Advanced Studies in Pure Mathematics 40, 2004
Representation Theory of Algebraic Groups and Quantum Groups
pp. 17–25

Hecke algebras with a finite number of indecomposable modules

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Abstract.

Recently, there has been progress in determining the representation type of the Hecke algebras of finite Weyl groups. We report on these results.

§1. Introduction

Recall that an Artin algebra A has *finite representation type* if A has finitely many isomorphism classes of indecomposable modules; otherwise, A has *infinite representation type.* In this short article, we report on a criterion for when the Hecke algebra of a finite Weyl group has finite representation type.

Let W be a finite Weyl group, K be an algebraically closed field and let q be a non-zero element of K. The K-algebra $\mathcal{H}_W(q)$ is the Hecke algebra associated with W.

First assume that $q = 1$. Then $\mathcal{H}_W(q)$ is the group algebra KW. Let l be the characteristic of K. It is well-known that if G is a finite group then the group algebra KG has finite representation type if and only if Sylow l–subgroups of G are cyclic; see [13] and [7]. In the case where W is a Weyl group, this implies the following.

Theorem 1. [4, Theorem 2] *Let W be a finite Weyl group. Then KW has finite representation type if and only if l^2 does not divide the order of W.*

Thus, we may assume that $q \neq 1$ in the rest of the paper. A criterion for $\mathcal{H}_W(q)$ to have finite representation type was conjectured by Uno [16]. To explain this, we recall the Poincaré polynomial of W.

Received March 13, 2002.

Definition 2. Let W be as above and let x be an indeterminate over K. Then the Poincaré polynomial $P_W(x)$ of W is the polynomial

$$P_W(x) = \sum_{w \in W} x^{l(w)} \in K[x],$$

where $l(w)$ is the length of $w \in W$.

The following is the conjecture of Uno's.

Conjecture 3. *(Conjecture–Theorem) Let $q \neq 1$ and $\mathcal{H}_W(q)$ be as above. Then $\mathcal{H}_W(q)$ has finite representation type if and only if $(x-q)^2$ does not divide $P_W(x)$.*

Uno's conjecture is now a theorem when W does not have a component of exceptional type. If W does have a component of exceptional type then the conjecture is known to be true under a mild assumption on the field K.

Let us explain the strategy used to prove the conjecture. Using the notion of complexity, we can reduce to the case where W is an irreducible Weyl group; see [4, Proposition 8]. We now proceed with a case–by–case analysis. When W is of type A the conjecture was already confirmed by Uno [16]. Uno also proved his conjecture for $\mathcal{H}_W(q)$ whenever W is a finite Coxeter group of rank two. For exceptional types, the conjecture has been proved by Miyachi [15] under the assumption that the characteristic of K is not too small; this uses computational results which had been obtained by Geck, Lux et al.

We now consider the cases where W is of type B or type D. Then, as is explained in [4], the conjecture is a corollary of [6, Theorem 1.4] (Theorem 4 below); see [4] and [6] for the details. Note that we excluded the case $q = -1$ in [6]. However, as we show below, a similar argument works in this case also and the main theorem [6, Theorem 1.4] is true when $q = -1$. In the next section, we explain the proof of this main theorem taking the case $q = -1$ as an example.

§2. Theorem 1.4 of [6] and the case $q = -1$

Recall that we are assuming that $q \neq 1$. Let W_n be the Weyl group of type B_n. Fix a non–negative integer f and let $\mathcal{H}_n = \mathcal{H}_{W_n}(q, -q^f)$ be the K–algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{array}{lll} (T_0-1)(T_0-q^f)=0, & (T_i+1)(T_i-q)=0, & \text{for } 1 \le i \le n-1, \\ T_0T_1T_0T_1 = T_1T_0T_1T_0, & T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i & \text{for } 1 \le i \le n-2, \end{array}$$
$$T_iT_j = T_jT_i, \text{ for } 0 \le i < j-1 \le n-2.$$

We are really considering the two parameter Hecke algebra of type B here; by a Morita equivalence argument the general two parameter case for type B reduces to considering the algebras above.

By renormalizing T_0 if necessary (see [6]) we may assume that q is a primitive e^{th} root of unity, where $e \geq 2$, and that $0 \leq f \leq \frac{e}{2}$. The main result of [6] asserts that the following is true.

Theorem 4 ([6, Theorem 1.4]). *Suppose that K is an algebraically closed field, $e \geq 2$ and that $0 \leq f \leq \frac{e}{2}$. Then $\mathcal{H}_n$ is of finite representation type if and only if $n < \min(e, 2f+4)$.*

In fact, in [6] Theorem 4 is proved only for the cases with $e \geq 3$; or, equivalently, when $q \neq \pm 1$. We first discuss the main ideas behind the proof of [6, Theorem 1.4]. We then illustrate how we use them in the argument by giving a proof of Theorem 4 in the case $q = -1$.

To prove that $\mathcal{H}_n$ has finite representation type if $n < \min(e, 2f+4)$ we used the combinatorics of path sequences together with the Jantzen-Schaper sum formula [14] for $\mathcal{H}_n$. Note that the case $q = -1$ (which was not considered in [6]), corresponds to $e = 2$; therefore, if $q = -1$ then $n < \min(e, 2f+4)$ only if $n = 1$. Thus, when $e = 2$ it is automatic that $\mathcal{H}_n$ has finite representation type if $n < \min(e, 2f+4)$.

We now consider the converse. To prove that $\mathcal{H}_n$ has infinite representation type when $n \geq \min(e, 2f+4)$ we rely on two theories. One is the Specht module theory developed by Dipper, James and Murphy [9]. The other is the description of the decomposition numbers of $\mathcal{H}_n$ as the coefficients of the canonical basis elements of a certain level 2 Fock space [1, 5]; we call this Fock space theory.

The Specht module theory provides us with a set of $\mathcal{H}_n$–modules, called Specht modules, indexed by bipartitions. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition of n and let S^λ be the corresponding Specht module. Then each S^λ is equipped with an invariant bilinear form. Let $\mathrm{rad}(S^\lambda)$ be the radical of the bilinear form and set $D^\lambda = S^\lambda/\mathrm{rad}(S^\lambda)$. Then the non–zero D^λ form a complete set of pairwise non–isomorphic $\mathcal{H}_n$–modules. Define P^λ to be the projective cover of $D^\lambda \neq 0$.

Let $\rhd$ be the dominance ordering on the set of bipartitions of n.

Proposition 5. [6, 3.12,3.13]

1. *If $D^\lambda \neq 0$ then S^λ is an indecomposable $\mathcal{H}_n$–module and D^λ is the unique head of S^λ.*
2. *Each projective $\mathcal{H}_n$–module P has a Specht filtration; thus, there exist bipartitions $\nu_1, \dots, \nu_k$ and a filtration*

$$P = P^k > P^{k-1} > \cdots > P^1 > P^0 = 0$$

such that $P^i/P^{i-1} \cong S^{\nu_i}$*, for* $1 < i \leq k$*, and* $i < j$ *whenever* $\nu_i \rhd \nu_j$*.*

3. *Suppose that* $P = P^\mu$ *for some bipartition* μ *with* $D^\mu \neq 0$*. Then the Specht filtration of* (2) *can be chosen so that*

$$d_{\lambda\mu} = \#\{\, 1 \leq i \leq k \mid \nu_i = \lambda \,\}.$$

In particular, if λ *is maximal in the dominance ordering such that* $d_{\lambda\mu} \neq 0$ *then* P^μ *has a submodule isomorphic to* S^λ*.*

The non–zero D^λ were classified by the first author in [2].

Now we turn to the Fock space theory. We begin by recalling the following theorem; see [3, Theorem 12.5] or [1], [5]. For the statement, let $\Lambda_0, \dots, \Lambda_{e-1}$ be the fundamental weights for the Kac–Moody Lie algebra $U(\widehat{sl}_e)$ and, for a dominant weight Λ, let $L(\Lambda)$ be the corresponding integrable highest weight module.

Theorem 6. *For* $i = 0, 1, \dots, e-1$ *there exist exact functors*

$$e_i, f_i : \mathcal{H}_n\text{-}\mathbf{mod} \longrightarrow \mathcal{H}_{n\pm1}\text{-}\mathbf{mod}$$

such that the operators induced by these, and suitably defined operators d *and* h_i*, for* $i = 0, 1, \dots, e-1$*, give* $\mathcal{K}_0 = \bigoplus_{n\geq 0} \mathcal{K}_0(\mathcal{H}_n\text{-}\mathbf{proj}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *the structure of a* $U(\widehat{sl}_e)$*–module. Moreover,* $\mathcal{K}_0 \cong L(\Lambda_0 + \Lambda_f)$ *as a* $U(\widehat{sl}_e)$*–module and if* K *is a field of characteristic zero then the principal indecomposable* $\mathcal{H}_n$*–modules correspond to elements of the Lusztig–Kashiwara canonical basis of* $L(\Lambda_0 + \Lambda_f)$ *under this isomorphism.*

As a consequence of this result, when K is a field of characteristic zero the decomposition numbers of $\mathcal{H}_n$ can be computed using the canonical basis of a certain v–deformed Fock space $\mathcal{F}_v = \mathcal{F}_v(\Lambda_0 + \Lambda_f)$; see [3] for details. In our case, the set of bipartitions form a basis of $\mathcal{F}_v$. Let $U_v(\widehat{sl}_e)$ be the quantum group of $U(\widehat{sl}_e)$; then $\mathcal{F}_v$ is a $U_v(\widehat{sl}_e)$–module. Let $L_v(\Lambda_0 + \Lambda_f)$ be the integrable highest weight module for $U_v(\widehat{sl}_e)$ of highest weight $\Lambda_0 + \Lambda_f$. Then, by definition, the canonical basis of $L(\Lambda_0 + \Lambda_f)$ is the canonical basis of $L_v(\Lambda_0 + \Lambda_f)$ specialized at $v = 1$.

The action of $U(\widehat{sl}_e)$ on the Fock space is the specialization at $v = 1$ of the action of $U_v(\widehat{sl}_e)$ on $\mathcal{F}_v$. In order to describe this let x and y be nodes of a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$. We say that x is *above* y if either (i) $x \in \lambda^{(1)}$ and $y \in \lambda^{(2)}$, or (ii) x and y are both in the same component of λ (i.e. in $\lambda^{(1)}$ or in $\lambda^{(2)}$), and x is above y. (We follow the English convention for describing partitions as Young diagrams.) For each $i \in \mathbb{Z}/e\mathbb{Z}$, write $\lambda \overset{i}{\longrightarrow} \mu$ if μ can be obtained by adding a single

i–node to λ; see [6]. Then the action of the Chevalley generator f_i of $U_v(\widehat{sl}_e)$ on $\mathcal{F}_v$ is given by

$$f_i\lambda = \sum_{\mu:\lambda \xrightarrow{i} \mu} v^{N_i^b(\mu/\lambda)}\mu,$$

where $N_i^b(\mu/\lambda)$ is the number of addable i–nodes below the node μ/λ minus the number of removable i–nodes below the node μ/λ. (The action of $f_i \in U(\widehat{sl}_e)$ on the Fock space is given by setting $v = 1$.)

The submodule of $\mathcal{F}_v$ generated by the empty bipartition is isomorphic to $L_v(\Lambda_0 + \Lambda_f)$ – the corresponding integrable highest weight module of $U_v(\widehat{sl}_e)$; this module becomes $L(\Lambda_0 + \Lambda_f)$ when we specialize v to 1. Denote the empty bipartition in $\mathcal{F}_v$ by $v_{\Lambda_0+\Lambda_f}$; then $L_v(\Lambda_0 + \Lambda_f) \cong U_v(\widehat{sl}_e)v_{\Lambda_0+\Lambda_f}$.

Corollary 7. [6, Corollary 3.16] *Suppose that $D^\mu \neq 0$ and that, in characteristic zero, $[P^\mu]$ corresponds to an element of the canonical basis which has the form $f_{i_1}^{(m_1)} \dots f_{i_l}^{(m_l)} v_{\Lambda_0+\Lambda_f}$ under the isomorphism of Theorem 6. Then P^μ has the same Specht filtration in positive characteristic as in characteristic zero.*

This corollary, together with the characterization of the canonical basis, implies that if

$$f_{i_1}^{(m_1)} \dots f_{i_l}^{(m_l)} v_{\Lambda_0+\Lambda_f} \in \lambda + \sum_\mu v\mathbb{Z}[v]\mu$$

in the Fock space $\mathcal{F}_v$ then the column of the decomposition matrix of $\mathcal{H}_n$ corresponding to λ does not depend on the characteristic of the base field K. Thus, the corollary gives us a way of applying Theorem 6 to compute decomposition numbers of $\mathcal{H}_n$ when K is a field of positive characteristic.

Using this, and the properties of the Specht modules listed above, we can prove that if $n \geq \min(e, 2f + 4)$ then $\mathcal{H}_n$ has infinite representation type. The reader can experience the flavour of the arguments of [6] from the following two lemmas which extend Theorem 4 to the case $q = -1$. Note that we only have to consider the cases $f = 0, 1$ since $0 \leq f \leq \frac{e}{2}$.

Lemma 8. *Assume that $q = -1$, $f = 1$ and $n \geq 2$. Then $\mathcal{H}_n$ has infinite representation type.*

Proof. By [6, Lemma 2.5] we may assume that $n = 2$. The defining relations of $\mathcal{H}_2$ are

$$T_0^2 - 1 = 0, \ \ (T_1 + 1)^2 = 0, \ \ (T_0T_1)^2 = (T_1T_0)^2.$$

Let $\lambda_1 = ((0),(1^2))$ and $\lambda_2 = ((1),(1))$. The Fock space has highest weight $\Lambda_0 + \Lambda_1$ and the decomposition matrix is as follows.

	λ_1	λ_2
$((0),(1^2))$	1	0
$((0),(2))$	1	0
$((1),(1))$	1	1
$((1^2),(0))$	0	1
$((2),(0))$	0	1

If M is a finite dimensional $\mathcal{H}_n$–module let $[M]$ denote the corresponding equivalence class in the Grothendieck group of $\mathcal{H}_n$ and let $\operatorname{Rad}(M)$ denote the radical of M. By the decomposition matrix above, we have $[P^{\lambda_1}] = 3[D^{\lambda_1}] + [D^{\lambda_2}]$ and $[P^{\lambda_2}] = [D^{\lambda_1}] + 3[D^{\lambda_2}]$. Observe that S^{λ_2} is indecomposable with head D^{λ_2} and socle D^{λ_1}. Since its dual module is indecomposable with head D^{λ_1} and socle D^{λ_2}, so that D^{λ_2} must appear in $\operatorname{Rad}(P^{\lambda_1})/\operatorname{Rad}^2(P^{\lambda_1})$. On the other hand, $\operatorname{Rad}(P^{\lambda_1})$ has a Specht filtration whose successive quotients are $S^{((0),(2))} = D^{\lambda_1}$ and S^{λ_2}. Thus D^{λ_1} must appear in $\operatorname{Rad}(P^{\lambda_1})/\operatorname{Rad}^2(P^{\lambda_1})$.

Using a similar argument we can prove that D^{λ_1} and D^{λ_2} must appear in $\operatorname{Rad}(P^{\lambda_2})/\operatorname{Rad}^2(P^{\lambda_2})$.

Considering the separation diagram, we conclude that the $\mathcal{H}_2$ has infinite representation type; see [6, Theorem 2.7]. □

Lemma 9. *Assume that $q = -1$, $f = 0$ and $n \geq 2$. Then $\mathcal{H}_n$ has infinite representation type.*

Proof. As before we may assume that $n = 2$. This time the defining relations of $\mathcal{H}_2$ are

$$(T_0 - 1)^2 = 0, \quad (T_1 + 1)^2 = 0, \quad (T_0T_1)^2 = (T_1T_0)^2.$$

Let $\lambda = ((0),(1^2))$. The element of the canonical basis corresponding to λ is given by

$$((0),(1^2)) + v((0),(2)) + v((1^2),(0)) + v^2((2),(0)).$$

The other element of the canonical basis corresponding to $((1),(1))$ is $((1),(1)) = f_0^{(2)}((0),(0))$. Thus, $[P^\lambda] = 4[D^\lambda]$. Looking at the defining relations, we can define a representation of $\mathcal{H}_2$ by

$$T_0 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} -1 & 0 & c \\ 0 & -1 & d \\ 0 & 0 & -1 \end{pmatrix}.$$

We choose $a, b, c, d \in K$ so that $ad - bc \neq 0$. Then this representation gives an indecomposable module with head D^λ and socle $D^\lambda \oplus D^\lambda$. Therefore, $\mathrm{End}_{\mathcal{H}_2}(P^\lambda) \not\simeq K[x]/(x^m)$ for any $m \geq 0$ (it has two independent generators); so we conclude that the $\mathcal{H}_2$ has infinite representation type by [6, Lemma 2.6]. □

§3. A result of Erdmann and Nakano

In this section, we assume that W has type A_{n-1}. Let e be the multiplicative order of q as before. Recall that an e–core is a partition which does not contain a removable e–hook. Then the blocks of $\mathcal{H}_W(q)$ are labelled by e–cores such that $n - |\kappa|$ is divisible by e. We denote by $\mathcal{B}_\kappa$ the block labelled by an e–core κ.

Artin algebras fall into three categories; finite, tame and wild. Erdmann and Nakano [10] have determined the representation type of the block algebras $\mathcal{B}_\kappa$.

Recall that if κ is an e–core then the e–**weight** of κ is

$$w(\kappa) := \frac{n - |\kappa|}{e}.$$

Theorem 10. [10, Theorem 1.2] *Maintain the notation above.*

(1) $\mathcal{B}_\kappa$ *is semisimple if and only if* $w(\kappa) = 0$.
(2) $\mathcal{B}_\kappa$ *has finite representation type (and is not semisimple) if and only if* $w(\kappa) = 1$.
(3) $\mathcal{B}_\kappa$ *has tame representation type if and only if* $e = 2$ *and* $w(\kappa) = 2$.
(4) $\mathcal{B}_\kappa$ *has wild representation type if and only if either* $e \geq 3$ *and* $w(\kappa) \geq 2$, *or* $e = 2$ *and* $w(\kappa) \geq 3$.

Generalization of this theorem to other types remains open.

§4. Appendix

The aim of the paper [6] was to determine when the two parameter Hecke algebra $\mathcal{H}_n(q, Q)$ of type B, which is defined by

$$\begin{array}{lll} (T_0 - 1)(T_0 + Q) = 0, & (T_i + 1)(T_i - q) = 0, & \text{for } 1 \leq i \leq n-1, \\ T_0T_1T_0T_1 = T_1T_0T_1T_0, & T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i, & \text{for } 1 \leq i \leq n-2, \\ \multicolumn{3}{c}{T_iT_j = T_jT_i \text{ for } 0 \leq i < j-1 \leq n-2,} \end{array}$$

has finite representation type. The Morita equivalence theorem of Dipper and James [8] implies that it is enough to consider the algebras $\mathcal{H}_n = \mathcal{H}_n(q, -q^f)$ of section 2, where $f \in \mathbb{Z}$. Recall that we assumed

$q \neq 1$ in section 2; however, as we now show, it is easy to determine when $\mathcal{H}_n(1,Q)$ has finite representation type.

Assume that $q = 1$. Then, as an algebra, $\mathcal{H}_n(1,Q)$ is isomorphic to the semidirect product of the commutative algebra $\mathcal{L}_n$ and the group algebra of the symmetric group KS_n, where

$$\mathcal{L}_n = \big(K[L]/(L^2 - (Q-1)L - Q)\big)^{\otimes n}$$

and S_n acts on $\mathcal{L}_n$ by conjugation in the natural way.

If $Q = -1$ and $n = 2$ then $\mathcal{L}_2 = \big(K[L]/(L+1)^2\big)^{\otimes 2}$ is the Kronecker algebra and $\mathcal{H}_2(1,Q) = \mathcal{L}_2 \oplus \mathcal{L}_2 T_1 \mathcal{L}_2$. Thus, $\mathcal{H}_n(1,-1)$ has infinite representation type when $n \geq 2$. Hence, we have proved the following.

Proposition 11. *Suppose that K is a field. Then $\mathcal{H}_n(1,-1)$ has finite representation type if and only if $n = 1$.*

If $Q \neq -1$ then the Dipper–James Morita equivalence theorem combined with Uno's proof of Conjecture 3 for type A gives the following.

Proposition 12. *Suppose that K is a field. Then $\mathcal{H}_n(1,Q)$ with $Q \neq -1$ has finite representation type if and only if $n < 2l$ where l is the characteristic of the base field.*

Remark 13. We can prove this statement without appealing to the Dipper–James Morita equivalence theorem. If $l \neq 2$ then

$$K[L]/(L^2 - (Q-1)L - Q) \simeq K \oplus K \simeq KC_2$$

and thus $\mathcal{H}_n(1,Q) \simeq KW_n$ where W_n is the Weyl group of type B_n. Therefore, by Theorem 1, $\mathcal{H}_n(1,Q)$ has finite representation type if and only if $n < 2l$.

Next assume that $l = 2$. Since KS_n is a factor algebra of $\mathcal{H}_n(1,Q)$, Theorem 1 again implies that $\mathcal{H}_n(1,Q)$ has infinite representation type when $n \geq 4$. Let $G_n = C_3 \wr \mathfrak{S}_n$. To prove that $\mathcal{H}_n(1,Q)$ has finite representation type when $n < 4$ it is enough to observe that there is a surjective homomorphism

$$KG_n = (K \oplus K \oplus K)^{\otimes n} KS_n \to (K \oplus K)^{\otimes n} KS_n = \mathcal{H}_n(1,Q).$$

By the remarks before Theorem 1, KG_n has finite representation type if $n < 4$; hence, $\mathcal{H}_n(1,Q)$ has finite representation type when $n < 4$.

References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m,1,n)$, J. Math. Kyoto Univ., **36** (1996), 789–808.

[2] ——, On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions, Osaka J. Math., **38** (2001), 827–837.

[3] ——, Representations of Quantum Algebras and Combinatorics of Young Tableaux, University Lecture Series **26** (2002), A.M.S..

[4] ——, Uno's conjecture on representation types of Hecke algebras, Algebraic Combinatorics and Quantum Groups, N. Jing (ed), 2003, World Scientific, 1-9.

[5] S. Ariki and A. Mathas, The number of simple modules of the Hecke algebras of type $G(r,1,n)$, Math. Z., **233** (2000), 601–623.

[6] ——, The representation type of Hecke algebras of type B, Adv. Math. **181** (2004), 134-159.

[7] V.M. Bondarenko and J.A. Drozd, The representation type of finite groups, J. Soviet Math., **20** (1982), 2515–2528.

[8] R. Dipper and G. James, Representations of Hecke algebras of type B_n, J. Algebra, **146** (1992), 454–481.

[9] R. Dipper, G. James and G. Murphy, Hecke algebras of type B_n at roots of unity, Proc. L.M.S., **70** (1995), 505–528.

[10] K. Erdmann and D.K. Nakano, Representation type of Hecke algebras of type A, Trans. A.M.S. **354** (2002), 275-285.

[11] M. Geck, Brauer trees of Hecke algebras, Comm. Alg., **20** (1992), 2937–2973.

[12] A. Gyoja and K. Uno, On the semisimplicity of Hecke algebras, J. Math. Soc. of Japan, **41** (1989), 75–79.

[13] D. Higman, Indecomposable representations at characteristic p, Duke Math. J., **21** (1954), 377–381.

[14] G. James and A. Mathas, The Jantzen sum formula for cyclotomic q–Schur algebras, Trans. A.M.S., **352** (2000), 5381–5404.

[15] H. Miyachi, Uno's conjecture for the exceptional Hecke algebras, in preparation.

[16] K. Uno, On representations of non–semisimple specialized Hecke algebras, J. Algebra, **149** (1992), 287–312.

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