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Representations of Lie algebras in positive characteristic

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About 50 years ago it was discovered that finite dimensional Lie algebras in positive characteristic only have finite dimensional irreducible representations. About 15 years ago the irreducible representations for the Lie algebra $\mathfrak{gl}_n$ were classified. About 5 years ago a conjecture was formulated that should lead to a calculation of the dimensions of these simple $\mathfrak{gl}_n$–modules if $p > n$. For Lie algebras of other reductive groups our knowledge is more restricted, but there has been some remarkable progress in this area over the last years. The purpose of this survey is to report on these developments and to update the earlier surveys [H3] and [J3].

Throughout this paper let K be an algebraically closed field of prime characteristic p. All Lie algebras over K will be assumed to be finite dimensional.

A General Theory

A.1. If $\mathfrak{g}$ is a Lie algebra over K, then we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$.

A *restricted Lie algebra* over K is a Lie algebra $\mathfrak{g}$ over K together with a map $\mathfrak{g} \to \mathfrak{g}$, $X \mapsto X^{[p]}$, often called the *p–th power map*, provided certain conditions are satisfied. The first condition says that for each $X \in \mathfrak{g}$ the element

$$\xi(X) = X^p - X^{[p]} \in U(\mathfrak{g})$$

actually belongs to the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. (Here X^p is the p–th power of X taken in $U(\mathfrak{g})$.) The other condition says that $\xi : \mathfrak{g} \to Z(\mathfrak{g})$ is semi-linear in the following sense: We have

$$\xi(X+Y) = \xi(X) + \xi(Y) \qquad \text{and} \qquad \xi(a\,X) = a^p\,\xi(X)$$

for all $X, Y \in \mathfrak{g}$ and $a \in K$.

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A.2. For example, let A be an associative algebra over K considered as a Lie algebra via $[X,Y] = XY - YX$. Then A becomes a restricted Lie algebra if we set $X^{[p]}$ equal to the p–th power of X taken in A. Indeed, if we write l_X and r_X for left and write multiplication by X in A (so we have, e.g., $l_X(Y) = XY$ for all $Y \in A$), then $\operatorname{ad}(X) = l_X - r_X$. Since l_X and r_X commute and since we are in characteristic p, we get $\operatorname{ad}(X)^p = (l_X)^p - (r_X)^p$. Now $(l_X)^p$ is clearly left multiplication by the p–th power of X that we have decided to denote by $X^{[p]}$; similarly for $(r_X)^p$. We get thus $\operatorname{ad}(X)^p = \operatorname{ad}(X^{[p]})$. Now the same argument used in $U(A)$ instead of A shows that also $\operatorname{ad}(X)^p = \operatorname{ad}(X^p)$, hence that $X^p - X^{[p]}$ commutes with each element of $\mathfrak{g}$. Therefore $\xi(X) = X^p - X^{[p]}$ belongs to $Z(A)$. It remains to check the semi-linearity of ξ: The identity $\xi(a\,X) = a^p\,\xi(X)$ is obvious. The proof of the additivity of ξ requires a formula due to Jacobson expressing $(X+Y)^p - X^p - Y^p$ in terms of commutators, see [Ja], § V.7.

On the other hand, if G is an algebraic group over K, then $\operatorname{Lie}(G)$ has a natural structure as a restricted Lie algebra: One can think of $\operatorname{Lie}(G)$ as the Lie algebra of certain invariant derivations, cf. [H2], 9.1; Then one defines $X^{[p]}$ as the p–th power of X taken as derivation. In case $G = \mathrm{GL}_n(K)$, then one gets thus on $\operatorname{Lie}(G)$ the same structure as from the identification of $\operatorname{Lie}(G)$ with the space $M_n(k)$ of all $(n \times n)$–matrices over K and from the construction in the preceding paragraph. If G is a closed subgroup of $\mathrm{GL}_n(K)$, then we can identify $\operatorname{Lie}(G)$ with a Lie subalgebra of $M_n(k)$ and the p–th power map of $\operatorname{Lie}(G)$ is the restriction of that on $M_n(k)$.

A.3. Let now $\mathfrak{g}$ be an arbitrary restricted Lie algebra over K. Denote by $Z_0(\mathfrak{g})$ the subalgebra of $Z(\mathfrak{g})$ generated by all $\xi(X) = X^p - X^{[p]}$ with $X \in \mathfrak{g}$. This subalgebra is often called the *p–centre* of $U(\mathfrak{g})$.

Let $X_1, X_2, \dots, X_n$ be a basis for $\mathfrak{g}$. Using the PBW-theorem one checks now easily: The algebra $Z_0(\mathfrak{g})$ is generated by all $X_i^p - X_i^{[p]}$ with $1 \le i \le n$; these elements are algebraically independent over K. Considered as a $Z_0(\mathfrak{g})$–module under left (= right) multiplication $U(\mathfrak{g})$ is free of rank $p^{\dim(\mathfrak{g})}$; all products

$$X_1^{m(1)} X_2^{m(2)} \dots X_n^{m(n)} \qquad \text{with } 0 \le m(i) < p \text{ for all } i$$

form a basis of $U(\mathfrak{g})$ over $Z_0(\mathfrak{g})$.

The first of these two claims can be restated as follows: The map ξ induces an isomorphism of algebras

$$S(\mathfrak{g}^{(1)}) \xrightarrow{\sim} Z_0(\mathfrak{g}).$$

Here we use the following convention: If V is a vector space over K, we denote by $V^{(1)}$ the vector space over K that is equal to V as an additive group, but where any $a \in K$ acts on $V^{(1)}$ as $a^{1/p}$ does on V. Now the semi-linearity of ξ means that ξ is a linear map $\mathfrak{g}^{(1)} \to Z_0(\mathfrak{g})$, hence induces an algebra homomorphism from the symmetric algebra $S(\mathfrak{g}^{(1)})$ to the commutative algebra $Z_0(\mathfrak{g})$. The claim in the preceding paragraph shows that this map is bijective.

It is now easy to deduce from the results above:

Proposition: *The centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a finitely generated algebra over K. Considered as a $Z(\mathfrak{g})$–module $U(\mathfrak{g})$ is finitely generated.*

This result actually generalises to all (finite dimensional!) Lie algebras over K.

A.4. Theorem: *Each simple $\mathfrak{g}$–module is finite dimensional. Its dimension is less than or equal to $p^{\dim(\mathfrak{g})}$.*

Proof: Choose $u_1, u_2, \dots, u_r \in U(\mathfrak{g})$ such that $U(\mathfrak{g}) = \sum_{i=1}^r Z(\mathfrak{g})u_i$. This is possible by the proposition; in fact, we may assume that $r \leq p^{\dim(\mathfrak{g})}$ as $U(\mathfrak{g})$ has that rank over the smaller subalgebra $Z_0(\mathfrak{g})$.

Let E be a simple $\mathfrak{g}$–module. Pick $v \in E$, $v \neq 0$. We have then $E = U(\mathfrak{g})v$, hence $E = \sum_{i=1}^r Z(\mathfrak{g})u_i v$. So E is finitely generated as a module over $Z(\mathfrak{g})$. Since $Z(\mathfrak{g})$ is a finitely generated K–algebra, hence a Noetherian ring, there exists a maximal submodule $E' \subset E$. The simple $Z(\mathfrak{g})$–module E/E' is then isomorphic to $Z(\mathfrak{g})/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset Z(\mathfrak{g})$. Now xE is a $\mathfrak{g}$–submodule of E for all $x \in Z(\mathfrak{g})$. If $x \in \mathfrak{m}$, then $x(E/E') = 0$, hence $xE \subset E'$ is a submodule different from E. As E is simple, this implies that $xE = 0$. We get thus that $\mathfrak{m}E = 0$.

A weak version of the Hilbert Nullstellensatz says that $\mathfrak{m}$ has codimension 1 in $Z(\mathfrak{g})$, hence that $Z(\mathfrak{g}) = K1 + \mathfrak{m}$. So $E = \sum_{i=1}^r Z(\mathfrak{g})u_i v$ implies $E = \sum_{i=1}^r K u_i v$, hence $\dim(E) \leq r \leq p^{\dim(\mathfrak{g})}$.

A.5. Thanks to Theorem A.4 we can now associate to $\mathfrak{g}$ the number

$$M(\mathfrak{g}) = \max\{\dim E \mid E \text{ a simple } \mathfrak{g}\text{–module}\}.$$

Zassenhaus gave in [Za] a ring theoretic interpretation of $M(\mathfrak{g})$. Denote by F_0 a field of fractions for $Z_0(\mathfrak{g})$. Then

$$\mathrm{Frac}(U(\mathfrak{g})) = U(\mathfrak{g}) \otimes_{Z_0(\mathfrak{g})} F_0$$

is a localisation of $U(\mathfrak{g})$. The map $u \mapsto u \otimes 1$ from $U(\mathfrak{g})$ to $\mathrm{Frac}(U(\mathfrak{g}))$ is injective because $U(\mathfrak{g})$ is free over $Z_0(\mathfrak{g})$; we use it to identify $U(\mathfrak{g})$ with a subring of $\mathrm{Frac}(U(\mathfrak{g}))$. Each non-zero element $u \in U(\mathfrak{g})$ is invertible in $\mathrm{Frac}(U(\mathfrak{g}))$ as u is integral over $Z_0(\mathfrak{g})$. Therefore $\mathrm{Frac}(U(\mathfrak{g}))$ is a division ring. It contains a field of fractions F for $Z(\mathfrak{g})$ and F is the centre of $\mathrm{Frac}(U(\mathfrak{g}))$. It now turns out that

$$M(\mathfrak{g})^2 = \dim_F \mathrm{Frac}(U(\mathfrak{g}))$$

cf. [Za], Thms. 1 and 6. On the other hand, we have $\dim_{F_0} \mathrm{Frac}(U(\mathfrak{g})) = p^{\dim(\mathfrak{g})}$ because $U(\mathfrak{g})$ is free of rank $p^{\dim(\mathfrak{g})}$ over $Z_0(\mathfrak{g})$. It follows that $p^{\dim(\mathfrak{g})} = M(\mathfrak{g})^2 \cdot \dim_{F_0} F$, hence that $M(\mathfrak{g})$ is a power of p.

A.6. In [VK], 1.2 Veisfeiler and Kats made a conjecture on the value of $M(\mathfrak{g})$. For each linear form $\chi \in \mathfrak{g}^*$ denote by $\mathfrak{g}_\chi$ its stabiliser in $\mathfrak{g}$ for the coadjoint action:

$$\mathfrak{g}_\chi = \{\, X \in \mathfrak{g} \mid X \cdot \chi = 0 \,\} = \{\, X \in \mathfrak{g} \mid \chi([X, \mathfrak{g}]) = 0 \,\}.$$

This is a restricted Lie subalgebra of $\mathfrak{g}$. For each $\chi \in \mathfrak{g}^*$ the bilinear form $(X,Y) \mapsto \chi([X,Y])$ on $\mathfrak{g}$ induces a non-degenerate alternating form on $\mathfrak{g}/\mathfrak{g}_\chi$; therefore the dimension of this quotient space is even. Set

$$r(\mathfrak{g}) = \min \{\, \dim \mathfrak{g}_\chi \mid \chi \in \mathfrak{g}^* \,\}.$$

Then also $\dim(\mathfrak{g}) - r(\mathfrak{g})$ is even and the conjecture says:

Conjecture ([VK]): $\qquad M(\mathfrak{g}) = p^{(\dim(\mathfrak{g}) - r(\mathfrak{g}))/2}.$

Work by Mil′ner and by Premet and Skryabin (see [Mi], Thm. 3 and [PS], Thm. 4.4) shows:

Theorem: *If there exists a linear form χ on $\mathfrak{g}$ such that $\mathfrak{g}_\chi$ is a toral subalgebra of $\mathfrak{g}$, then this conjecture holds.*

(A subalgebra $\mathfrak{h}$ of a restricted Lie algebra is called toral if it is commutative and if the p–power map $X \mapsto X^{[p]}$ restricts to a bijective map $\mathfrak{h} \to \mathfrak{h}$. This means that $\mathfrak{h}$ is isomorphic as a restricted Lie algebra to the Lie algebra of a torus.)

A.7. Let E be a simple $\mathfrak{g}$–module. Since $\dim(E) < \infty$, Schur's lemma implies that each element in $Z(\mathfrak{g})$ acts as multiplication by a scalar on E. This applies in particular to all $\xi(X) = X^p - X^{[p]}$ with $X \in \mathfrak{g}$. Using the semi-linearity of ξ one shows now that there exists a linear form $\chi_E \in \mathfrak{g}^*$ with

$$(X^p - X^{[p]})_{|E} = \chi_E(X)^p \,\mathrm{id}_E \qquad \text{for all } X \in \mathfrak{g}.$$

One calls χ_E the *p–character* of E.

For each $\chi \in \mathfrak{g}^*$ set

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(X^p - X^{[p]} - \chi(X)^p 1 \mid X \in \mathfrak{g}).$$

This is a finite dimensional algebra over K of dimension $p^{\dim(\mathfrak{g})}$. If X_1, $X_2, \dots, X_n$ is a basis for $\mathfrak{g}$, then the classes of all $X_1^{m(1)} X_2^{m(2)} \dots X_n^{m(n)}$ with $0 \le m(i) < p$ for all i are a basis for $U_\chi(\mathfrak{g})$. One calls $U_\chi(\mathfrak{g})$ a *reduced enveloping algebra* of $\mathfrak{g}$. (The special case $U_0(\mathfrak{g})$ is usually called the *restricted enveloping algebra* of $\mathfrak{g}$.)

Each simple $U_\chi(\mathfrak{g})$–module (for any χ) is in a natural way a simple $\mathfrak{g}$–module. The discussion in the first paragraph of this subsection shows: *Each simple $\mathfrak{g}$–module is a simple $U_\chi(\mathfrak{g})$–module for exactly one $\chi \in \mathfrak{g}^*$.*

A.8. If $\gamma : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ as a restricted Lie algebra (i.e., a Lie algebra automorphism with $\gamma(X^{[p]}) = \gamma(X)^{[p]}$ for all $X \in \mathfrak{g}$), then γ induces an isomorphism

$$U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{\gamma\cdot\chi}(\mathfrak{g})$$

where $(\gamma \cdot \chi)(X) = \chi(\gamma^{-1}(X))$.

In particular, if $\mathfrak{g} = \mathrm{Lie}(G)$ for some algebraic group G over K, then any $g \in G$ acts via the adjoint action $\mathrm{Ad}(g)$ on $\mathfrak{g}$. Each $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$ as a restricted Lie algebra. So we get an isomorphism $U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{g\cdot\chi}(\mathfrak{g})$ where $g \cdot \chi$ refers to the coadjoint action of g. This implies: If we know all simple $\mathfrak{g}$–modules with a given p–character χ, then we know also all simple $\mathfrak{g}$–modules with a p–character in the coadjoint orbit $G \cdot \chi$.

B Reductive Lie Algebras

B.1. Assume from now on that G is a connected reductive algebraic group over K with a maximal torus T and a Borel subgroup $B^+ \supset T$. Denote the Lie algebras of G, T, B^+ by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}^+$ respectively.

We denote by X the character group of T and set $R \subset X$ equal to the set of roots of G relative to T. We denote by R^+ the set of roots of B^+ relative to T; this is a system of positive roots in R.

For each $\alpha \in R$ let $\mathfrak{g}_\alpha$ denote the corresponding root subspace in $\mathfrak{g}$. Set $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) equal to the sum of all $\mathfrak{g}_\alpha$ with $\alpha \in R^+$ (resp. with $-\alpha \in R^+$). We have then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$.

Fix for each α a basis vector X_α for $\mathfrak{g}_\alpha$. These elements satisfy $X_\alpha^{[p]} = 0$, e.g., since $X_\alpha^{[p]} \in \mathfrak{g}_{p\alpha} = 0$. On the other hand, the Lie algebra $\mathfrak{h}$ of the torus T has a basis $H_1, H_2, \dots, H_m$ with $H_i^{[p]} = H_i$ for all i.

If $\mathfrak{a}$ is a restricted Lie subalgebra of $\mathfrak{g}$, then we shall usually write $U_\chi(\mathfrak{a}) = U_{\chi|_\mathfrak{a}}(\mathfrak{a})$ for all $\chi \in \mathfrak{g}^*$.

B.2. For each $Y \in \mathfrak{g}$ one can find $g \in G$ with $\mathrm{Ad}(g)(Y) \in \mathfrak{b}^+$, cf. [Bo], Prop. 14.25. One should have analogously:

$$\text{For each } \chi \in \mathfrak{g}^* \text{ there exists } g \in G \text{ with } (g \cdot \chi)(\mathfrak{n}^+) = 0. \qquad (*)$$

This was proved in [KW], Lemma 3.2 for almost simple G except for the case where $G = \mathrm{SO}_{2n+1}$ and $p = 2$. Their argument can be extended to prove $(*)$ whenever the derived group of G is simply connected.

In many cases there exists a non-degenerate G–invariant bilinear form on $\mathfrak{g}$. We can use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $Y \in \mathfrak{g}$ correspond to $\chi \in \mathfrak{g}^*$. Choose $g \in G$ with $\mathrm{Ad}(g)(Y) \in \mathfrak{b}^+$. Then the image $g \cdot \chi$ of $\mathrm{Ad}(g)(Y)$ vanishes on $(\mathfrak{b}^+)^\perp = \mathfrak{n}^+$. So we get $(*)$ in this case.

Suppose that G satisfies $(*)$. It then follows from A.8 that it suffices to determine the simple $U_\chi(\mathfrak{g})$–modules for all χ with $\chi(\mathfrak{n}^+) = 0$ if we want to describe all simple $\mathfrak{g}$–modules.

B.3. Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$. Then each X_α with $\alpha \in R^+$ acts nilpotently on any $U_\chi(\mathfrak{n}^+)$–module since $\chi(X_\alpha) = 0$ and $X_\alpha^{[p]} = 0$. This implies (using an inductive argument) for each $U_\chi(\mathfrak{n}^+)$–module M

$$M \neq 0 \implies M^{\mathfrak{n}^+} \neq 0.$$

(We write generally $M^{\mathfrak{a}}$ for the space of fixed points in a module M over a Lie algebra $\mathfrak{a}$.)

If M is a $U_\chi(\mathfrak{b}^+)$–module, then $\mathfrak{h}$ stabilises $M^{\mathfrak{n}^+}$ as $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}^+$. Since $\mathfrak{h}$ is commutative, it then has a common eigenvector in $M^{\mathfrak{n}^+}$ provided $M \neq 0$. So we get in this case some $v \in M$, $v \neq 0$ and some $\mu \in \mathfrak{h}^*$ with $Hv = \mu(H)v$ for all $H \in \mathfrak{h}$ and with $Yv = 0$ for all $Y \in \mathfrak{n}^+$.

Each $\lambda \in \mathfrak{h}^*$ defines a one dimensional $\mathfrak{b}^+$–module K_λ where $\mathfrak{n}^+$ acts as 0 and where each $H \in \mathfrak{h}$ acts as $\lambda(H)$. Then K_λ is a $U_\chi(\mathfrak{b}^+)$–module if and only if $\lambda \in \Lambda_\chi$ where we set

$$\Lambda_\chi = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H)^p - \lambda(H^{[p]}) = \chi(H)^p \text{ for all } H \in \mathfrak{h} \}.$$

Now the linear function μ in the preceding paragraph has to belong to Λ_χ because M is a $U_\chi(\mathfrak{b}^+)$–module.

Recall that $\mathfrak{h}$ has a basis $H_1, H_2, \dots, H_m$ with $H_i^{[p]} = H_i$ for all i. The semi-linearity of the map $H \mapsto H^p - H^{[p]}$ implies that some $\lambda \in \mathfrak{h}^*$ belongs to Λ_χ if and only if

$$\chi(H_i)^p = \lambda(H_i)^p - \lambda(H_i^{[p]}) = \lambda(H_i)^p - \lambda(H_i)$$

for all i. Given χ, this shows that each $\lambda(H_i)$ can take exactly p distinct values. This implies that

$$|\Lambda_\chi| = p^{\dim(\mathfrak{h})}.$$

B.4. Let again $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$. For each $\lambda \in \Lambda_\chi$ we can now consider the $U_\chi(\mathfrak{b}^+)$–module K_λ and the induced $U_\chi(\mathfrak{g})$–module

$$Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}^+)} K_\lambda$$

which is often called a *baby Verma module.*

We have

$$\dim Z_\chi(\lambda) = p^N \qquad \text{where } N = |R^+|.$$

If we set $v_\lambda = 1 \otimes 1$ and choose a numbering $\alpha_1, \alpha_2, \dots, \alpha_N$ of the positive roots, then all

$$X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$$

with $0 \le m(i) < p$ for all i form a basis for $Z_\chi(\lambda)$.

If M is a non-zero $U_\chi(\mathfrak{g})$–module, then the discussion in B.3 shows that there exists some $\lambda \in \Lambda_\chi$ with $\operatorname{Hom}_{\mathfrak{b}^+}(K_\lambda, M) \neq 0$. The universal property of the tensor product implies then $\operatorname{Hom}_{\mathfrak{g}}(Z_\chi(\lambda), M) \neq 0$. We get therefore (as observed in [Ru]):

Lemma: *If E is a simple $U_\chi(\mathfrak{g})$–module, then E is a homomorphic image of some $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$.*

B.5. Let us assume that the derived group of G is simply connected. If the p–character χ of a simple $\mathfrak{g}$–module E satisfies $\chi(\mathfrak{n}^+) = 0$, then the results in B.4 imply that $\dim E \le \dim Z_\chi(\lambda)$ for a suitable λ, hence $\dim E \le p^N$. By our assumption G satisfies B.2(*), so this inequality holds for all simple $\mathfrak{g}$–modules E. On the other hand, one knows that G has a Steinberg module that is irreducible of dimension p^N and remains irreducible under restriction to $\mathfrak{g}$. This shows (in the notation from A.5) that

$$M(\mathfrak{g}) = p^N.$$

Note that this result is compatible with the conjecture mentioned in A.6. Since $\dim(\mathfrak{g}) = 2N + \dim(\mathfrak{h})$, we just have to check that $\dim(\mathfrak{h})$ is the minimal dimension of all $\mathfrak{g}_\chi$ with $\chi \in \mathfrak{g}^*$. Well, our assumption on the derived group of G implies that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ for all $\alpha \in R$. Therefore we can find $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0 = \chi(\mathfrak{n}^-)$ and $\chi([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \neq 0$ for all $\alpha \in R$. Then χ satisfies $\mathfrak{g}_\chi = \mathfrak{h}$. (This shows, by the way, that the assumption in Theorem A.6 is satisfied.) One can now use semi-continuity arguments to show that $\dim \mathfrak{g}_\chi \ge \dim(\mathfrak{h})$ for all χ. (The union of the orbits $G \cdot \chi$ with $\mathfrak{g}_\chi = \mathfrak{h}$ is dense in $\mathfrak{g}^*$.)

B.6. Let us make from now on the following simplifying assumptions:

(H1) The derived group of G is simply connected.

(H2) The prime p is good for G.

(H3) There exists a non-degenerate G–invariant bilinear form on $\mathfrak{g}$.

The assumption (H2) excludes $p = 2$ if R has a component not of type A, it excludes $p = 3$ if R has a component of exceptional type, and it excludes $p = 5$ if R has a component of type E_8. If G is almost simple and and if (H2) holds, then (H3) holds unless R has type A_n with $p \mid n+1$. Note that $G = \mathrm{GL}_n$ satisfies all three conditions for all n and p: In (H3) one can take the bilinear form $(Y, Z) = \mathrm{trace}(YZ)$ on $\mathrm{Lie}(G) = M_n(K)$.

One nice aspect of (H1)–(H3) is the following: With G also each Levi subgroup of G satisfies these hypotheses.

B.7. Premet has shown in [Pr], Thm. 3.10 (see also [PS], Thm. 5.6) under our assumptions or slightly weaker ones (proving a conjecture from [VK], 3.5):

Theorem: *Let $\chi \in \mathfrak{g}^*$. Then $p^{\dim(G\cdot\chi)/2}$ divides $\dim(M)$ for each finite dimensional $U_\chi(\mathfrak{g})$–module M.*

(It turns out that under our assumption each orbit $G\cdot\chi$ has an even dimension; so the claim makes sense.) For an introduction to Premet's original proof one may also compare [J3], sections 7 and 8.

B.8. Let $\psi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be an isomorphism of G–modules induced by a bilinear form as in (H3). We can use ψ to transport notions like *nilpotent* or *semi-simple* from $\mathfrak{g}$ to $\mathfrak{g}^*$ and call $\chi \in \mathfrak{g}^*$ nilpotent (or semi-simple) if $\psi^{-1}(\chi)$ is so. One can also define these notions intrinsically saying that χ is semi-simple if and only if the orbit $G\cdot\chi$ is closed if and only if there exists $g \in G$ with $(g\cdot\chi)(\mathfrak{n}^+ \oplus \mathfrak{n}^-) = 0$. And χ is nilpotent if and only if $0 \in \overline{G\cdot\chi}$ if and only if there exists $g \in G$ with $(g\cdot\chi)(\mathfrak{b}^+) = 0$.

A general $\chi \in \mathfrak{g}^*$ has then a Jordan decomposition $\chi = \chi_s + \chi_n$ with χ_s semi-simple and χ_n nilpotent such that $\psi^{-1}(\chi) = \psi^{-1}(\chi_s) + \psi^{-1}(\chi_n)$ is the Jordan decomposition in $\mathfrak{g}$. (Again, this can be defined directly in $\mathfrak{g}^*$, see [KW], section 3.)

Consider a Jordan decomposition $\chi = \chi_s + \chi_n$ as above and set $\mathfrak{l} = \mathfrak{g}_{\chi_s}$. Our assumption (H2) implies that there exists a Levi subgroup L of G with $\mathfrak{l} = \mathrm{Lie}(L)$. (Replacing χ by an element in $G\cdot\chi$, one may assume that $\chi_s(\mathfrak{n}^+) = 0 = \chi_s(\mathfrak{n}^-)$. Then $\mathfrak{l}$ is equal to the sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\chi_s([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = 0$. The set of all α with this property is conjugate under the Weyl group to a set of the form $R \cap \mathbf{Z}I$ where I is a subset of the set of simple roots; here one uses that p is good.) There exists then a parabolic subgroup P in G such that P is the semi-direct product of its unipotent radical U_P and of L. Then $\mathfrak{p} = \mathrm{Lie}(P)$ satisfies $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u} = \mathrm{Lie}(U_P)$.

We have now $\chi(\mathfrak{u}) = 0$. (This follows from the fact that $\psi^{-1}(\chi_s)$ and $\psi^{-1}(\chi_n)$ commute.) If we extend a $U_\chi(\mathfrak{l})$–module to a $\mathfrak{p}$–module letting the ideal $\mathfrak{u}$ act via 0, then we get therefore a $U_\chi(\mathfrak{p})$–module. Now one can show:

Theorem: *The functors $V \mapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V$ and $M \mapsto M^{\mathfrak{u}}$ are inverse equivalences of categories between $\{\, U_\chi(\mathfrak{l})$–modules$\,\}$ and $\{\, U_\chi(\mathfrak{g})$–modules$\,\}$. They induce bijections of isomorphism classes of simple modules.*

This goes back to Veisfeiler and Kats who showed in [VK], Thm. 2 that the simple $U_\chi(\mathfrak{g})$–modules are the $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} E$ with E a simple $U_\chi(\mathfrak{l})$–module. The more precise statement here is due to Friedlander and Parshall, see [FP1], Thm. 3.2 and Thm. 8.5. (See also [J3], 7.4 for a proof of this result based on a theorem of Premet.)

The unipotent group U_P satisfies $U_\chi(\mathfrak{u})^{\mathfrak{u}} = K1$. Using the PBW theorem one can now check that the equivalence of categories $M \mapsto M^{\mathfrak{u}}$ take $U_\chi(\mathfrak{g})$ to a direct sum of p^d copies of $U_\chi(\mathfrak{l})$ where $d = \dim(\mathfrak{g}/\mathfrak{p})$. This implies that $U_\chi(\mathfrak{g})$ is isomorphic to the matrix ring $M_{p^d}(U_\chi(\mathfrak{l}))$.

B.9. Theorem B.8 reduces the problem of finding the simple $\mathfrak{g}$–modules to the investigation of the simple $U_\chi(\mathfrak{g})$–modules for $\chi \in \mathfrak{g}^*$ with $\mathfrak{g}_{\chi_s} = \mathfrak{g}$ and to the analogous problem for Lie algebras of smaller reductive groups that again satisfy (H1)–(H3).

By definition $\mathfrak{g}_{\chi_s} = \mathfrak{g}$ means that $\chi_s([\mathfrak{g}, \mathfrak{g}]) = 0$. Under our assumptions $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of the derived group of G. So $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a restricted Lie algebra. Let E be a simple $U_\gamma(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$–module where $\gamma(Y + [\mathfrak{g}, \mathfrak{g}]) = \chi_s(Y)$ for all $Y \in \mathfrak{g}$. Then E has dimension 1 because $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is commutative, and E is a $U_{\chi_s}(\mathfrak{g})$–module when considered as a $\mathfrak{g}$–module. Now $V \mapsto V \otimes E$ and $V' \mapsto V' \otimes E^*$ are inverse equivalences of categories between $\{\, U_{\chi_n}(\mathfrak{g})$–modules$\,\}$ and $\{\, U_\chi(\mathfrak{g})$–modules$\,\}$.

This shows that it suffices to study $U_{\chi_n}(\mathfrak{g})$–modules. So we have a reduction to the case where χ is nilpotent.

B.10. Before we investigate the nilpotent case, let us look at a special case of Theorem B.8. Suppose that χ is *regular semi-simple*, i.e., that $\chi = \chi_s$ and that $\dim(\mathfrak{g}_\chi) = \dim(\mathfrak{h})$. Replacing χ by a conjugate under G, we may assume that $\mathfrak{g}_\chi = \mathfrak{h}$. This means that $\chi(\mathfrak{n}^+) = 0 = \chi(\mathfrak{n}^-)$ and that $\chi([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \neq 0$ for all $\alpha \in R$. We have $\mathfrak{l} = \mathfrak{h}$ in the notation of B.8 and may choose $P = B^+$. So Theorem B.8 says in this case that all $U_\chi(\mathfrak{g})$–modules are semi-simple (since $U_\chi(\mathfrak{h})$–modules are so) and that the simple $U_\chi(\mathfrak{g})$–modules are the $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$. These $Z_\chi(\lambda)$ are pairwise non-isomorphic.

C Nilpotent Forms

We keep throughout the assumptions and notations from B.1 and B.6.

C.1. By B.8 and B.9 we may restrict to the case where χ is nilpotent. Replacing χ by some $g \cdot \chi$ with $g \in G$, we may assume that $\chi(\mathfrak{b}^+) = 0$. We have then in particular $\Lambda_\chi = \Lambda_0$.

One can identify the multiplicative group over K with GL_1, hence its Lie algebra with $M_1(K)$. This Lie algebra has basis H equal to the (1×1)–matrix (1). This shows that $H^{[p]} = H$ and that each character $\varphi_n : t \mapsto t^n$ with $n \in \mathbf{Z}$ of the multiplicative group has tangent map $d\varphi_n$ mapping H to n. A linear form λ on $M_1(K)$ satisfies $\lambda(H)^p - \lambda(H^{[p]}) = 0$ if and only if $\lambda(H) \in \mathbf{F}_p$ if and only if $\lambda = d\varphi_n$ for some n. Furthermore, we have $d\varphi_n = d\varphi_m$ if and only if $n \equiv m \pmod{p}$.

This implies now for T, a direct product of multiplicative groups, that
$$\Lambda_0 = \{\, d\lambda \mid \lambda \in X \,\}$$
and that $d\lambda = d\mu$ if and only if $\lambda \equiv \mu \pmod{pX}$.

We shall often write K_μ instead of $K_{d\mu}$ and $Z_\chi(\mu)$ instead of $Z_\chi(d\mu)$ for $\mu \in X$ and $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. We have to keep in mind that then $Z_\chi(\mu) = Z_\chi(\mu + p\nu)$ for all $\mu, \nu \in X$.

Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. We know by B.4 that each simple $U_\chi(\mathfrak{g})$–module is the homomorphic image of some $Z_\chi(\mu)$ with $\mu \in X$. The problem now is that μ is not necessarily uniquely determined; we shall see an example of this in a moment. Furthermore, some $Z_\chi(\mu)$ may have more than one simple homomorphic image. In fact, there exist even $Z_\chi(\mu)$ that are decomposable, see [J3], 6.9.

C.2. Fix $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. For each finite dimensional $U_\chi(\mathfrak{g})$–module M denote by $[M]$ the class of M in the Grothendieck group of all finite dimensional $U_\chi(\mathfrak{g})$–modules. If $E_1, E_2, \dots, E_r$ is a system of representatives for the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules, then $[E_1], [E_2], \dots, [E_r]$ is a basis over $\mathbf{Z}$ for this Grothendieck group. An arbitrary M then satisfies $[M] = \sum_{i=1}^r [M : E_i][E_i]$ where $[M : E_i]$ is the multiplicity of E_i as a composition factor of M.

Let W denote the Weyl group of G with respect to T. For each $\alpha \in R$ denote by $s_\alpha \in W$ the corresponding reflection given by $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$ where $\alpha^\vee$ is the coroot to α. We shall often use the "dot action" of W on X, given by $w \bullet \mu = w(\mu + \rho) - \rho$ where $\rho \in X \otimes_{\mathbf{Z}} \mathbf{Q}$ is half the sum of all positive roots.

Proposition: *We have* $[Z_\chi(w\bullet\lambda)] = [Z_\chi(\lambda)]$ *for all* $w \in W$ *and* $\lambda \in X$.

This was first shown in [H1], Thm. 2.2 in case $\chi = 0$; the proof in that case generalises. It suffices to take $w = s_\alpha$ with α a simple root. Let d denote the integer with $\langle \lambda + \rho, \alpha^\vee\rangle \equiv d \pmod p$ and $0 \le d < p$. Then $s_\alpha \bullet \lambda \equiv \lambda - d\alpha \pmod{pX}$; so we have to show that $[Z_\chi(\lambda - d\alpha)] = [Z_\chi(\lambda)]$.

This is trivial if $d = 0$. If $d > 0$, then rank 1 calculations show that there exists a homomorphism of $\mathfrak{g}$–modules $\varphi : Z_\chi(\lambda - d\alpha) \to Z_\chi(\lambda)$ given by $\varphi(v_{\lambda-d\alpha}) = X_{-\alpha}^d v_\lambda$. (We use here notations like v_λ as in B.4.)

If $\chi(X_{-\alpha}) \neq 0$, then $X_{-\alpha}^p - X_{-\alpha}^{[p]} = X_{-\alpha}^p$ acts as the non-zero scalar $\chi(X_{-\alpha})^p$ on $Z_\chi(\lambda)$. It then follows that

$$v_\lambda = \chi(X_{-\alpha})^{-p} X_{-\alpha}^p v_\lambda = \chi(X_{-\alpha})^{-p} X_{-\alpha}^{p-d} \varphi(v_{\lambda-d\alpha})$$

belongs to the image of φ. Therefore φ is surjective, hence bijective by dimension comparison. So in this case φ is an isomorphism $Z_\chi(\lambda - d\alpha) \xrightarrow{\sim} Z_\chi(\lambda)$.

If $\chi(X_{-\alpha}) = 0$, then one checks — working with bases as in B.4 such that $\alpha_N = \alpha$ — that the kernel of φ is generated by $X_{-\alpha}^{p-d} v_{\lambda-d\alpha}$. Furthermore there is a homomorphism ψ from $Z_\chi(\lambda - p\alpha) = Z_\chi(\lambda)$ to $Z_\chi(\lambda - d\alpha)$ with $\psi(v_\lambda) = X_{-\alpha}^{p-d} v_{\lambda-d\alpha}$. One gets then $\ker(\varphi) = \operatorname{im}(\psi)$ and $\ker(\psi) = \operatorname{im}(\varphi)$, hence

$$[Z_\chi(\lambda)] = [\ker(\psi)] + [\operatorname{im}(\psi)] = [\operatorname{im}(\varphi)] + [\ker(\varphi)] = [Z_\chi(\lambda - d\alpha)]$$

as claimed.

C.3. If χ (as in C.2) satisfies $\chi(X_{-\alpha}) \neq 0$ for *all* simple roots α, then the (sketched) proof of Proposition C.2 shows that $Z_\chi(w \bullet \lambda) \simeq Z_\chi(\lambda)$ for all $w \in W$ and $\lambda \in X$.

In this case χ is "regular nilpotent", i.e., satisfies $\dim(G \cdot \chi) = 2\dim(\mathfrak{g}/\mathfrak{b}) = 2N$. (If χ corresponds to $Y \in \mathfrak{g}$ under an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ as in B.8, then $Y = \sum_{\alpha\in R^+} a_\alpha X_\alpha$ with suitable $a_\alpha \in K$ and with $a_\alpha \neq 0$ for all simple roots α. Such elements in $\mathfrak{g}$ are regular nilpotent, see [J6], 6.7(1).)

Now Theorem B.7 says in this case that p^N divides the dimension of each $U_\chi(\mathfrak{g})$–module. Since all $Z_\chi(\lambda)$ have this dimension, they have to be simple. We have thus shown most of the following result (proved in [FP1], 4.2/3 for certain types, in [FP2], 2.2–4 in general under slightly more restrictive conditions on p; for $G = \mathrm{SL}_n$ see also [P1], Thm. 5):

Proposition: *Let* $\chi \in \mathfrak{g}^*$ *with* $\chi(\mathfrak{b}^+) = 0$ *and* $\chi(X_{-\alpha}) \neq 0$ *for all simple roots* α. *Then each* $Z_\chi(\lambda)$ *with* $\lambda \in X$ *is simple and each simple*

$U_\chi(\mathfrak{g})$–module is isomorphic to some $Z_\chi(\lambda)$ with $\lambda \in X$. Given $\lambda, \mu \in X$ we have $Z_\chi(\lambda) \simeq Z_\chi(\mu)$ if and only if $\lambda \in W \bullet \mu + pX$.

C.4. The only argument missing above is the proof of the claim: If $Z_\chi(\lambda) \simeq Z_\chi(\mu)$, then $\lambda \in W \bullet \mu + pX$. For this one looks at the subalgebra $U(\mathfrak{g})^G$ of all $\mathrm{Ad}(G)$–invariant elements in $U(\mathfrak{g})$. This subalgebra is contained in the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. If we were working in characteristic 0, then $U(\mathfrak{g})^G$ would be all of $Z(\mathfrak{g})$. In our present set-up, however, there are many elements of the form $Y^p - Y^{[p]}$ with $Y \in \mathfrak{g}$ that belong to $Z(\mathfrak{g})$, but not to $U(\mathfrak{g})^G$. One can show that $Z(\mathfrak{g})$ is generated by $U(\mathfrak{g})^G$ and $Z_0(\mathfrak{g})$, cf. [BG], Thm. 3.5. (However, this does not imply that the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$ maps $U(\mathfrak{g})^G$ onto the centre of $U_\chi(\mathfrak{g})$; see the counter-example by Premet in [BG], 3.17.)

One checks now that there exists for each $u \in U(\mathfrak{g})^G$ and each $\lambda \in X$ a scalar $\mathrm{cen}_\lambda(u) \in K$ such that $u\, v_\lambda = \mathrm{cen}_\lambda(u)\, v_\lambda$ in $Z_\chi(\lambda)$ for all $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^*) = 0$. It then follows u acts as multiplication with $\mathrm{cen}_\lambda(u)$ on all of $Z_\chi(\lambda)$, hence also on all composition factors of $Z_\chi(\lambda)$. Each cen_λ is an algebra homomorphism from $U(\mathfrak{g})^G$ to K. Now one has analogously to the Harish-Chandra theorem in characteristic 0:

Proposition: *If $\lambda, \mu \in X$, then $\mathrm{cen}_\lambda = \mathrm{cen}_\mu$ if and only if $\lambda \in W \bullet \mu + pX$.*

In fact, one has as in characteristic 0 a Harish-Chandra isomorphism $U(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{h})^W$. This was first proved in [H1], Thm. 3.1 for large p (larger than the Coxeter number) and in [KW] for almost simple G. (The arguments there extend to the present set-up. See [J3], 9.6 for a proof that uses reduction modulo p techniques, as in [H1].)

C.5. Proposition C.4 has an obvious corollary for the description of the blocks of $U_\chi(\mathfrak{g})$: If two simple $U_\chi(\mathfrak{g})$–modules E and E' belong to the same block, then $U(\mathfrak{g})^G$ has to act via the same character on both E and E'. So, if E is a composition factor of $Z_\chi(\lambda)$ and if E' is one of $Z_\chi(\lambda')$, and if E, E' belong to the same block, then $\lambda' \in W \bullet \lambda + pX$. But then Proposition C.2 implies that E' is also a composition factor of $Z_\chi(\lambda)$. This proves the implications "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (ii)" in:

Proposition: *Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. Let E and E' be simple $U_\chi(\mathfrak{g})$–modules. Then the following are equivalent:*

(i) *E and E' belong to the same block of $U_\chi(\mathfrak{g})$.*

(ii) *$U_\chi(\mathfrak{g})^G$ acts via the same character on E and E'.*

(iii) *There exists $\lambda \in X$ such that both E and E' are composition factors of $Z_\chi(\lambda)$.*

Remark: If (i) or (ii) holds for one χ, then it holds for all $g \cdot \chi$ with $g \in G$. Therefore the equivalence of (i) and (ii) holds for all nilpotent $\chi \in \mathfrak{g}^*$, not only for those with $\chi(\mathfrak{b}^+) = 0$.

Note that the implication "(iii) $\Rightarrow$ (i)" [that we did not prove here] is obvious in the cases where all $Z_\chi(\lambda)$ are indecomposable. This holds for $\chi = 0$ by [H1], Prop. 1.5, and more generally for χ in standard Levi form, see D.1 below. (It actually suffices to find for each $\lambda \in X$ one $\lambda' \in W \bullet \lambda + pX$ such that $Z_\chi(\lambda')$ is indecomposable; that is in many additional cases possible, for example always when R has no component of exceptional type, see [J5], C.3 and H.1.)

The first general proof of "(ii) $\Rightarrow$ (i)" was given in [BG], Thm. 3.18 (assuming $p > 2$). A more direct proof (that works also for $p = 2$) is due to Gordon, see [Go], Thm. 3.6. Let m_0 denote the number of orbits of W on X/pX with respect to the dot action. The implication "(i) $\Rightarrow$ (iii)" shows that the number of blocks of $U_\chi(\mathfrak{g})$ is at least equal to m_0 for all $\chi \in (\mathfrak{b}^+)^\perp = \{\chi \in \mathfrak{g}^* \mid \chi(\mathfrak{b}^+) = 0\}$. So the implication "(iii) $\Rightarrow$ (i)" is equivalent to the claim that each $U_\chi(\mathfrak{g})$ has at most m_0 blocks. Now one checks for each m that the set

$$D_m = \{\, \chi \in (\mathfrak{b}^+)^\perp \mid U_\chi(\mathfrak{g}) \text{ has at most } m \text{ blocks} \,\}$$

is closed in $(\mathfrak{b}^+)^\perp$. Proposition C.3 implies that all regular nilpotent elements in $(\mathfrak{b}^+)^\perp$ belong to D_{m_0}. As these elements are dense in $(\mathfrak{b}^+)^\perp$, we get $D_{m_0} = (\mathfrak{b}^+)^\perp$, hence the claim.

C.6. Semi-continuity arguments like the one used in C.5 can be used for many purposes in the present theory. They often rely on the following observation: Fix $\lambda \in X$ and choose a numbering $\alpha_1, \alpha_2, \dots, \alpha_N$ of the positive roots. For all $\chi \in (\mathfrak{b}^+)^\perp$ and $\mathbf{m} = (m(1), m(2), \dots, m(N))$ in $\mathbf{Z}^N$ with $0 \le m(i) < p$ for all i let $z_{\mathbf{m},\chi}$ denote the basis element $X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$ of $Z_\chi(\lambda)$, cf. B.4. There are then for all $Y \in \mathfrak{g}$ and all $\mathbf{m}$, $\mathbf{n}$ elements $c_{\mathbf{n},\mathbf{m}}(Y,\chi) \in K$ such that

$$Y\, z_{\mathbf{m},\chi} = \sum_{\mathbf{n}} c_{\mathbf{n},\mathbf{m}}(Y,\chi)\, z_{\mathbf{n},\chi}$$

cf. the proof of A.7(2) in [J5].

Each $c_{\mathbf{n},\mathbf{m}} : \mathfrak{g} \times (\mathfrak{b}^+)^\perp \to K$ is a linear function of $Y \in \mathfrak{g}$ and a polynomial function of $\chi \in (\mathfrak{b}^+)^\perp$. Using this one can check that both

$$\{\, \chi \in (\mathfrak{b}^+)^\perp \mid Z_\chi(\lambda) \text{ is simple} \,\} \text{ and } \{\, \chi \in (\mathfrak{b}^+)^\perp \mid Z_\chi(\lambda) \text{ is projective} \,\}$$

are open subsets of $(\mathfrak{b}^+)^\perp$. They are also $\mathrm{Ad}(T)$–stable. It is easy to see that $0 \in \overline{\mathrm{Ad}(T) \cdot \chi}$ for all $\chi \in (\mathfrak{b}^+)^\perp$. So if $Z_\chi(\lambda)$ is not simple (or not projective) for some $\chi \in (\mathfrak{b}^+)^\perp$, then also $Z_0(\lambda)$ is not simple (or not projective). Now classical results on the Steinberg module in the restricted case (cf. [J1], II.3.18 and II.10.2) yield:

Proposition: *Let $\lambda \in X$ with $\langle \lambda + \rho, \alpha^\vee \rangle \equiv 0 \pmod p$ for all roots α. Then $Z_\chi(\lambda)$ is simple and projective for all $\chi \in (\mathfrak{b}^+)^\perp$.*

Kac seems to have had in mind a proof along the lines indicated above when he claimed the simplicity of $Z_\chi(\lambda)$ for these λ in [Ka]. Proofs of these results appeared in [FP2], Thms. 4.1/2. Compare also [BG], Cor. 3.11.

Remark: Let me indicate how one can prove that the two sets above are open in $(\mathfrak{b}^+)^\perp$ or rather that their complements are closed.

All $Z_\chi(\lambda)$ have dimension $r = p^N$; we have bijections $f_\chi : K^r \to Z_\chi(\lambda)$ mapping a family $(a_{\mathbf{m}})_{\mathbf{m}}$ to $\sum_{\mathbf{m}} a_{\mathbf{m}} z_{\mathbf{m},\chi}$. Set N_d equal to the set of all $\chi \in (\mathfrak{b}^+)^\perp$ for which $Z_\chi(\lambda)$ has a submodule of dimension d. We want to show that each N_d with $0 < d < r$ is closed in $(\mathfrak{b}^+)^\perp$. Let $\mathbf{G}_{d,r}$ denote the Grassmannian of all d–dimensional subspaces of K^r. The description of the action of $Y \in \mathfrak{g}$ on $z_{\mathbf{m},\chi}$ implies that the set $M_d(Y)$ of all $(V, \chi) \in \mathbf{G}_{d,r} \times (\mathfrak{b}^+)^\perp$ with $Y f_\chi(V) \subset f_\chi(V)$ is closed. Hence so is the intersection M_d of all $M_d(Y)$ with $Y \in \mathfrak{g}$. Now the second projection maps M_d onto N_d, and this image is closed because $\mathbf{G}_{d,r}$ is a complete variety.

In order to get the claim on projectivity I shall use support varieties. Set $\mathcal{N}_p(\mathfrak{g})$ equal to the set of all $x \in \mathfrak{g}$ with $x^{[p]} = 0$. This is a closed and conic subvariety of $\mathfrak{g}$. So the image $\mathbf{P}\mathcal{N}_p(\mathfrak{g})$ of $\mathcal{N}_p(\mathfrak{g}) \setminus \{0\}$ in the projective space $\mathbf{P}(\mathfrak{g})$ is closed. For any $U_\chi(\mathfrak{g})$–module M set $\Phi_{\mathfrak{g}}(M)$ equal to the set of all $Kx \in \mathbf{P}\mathcal{N}_p(\mathfrak{g})$ such that the rank of $x - \chi(x)$ acting on M is strictly less than $(p-1)\dim(M)/p$. Then M is a projective $U_\chi(\mathfrak{g})$–module if and only if $\Phi_{\mathfrak{g}}(M) = \emptyset$, see [FP1], Thm. 6.4.

Now $Kx \in \mathbf{P}\mathcal{N}_p(\mathfrak{g})$ belongs to $\Phi_{\mathfrak{g}}(Z_\chi(\lambda))$ if and only if all $(m \times m)$–minors with $m = (p-1)p^{N-1}$ of the matrix of $x - \chi(x)$ with respect to the $z_{\mathbf{m},\chi}$ are 0. Therefore the set of all $(Kx, \chi) \in \mathbf{P}\mathcal{N}_p(\mathfrak{g}) \times (\mathfrak{b}^+)^\perp$ with $Kx \in \Phi_{\mathfrak{g}}(Z_\chi(\lambda))$ is closed. Hence so is its image under the second projection since $\mathbf{P}\mathcal{N}_p(\mathfrak{g})$ is a complete variety. That image is exactly the set of all $\chi \in (\mathfrak{b}^+)^\perp$ such that $Z_\chi(\lambda)$ is not projective.

D Standard Levi Form

We keep throughout the same assumptions and notations as in the preceding section.

D.1. A linear form $\chi \in \mathfrak{g}^*$ is said to have *standard Levi form* if $\chi(\mathfrak{b}^+) = 0$ and if there exists a subset I of the set of simple roots such that $\chi(X_{-\alpha}) \neq 0$ for all $\alpha \in I$ while $\chi(X_{-\beta}) = 0$ for all $\beta \in R^+ \setminus I$. (This definition goes back to [FP2], 3.1.)

If χ has standard Levi form, then $\chi([\mathfrak{n}^-, \mathfrak{n}^-]) = 0$ and $\chi(\mathfrak{n}^{-[p]}) = 0$. This implies that χ defines a one-dimensional $\mathfrak{n}^-$–module that is then a $U_\chi(\mathfrak{n}^-)$–module. Since $\mathfrak{n}^-$ is unipotent, this is the only simple $U_\chi(\mathfrak{n}^-)$–module (up to isomorphism), cf. [J3], 3.3. It then follows that $U_\chi(\mathfrak{n}^-)$ is the projective cover of this simple module, hence has a unique maximal submodule. Each $Z_\chi(\lambda)$ with $\lambda \in X$ is isomorphic to $U_\chi(\mathfrak{n}^-)$ as an $\mathfrak{n}^-$–module. Any proper $\mathfrak{g}$–submodule of $Z_\chi(\lambda)$ is then contained in that unique maximal $\mathfrak{n}^-$–submodule. Taking the sum of all these $\mathfrak{g}$–submodules we see:

Lemma: *If $\chi \in \mathfrak{g}^*$ has standard Levi form, then each $Z_\chi(\lambda)$ with $\lambda \in X$ has a unique maximal submodule.*

We then denote by $L_\chi(\lambda)$ the unique simple quotient of $Z_\chi(\lambda)$. Lemma B.4 tells us now that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $L_\chi(\lambda)$ with $\lambda \in X$. However, λ will not be unique: We have at least $L_\chi(\lambda) \simeq L_\chi(\lambda + p\mu)$ for all $\mu \in X$; but there may be additional isomorphisms.

D.2. Before returning to the question when $L_\chi(\lambda) \simeq L_\chi(\lambda')$ in D.1, let us look at the special case $\chi = 0$ that clearly has standard Levi form.

This was the first case to be investigated. If V is a G–module (i.e., a vector space over K with a representation $G \to \mathrm{GL}(V)$ that is a homomorphism of algebraic groups), then V becomes a $\mathfrak{g}$–module taking the tangent map at 1 of the representation. One gets thus $U_0(\mathfrak{g})$–modules.

The simple G–modules are classified by their highest weight. There is for each $\lambda \in X$ with $0 \leq \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in R^+$ a simple G–module $L(\lambda)$ with highest weight λ; each simple G–module is isomorphic to exactly one of these $L(\lambda)$, cf. [J1], II.2.7. Now Curtis proved in [Cu]:

Theorem: a) *The $L(\lambda)$ with $\lambda \in X$ and $0 \leq \langle \lambda, \alpha^\vee \rangle < p$ for all simple roots α are simple also as $\mathfrak{g}$–modules.*

b) *Each simple $U_0(\mathfrak{g})$–module is isomorphic to one of these $L(\lambda)$. If $\lambda, \lambda' \in X$ both satisfy the condition in* a)*, then $L(\lambda)$ and $L(\lambda')$ are isomorphic as $\mathfrak{g}$–modules if and only if $\lambda - \lambda' \in pX$.*

Note: if $\lambda, \lambda' \in X$ both satisfy the condition in a) and if $\lambda - \lambda' \in pX$, then $\langle \lambda, \alpha^\vee \rangle = \langle \lambda', \alpha^\vee \rangle$ for all $\alpha \in R$; in case G is semi-simple this implies that $\lambda = \lambda'$.

If one compares with the set up in D.1, specialised to the case $\chi = 0$, then we get that $L(\lambda)$ with λ as in a) is isomorphic as a $\mathfrak{g}$–module to $L_0(\lambda)$. And we get for all $\mu, \nu \in X$ that $L_0(\mu) \simeq L_0(\nu)$ if and only if $\mu \equiv \nu \pmod{pX}$.

D.3. Return to the more general situation in D.1 and consider $\chi \in \mathfrak{g}^*$ in standard Levi form with a set I of simple roots as in the definition. The (sketched) proof of Proposition C.2 shows that $Z_\chi(\lambda) \simeq Z_\chi(w \bullet \lambda)$ for all $\lambda \in X$ and $w \in W_I$ where W_I is the subgroup of the Weyl group W generated by all reflections s_α with $\alpha \in I$. We get thus one direction of:

Proposition: *Suppose that χ has standard Levi form and that $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. Let $\lambda, \mu \in X$. Then $L_\chi(\mu) \simeq L_\chi(\lambda)$ if and only if $\mu \in W_I \bullet \lambda + pX$.*

The "only if" part is proved in [FP2], 3.2/4; for $G = \mathrm{SL}_n$ see also [P1], Thm. 3. Generalisations were found by Shen Guangyu and by Nakano, cf. [J3], 10.7.

D.4. The classification above of the simple $U_\chi(\mathfrak{g})$–modules for χ in standard Levi form leads immediately to a classification of the simple $U_\chi(\mathfrak{g})$–modules for χ in the G–orbit of some χ' in standard Levi form. Consider an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ arising from a bilinear form as in (H3). If χ has standard Levi form and if $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$, then χ corresponds under this isomorphism to some $Y \in \mathfrak{g}$ that is regular nilpotent in the Levi factor $\mathfrak{g}_I$ of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\alpha \in R \cap \mathbf{Z}I$.

If $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$, then the classification of nilpotent orbits in $\mathfrak{g}$ by the Jordan normal form shows that each nilpotent element in $\mathfrak{g}$ is conjugate to a regular nilpotent one in some Levi factor, hence each nilpotent linear form on $\mathfrak{g}$ conjugate to one in standard Levi form. Therefore Proposition D.3 and the earlier reductions yield a complete classification of all simple $\mathfrak{g}$–modules in case G is isomorphic to some GL_n (or a product of such groups).

Also for R of type B_2 or B_3 each nilpotent linear form on $\mathfrak{g}$ is conjugate to one in standard Levi form. This is no longer true for the other types. In those cases Proposition D.3 does not yield a complete classification of all simple modules.

D.5. Let us stay with χ in standard Levi form. Having achieved a classification of the simple modules one would like to know more about their structure; at least their dimensions ought to be determined. In the case $\chi = 0$ Lusztig's conjecture in [L1] on formal characters of simple G–modules yields via Theorem D.2 a conjecture for simple $\mathfrak{g}$–modules that determines (among other things) their dimensions provided p is not too small. One may hope that p greater than the Coxeter number of R will do. It is known that Lusztig's conjecture is true for p larger than an unknown bound that depends on the root system, see [AJS].

There is a similar conjecture for any χ in standard Levi form. In order to formulate it, we have to replace the category of $U_\chi(\mathfrak{g})$–modules by a certain category of graded $U_\chi(\mathfrak{g})$–modules.

Fix $\chi \in \mathfrak{g}^*$ in standard Levi form and set $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. The enveloping algebra $U(\mathfrak{g})$ is in a natural way $\mathbf{Z}R$–graded such that each X_α has degree α and each $H \in \mathfrak{h}$ degree 0. However, it will now be more useful to regard $U(\mathfrak{g})$ as $\mathbf{Z}R/\mathbf{Z}I$–graded such that each X_α has degree $\alpha + \mathbf{Z}I$ and each $H \in \mathfrak{h}$ degree $0 + \mathbf{Z}I$. This has the advantage that the kernel of the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$ is homogeneous: It is generated by all $H^p - H^{[p]}$ with $H \in \mathfrak{h}$ — homogeneous of degree $0+\mathbf{Z}I$ — and by all $X_\alpha^p - \chi(X_\alpha)^p$ with $\alpha \in R$ — homogeneous of degree $p\alpha + \mathbf{Z}I$ since $p\alpha \in \mathbf{Z}I$ whenever $\chi(X_\alpha) \neq 0$.

We get now a $\mathbf{Z}R/\mathbf{Z}I$–grading on $U_\chi(\mathfrak{g})$. It will be convenient to regard this as a grading by the larger group $X/\mathbf{Z}I$ and now to study $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules. For example, we can give each $Z_\chi(\lambda)$ with $\lambda \in X$ a grading such that each basis element $X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$ as in B.4 is homogeneous of degree $\lambda - \sum_{i=1}^N m(i)\alpha_i + \mathbf{Z}I$. Denote $Z_\chi(\lambda)$ with this grading by $\widehat{Z}_\chi(\lambda)$.

General results on graded modules (cf. [J4], 1.4/5) imply that the radical of any $Z_\chi(\lambda)$ is a graded submodule. It follows that each $L_\chi(\lambda)$ has a grading such that the image of v_λ in $L_\chi(\lambda)$ is homogeneous of degree $\lambda + \mathbf{Z}I$. Denote this graded module by $\widehat{L}_\chi(\lambda)$. Note that any $\widehat{Z}_\chi(\lambda+p\nu)$ is just $\widehat{Z}_\chi(\lambda)$ with the grading shifted by $p\nu+\mathbf{Z}I$, similarly for $\widehat{L}_\chi(\lambda+p\nu)$ and $\widehat{L}_\chi(\lambda)$. If $\nu \in \mathbf{Z}I$, then we still have $\widehat{Z}_\chi(\lambda+p\nu) \simeq \widehat{Z}_\chi(\lambda)$; but this is no longer true when $\nu \notin \mathbf{Z}I$. One can now show (see [J3], 11.9):

Proposition: *Let $\lambda, \mu \in X$. Then*

$$\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu) \iff \widehat{Z}_\chi(\lambda) \simeq \widehat{Z}_\chi(\mu) \iff \lambda \in W_I \bullet \mu + p\mathbf{Z}I.$$

D.6. Keep the assumptions on χ and I until the end of Section D. We began in D.5 with a discussion of graded $U_\chi(\mathfrak{g})$–modules. However, we shall not consider all possible modules of this type. If $M = \bigoplus_{\gamma \in X/\mathbf{Z}I} M_\gamma$ is an $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module, then each M_γ is $\mathfrak{h}$–stable, hence a direct sum of weight spaces for $\mathfrak{h}$. We now make the additional condition:

If $\gamma = \mu + \mathbf{Z}I$ with $\mu \in X$, then all weights of $\mathfrak{h}$ on M_γ have the form $d(\mu+\nu)$ with $\nu \in \mathbf{Z}I$. We denote by $\mathcal{C}$ the category of all $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules satisfying this condition.

(In case $\chi = 0$ we have $I = \emptyset$ and deal here with X–graded $U_0(\mathfrak{g})$–modules $M = \bigoplus_{\mu\in X} M_\mu$. In this case the extra condition means that $\mathfrak{h}$ acts on each M_μ via $d\mu$. It follows that we can identify $\mathcal{C}$ for $\chi = 0$ with the category of G_1T–modules as in [J1], II.9.)

Now one easily checks (in the general case) that all $\widehat{Z}_\chi(\lambda)$ belong to $\mathcal{C}$. So do all their composition factors, in particular all $\widehat{L}_\chi(\lambda)$. An argument like that in B.4 shows that each simple module in $\mathcal{C}$ is isomorphic to some $\widehat{L}_\chi(\lambda)$ with $\lambda \in X$, cf. [J4], 2.5.

Set

$$C_I = \{\, \lambda \in X \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p \text{ for all } \alpha \in R^+ \cap \mathbf{Z}I \,\}.$$

This is a fundamental domain for the dot action on X of the affine reflection group generated by W_I and by the translations by all $p\alpha$ with $\alpha \in I$. Therefore Proposition D.5 implies:

Corollary: *Each simple module in $\mathcal{C}$ is isomorphic to exactly one $\widehat{L}_\chi(\lambda)$ with $\lambda \in C_I$.*

D.7. Let W_p denote the affine Weyl group of R, generated by W and the translations by elements in $p\mathbf{Z}R$. It is also generated by all affine reflections $s_{\alpha,rp}$ with $\alpha \in R$ and $r \in \mathbf{Z}$ where $s_{\alpha,rp}(\mu) = \mu - (\langle \mu, \alpha^\vee \rangle - rp)\alpha$. We shall usually consider the dot action of W_p on X where $w \bullet \mu = w(\mu + \rho) - \rho$.

Let $\le$ denote the usual order relation on X: We have $\mu \le \nu$ if and only if $\nu - \mu \in \sum_{\alpha\in R^+} \mathbf{N}\alpha$.

The reflections $s_{\alpha,mp} \in W_p$ are used to define another order relation on X that will be denoted by $\uparrow$. If $\alpha \in R^+$, $m \in \mathbf{Z}$, and $\lambda \in X$, then we say that $s_{\alpha,mp} \bullet \lambda \uparrow \lambda$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \ge mp$. In general $\uparrow$ is defined as the transitive closure of this relation. It is then clear that $\mu \uparrow \lambda$ implies $\mu \le \lambda$ and $\mu \in W_p \bullet \lambda$. (Cf. [J1], II.6.4.)

One has now a strong linkage principle, see [J3], 11.11 and [J4], 4.5:

Proposition: *Let $\lambda, \mu \in C_I$. If $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)] \ne 0$, then $\mu \uparrow \lambda$.*

This is analogous to classical results for G–modules or G_1T–modules (the case $\chi = 0$ here), cf. [J1], II.6.16 and II.9.12.

We have furthermore (see [J4], 2.8(1))

$$[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\lambda)] = 1 \qquad \text{for all } \lambda \in C_I.$$

It follows that the (infinite) "decomposition matrix" of all $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ is lower triangular with ones on the diagonal (with respect to some total ordering on X that refines $\leq$. This will be crucial for us when we shall want to use information on this decomposition matrix to determine characters and dimensions of the simple modules, cf. the discussion in D.12 below. Without introducing the grading this would not work because Proposition C.2 says that all $Z_\chi(\lambda)$ in a block have the same composition factors.

D.8. One possible proof of Proposition D.7 involves a sum formula for a certain filtration: Each $\widehat{Z}_\chi(\lambda)$ has a filtration

$$\widehat{Z}_\chi(\lambda) \supset \widehat{Z}_\chi(\lambda)^1 \supset \widehat{Z}_\chi(\lambda)^2 \supset \cdots$$

with $\widehat{Z}_\chi(\lambda)/\widehat{Z}_\chi(\lambda)^1 = \widehat{L}_\chi(\lambda)$ and $\widehat{Z}_\chi(\lambda)^i = 0$ for $i \gg 0$ such that for all simple modules E in $\mathcal{C}$

$$\sum_{i>0} [\widehat{Z}_\chi(\lambda)^i : E] = \sum_\beta \Big(\sum_{i\geq 0} [\widehat{Z}_\chi(\lambda-(ip+n_\beta)\beta) : E] - \sum_{i>0} [\widehat{Z}_\chi(\lambda-ip\beta) : E]\Big)$$

where n_β is the integer with $0 < n_\beta \leq p$ and $\langle \lambda+\rho, \beta^\vee\rangle \equiv n_\beta \pmod p$ and where we sum over all $\beta \in R^+ \setminus \mathbf{Z}I$ with $n_\beta < p$. This formula is proved in [J4], 3.10 generalising the case $\chi = 0$ treated in [AJS], 6.6. (The infinite sum in the formula makes sense because one can check for each E that there are only finitely many non-zero terms in the sum.)

D.9. The sum formula in D.8 is also one of the main tools in determining all multiplicities $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ in "easy" cases. Other important tools used there are Premet's theorem B.7, translation functors, and indecomposable projective modules. Projective modules will be discussed later on, see G.1. The translation functors on $\mathcal{C}$ are defined in a way similar to the one used for G– or G_1T–modules (for these compare [J1], II.7 and II.9.19) and they have similar properties: Translation from a "regular weight" (i.e., one with trivial stabiliser in W_p) to an arbitrary weight takes a baby Verma module to a baby Verma module, and it takes a simple module to a simple module or to 0, and one knows, when one gets 0, see [J4], 4.9 and 4.11.

Using such techniques I have been able to determine all $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ in the following cases (sometimes under additional restrictions on p):

G	A_n	B_n	B_2	A_{n+1}	G_2	D_n
I	A_{n-1}	B_{n-1}	A_1	$A_1 \times A_{n-1}$	$\widetilde{A}_1$	D_{n-1}

Here G is assumed to be semi-simple and simply connected of the type mentioned in the first row. The second row then describes the type of the root system $R \cap \mathbf{Z}I$. In the cases of two root lengths I of type A_1 (resp. $\widetilde{A}_1$) means that I consists of a *long* (resp. *short*) simple root. For $n = 2$ the B_{n-1} under B_n has to be interpreted as $\widetilde{A}_1$.

The first two cases in the table were treated in [J2]; there the only restrictions on p are those imposed by (H3) or (H2): One has $p \nmid n+1$ for type A_n, and $p \neq 2$ for type B_n. (If one works in the first case with GL_{n+1} instead of SL_{n+1}, then no restriction on p is needed according to [GP], Rem. 9.3.) The case (B_2, A_1) was first dealt with (for $p \neq 2$) by brute force calculations in an Aarhus preprint (1997:13); in [J4], 5.2–10 this case is treated for $p > 3$ using the general ideas mentioned above. In the remaining cases I assume that p is larger than the Coxeter number of R. Files with the lengthy calculations in the A_{n+1} and D_n cases are available from me upon request.

D.10. The explicit results referred to in D.9 confirm in each case a conjecture by Lusztig. As in the case $\chi = 0$ (see D.5) one will have to expect some restrictions on p for this conjecture to be true; one may also here hope that p greater than the Coxeter number of R will suffice.

The conjecture says that any multiplicity $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ with $\lambda, \mu \in C_I$ is the value of a certain polynomial at 1 or -1 (depending on some normalisation of the polynomial). These polynomials were first constructed in [L2] where Lusztig then (in 13.17) expresses his hope that they would play a role similar to that of some previously constructed polynomials in the case $\chi = 0$. (See also the explicit formulation in [J3], 11.24.)

As in other situations one may speculate whether also the coefficients of these polynomials have a representation theoretic interpretation. For example, one may ask whether they yield the multiplicities in the factors of subsequent terms in the filtration mentioned in D.8 or in the radical filtration of $\widehat{Z}_\chi(\lambda)$. [These filtrations may well coincide.]

Set

$$A_0 = \{\, \lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+ \,\}.$$

This is the set of integral weights in the interior of the "first dominant alcove" with respect to W_p. This is the intersection of X with the "first dominant alcove over $\mathbf{R}$" as in [J1], II.6.2(6). Let us assume that $A_0 \neq \emptyset$, i.e., that p is at least equal to the Coxeter number, cf. [J1], II.6.2(10).

We call sets of the form $w \bullet A_0$ with $w \in W_p$ alcoves (in X). We denote the set of all alcoves by $\mathfrak{A}$; then $w \mapsto w \bullet A_0$ is a bijection from W_p onto $\mathfrak{A}$. Set $\mathfrak{A}_I$ equal to the set of all alcoves $A \in \mathfrak{A}$ with $A \subset C_I$.

Choose $\lambda_0 \in A_0$. For each $A \in \mathfrak{A}$ let λ_A denote the unique element in $A \cap W_p \bullet \lambda_0$; if $A = w \bullet A_0$ with $w \in W_p$, then $\lambda_A = w \bullet \lambda_0$. We have now $C_I \cap W_p \bullet \lambda_0 = \{\lambda_A \mid A \in \mathfrak{A}_I\}$. Lusztig's conjecture in the version of [L5], 17.3 says:

Conjecture ([L5]): $[\widehat{Z}_\chi(\lambda_A) : \widehat{L}_\chi(\lambda_B)] = \pi_{B,A}(-1)$ for all $A, B \in \mathfrak{A}_I$.

Here the $\pi_{B,A}$ are Lusztig's "periodic polynomials" normalised as in [L5], 9.17.

Thanks to the linkage principle the conjecture would determine all composition factors of all $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I \cap W_p \bullet \lambda_0$. Then translation functors would yield all composition factors of all $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I$.

Since baby Verma modules are finite dimensional, there are for each $A \in \mathfrak{A}_I$ only finitely many $B \in \mathfrak{A}_I$ with $[\widehat{Z}_\chi(\lambda_A) : \widehat{L}_\chi(\lambda_B)] \neq 0$. According to Conj. 9.20(b) in [L5] the $\pi_{B,A}$ are expected to have alternating signs. Therefore $\pi_{B,A} \neq 0$ should imply $\pi_{B,A}(-1) \neq 0$. So given A there should be only finitely many $B \in \mathfrak{A}_I$ with $\pi_{B,A} \neq 0$. That is known to hold in many examples, but is only conjectured by Lusztig in general, see [L5], 12.7/8. Also, it is not clear whether there is a good algorithm for computing the $\pi_{B,A}$, cf. [L2], 13.19.

D.11. The formal character of a finite dimensional $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module $M = \bigoplus_{\tau \in X/\mathbf{Z}I} M_\tau$ is defined as

$$\operatorname{ch} M = \sum_{\tau \in X/\mathbf{Z}I} \dim(M_\tau)\, e(\tau) \in \mathbf{Z}[X/\mathbf{Z}I]$$

where the $e(\tau)$ with $\tau \in X/\mathbf{Z}I$ form the canonical basis of the group ring $\mathbf{Z}[X/\mathbf{Z}I]$.

Note that $\mathbf{Z}R/\mathbf{Z}I$ is a free abelian group of finite rank: The cosets of the simple roots not in I are a basis. Therefore the group ring $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ is a localised polynomial ring, hence an integral domain. The larger group ring $\mathbf{Z}[X/\mathbf{Z}I]$ contains $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ as a subring and is a free module over this subring.

The standard basis of $\widehat{Z}_\chi(\lambda)$ shows that

$$\operatorname{ch} \widehat{Z}_\chi(\lambda) = p^{N(I)}\, e(\lambda + \mathbf{Z}I) \prod_{\alpha \in R^+ \setminus \mathbf{Z}I} \frac{1 - e(-p\alpha)}{1 - e(-\alpha)} \tag{1}$$

for all $\lambda \in X$, where $N(I) = |R^+ \cap \mathbf{Z}I|$. It is then clear that

$$\operatorname{ch} \widehat{Z}_\chi(\mu) = e(\mu - \lambda + \mathbf{Z}I)\, \operatorname{ch} \widehat{Z}_\chi(\lambda) \tag{2}$$

for all $\mu, \lambda \in X$. On the other hand, we have

$$\widehat{L}_\chi(\lambda + p\nu) = e(p\nu + \mathbf{Z}I)\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) \tag{3}$$

for all $\lambda, \nu \in X$ because adding $p\nu$ only amounts to a shift of the grading.

We get from the results in D.7 that for all $\lambda \in C_I$

$$\mathrm{ch}\,\widehat{Z}_\chi(\lambda) = \mathrm{ch}\,\widehat{L}_\chi(\lambda) + \sum_{\mu<\lambda, \mu\in C_I} [\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]\,\mathrm{ch}\,\widehat{L}_\chi(\mu).$$

It follows that we can write each $\mathrm{ch}\,\widehat{L}_\chi(\lambda)$ as a (usually infinite) linear combination of the form $\sum_{\mu\leq\lambda} a_{\lambda\mu}\,\mathrm{ch}\,\widehat{Z}_\chi(\mu)$ with all $a_{\lambda\mu} \in \mathbf{Z}$ and $a_{\lambda\lambda} = 1$. Such infinite sums make sense because each $e(\tau)$ with $\tau \in X/\mathbf{Z}I$ occurs only in finitely many $\mathrm{ch}\,\widehat{Z}_\chi(\mu)$, $\mu \in C_I$ with a non-zero coefficient.

Here is an easy example that shows also how one may replace an infinite sum by a finite one. Consider $G = \mathrm{SL}_2$ and $\chi = 0$. Denote the only positive root by α. Consider $\lambda \in X$ and an integer d with $\langle\lambda+\rho, \alpha^\vee\rangle \equiv d \pmod p$ and $0 < d < p$. It is easy to show (and a classical result) for all $i \in \mathbf{Z}$ that $[\widehat{Z}_\chi(\lambda - ip\alpha)] = [\widehat{L}_\chi(\lambda - ip\alpha)] + [\widehat{L}_\chi(\lambda - (ip+d)\alpha)]$ and $[\widehat{Z}_\chi(\lambda - (ip+d)\alpha)] = [\widehat{L}_\chi(\lambda - (ip+d)\alpha)] + [\widehat{L}_\chi(\lambda - (i+1)p\alpha)]$ in the Grothendieck group. It follows that

$$\mathrm{ch}\,\widehat{L}_\chi(\lambda) = \sum_{i\geq 0} \Big(\mathrm{ch}\,\widehat{Z}_\chi(\lambda - ip\alpha) - \mathrm{ch}\,\widehat{Z}_\chi(\lambda - (ip+d)\alpha)\Big).$$

Comparing with the analogous formula for $\mathrm{ch}\,\widehat{L}_\chi(\lambda - p\alpha)$ one gets using (2) and (3)

$$\begin{aligned}(1 - e(-p\alpha))\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) &= \mathrm{ch}\,\widehat{L}_\chi(\lambda) - \mathrm{ch}\,\widehat{L}_\chi(\lambda - p\alpha)\\ &= \mathrm{ch}\,\widehat{Z}_\chi(\lambda) - \mathrm{ch}\,\widehat{Z}_\chi(\lambda - d\alpha) = (1 - e(-d\alpha))\ \mathrm{ch}\,\widehat{Z}_\chi(\lambda).\end{aligned}$$

Now plug in (1):

$$(1 - e(-p\alpha))\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) = \frac{(1 - e(-d\alpha))\,(1 - e(-p\alpha))}{1 - e(-\alpha)}\ e(\lambda)$$

and cancel the common factor $1 - e(-p\alpha)$:

$$\mathrm{ch}\,\widehat{L}_\chi(\lambda) = \frac{1 - e(-d\alpha)}{1 - e(-\alpha)}\ e(\lambda) = \sum_{j=0}^{d-1} e(\lambda - j\alpha).$$

Note that such a cancellation is quite generally permitted because we work in a free module over an integral domain, see above.

D.12. I want to conclude this section by showing that it will always be possible to get each $\operatorname{ch} \widehat{L}_\chi(\lambda)$ as a finite sum modulo two conjectures by Lusztig. For this I need some properties of the polynomials $\pi_{B,A}$ and therefore have to look closer at Lusztig's constructions.

Lusztig works with the localised polynomial ring $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ in the indeterminate v and he considers the free $\mathcal{A}$–module $\mathbf{M}_c$ with basis $\mathfrak{A}_I$ as well as two completions, $\mathbf{M}_\leq$ and $\mathbf{M}_\geq$, of $\mathbf{M}_c$. In order to define them, we need an order relation on $\mathfrak{A}$: We set $A \leq B$ if and only if $\lambda_A \uparrow \lambda_B$ in the set-up of D.10. (This is the order relation on the alcoves denoted by $\uparrow$ in [J1], II.6.5.)

The support of a function $f : \mathfrak{A}_I \to \mathcal{A}$ is the set $\{A \in \mathfrak{A}_I \mid f(A) \neq 0\}$. We identify $\mathbf{M}_c$ with the $\mathcal{A}$–module of all f that have finite support. Then $\mathbf{M}_\leq$ (resp. $\mathbf{M}_\geq$) consists of all functions $f : \mathfrak{A}_I \to \mathcal{A}$ whose support is bounded above (resp. below) relative to $\leq$. We write such functions as formal sums $f = \sum_{A \in \mathfrak{A}_I} f(A)A$.

In [L5], 9.17/19 Lusztig introduces (for each $B \in \mathfrak{A}_I$) elements $B_\leq = \sum_{A \leq B} \tilde{\pi}_{A,B} A \in \mathbf{M}_\leq$ and $B_\geq = \sum_{A \geq B} \pi_{B,A} A \in \mathbf{M}_\geq$ and $\check{B}_\leq = \sum_{A \leq B} \check{\pi}_{A,B} A \in \mathbf{M}_\leq$ where the coefficients ($\tilde{\pi}_{A,B}$ etc.) belong to $\mathbf{Z}[v^{-1}]$. More precisely, we have $\tilde{\pi}_{B,B} = 1$ and $\tilde{\pi}_{A,B} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $A \neq B$, similarly for the other polynomials. Lemma 11.7(b) in [L5] says $\partial(\check{C}_\leq \| B_\geq) = \delta_{C,B}$ which means by the definition in [L5], 11.6 that

$$\sum_A \pi_{B,A} \check{\pi}_{A,C} = \delta_{C,B} \tag{1}$$

for all $B, C \in \mathfrak{A}_I$.

The fundamental domain property of C_I shows for all $A \in \mathfrak{A}_I$ and $\mu \in \mathbf{Z}R$ that there exists an alcove $\gamma_\mu(A) \in \mathfrak{A}_I$ such that $\gamma_\mu(A) = w \bullet (A + p\mu) + p\nu$ for suitable $w \in W_I$ and $\nu \in \mathbf{Z}I$. If $\mu \in \mathbf{Z}I$, then obviously $\gamma_\mu(A) = A$ for all A. It follows that we get an action of $\mathbf{Z}R/\mathbf{Z}I$ on $\mathfrak{A}_I$. We denote this action by $*$ and write $\tau * A = \gamma_\mu(A)$ if $\tau = \mu + \mathbf{Z}I$. This action preserves $\leq$, see [L2], 2.12(c).

The element $B_\geq \in \mathbf{M}_\geq$ is determined by the form of the coefficients stated above together with its invariance under a certain involution on $\mathbf{M}_\geq$. This involution commutes with the action of $\mathbf{Z}R/\mathbf{Z}I$ described above. Using this one can check that $(\tau * B)_\geq = \tau * (B_\geq)$ for all B, hence that $\pi_{\tau * B, \tau * A} = \pi_{B,A}$ for all B, A. Similar results hold for the other polynomials. We shall use below the corresponding result for the $\check{B}_\leq$:

$$\check{\pi}_{\tau * A, \tau * B} = \check{\pi}_{A,B}. \tag{2}$$

In [L5], 8.3 Lusztig constructs a homomorphism $\mu'_I : \mathbf{Z}R \to \mathbf{Z}$ such that $\mu'_I(\alpha) = -2$ for all $\alpha \in I$ and such that there are integers c_α with

$\mu'_I(\lambda) = \sum_{\alpha\in I} c_\alpha\langle\lambda,\alpha^\vee\rangle$ for all $\lambda \in \mathbf{Z}R$. Then μ'_I induces a homomorphism $\tau \mapsto \varepsilon_\tau$ from $\mathbf{Z}R/\mathbf{Z}I$ to $\{\pm 1\}$ such that $\varepsilon_{\lambda+\mathbf{Z}I} = (-1)^{\mu'_I(\lambda)}$ for all λ.

Lusztig extends the $\mathcal{A}$-module structure on $\mathbf{M}_c$, $\mathbf{M}_\leq$, and $\mathbf{M}_\geq$ to the group algebra $\mathcal{A}[\mathbf{Z}R/\mathbf{Z}I]$ by setting

$$e(\tau)\,A = \varepsilon_\tau\,(\tau * A)$$

for all $A \in \mathfrak{A}_I$ and $\tau \in \mathbf{Z}R/\mathbf{Z}I$.

There is a certain element $z = \sum_\tau z_\tau e(\tau) \in \mathcal{A}[\mathbf{Z}R/\mathbf{Z}]$ with $B_\leq = z\check{B}_\leq$, see [L5], 9.19. This means that

$$B_\leq = \sum_A\sum_\tau \check{\pi}_{A,B}\varepsilon_\tau z_\tau(\tau * A) = \sum_A\sum_\tau \check{\pi}_{(\tau *)^{-1}A,B}\varepsilon_\tau z_\tau A,$$

hence

$$\tilde{\pi}_{A,B} = \sum_\tau \varepsilon_\tau z_\tau \check{\pi}_{(\tau *)^{-1}A,B} = \sum_\tau \varepsilon_\tau z_\tau \check{\pi}_{A,\tau * B} \tag{3}$$

where we used (2) for the last step. Combining (3) and (1) we get now

$$\sum_A \pi_{C,A}\tilde{\pi}_{A,B} = \sum_\tau \varepsilon_\tau z_\tau \delta_{C,\tau * B}. \tag{4}$$

If now Lusztig's conjecture as in D.10 holds, then we get for all $B \in \mathfrak{A}_I$ (in the Grothendieck group)

$$\sum_{A\leq B} \tilde{\pi}_{A,B}(-1)\,[\widehat{Z}_\chi(\lambda_A)] = \sum_\tau \varepsilon_\tau z_\tau(-1)\,[\widehat{L}_\chi(\lambda_{\tau * B})]. \tag{5}$$

This means on the character level for each B

$$\sum_{A\leq B} \tilde{\pi}_{A,B}(-1)\ \mathrm{ch}(\widehat{Z}_\chi(\lambda_A)) = \mathrm{ch}(\widehat{L}_\chi(\lambda_B)) \sum_\tau \varepsilon_\tau z_\tau(-1)e(p\tau). \tag{6}$$

There is a ring homomorphism $\varphi : \mathcal{A}[\mathbf{Z}R/\mathbf{Z}I] \to \mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ with $\varphi(v) = -1$ and $\varphi(e(\tau)) = \varepsilon_\tau e(p\tau)$ for all τ. In this notation the right hand side in (6) is equal to $\varphi(z)\ \mathrm{ch}(\widehat{L}_\chi(\lambda_B))$.

The element z is introduced in [L5], Lemma 8.15. By that lemma (compare also the proof of Lemma 3.2.a in [L5]) z is a product of factors of the form $1 - v^m e(\tau)$ with $m \in \mathbf{Z}$ and $\tau \in (\mathbf{Z}R/\mathbf{Z}I) \setminus 0$, in particular with $e(\tau) \neq \pm 1$. It follows that $\varphi(z) \neq 0$. The characters in (6) belong to a free module over $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$. Therefore $\mathrm{ch}(\widehat{L}_\chi(\lambda_B))$ is uniquely determined by $\varphi(z)\,\mathrm{ch}(\widehat{L}_\chi(\lambda_B))$ and we can use (6) to compute it. Finally, Lusztig conjectures that for each B one should have only finitely many A with $\tilde{\pi}_{A,B} \neq 0$, see [L5], 12.7/8. So the left hand side in (6) should be a finite sum.

E Representations and Springer Fibres

We keep throughout the same assumptions and notations as in the preceding section. Let h denote the Coxeter number of the root system R.

E.1. Set $\mathcal{B} = G/B^+$; this is the flag variety of G. It can be identified via $gB^+ \mapsto gB^+g^{-1}$ with the set of all Borel subgroups of G, or via $gB^+ \mapsto \mathrm{Ad}(g)(\mathfrak{b}^+)$ with the set of all Borel subalgebras of $\mathfrak{g}$.

For each $\chi \in \mathfrak{g}^*$ set

$$\mathcal{B}_\chi = \{\, gB^+ \in \mathcal{B} \mid \chi(\mathrm{Ad}(g)(\mathfrak{b}^+)) = 0 \,\}.$$

This is a closed subvariety of $\mathcal{B}$. Using the identification above we can think of $\mathcal{B}_\chi$ as the set of all Borel subalgebras of $\mathfrak{g}$ contained in the kernel of χ. Note that $\mathcal{B}_\chi \neq \emptyset$ if and only if $(g^{-1} \cdot \chi)(\mathfrak{b}^+) = 0$ for some $g \in G$ if and only if χ is nilpotent, cf. B.8.

Suppose that we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using a bilinear form as in (H3) and that $Y \in \mathfrak{g}$ corresponds to some nilpotent $\chi \in \mathfrak{g}^*$. Then we get $\mathcal{B}_\chi = \mathcal{B}_Y$ where $\mathcal{B}_Y$ is the set of all gB^+ with $Y \in \mathrm{Ad}(g)(\mathfrak{n}^+)$. So $\mathcal{B}_\chi$ is equal to the fibre over Y in Springer's resolution of the nilpotent cone in $\mathfrak{g}$, cf. [J6], 6.6(3). We therefore call $\mathcal{B}_\chi$ the *Springer fibre* of χ.

The Springer resolution and its fibres have played an important role in other parts of representation theory, e.g., in Springer's theory of Weyl group representations. Humphreys has suggested for some time (cf. [H3], 23) that there might be connections between the theory of $U_\chi(\mathfrak{g})$–modules and the geometry of $\mathcal{B}_\chi$. This motivated geometric constructions of representations by Mirković and Rumynin in [MR]. Lusztig made then some explicit conjectures involving $\mathcal{B}_\chi$ (in [L3], [L4], and [L5]) part of which has now been proved by Bezrukavnikov, Mirković, and Rumynin in [BMR]. We shall describe their main results in this section.

E.2. Recall from C.2 the subalgebra $U(\mathfrak{g})^G$ of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and the algebra homomorphisms $\mathrm{cen}_\lambda : U(\mathfrak{g})^G \to K$. For any nilpotent χ and any λ the simple $U_\chi(\mathfrak{g})$–modules on which $U(\mathfrak{g})^G$ acts via cen_λ are the simple modules in one block of $U_\chi(\mathfrak{g})$, see C.5.

For any $\chi \in \mathfrak{g}^*$ let U_χ^0 denote the quotient algebra of $U_\chi(\mathfrak{g})$ by the ideal generated by the image of $\ker(\mathrm{cen}_0)$ under the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$. So the simple U_χ^0–modules are for nilpotent χ the simple $U_\chi(\mathfrak{g})$–modules in one specific block of $U_\chi(\mathfrak{g})$.

Now one of the main results in [BMR] (Cor. 5.2.2 and Section 6) is:

Theorem: *Suppose that $p > 2h - 2$. Let $\chi \in \mathfrak{g}^*$ be nilpotent. Then the number of simple U_χ^0–modules is equal to the rank of the Grothendieck* *group of the category of coherent sheaves on the Springer fibre $\mathcal{B}_\chi$. This rank is also equal to the dimension of the l–adic cohomology of $\mathcal{B}_\chi$.*

Here h is the Coxeter number of R; one should expect that the theorem should extend to all $p > h$. Take for example χ in standard Levi form and I as in the definition in D.1. Then Proposition D.3 shows that the number of simple U_χ^0–modules is equal to $|W/W_I|$ in case $p > h$. This is compatible with the present theorem. For $p \leq h$ the number gets smaller indicating that the theorem cannot extend to such p.

The theorem proves (for $p > 2h - 2$) a conjecture of Lusztig. In his formulation the conjecture actually did not involve $\mathcal{B}_\chi$, but an analogue over $\mathbf{C}$: Let $G_{\mathbf{C}}$ denote the connected reductive group over $\mathbf{C}$ with the same root data as G, set $\mathfrak{g}_{\mathbf{C}} = \mathrm{Lie}(G_{\mathbf{C}})$ and let $\mathcal{B}^{\mathbf{C}}$ denote the flag variety of $G_{\mathbf{C}}$. Under our assumption on p we have a bijection between the set of nilpotent orbits in $\mathfrak{g}$ and the set of those in $\mathfrak{g}_{\mathbf{C}}$, hence also a bijection between the set of nilpotent orbits in $\mathfrak{g}^*$ and the set of those in $\mathfrak{g}_{\mathbf{C}}^*$. Choose a nilpotent $\chi(\mathbf{C}) \in \mathfrak{g}_{\mathbf{C}}^*$ in the orbit corresponding to that of χ. Then Lusztig predicts that the number of simple U_χ^0–modules should be equal to the dimension of the ordinary cohomology of $\mathcal{B}_{\chi(\mathbf{C})}^{\mathbf{C}}$ with coefficients in a field of characteristic 0. However, that dimension is equal to the dimension of the l–adic cohomology of $\mathcal{B}_\chi$, thanks to work by Lusztig, cf. [BMR], 6.4.3.

E.3. Let $\chi \in \mathfrak{g}^*$ be nilpotent. Theorem E.2 yields for $p > 2h - 2$ the number of simple $U_\chi(\mathfrak{g})$–modules in a specific block of $U_\chi(\mathfrak{g})$. One may ask about the remaining blocks.

Consider first $\lambda \in X$ such that $\lambda + pX$ has trivial stabiliser for the dot action on X/pX. Then the number of simple $U_\chi(\mathfrak{g})$–modules in the block determined by cen_λ is equal to the number of simple U_χ^0–modules. In fact there are translation functors that are equivalences between the blocks determined by cen_λ and by cen_0 and that induce inverse bijections between the two sets of simple modules, cf. [J5], Prop. B.5.

It is not clear what will happen when $\lambda + pX$ has a non-trivial stabiliser for the dot action on X/pX. Suppose that $p > h$. If χ has standard Levi form (or if $G \cdot \chi$ contains an element in standard Levi form), then the translation functor from the block of 0 to the block of λ takes simple modules to 0 or to simple modules; we get thus each simple module in the block of λ from exactly one simple module (up to isomorphism) in the block of 0. (This follows from the analogous result in the graded case mentioned in D.9.) We get the same behaviour in the subregular nilpotent cases in Section F. One may speculate whether such results generalise. Maybe further work in the spirit of [BMR] will lead to some answers to these questions.

E.4. Set U^0 equal to the quotient of $U(\mathfrak{g})$ by the ideal generated by $\ker(\text{cen}_0)$. So we can describe any U^0_χ also as the quotient of U^0 by the ideal generated by the images in U^0 of all $Y^p - Y^{[p]} - \chi(Y)^p$ with $Y \in \mathfrak{g}$.

The work in [BMR] leading to Theorem E.2 is inspired by the paper [BB1]. There Beilinson and Bernstein show that the analogue to U^0 over $\mathbf{C}$ is isomorphic to the algebra of global sections of the sheaf $\mathcal{D}_{\mathbf{C}}$ of differential operators on the flag variety, and that the global section functor induces an equivalence of categories from (certain) sheaves of $\mathcal{D}_{\mathbf{C}}$–modules to U^0–modules. (This result is one of the main steps in Beilinson's and Bernstein's proof of the Kazhdan-Lusztig conjecture on characters of simple highest weight modules over $\mathfrak{g}_{\mathbf{C}}$.)

The generalisation of this result to prime characteristic involves two major changes: One has to replace differential operators by "*crystalline*" *differential operators* (using the terminology of [BMR]), and one gets at the end not an equivalence of categories, but an equivalence of derived categories.

E.5. If A is the algebra of regular functions on a smooth affine variety over K, then the algebra of "crystalline" differential operators on A is an algebra generated by A and the Lie algebra $\text{Der}_K(A)$ of all K–linear derivations of A. One imposes some obvious relations, e.g., $D\,a - a\,D = D(a)$ for all $a \in A$ and $D \in \text{Der}_K(A)$.

For example, if A is a polynomial ring $A = K[X_1, X_2, \dots, X_n]$, then the algebra of "crystalline" differential operators on A is a free module over A with basis all monomials in the partial derivatives $\partial_i = \partial/\partial X_i$, $1 \le i \le n$. This algebra acts on A via "true" differential operators, but this action is not faithful because (e.g.) any $(\partial_i)^p$ acts as 0.

This construction has a sheaf version that leads to a sheaf of "crystalline" differential operators $\mathcal{D}_Y$ on any smooth variety Y over K. One replaces A by the sheaf $\mathcal{O}_Y$ of regular functions on Y, and $\text{Der}_K(A)$ by the tangent sheaf $\mathcal{T}_Y$, cf. [BMR], 1.2. This is a special case of a construction from [BB2], 1.2.5.

E.6. Consider now $Y = \mathcal{B}$, the flag variety, and set $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$. The action of G on $\mathcal{B} = G/B^+$ defines a Lie algebra homomorphism from $\mathfrak{g}$ to the global sections of the tangent sheaf $\mathcal{T}_{\mathcal{B}}$. This induces by construction a homomorphism from the enveloping algebra $U(\mathfrak{g})$ to the algebra $H^0(\mathcal{B}, \mathcal{D})$ of global sections of $\mathcal{D}$.

This homomorphism factors over U^0 ([BMR], 3.1.7) and induces for good p an isomorphism

$$U^0 \xrightarrow{\sim} H^0(\mathcal{B}, \mathcal{D}) \tag{1}$$

see [BMR], Prop. 3.3.1. The proof uses a transition to the associated graded algebras; this technique shows also that $H^i(\mathcal{B}, \mathcal{D}) = 0$ for all $i > 0$.

Denote by $\text{mod}_c(\mathcal{D})$ the category of coherent $\mathcal{D}$–modules; here coherent means locally finitely generated. Write $\text{mod}_{fg}(U^0)$ for the category of finitely generated U^0–modules. For each coherent $\mathcal{D}$–module $\mathcal{F}$ the space $H^0(\mathcal{B}, \mathcal{F})$ of global sections is via (1) a U^0–module that turns out to be finitely generated. One gets thus a functor that induces a functor on the bounded derived categories $D^b(\text{mod}_c(\mathcal{D})) \to D^b(\text{mod}_{fg}(U^0))$ because $H^i(\mathcal{B}, ?) = 0$ for $i > \dim(\mathcal{B})$. Now the first main result in [BMR] (Thm. 3.2) says:

Theorem: *Suppose that $p > 2h - 2$. Then the functor*

$$D^b(\text{mod}_c(\mathcal{D})) \to D^b(\text{mod}_{fg}(U^0)) \tag{2}$$

is an equivalence of derived categories.

As pointed out in [BMR] (Remarks 1 and 2 following Thm. 3.2) this theorem does not hold without going over to the derived categories; it also will not hold for small p.

The equivalence in the opposite direction in (2) is induced by the "localisation functor" that takes a U^0–module M to the $\mathcal{D}$–module $\mathcal{D} \otimes_{U^0} M$. For this to work one first has to show that the derived functor $\mathcal{D} \,{}^L\!\otimes_{U^0}$ takes $D^b(\text{mod}_{fg}(U^0))$ to the *bounded* derived category $D^b(\text{mod}_c(\mathcal{D}))$, see [BMR], Prop. 3.8.1. One then wants to show that the compositions of these functors are isomorphic to the identity. This is checked only on certain special objects; the proof that it suffices to look at these special objects requires the bound on p in the theorem.

E.7. The transition from Theorem E.6 to Theorem E.2 involves two main steps. Denote by $\mathcal{D}_\chi$ the restriction of $\mathcal{D}$ to $\mathcal{B}_\chi \subset \mathcal{B}$. If we were lucky, then the equivalence in E.6(2) would induce an equivalence between $D^b(\text{mod}_c(\mathcal{D}_\chi))$ and $D^b(\text{mod}_{fg}(U^0_\chi))$ for each nilpotent $\chi \in \mathfrak{g}^*$, and the derived category $D^b(\text{mod}_c(\mathcal{D}_\chi))$ would also be equivalent to the derived category of coherent modules over the structure sheaf of $\mathcal{B}_\chi$. This is almost true: It holds when we replace $\mathcal{B}_\chi$ by a formal neighbourhood, see [BMR], Thm. 5.2.1. That however does not matter in the end because this transition does not change the Grothendieck groups of the categories involved, cf. the proof of Cor. 4.1.4 in [BMR]. Finally one has to observe that the Grothendieck groups of these categories and those of their bounded derived categories coincide.

F The Subregular Nilpotent Case

In addition to our earlier conventions, we assume in this section that G is almost simple.

F.1. Since we assume G to be almost simple there is exactly one nilpotent orbit in $\mathfrak{g}$ of dimension $2(N-1)$ where $N = |R^+|$. Thanks to our general assumption (H3) there is also only one nilpotent orbit in $\mathfrak{g}^*$ of this dimension. We call these orbits the *subregular nilpotent* orbits (in $\mathfrak{g}$ or $\mathfrak{g}^*$).

If R is of type A_n or B_n for some n, then we can choose χ in the subregular nilpotent orbit in $\mathfrak{g}^*$ such that χ has standard Levi form. In these cases the simple $U_\chi(\mathfrak{g})$–modules as well as the structure of the baby Verma modules were determined in [J2].

For R not of type A_n or B_n, then the subregular nilpotent orbit in $\mathfrak{g}^*$ does not contain an element in standard Levi form. In these cases many results on $U_\chi(\mathfrak{g})$–modules were proved in [J5]. The results were most detailed in the simply laced cases (i.e., for R of type D_n or E_n) and proved in these special cases Lusztig's conjectures from [L4], 2.4/6 and [L5], 17.2. For other types the results in [J5] are less complete and can now be improved using the results from [BMR].

Thoughout this section we usually say "simple modules" when we mean "isomorphism classes of simple modules".

F.2. Choose some χ in the subregular nilpotent orbit in $\mathfrak{g}^*$ such that $\chi(\mathfrak{b}^+) = 0$. We pick some $\lambda \in X$ such that $0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p$ for all $\alpha \in R^+$. Each orbit of W on X/pX has a representative of this form. So we describe an arbitrary block of $U_\chi(\mathfrak{g})$ if we describe the block "of λ", i.e., the block where $U(\mathfrak{g})^G$ acts on all simple modules via cen_λ, cf. C.5. We know also that the simple modules in this block are exactly the composition factors of $Z_\chi(\lambda)$.

We call λ *regular* if p does not divide any $\langle \lambda+\rho, \alpha^\vee \rangle$, or, equivalently, if $0 < \langle \lambda + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$, i.e., if λ belongs to the interior A_0 of the first dominant alcove as in D.10. We shall assume throughout that $p > h$. This implies that regular λ do exist.

Denote the set of all simple roots by Π and denote by α_0 the largest short root. Set $m_{-\alpha_0} = 1$ and define positive integers $(m_\alpha)_{\alpha \in \Pi}$ by $\alpha_0^\vee = \sum_{\alpha \in \Pi} m_\alpha \alpha^\vee$. Let $(\varpi_\alpha)_{\alpha \in \Pi}$ denote the fundamental weights.

If λ is regular, then we associate to each $U_\chi(\mathfrak{g})$–module M in the block of λ an invariant $\kappa(M)$ defined by

$$\kappa(M) = \{\, \alpha \in \Pi \cup \{0\} \mid T_\lambda^{\varpi_\alpha - \rho}(M) \neq 0 \,\}$$

where we set $\varpi_0 = 0$. Here each T_λ^μ is a translation functor, cf. [J5], B.2.

F.3. Assume in this subsection that R is simply laced, i.e., of type A_n, D_n, or E_n. In type E_8 the results stated below are proved only for $p > h+1$; this additional restriction should be unnecessary.

Suppose first that λ is regular. Then the block of λ contains $|\Pi|+1$ simple modules. We can denote these simple modules by L_α^λ with $\alpha \in \Pi \cup \{-\alpha_0\}$ such that

$$\dim(L_\alpha^\lambda) = \begin{cases} \langle \lambda+\rho, \alpha^\vee\rangle\, p^{N-1}, & \text{if } \alpha \in \Pi, \\ (p - \langle \lambda+\rho, \alpha_0^\vee\rangle)\, p^{N-1}, & \text{if } \alpha = -\alpha_0, \end{cases} \tag{1}$$

(recall that $N = |R^+|$) and

$$\kappa(L_\alpha^\lambda) = \begin{cases} \{\alpha\}, & \text{if } \alpha \in \Pi, \\ \Pi \cup \{0\}, & \text{if } \alpha = -\alpha_0. \end{cases} \tag{2}$$

We have

$$[Z_\chi(\lambda) : L_\alpha^\lambda] = m_\alpha \tag{3}$$

for all $\alpha \in \Pi \cup \{-\alpha_0\}$. Denote by Q_α^λ the projective cover of L_α^λ in the category of all $U_\chi(\mathfrak{g})$–modules. Then we have

$$[Q_\alpha^\lambda : L_\beta^\lambda] = |W|\, m_\alpha\, m_\beta \tag{4}$$

for all $\alpha, \beta \in \Pi \cup \{-\alpha_0\}$. Let us write Ext for Ext-groups in the category of all $U_\chi(\mathfrak{g})$–modules. One gets now

$$\operatorname{Ext}^1(L_\alpha^\lambda, L_\beta^\lambda) \simeq \begin{cases} K, & \text{if } \langle \beta, \alpha^\vee\rangle < 0, \\ 0, & \text{if } \langle \beta, \alpha^\vee\rangle = 0, \end{cases} \tag{5}$$

for all $\alpha, \beta \in \Pi \cup \{-\alpha_0\}$ with $\alpha \neq \beta$ — except that we have to replace K by K^2 if R has type A_1. The size of the Ext group in case $\alpha = \beta$ is unknown to me; one can show that it is non-zero in most cases.

The strategy for the proof of these results in [J5] is as follows: Using homomorphisms between baby Verma modules one constructs a chain of submodules in $Z_\chi(\lambda)$ where the dimensions and κ–invariants are known for the quotients of subsequent terms. Then translation functors and Premet's theorem are used to show that these quotients are simple. Next a deformation argument yields that simple modules with the same κ–invariant are isomorphic to each other. Thus one gets the classification of the simple modules and (1)–(3), cf. [J5], Thm. F.5. Quite general arguments imply in our case that $\dim(Q_\alpha^\lambda) = p^N |W|\, [Z_\chi(\lambda) : L_\alpha^\lambda]$, see

G.3(1) below. Inductive arguments using translation functors show in the present case that $[Q] = (\dim(Q)/p^N)[Z_\chi(\lambda)]$ in the Grothendieck group for all projective modules Q. Now (4) follows from (3), see [J5], Thm. G.6. The proof of (5) uses an idea of Vogan, relating Ext groups to the structure of simple modules translated through a wall, see [J5], Prop. H.12.

The statement about the number of simple modules as well as the result (4) on the Cartan matrix of the block prove conjectures by Lusztig from [L4], 2.4/6 and [L5], 17.2.

Drop now the assumption that λ should be regular, but continue to assume that $\langle \lambda+\rho, \alpha^\vee\rangle < p$. Then the number of simple modules in the block of λ is $1+|\{\alpha \in \Pi \mid \langle\lambda+\rho,\alpha^\vee\rangle > 0\}|$. One can denote these simple modules by L_α^λ with $\alpha = -\alpha_0$ or with $\alpha \in \Pi$ with $\langle\lambda+\rho,\alpha^\vee\rangle > 0$. Then (1) and (3) above still hold as long as we only allow α from this new parameter set. So does (4) if we replace $|W|$ by $|W \bullet (\lambda + pX)|$. Also (2) survives if we modify the definition of κ appropriately. If $\mu \in A_0$, then T_μ^λ takes L_α^μ to L_α^λ if this module is defined, to 0 otherwise. I have no information about Ext groups between simple modules when λ is not regular (except for R of type A_n where one can consult [J2], Prop. 2.19). These results are actually proved also for $p < h$ (except in type E_8) as long as (H2) and (H3) hold.

Suppose on the other hand that $\langle\lambda+\rho,\alpha^\vee\rangle = p$ for some $\alpha \in R^+$. Then we have in particular $\langle\lambda+\rho,\alpha_0^\vee\rangle = p$. In this case the simple modules should be parametrised by the simple roots α with $\langle\lambda+\rho,\alpha^\vee\rangle > 0$, and (1), (3), (4) should still hold with the appropriate modifications as in the preceding paragraph. This is true for R of type A_n and D_n, but open in type E_n.

F.4. As mentioned before, the classification of the simple modules in the regular case in F.3 was the first evidence besides the case of standard Levi form that Theorem E.2 should hold. To see this, let us describe $\mathcal{B}_\chi$ explicitly for subregular nilpotent χ.

For R of type A_n, D_n, or E_n the variety $\mathcal{B}_\chi$ has $|\Pi|$ irreducible components $(\ell_\alpha)_{\alpha\in\Pi}$. Each component ℓ_α is isomorphic to the projective line $\mathbf{P}^1$. Two distinct components ℓ_α and ℓ_β do not intersect if $\langle\beta,\alpha^\vee\rangle = 0$; they intersect transversally in one point if $\langle\beta,\alpha^\vee\rangle < 0$.

This description shows that $H^0(\mathcal{B}_\chi, \mathbf{Q}_l) \simeq \mathbf{Q}_l$ because $\mathcal{B}_\chi$ is connected, and that $\dim H^2(\mathcal{B}_\chi, \mathbf{Q}_l) = |\Pi|$ because all irreducible components of $\mathcal{B}_\chi$ have dimension 1 and there are $|\Pi|$ of them. This fact implies also that $H^i(\mathcal{B}_\chi, \mathbf{Q}_l) = 0$ for $i > 2$. Finally, $H^1(\mathcal{B}_\chi, \mathbf{Q}_l) = 0$ follows (e.g.) from the fact that one can pave $\mathcal{B}_\chi$ by affine lines and one point.

For the remaining root systems one can reduce the description of $\mathcal{B}_\chi$ to the cases already considered: If R has type B_n (resp. C_n, F_4, G_2), then $\mathcal{B}_\chi$ is isomorphic as variety to the analogous object for a group of type A_{2n-1} (resp. D_{n+1}, E_6, D_4). So again each irreducible component of $\mathcal{B}_\chi$ isomorphic to the projective line $\mathbf{P}^1$. But their number is now larger than $|\Pi|$. In types B_n, C_n, and F_4 one has now one component for each short simple root, but two components for each long simple root. In type G_2 there are one component for the short simple root and three components for the long simple root.

In any case the centraliser $\overline{G}_\chi$ of χ in the adjoint group $\overline{G}$ of G acts on $\mathcal{B}_\chi$, hence permutes the irreducible components of $\mathcal{B}_\chi$. The connected component $\overline{G}_\chi^0$ of 1 in $\overline{G}_\chi$ has to stabilise each component of $\mathcal{B}_\chi$, so the *component group* $C(\chi) = \overline{G}_\chi/\overline{G}_\chi^0$ acts as permutation group on the irreducible components of $\mathcal{B}_\chi$. For R of type A_n, D_n, or E_n the group $C(\chi)$ is trivial. For R of type B_n, C_n (with $n \geq 2$ in both cases), or F_4 the group $C(\chi)$ is cyclic of order 2; the nontrivial element interchanges for each long simple root α the two components belonging to α and it fixes the components belonging to short simple roots. For R of type G_2 the group $C(\chi)$ is isomorphic to the symmetric group S_3; it fixes the component belonging to the short simple root and acts as full permutation group on the three components belonging to the long simple root.

The action of $\overline{G}_\chi$ on $\mathcal{B}_\chi$ induces an action of $C(\chi)$ on $H^\bullet(\mathcal{B}_\chi, \mathbf{Q}_l)$. This action is trivial on $H^0(\mathcal{B}_\chi, \mathbf{Q}_l)$. On the other hand, $H^2(\mathcal{B}_\chi, \mathbf{Q}_l)$ has a basis indexed by the irreducible components of $\mathcal{B}_\chi$ and $C(\chi)$ permutes these basis elements in the same way as it permutes the irreducible components.

F.5. Assume in this subsection that R is of type B_n, C_n (with $n \geq 2$ in both cases), or F_4. Assume that $p > 2h - 2$ if R has type C_n with $n \geq 3$ or F_4, i.e., $p > 4n - 2$ in type C_n and $p > 22$ in type F_4. We need this bound so that we can apply the results from [BMR]. Let Π_s (resp. Π_l) denote the set of all short (resp. long) simple roots.

We look again first at the case where λ is regular. Then the block of λ contains $|\Pi_s| + 2|\Pi_l| + 1$ simple modules. We can denote them by L_α^λ with $\alpha \in \Pi_s \cup \{-\alpha_0\}$ and $L_{\alpha,1}^\lambda$, $L_{\alpha,2}^\lambda$ with $\alpha \in \Pi_l$ such that F.3(1)–(3) also hold here for all short α, i.e., for $\alpha \in \Pi_s \cup \{-\alpha_0\}$, and such that for all $\alpha \in \Pi_l$ and all $i \in \{1, 2\}$

$$\dim(L_{\alpha,i}^\lambda) = \langle \lambda + \rho, \alpha^\vee \rangle \, p^{N-1} \tag{1}$$

and

$$\kappa(L^\lambda_{\alpha,i}) = \{\alpha\} \tag{2}$$

and

$$[Z_\chi(\lambda) : L^\lambda_{\alpha,i}] = \frac{m_\alpha}{2}\,. \tag{3}$$

In [J5] I had to leave open the question whether in types C_n and F_4 the two simple modules $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ corresponding to some $\alpha \in \Pi_l$ are isomorphic to each other or not; the results from [BMR] yielding the total number of simple modules give the answer: they are not.

Denote by Q^λ_α the projective cover of L^λ_α for short α and by $Q^\lambda_{\alpha,i}$ the projective cover of $L^\lambda_{\alpha,i}$ for long α. Then F.3(4) holds for short α and β. If α is short and β is long, then one has for all $i \in \{1,2\}$

$$[Q^\lambda_\alpha : L^\lambda_{\beta,i}] = [Q^\lambda_{\beta,i} : L^\lambda_\alpha] = |W|\,\frac{m_\alpha\, m_\beta}{2}\,. \tag{4}$$

For the remaining Cartan invariants I can only state the obvious:

Conjecture :
$$[Q^\lambda_{\alpha,i} : L^\lambda_{\beta,j}] = |W|\,\frac{m_\alpha\, m_\beta}{4}$$

for all long α and β and all $i,j \in \{1,2\}$. (This should be a special case of Lusztig's conjecture in [L5], 17.2.) At least in type B_n this formula holds thanks to the results in [J2].

If both α and β are short with $\alpha \neq \beta$, then F.3(5) extends to the present situation. If α is short and β is long, then one gets for all $i \in \{1,2\}$

$$\operatorname{Ext}^1(L^\lambda_\alpha, L^\lambda_{\beta,i}) \simeq \operatorname{Ext}^1(L^\lambda_{\beta,i}, L^\lambda_\alpha) \simeq \begin{cases} K, & \text{if } \langle\beta,\alpha^\vee\rangle < 0,\\ 0, & \text{if } \langle\beta,\alpha^\vee\rangle = 0. \end{cases} \tag{5}$$

Finally, one can choose the numbering of the $L^\lambda_{\alpha,i}$ such that for all long α and β with $\alpha \neq \beta$ and all $i,j \in \{1,2\}$

$$\operatorname{Ext}^1(L^\lambda_{\alpha,i}, L^\lambda_{\beta,j}) \simeq \begin{cases} K, & \text{if } \langle\beta,\alpha^\vee\rangle < 0 \text{ and } i=j,\\ 0, & \text{if } \langle\beta,\alpha^\vee\rangle = 0 \text{ or } i \neq j. \end{cases} \tag{6}$$

I do not know what $\dim \operatorname{Ext}^1(L^\lambda_{\alpha,1}, L^\lambda_{\alpha,2})$ is for long α. Furthermore, usually the simple modules have non-trivial self-extensions, but as in F.3 I do not know how big the Ext group is. (For all of this, see [J5], Prop. H.12.)

If we twist a $U_\chi(\mathfrak{g})$–module with $\operatorname{Ad}(g)$ for some $g \in \overline{G}_\chi$, the centraliser of χ in the adjoint group of G, then we get again a $U_\chi(\mathfrak{g})$–module. If the original module was simple, then so is the twisted one. If $U(\mathfrak{g})^G$

acts via cen_λ on the original module, then also on the new one. We get thus an action of the component group $C(\chi)$ as in F.4 on the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules in the block of λ. In our present situation $C(\chi)$ is cyclic of order 2 and the non-trivial element in $C(\chi)$ interchanges $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for all long α, and it fixes all L^λ_β with β short, see [J5], Prop. F.7.

For non-regular λ the situation should be similar to the one described in F.3. If we still have $\langle \lambda+\rho, \beta^\vee\rangle < p$ for all $\beta \in R^+$, then one gets simple modules L^λ_α and $L^\lambda_{\alpha,i}$ more or less as above except that we drop all α with $\langle \lambda+\rho, \alpha^\vee\rangle = 0$. Each simple module is isomorphic to one of these. The only problem is that we no longer can be sure that $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for long α are not isomorphic to each other because [BMR] does not tell us the number of simple modules in the non-regular case. However, in the case where there is only one $\beta \in R^+$ such that p divides $\langle \lambda+\rho, \beta^\vee\rangle$, then one can show that the simple modules $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for long α are not isomorphic, see [J5], Lemma H.11.b.

F.6. Consider now R of type G_2 and assume that $p > 2h-2 = 10$. Denote the simple roots by α_1 (short) and α_2 (long). Let λ be regular. Then [J5], D.11 shows: The block of λ contains a simple module $L^\lambda_{-\alpha_0}$ with κ–invariant $\Pi \cup \{0\}$ and dimension $(p - \langle \lambda+\rho, \alpha_0^\vee\rangle)\, p^{N-1}$; this is the only simple module in this block with κ–invariant $\Pi\cup\{0\}$. There are one or two simple module in this block with κ–invariant $\{\alpha_1\}$; this module or these modules have dimension $\langle \lambda+\rho, \alpha_1^\vee\rangle\, p^{N-1}$. There is at least one simple module with κ–invariant $\{\alpha_2\}$ and dimension $\langle \lambda+\rho, \alpha_2^\vee\rangle\, p^{N-1}$. All remaining simple modules have κ–invariant $\{\alpha_2\}$ or $\emptyset$.

Now [BMR] tells us that there are five simple modules. We know also that the component group $C(\chi) \simeq S_3$ permutes these modules. This action preserves the κ–invariant by [J5], D.5(3), and is independent of the choice of λ if we use translation functors to identify the simple modules in different blocks for regular weights, see [J5], B.13(2).

The possible orbit sizes of $C(\chi)$ permuting the five simple modules are 1, 2, and 3. There have to be at least three orbits. If one of the orbits has size 3, then there are exactly three orbits; the other two orbits have cardinality 1. If there is no orbit of size 3, then the alternating subgroup in $C(\chi) \simeq S_3$ has to fix all simple modules.

We have by [BMR] an isomorphism between the Grothendieck group of the block of 0 and the Grothendieck group $K_0(\mathcal{B}_\chi)$ of coherent sheaves on $\mathcal{B}_\chi$. By the naturality of the construction in [BMR] one may hope that this isomorphism is compatible with the actions of $\overline{G}_\chi$.

Let us assume that this is true. Then the action of $\overline{G}_\chi$ on $K_0(\mathcal{B}_\chi)$ factors over $C(\chi)$ since this holds for the action on the block side.

Denote by Y the irreducible component of $\mathcal{B}_\chi$ corresponding to the short simple root, and by U the (open) complement of Y. The group $\overline{G}_\chi$ stabilises Y and hence also U. We have then a surjective map $K_0(\mathcal{B}_\chi) \to K_0(U)$ that is compatible with the action of $\overline{G}_\chi$. Now U is isomorphic to a disjoint union of three affine lines. This implies that $K_0(U) \simeq \mathbf{Z}^3$. Furthermore, the action of $\overline{G}_\chi$ on $K_0(U)$ has to factor over $C(\chi)$, and the action of $C(\chi)$ has to permute the three summands isomorphic to $\mathbf{Z}$ in $K_0(U)$ because $C(\chi)$ permutes the three affine lines. Therefore $C(\chi)$ acts faithfully on $K_0(U)$, hence on $K_0(\mathcal{B}_\chi)$.

This should now imply that also the action of $C(\chi)$ on the Grothendieck group of the block is faithful. As pointed out above this means that there are three orbits, and that they have cardinality 1, 1, and 3. This holds at first for $\lambda = 0$ and then for all λ using translation functors. It then follows that there is only one simple module, say $L^\lambda_{\alpha_1}$, with κ–invariant $\{\alpha_1\}$, and that there are three simple modules, say $L^\lambda_{\alpha_2,1}$, $L^\lambda_{\alpha_2,2}$, $L^\lambda_{\alpha_2,3}$, with κ–invariant $\{\alpha_2\}$, all of dimension $\langle \lambda + \rho, \alpha_2^\vee \rangle \, p^{N-1}$. One gets then (using [J5], D.11 once more) that

$$[Z_\chi(\lambda) : L^\lambda_{\alpha_1}] = 2$$

and

$$[Z_\chi(\lambda) : L^\lambda_{-\alpha_0}] = 1 = [Z_\chi(\lambda) : L^\lambda_{\alpha_2,i}]$$

for all i.

G Projective Modules

We keep the previous assumptions as in Sections D and E.

G.1. Each $U_\chi(\mathfrak{g})$ with $\chi \in \mathfrak{g}^*$ is a finite dimensional algebra. So there is a bijection between the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules and the set of isomorphism classes of indecomposable projective $U_\chi(\mathfrak{g})$–modules. Any simple $U_\chi(\mathfrak{g})$–module E corresponds to its projective cover Q_E, the unique (up to isomorphism) indecomposable projective $U_\chi(\mathfrak{g})$–module with $E \simeq Q_E/\mathrm{rad}(Q_E)$. The matrix of all $[Q_E : E']$ with E, E' running over representatives for the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules is then the *Cartan matrix* of $U_\chi(\mathfrak{g})$.

G.2. Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form. Denote the projective cover of any $L_\chi(\lambda)$ as in D.1 by $Q_\chi(\lambda)$. Recall now the graded category $\mathcal{C}$ from D.5/6. Each simple module $\widehat{L}_\chi(\lambda)$ in $\mathcal{C}$ has a projective cover $\widehat{Q}_\chi(\lambda)$ in $\mathcal{C}$. If we forget the grading, then $\widehat{Q}_\chi(\lambda)$ is isomorphic to $Q_\chi(\lambda)$, cf. [J4], 1.4.

Set $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. Recall from D.6 the definition of C_I and the fact that $\lambda \mapsto \widehat{L}_\chi(\lambda)$ yields a bijection between C_I and the set of isomorphism classes of simple modules in $\mathcal{C}$. One can show (see [J4], Prop. 2.9) that each $\widehat{Q}_\chi(\lambda)$ with $\lambda \in C_I$ has a filtration with factors $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I$, each $\widehat{Z}_\chi(\mu)$ occurring $|W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]$ times. (Note that there is a misprint in [J4], page 154, line −2: Replace $|W_I(\lambda + pX)|$ by $|W_I \bullet (\lambda + pX)|$.) This yields in the Grothendieck group of $\mathcal{C}$ using Proposition D.7

$$[\widehat{Q}_\chi(\lambda)] = \sum_{\mu \in C_I,\, \lambda \uparrow \mu} |W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]\,[\widehat{Z}_\chi(\mu)]. \tag{1}$$

We have in the ungraded category $[Z_\chi(\mu)] = [Z_\chi(\lambda)]$ for all $\mu \in W \bullet \lambda + pX$, hence for all μ with $\lambda \uparrow \mu$, see Proposition C.2. So (1) implies that in the Grothendieck group of the ungraded category

$$[Q_\chi(\lambda)] = [Z_\chi(\lambda)] \sum_{\mu \in C_I,\, \lambda \uparrow \mu} |W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]. \tag{2}$$

It follows since $\dim Z_\chi(\lambda) = p^N$ with $N = |R^+|$

$$[Q_\chi(\lambda)] = \frac{\dim(Q_\chi(\lambda))}{p^N}[Z_\chi(\lambda)]. \tag{3}$$

So all rows in the Cartan matrix belonging to a fixed block are proportional to each other. Therefore the Cartan matrix has determinant 0 unless each block contains only one simple module. The latter is the case if and only if I consists of all simple roots, i.e., if and only if χ is regular nilpotent. In that special case all composition factors of $Q_\chi(\lambda)$ are isomorphic to $L_\chi(\lambda) = Z_\chi(\lambda)$, and there are $|W_I \bullet (\lambda + pX)|$ of them.

G.3. For an arbitrary nilpotent $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$ one knows less about the projective indecomposable modules. If E is a simple $U_\chi(\mathfrak{g})$–module in the block of some $\lambda \in X$, then one can show that

$$\dim(Q_E) = p^N\,|W \bullet (\lambda + pX)|\,[Z_\chi(\lambda) : E] \tag{1}$$

see [J5], B.12(2). For χ in standard Levi form this can be deduced from G.2(2). In general (for χ not in standard Levi form) it is not known whether Q_E has a filtration with factors of the form $Z_\chi(\mu)$ that would "explain" (1) and that would allow us to generalise G.2(3). So we have at present only a

Hope: $$[Q_E] = \frac{\dim Q_E}{p^N}\,[Z_\chi(\lambda)]$$

for all simple E in the block of some λ. Results in the subregular nilpotent case (see F.3 and F.5) support this hope.

G.4. Let $\chi \in \mathfrak{g}^*$ be nilpotent and let $\lambda \in X$ be regular. (This means as in F.2 that p does not divide any $\langle \lambda + \rho, \alpha^\vee \rangle$ with $\alpha \in R$. So we assume that $p \geq h$.)

Lusztig has a general conjecture on the Cartan matrix of the block of λ of $U_\chi(\mathfrak{g})$, see [L4], 2.4 and [L5], 17.2. Actually, the conjecture is stated for a graded version of that category. One chooses a maximal torus T_0 in the centraliser of χ in G. Then the adjoint action of T_0 on $U(\mathfrak{g})$ induces one on $U_\chi(\mathfrak{g})$. This leads then to a grading of $U_\chi(\mathfrak{g})$ by the character group $X(T_0)$ of T_0. One now considers $X(T_0)$–graded $U_\chi(\mathfrak{g})$–modules satisfying a compatibility condition analogous to the one in D.6; finally one takes a block of this category where $U(\mathfrak{g})^G$ acts on all simple modules via cen_λ.

Now Lusztig's conjecture predicts that the simple modules in this block should be in bijection with a certain basis of some equivariant K–group of some variety (a Slodowy variety) related to $\mathcal{B}_\chi$, and that the entry in the Cartan matrix corresponding to two simple modules should be given by a certain bilinear form evaluated at the two corresponding basis elements. (See [L5] for more details.)

In the case where χ is subregular nilpotent and G almost simple not of type A_n or B_n, then $T_0 = 1$, so we can forget about the grading. If G is almost simple of type D_n or E_n, then [L4], 2.6 gives an explicit description of the conjectured Cartan matrix that since has turned out to be correct, see F.3(4).

Suppose that χ has standard Levi form (for arbitrary G). In this case one can choose $T_0 \subset T$ with $X(T_0) = \mathbf{Z}R/\mathbf{Z}I$. In this case the category described above is a direct summand of the category $\mathcal{C}$ from D.6 where gradings by $X/\mathbf{Z}I$ are permitted. However, this smaller category contains the block corresponding to $\lambda = 0$; all other blocks for regular λ are equivalent to this one (assuming $p \geq h$). For such λ Lusztig's conjecture on the Cartan matrix will follow using G.2(1) from Lusztig's conjecture, mentioned in D.10, on the composition factors of the $\widehat{Z}_\chi(\mu)$, see [L5], 17.3.

G.5. Consider the case $\chi = 0$. So the graded category $\mathcal{C}$ from D.6 is the same as that of all G_1T–modules as in [J1], II.9. Let us now drop in this case the index χ and write $\widehat{L}(\lambda)$, $\widehat{Z}(\lambda)$, $\widehat{Q}(\lambda)$ instead of $\widehat{L}_\chi(\lambda)$, etc. So these objects are parametrised by $\lambda \in X$. Each $\widehat{Q}(\lambda)$ has a filtration with factors of the form $\widehat{Z}(\mu)$ such that each $\widehat{Z}(\mu)$ occurs $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$

times. We have in particular

$$[\widehat{Q}(\lambda):\widehat{L}(\nu)] = \sum_{\mu\in X} [\widehat{Z}(\mu):\widehat{L}(\lambda)]\,[\widehat{Z}(\mu):\widehat{L}(\nu)] \tag{1}$$

for all $\lambda, \nu \in X$.

There is now the older conjecture by Lusztig predicting all $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$ for $p > h$ and proved in [AJS] for $p \gg 0$. And there is the newer one that predicts all $[\widehat{Q}(\lambda) : \widehat{L}(\nu)]$ for $p > h$, and that follows, according to Lusztig, from the older one using (1). Now I want to point out that in this case ($\chi = 0$) conversely the older conjecture follows from the newer one. Since Lusztig has already shown the other direction, it suffices to show that the Cartan matrix determines the decomposition matrix. More precisely:

Claim: *Let $\lambda \in A_0$. If we know all $[\widehat{Q}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $x, w \in W_p$, then we know all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $x, w \in W_p$.*

Here $A_0 \subset X$ is as in D.10 the interior of the first dominant alcove inside X, and W_p is the affine Weyl group as in D.7.

G.6. Before we can start proving the claim in G.5 we need some facts on the alcove geometry. Let X_+ denote the set of all dominant weights in X. For each $\mu \in X$ there exists $w \in W$ such that $w\mu$ is antidominant (or, equivalently, such that $-w\mu \in X_+$) and $w\mu$ is unique with this property. Moving the origin to another point, we get:

Fix $\nu \in X$. For each $\mu \in X$ there exists $w \in W_p$ such that $w \bullet (p\nu - \rho) = p\nu - \rho$ and $p\nu - \rho - w \bullet \mu \in X_+$. The weight $w \bullet \mu$ is uniquely determined by this condition; we shall denote it by $a_\nu(\mu)$.

Indeed: There exists $x \in W$ with $x(p\nu - \mu - \rho) \in X_+$. Set $\gamma = \nu - x(\nu) \in \mathbf{Z}R$ and set $w \in W_p$ equal to the composition of first x, then translation with $p\gamma$. Then

$$w \bullet (p\nu - \rho) = x(p\nu) + p\gamma - \rho = p\nu - \rho$$

and

$$p\nu - \rho - w \bullet \mu = p\nu - w(\mu + \rho) = p\nu - p\gamma - x(\mu + \rho) = x(p\nu - \mu - \rho) \in X_+.$$

This yields the existence; the uniqueness is left as an exercise.

Keep ν as above. A trivial remark: If $z \in W_p$, then:

$$z \bullet (p\nu - \rho) = p\nu - \rho \implies a_\nu(z \bullet \mu) = a_\nu(\mu) \text{ for all } \mu \in X. \tag{1}$$

(If $a_\nu(\mu) = w \bullet \mu$ with $w \in W_p$ and $w \bullet (p\nu - \rho) = p\nu - \rho$ then $a_\nu(\mu) = (wz^{-1}) \bullet (z \bullet \mu)$ and $(wz^{-1}) \bullet (p\nu - \rho) = p\nu - \rho$.)

We claim that next

$$a_\nu(\mu) \uparrow \mu \qquad \text{for all } \mu \in X. \tag{2}$$

It is well known that $x \bullet \mu' \uparrow \mu'$ for all $x \in W$ if $\mu' + \rho$ is dominant. One has similarly $\mu' \uparrow x \bullet \mu'$ for all $x \in W$ if $\mu' + \rho$ is antidominant. We can apply this to $a_\nu(\mu) - p\nu$ since $a_\nu(\mu) + \rho - p\nu$ is antidominant. There are $x \in W$ and $\gamma \in \mathbf{Z}R$ with $a_\nu(\mu) = p\gamma + x \bullet \mu$ and $\gamma = \nu - x(\nu)$. Now we get

$$a_\nu(\mu) - p\nu \uparrow x^{-1} \bullet (a_\nu(\mu) - p\nu) = x^{-1}(p\gamma + x(\mu + \rho) - p\nu) - \rho = \mu - p\nu$$

hence (2).

Claim: *If ν and μ are antidominant, then*

$$a_\nu(z \bullet \mu) \uparrow a_\nu(\mu) \qquad \textit{for all } z \in W. \tag{3}$$

This follows by induction on the length of z from the following claim, where ν is assumed to be antidominant and μ is arbitrary: Let $\alpha \in R$. Then

$$\mu \uparrow s_\alpha \bullet \mu \implies a_\nu(s_\alpha \bullet \mu) \uparrow a_\nu(\mu). \tag{4}$$

Proof: We may assume that $\alpha > 0$. We have $s_\alpha \bullet \mu = \mu + m\alpha$ with $m = -\langle \mu + \rho, \alpha^\vee \rangle \geq 0$.

There exists $w \in W_p$ with $w \bullet (p\nu - \rho) = p\nu - \rho$ and $a_\nu(\mu) = w \bullet \mu$. Set $\mu' = w \bullet (s_\alpha \bullet \mu)$. Then (1) says that $a_\nu(\mu') = a_\nu(s_\alpha \bullet \mu)$; so we want to show that $a_\nu(\mu') \uparrow a_\nu(\mu)$.

There are $x \in W$ and $\gamma \in \mathbf{Z}R$ such that w is the composition of first x and then translation by $p\gamma$. The assumption $w \bullet (p\nu - \rho) = p\nu - \rho$ implies that $\gamma = \nu - x(\nu)$.

Set $\beta = x(\alpha)$. We have then $ws_\alpha w^{-1} = s_{\beta,rp}$ with $r = \langle \gamma, \beta^\vee \rangle$. Now $\mu' = ws_\alpha \bullet \mu = s_{\beta,rp} \bullet (w \bullet \mu) = s_{\beta,rp} \bullet a_\nu(\mu)$ shows that $a_\nu(\mu) \uparrow \mu'$ or $\mu' \uparrow a_\nu(\mu)$. On the other hand, we have

$$ws_\alpha \bullet \mu = p\gamma + x \bullet (\mu + m\alpha) = p\gamma + x \bullet \mu + m\beta = a_\nu(\mu) + m\beta.$$

If $\beta < 0$, then $m \geq 0$ shows that $\mu' \uparrow a_\nu(\mu)$, hence $a_\nu(s_\alpha \bullet \mu) = a_\nu(\mu') \uparrow \mu' \uparrow a_\nu(\mu)$ using (2).

Suppose now that $\beta > 0$. Set $n = \langle \nu, \beta^\vee \rangle \in \mathbf{Z}$. It is clear that $s_{\beta,np} \bullet (p\nu - \rho) = p\nu - \rho$. So we have $a_\nu(\mu') = a_\nu(s_{\beta,np} \bullet \mu')$. Note that

$$s_{\beta,np} \bullet \mu' = s_{\beta,np} s_{\beta,rp} \bullet a_\nu(\mu) = a_\nu(\mu) + p(n - r)\beta$$

and

$$n - r = \langle \nu - \gamma, \beta^\vee \rangle = \langle x(\nu), \beta^\vee \rangle = \langle \nu, x^{-1}(\beta)^\vee \rangle = \langle \nu, \alpha^\vee \rangle.$$

Since ν is antidominant, the last equation yields $n-r \leq 0$, hence $a_\nu(\mu)+p(n-r)\beta \uparrow a_\nu(\mu)$. Now we get

$$a_\nu(s_\alpha \bullet \mu) = a_\nu(s_{\beta,np} \bullet \mu') \uparrow s_{\beta,np} \bullet \mu' = a_\nu(\mu) + p(n-r)\beta \uparrow a_\nu(\mu)$$

as claimed.

G.7. (*Proof of the claim in G.5*) Our multiplicities are periodic: We have $[\widehat{Z}(\mu + p\nu) : \widehat{L}(\mu' + p\nu)] = [\widehat{Z}(\mu) : \widehat{L}(\mu')]$ for all $\mu, \mu', \nu \in X$ since adding $p\nu$ amounts just to a shift of the grading. Each $\mu \in X$ can be written in the form $\mu = \mu_0 + p\mu_1$ with $\mu_1 \in X$ and $\mu_0 \in X_1$ where

$$X_1 = \{\, \mu \in X \mid 0 \leq \langle \mu, \beta^\vee \rangle < p \text{ for all simple roots } \beta\}.$$

Actually, it will be more convenient for us to replace X_1 by $X_1 - p\rho$.

Set $\mathcal{W} = \{w \in W_p \mid w \bullet \lambda \in X_1 - p\rho\}$. By the observation above we get all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ if we know them for all $x \in W_p$ and $w \in \mathcal{W}$. We get from [J1], II.9.13 that

$$[\widehat{Z}(yx \bullet \lambda) : \widehat{L}(w \bullet \lambda)] = [\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)] \tag{1}$$

for all $w \in \mathcal{W}$, $x \in W_p$, and $y \in W$. It is therefore enough to know all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $w \in \mathcal{W}$ and $x \in W_p$ such that $x \bullet \lambda$ is antidominant.

Set $\mathcal{W}'$ equal to the set of all $x \in W_p$ such that $x \bullet \lambda$ is antidominant and such that there exists $w \in \mathcal{W}$ with $w \bullet \lambda \uparrow x \bullet \lambda$. This is a finite set containing $\mathcal{W}$. We have to show that the Cartan matrix determines all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $w \in \mathcal{W}$ and $x \in \mathcal{W}'$.

Use the notation $d(C)$ for any alcove C as in [J1], II.6.6(1). Choose a numbering $\mathcal{W}' = \{w_1, w_2, \dots, w_r\}$ such that $d(w_i \bullet A_0) < d(w_j \bullet A_0)$ implies $j < i$. It follows that $w_i \bullet \lambda \uparrow w_j \bullet \lambda$ implies $j \leq i$, cf. [J1], II.6.6. In particular, if $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \neq 0$, then $j \leq i$.

We use induction on i. For $i = 1$ observe that the only relevant multiplicity is $[\widehat{Z}(w_1 \bullet \lambda) : \widehat{L}(w_1 \bullet \lambda)] = 1$. Let now $i > 1$ and assume that the Cartan matrix determines all $[\widehat{Z}(w \bullet \lambda) : \widehat{L}(w_s \bullet \lambda)]$ with $s < i$ and $w \in W_p$. There exists a weight ν_i with $w_i \bullet \lambda \in p\nu_i - \rho - X_1$. Since $w_i \bullet \lambda$ is antidominant, so is ν_i. Now $w_i \bullet \lambda - p\nu_i \in -\rho - X_1$ show that there exists $w_m \in \mathcal{W}$ with $w_i \bullet A_0 = p\nu_i + w_m \bullet A_0$. If $\nu_i \neq 0$, then $d(w_m \bullet A_0) > d(w_i \bullet A_0)$, hence $m < i$. There exists $\lambda' \in A_0$ with $w_i \bullet \lambda - p\nu_i = w_m \bullet \lambda'$. We know by induction (and translation) all $[\widehat{Z}(w \bullet \lambda') : \widehat{L}(w_m \bullet \lambda')]$ and get by periodicity all $[\widehat{Z}(w \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$.

So suppose that $\nu_i = 0$, i.e., that $w_i \in \mathcal{W}$. We have in the Grothendieck group

$$[\widehat{Q}(w_i \bullet \lambda)] = \sum_{j \leq i} [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(xw_j \bullet \lambda)]. \tag{2}$$

Here we have used (1).

We want to show by induction on l that all $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ are determined. As $[\widehat{Z}(w_i \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] = 1$, we may assume $l < i$. Suppose first that $w_l \in \mathcal{W}$. Then $[\widehat{Q}(w_i \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$ is by (1) and (2) equal to

$$\sum_{j \leq i} |W| \, [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)][\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]. \tag{3}$$

By our induction on i we know all $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$. This multiplicity is 0 unless $j \leq l$. If $j < l$, then we know $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ by our induction on l. So the only unknown summand in (3) is the one for $j = l$ where we get $|W| \, [\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$. This shows that $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ is determined by the Cartan matrix.

Now turn to the case where $w_l \notin \mathcal{W}$. There exists an antidominant weight $\nu \neq 0$ such that $w_l \bullet \lambda \in p\nu - \rho - X_1$. In this case [J1], II.9.13 implies for all μ that

$$[\widehat{Z}(\mu) : \widehat{L}(w_l \bullet \lambda)] = [\widehat{Z}(a_\nu(\mu)) : \widehat{L}(w_l \bullet \lambda)].$$

So we get from (2) that $[\widehat{Q}(w_i \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$ is equal to

$$\sum_{j \leq i} [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(a_\nu(xw_j \bullet \lambda)) : \widehat{L}(w_l \bullet \lambda)]. \tag{4}$$

Again we know by our induction on i all $[\widehat{Z}(a_\nu(xw_j \bullet \lambda)) : \widehat{L}(w_l \bullet \lambda)]$. If this multiplicity is non-zero, then $w_l \bullet \lambda \uparrow a_\nu(xw_j \bullet \lambda)$. Since ν and $w_j \bullet \lambda$ are antidominant, we get then from G.6(3) that $w_l \bullet \lambda \uparrow a_\nu(w_j \bullet \lambda)$, hence from G.6(2) that $w_l \bullet \lambda \uparrow w_j \bullet \lambda$ and $j \leq l$. If $j < l$, then we know $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ by our induction on l. So the only unknown summand in (4) is the one for $j = l$ where we get

$$[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(xw_l \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)].$$

(We do not need a_ν any longer.) Now the sum over x is positive since all terms are non-negative and the one for $x = 1$ is equal to one. We see as above that $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ is determined by the Cartan matrix.

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