoplus_{d \in D} J_d$ where J_d is isomorphic to J' canonically. In particular, J is an intersection cohomology complex on $\mathcal{U}$ (since J' is). It is enough to show that the W-action on J satisfies $wJ_d = J_{wd}$ for $w \in W$ and the restriction to W' of the W-action on J_{d_0} is just the W'-action on J'. It is enough to check this over the open dense subset of semisimple elements of L that are regular in G. This is obvious. The lemma is proved.

2.5. Let $t \in \mathcal{Z}_r$. Let u be as in 1.2. Then $ut \in \mathcal{U}$, hence Lemma 2.4 is applicable, so that the W-modules $H^n(\mathcal{B}_{ut})$, $\mathrm{ind}_{W'}^W(H^n(\mathcal{B}'_{ut}))$ are isomorphic. Since $t \in \mathcal{Z}$, $H^n(\mathcal{B}'_{ut})$ may be identified with $H^n(\mathcal{B}'_u)$ as a W'-module. This completes the proof of Proposition 1.4.

§3. Proof of Proposition 1.5

3.1. To prove 1.5, it suffices to show that, if $w \in W$, then

(a) $$\mathrm{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \mathrm{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_{\tau u}))$$

for any $t \in \mathcal{Z}, \tau \in \mathcal{Z}_r$. The right hand side is independent of the choice of τ. Indeed, for $\tau \in \mathcal{Z}_r$ we have

$$\operatorname{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_{\tau u})) = \operatorname{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}}))$$

where $\mathcal{B}_{\mathcal{Z}} = \{B \in \mathcal{B}; \mathcal{Z} \subset B\}$, since $\mathcal{B}_{\tau u} = \mathcal{B}_u \cap \mathcal{B}_\tau = \mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}}$. (A special case of Lemma 2.2(c).)

Lemma 3.2. *Let Γ be a finite group, let E, E' be two finite dimensional representations of Γ over a field C of characteristic 0. Assume that the function $\phi : \Gamma \to C$ defined by $\phi(\gamma) = \operatorname{tr}(\gamma, E) - \operatorname{tr}(\gamma, E')$ is integer valued. Let $x, y \in \Gamma$ be such that $xy = yx$ and $y^f = 1$ where f is a prime number. Then $\phi(xy) - \phi(x) \in f\mathbf{Z}$.*

Let $\bar{C}$ be an algebraic closure of C, let $\xi_1, \ldots, \xi_n$ (resp. $\xi'_1, \ldots, \xi'_m$) be the eigenvalues of $y : E \to E$ (resp. $y : E' \to E'$). Let α_i be the trace of x on the ξ_i-generalized eigenspace of $y : E \to E$; let α'_j be the trace of x on the ξ'_j-generalized eigenspace of $y : E' \to E'$. Then

$$\phi(x) = \sum_i \alpha_i - \sum_j \alpha'_j, \phi(xy) = \sum_i \alpha_i \xi_i - \sum_j \alpha'_j \xi'_j.$$

Let C' be the subfield of $\bar{C}$ generated by $\xi_i, \alpha_i, \xi'_j, \alpha'_j$. (An algebraic number field.) Let A' be the ring of integers of C' and let $\mathfrak{m}$ be a prime ideal of A' such that $\mathfrak{m} \cap \mathbf{Z} = f\mathbf{Z}$. Let $\bar{\xi}_i$ be the image of ξ_i in $A'/\mathfrak{m}$ (a finite field of characteristic f). Since $\xi_i^f = 1$ we have $\bar{\xi}_i^f = 1$ in $A'/\mathfrak{m}$ hence $\bar{\xi}_i = 1$, that is $\xi_i - 1 \in \mathfrak{m}$. Similarly, $\xi'_j - 1 \in \mathfrak{m}$. It follows that $\sum_i \alpha_i \xi_i - \sum_j \alpha'_j \xi'_j - \sum_i \alpha_i + \sum_j \alpha'_j \in \mathfrak{m}$, that is $\phi(xy) - \phi(x) \in \mathfrak{m}$. Hence $\phi(xy) - \phi(x) \in \mathfrak{m} \cap \mathbf{Z} = f\mathbf{Z}$. The lemma is proved.

3.3. Let N, h_a be as in 2.2(a). We have $h_a = \tilde{h}_a^{e_a}$ where $\tilde{h}_a : \mathcal{Z} \to \mathbf{k}^*$ is a non-trivial character whose kernel is connected and $e_a \geq 1$ is an integer.

Let $\mathcal{P}$ be the set of all prime numbers f such that $f > N$ and f does not divide $pe_1 e_2 \ldots e_N$.

Let u be as in 1.2 and let $t \in \mathcal{Z}$. We choose a finite subfield F_q of $\mathbf{k}$ with q elements and an F_q-split rational structure on G with Frobenius map $F : G \to G$ such that $F(u) = u, F(t) = t, F(L) = L, F(P) = P$.

Lemma 3.4. *Let $f \in \mathcal{P}$. Let s_0 be the smallest integer ≥ 1 such that $q^{s_0} - 1$ is divisible by f, that is the order of q in $\mathbf{Z}/f\mathbf{Z}$. If $s \geq 1$ is divisible by s_0, then there exists $y \in t^{-1}\mathcal{Z}_r$ such that $F^s(y) = y$ and $y^f = 1$.*

Let $d = \dim \mathcal{Z}$. If $d = 0$ then $\mathcal{Z}_r = \mathcal{Z}$ and we may take $y = 1$. Assume now that $d \geq 1$. The number of elements $y \in \mathcal{Z}^{F^s}$ such that $y^f = 1$ is f^d. For any $a \in [1, N]$, let n_a be the number of elements $y \in t^{-1}\ker(h_a)$ such that $y^f = 1$. Let n'_a be the number of elements $y \in \ker(h_a)$ such that $y^f = 1$. Clearly, n_a equals either n'_a or 0. Since f does not divide e_a, we have

$$n'_a = \sharp(y \in \ker(\tilde{h}_a); y^f = 1) = f^{d-1}.$$

Thus n_a equals either f^{d-1} or 0. It follows that

$$\sharp(y \in \mathcal{Z}^{F^s}; y^f = 1, y \notin \bigcup_{a=1}^{N} t^{-1}\ker(h_a)) \geq f^d - Nf^{d-1}$$
$$= f^{d-1}(f - N) > 0.$$

The lemma is proved.

3.5. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of F on $\bigoplus_{i;i \text{ even}} H^i(\mathcal{B}_{tu})$ (in an algebraic closure of $\mathbf{Q}_l$) and let $\lambda'_1, \ldots, \lambda'_{n'}$ be the eigenvalues of F on $\bigoplus_{i;i \text{ odd}} H^i(\mathcal{B}_{tu})$. Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of F on $\bigoplus_{i;i \text{ even}} H^i(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})$ and let $\mu'_1, \ldots, \mu'_{m'}$ be the eigenvalues of F on $\bigoplus_{i;i \text{ odd}} H^i(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})$. For any $s \geq 1$ we have

(a) $$\operatorname{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \sum_i \lambda_i^s a_i - \sum_j {\lambda'_j}^s a'_j,$$
$$\operatorname{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \sum_i a_i - \sum_j a'_j,$$

(b) $$\operatorname{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) = \sum_h \mu_h^s b_h - \sum_k {\mu'_k}^s b'_k,$$
$$\operatorname{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) = \sum_h b_h - \sum_k b'_k.$$

Here a_i, a'_j, b_h, b'_k are integers. (They are traces of $w \in W$ on some W-module.) Let C be the algebraic number field generated by $\lambda_i, \lambda'_j, \mu_h, \mu'_k$. Let A be the ring of integers of C. Let $\mathcal{I}$ be the set of all non-zero prime ideals $\mathfrak{p}$ of A which contain at least one of the elements $\lambda_i, \lambda'_j, \mu_h, \mu'_k$. Note that $\mathcal{I}$ is a finite set. Let $\bar{\mathcal{I}}$ be the set of prime numbers f such that $\mathfrak{p} \cap \mathbf{Z} = f\mathbf{Z}$ for some $\mathfrak{p} \in \mathcal{I}$. Note that $\bar{\mathcal{I}}$ is a finite set. Hence $\mathcal{P} - \bar{\mathcal{I}}$ is an infinite set. Let $f \in \mathcal{P} - \bar{\mathcal{I}}$. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap \mathbf{Z} = f\mathbf{Z}$. Then $\mathfrak{p}$ does not contain any of the elements

$\lambda_i, \lambda'_j, \mu_h, \mu'_k$. Hence these elements have non-zero images in $A/\mathfrak{p}$ (a finite field of cardinal f^c where $c \geq 1$). Hence if s is divisible by $f^c - 1$, then $\lambda_i^s - 1 \in \mathfrak{p}, \lambda_j'^s - 1 \in \mathfrak{p}, \mu_h^s - 1 \in \mathfrak{p}, \mu_k'^s - 1 \in \mathfrak{p}$. It follows that for such s we have

(c) $$\sum_i \lambda_i^s a_i - \sum_j \lambda_j'^s a'_j = \sum_i a_i - \sum_j a'_j \mod \mathfrak{p},$$

(d) $$\sum_h \mu_h^s b_h - \sum_k \mu_k'^s b'_k = \sum_h b_h - \sum_k b'_k \mod \mathfrak{p}.$$

According to [L2], if s is large enough, we have

$$\mathrm{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \mathrm{tr}(tu, R_{w,F^s}),$$

and for any $\tau \in \mathcal{Z}_\tau^{F^s}$, we have

$$\mathrm{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) = \mathrm{tr}(\tau u, R_{w,F^s}),$$

where R_{w,F^s} is the virtual representation of G^{F^s} over $\mathbf{Q}_l$ associated in [DL] to an F^s-stable maximal torus in G corresponding to w.

We can choose s large enough (as above) and so that s is divisible by $f^c - 1$ and by s_0 (see Lemma 3.4). By 3.4, we can find $\tau \in \mathcal{Z}_\tau^{F^s}$ such that τt^{-1} has order f. We can apply Lemma 3.2 with $\Gamma = G^{F^s}$, $x = tu, y = \tau t^{-1}$ and we obtain

$$\mathrm{tr}(\tau u, R_{w,F^s}) = \mathrm{tr}(tu, R_{w,F^s}) \mod f\mathbf{Z}.$$

Hence

$$\mathrm{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \mathrm{tr}(F^s w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) \mod f\mathbf{Z}$$

and therefore, by (a),(b),

$$\sum_i \lambda_i^s a_i - \sum_j \lambda_j'^s a'_j = \sum_h \mu_h^s b_h - \sum_k \mu_k'^s b'_k \mod f\mathbf{Z}.$$

Using now (c),(d) and the inclusion $f\mathbf{Z} \subset \mathfrak{p}$ we deduce

$$\sum_i a_i - \sum_j a'_j = \sum_h b_h - \sum_k b'_k \mod \mathfrak{p}.$$

Since the left hand side is an integer and $\mathfrak{p} \cap \mathbf{Z} = f\mathbf{Z}$, we deduce

$$\sum_i a_i - \sum_j a'_j = \sum_h b_h - \sum_k b'_k \mod f\mathbf{Z}.$$

Thus the integer $\sum_i a_i - \sum_j a'_j - \sum_h b_h + \sum_k b'_k$ is divisible by infinitely many prime numbers (those in $\mathcal{P} - \bar{\mathcal{I}}$) hence it is 0. Thus, $\sum_i a_i - \sum_j a'_j = \sum_h b_h - \sum_k b'_k$ hence

$$\mathrm{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_{tu})) = \mathrm{tr}(w, \sum_n (-1)^n H^n(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})).$$

The proposition follows.

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