Advanced Studies in Pure Mathematics 39, 2004
Stochastic Analysis on Large Scale Interacting Systems
pp. 377–395

Zero-Range-Exclusion Particle Systems

Kôhei Uchiyama

§1. Introduction

Let $\mathbf{T}_N$ denote the one-dimensional discrete torus $\mathbf{Z}/N\mathbf{Z}$ represented by $\{1, ..., N\}$. The zero-range-exclusion process that we are to introduce and study in this article is a Markov process on the state space $\mathcal{X}^N := \mathbf{Z}_+^{\mathbf{T}_N}$ ($\mathbf{Z}_+ = \{0, 1, 2, ...\}$). Denote by $\eta = (\eta_x, x \in \mathbf{T}_N)$ a generic element of $\mathcal{X}^N$, and define

$$\xi_x = \mathbf{1}(\eta_x \geq 1)$$

(namely, ξ_x equals 0 or 1 according as η_x is zero or positive). The process is regarded as a 'lattice gas' of particles having energy. The site x is occupied by a particle if $\xi_x = 1$ and vacant otherwise. Each particle has energy, represented by η_x, which takes discrete values $1, 2, \ldots$ If y is a nearest neighbor site of x and is vacant, a particle at site x jumps to y at rate $c_{\text{ex}}(\eta_x)$, where c_{ex} is a positive function of $k = 1, 2, \ldots$ Between two neighboring particles the energies are transferred unit by unit according to the same stochastic rule as that of the zero-range processes. In this article we shall give some results related to the hydrodynamic scaling limit for this model.

To give a formal definition of the infinitesimal generator of the process we introduce some notations. Let $b = (x, y)$ be an oriented bond of $\mathbf{T}_N$, namely x and y are nearest neighbor sites of $\mathbf{T}_N$, and (x, y) stands for an ordered pair of them. Define the *exclusion* operator π_b and *zero-range* operator ∇_b attached to b which act on $f \in C(\mathcal{X}^N)$ by

$$\pi_b f(\eta) = f(S^b_{\text{ex}}\eta) - f(\eta) \quad \text{and} \quad \nabla_b f(\eta) = f(S^b_{\text{zr}}\eta) - f(\eta)$$

where the transformation $S^b_{\text{ex}} : \mathcal{X}^N \mapsto \mathcal{X}^N$ is defined by

$$(S^b_{\text{ex}}\eta)_z = \begin{cases} \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases}$$

Received December 26, 2002.
Revised March 24, 2003.

if $\xi_x = 1$ and $\xi_y = 0$; and $S^b_{\rm zr}\eta$ by

$$(S^b_{\rm zr}\eta)_z = \begin{cases} \eta_x - 1, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{otherwise}, \end{cases}$$

if $\eta_x \geq 2$ and $\xi_y = 1$; and in the remaining case of η, both $S^b_{\rm ex}\eta$ and $S^b_{\rm zr}\eta$ are set to be η, namely

$$S^b_{\rm ex}\eta = \eta \quad \text{if} \quad \xi_x(1-\xi_y) = 0,$$
$$S^b_{\rm zr}\eta = \eta \quad \text{if} \quad \mathbf{1}(\eta_x \geq 2)\xi_y = 0.$$

Let $c_{\rm ex}$ and $c_{\rm zr}$ be two non-negative functions on $\mathbf{Z}_+$ and define for $b = (x, y)$

$$L_b = c_{\rm ex}(\eta_x)\pi_b + c_{\rm zr}(\eta_x)\nabla_b.$$

Let $\mathbf{T}^*_N$ denote the set of all oriented bonds in $\mathbf{T}_N$:

$$\mathbf{T}^*_N = \{b = (x, y) : x, y \in \mathbf{T}_N, |x - y| = 1\}.$$

Then the infinitesimal generator L_N of our Markovian particle process on $\mathbf{T}_N$ is given by

$$L_N = \sum_{b \in \mathbf{T}^*_N} L_b.$$

It is assumed that for some positive constant a_0, $c_{\rm ex}(k) \geq a_0$ for $k \geq 1$ and $c_{\rm zr}(k) \geq a_0$ for $k \geq 2$. This especially implies that the lattice gas on $\mathbf{T}_N$ with both the number of particles and the total energy being given is ergodic. We call the Markov process generated by L_N the *zero-range-exclusion* process. For the sake of convenience we set

$$c_{\rm ex}(0) = 0 \quad \text{and} \quad c_{\rm zr}(0) = c_{\rm zr}(1) = 0.$$

We need some technical conditions on the functions $c_{\rm ex}$ and $c_{\rm zr}$: there exist positive constants a_1, a_2, a_3, a_4 and an integer k_0 such that

$$|c_{\rm zr}(k) - c_{\rm zr}(k+1)| \leq a_1 \quad \text{for all } k \geq 1; \tag{1}$$

$$c_{\rm zr}(k) - c_{\rm zr}(l) \geq a_2 \quad \text{whenever } k \geq l + k_0; \tag{2}$$

$$a_3 k \leq c_{\rm ex}(k) \leq a_4 k \quad \text{for all } k \geq 1. \tag{3}$$

These conditions are imposed mainly for guaranteeing an estimate of the spectral gaps for the local processes ([4]). The conditions (1) and

(2) are the same as in the paper [2] where is carried out an estimation of the spectral gap for the zero-range processes.

We shall also write $\pi_{x,y}$, $S_{\text{ex}}^{x,y}$, $L_{x,y}$, etc. for π_b, S_{ex}^b, L_b, etc.

Grand Canonical Measures and Dirichlet Form.

For a pair of constants $0 < p < 1$ and $\rho > p$ let $\nu_{p,\rho} = \nu_{p,\rho}^{\mathbf{T}_N}$ denote the product probability measure on $\mathcal{X}^N$ whose marginal laws are given by

$$\nu_{p,\rho}(\{\eta : \eta_x = l\}) := \begin{cases} 1-p & \text{if } l = 0, \\ \dfrac{p}{Z_{\lambda(p,\rho)}} & \text{if } l = 1, \\ \dfrac{p}{Z_{\lambda(p,\rho)}} \cdot \dfrac{\big(\lambda(p,\rho)\big)^{l-1}}{c_{\text{zr}}(2)c_{\text{zr}}(3)\cdots c_{\text{zr}}(l)} & \text{if } l \geq 2, \end{cases}$$

for all x. Here $Z_\lambda := 1 + \sum_{l=2}^{\infty} \dfrac{\lambda^{l-1}}{c_{\text{zr}}(2)c_{\text{zr}}(3)\cdots c_{\text{zr}}(l)}$ and $\lambda(p,\rho)$ is a positive constant depending on p and ρ and determined uniquely by the relation $E^{\nu_{p,\rho}}[\eta_x] = \rho$, where $E^{\nu_{p,\rho}}$ denotes the expectation under the law $\nu_{p,\rho}$. Clearly $E^{\nu_{p,\rho}}[\xi_x] = p$. The lattice gas is reversible relative to the measures $\nu_{p,\rho}$ (namely L_N is symmetric relative to each of them).

It is convenient to introduce the transformations S^b, $b = (x,y)$ which acts on $\eta \in \mathcal{X}^N$ according to

$$S^b\eta = \begin{cases} S_{\text{ex}}^b\eta & \text{if } \xi_y = 0, \\ S_{\text{zr}}^b\eta & \text{if } \xi_y = 1, \end{cases}$$

and the operators

$$\Gamma_b = \xi_x\pi_b + \mathbf{1}(\eta_x \geq 2)\nabla_b \qquad\qquad (b = (x,y)).$$

The latter may also be defined by $\Gamma_b f(\eta) = f(S^b\eta) - f(\eta)$ $(f \in C(\mathcal{X}^N))$. Let $\tau_x\eta$ be the configuration $\eta \in \mathcal{X}$ viewed from x, namely $(\tau_x\eta)_y = \eta_{x+y}$. We let it also act on a function f of η according to $\tau_x f(\eta) = f(\tau_x\eta)$. Setting

$$c_{01}(\eta) = c_{\text{ex}}(\eta_0)(1-\xi_1) + c_{\text{zr}}(\eta_0)\xi_1;$$
$$c_{10}(\eta) = c_{\text{ex}}(\eta_1)(1-\xi_0) + c_{\text{zr}}(\eta_1)\xi_0;$$

and $c_{x,x+1} = \tau_x c_{01}$, $c_{x+1,x} = \tau_x c_{10}$, we can write

$$L_b = c_b\Gamma_b.$$

The Dirichlet form is then given by

$$\mathcal{D}^{p,\rho}\{f\} = \sum_{b\in\mathbf{T}_N^*} E^{\nu_{p,\rho}}[(\Gamma_b f)^2 c_b].$$

(Functions f of configuration η will be always real in this article.)

Diffusion Coefficient Matrix.

Following Varadhan [7] we define the diffusion coefficient matrix. First we introduce some notations. Let $\mathcal{X}$ denote $\mathbf{Z}_+^{\mathbf{Z}}$, the set of all configurations on $\mathbf{Z}$ and $\mathcal{F}_c$ the set of all local functions on $\mathcal{X}$ (namely, $f \in \mathcal{F}_c$ if f depends only on a finite number of coordinates of $\eta \in \mathcal{X}$). For $f \in \mathcal{F}_c$ we use the symbol $\tilde{f}$ to represent the formal sum $\sum_x \tau_x f$. It has meaning if Γ_{01} is acted:

$$\Gamma_{01}\tilde{f} = \sum_x \Gamma_{01}\tau_x f = \sum_x \tau_x \Gamma_{x,x+1} f,$$

where the infinite sums are actually finite sums. Let $\chi(p,\rho)$ denote the covariance matrix of ξ_0 and η_0 under $\nu_{p,\rho}$:

$$\chi(p,\rho) = \begin{pmatrix} (1-p)p & (1-p)\rho \\ (1-p)\rho & E^{\nu_{p,\rho}}|\eta_0-\rho|^2 \end{pmatrix}$$

For each $0 < p < 1, \rho > p$, let $\hat{c}(p,\rho) = (\hat{c}^{i,j}(p,\rho))_{1\le i,j\le 2}$ denote a 2×2 symmetric matrix whose quadratic form is defined by the following variational formula:

$$\begin{aligned} \underline{\alpha}\cdot\hat{c}(p,\rho)\underline{\alpha} &= \hat{c}^{11}(p,\rho)\alpha^2 + 2\hat{c}^{12}(p,\rho)\alpha\beta + \hat{c}^{22}(p,\rho)\beta^2 \\ &= \inf_{f\in\mathcal{F}_c} E^{\nu_{p,\rho}}\left[\left(\Gamma_{01}\{\alpha\xi_0+\beta\eta_0+\tilde{f}\}\right)^2 c_{01}\right] \end{aligned}$$

where $\underline{\alpha} = (\alpha,\beta)^T$, a two-dimensional real column vector (T indicates the transpose), and $\cdot$ indicates the inner product in $\mathbf{R}\times\mathbf{R}$. Then the diffusion coefficient matrix is defined by

$$D(p,\rho) = \hat{c}(p,\rho)\chi^{-1}(p,\rho),$$

where $\chi^{-1}(p,\rho)$ is the inverse matrix of $\chi(p,\rho)$. The two eigen-values of D are positive (cf. Section 5) and D is diagonalizable.

Let $\nabla^-\xi$ and $\nabla^-\eta$ be the particle and energy gradients:

$$\nabla^-\xi = \xi_0 - \xi_1 \quad \text{and} \quad \nabla^-\eta = \eta_0 - \eta_1$$

and w_{01}^P and w_{01}^E the particle and energy currents, respectively, from the site 0 to the site 1 :

$$w_{01}^P = -L_{\{0,1\}}\{\xi_0\} \quad \text{and} \quad w_{01}^E = -L_{\{0,1\}}\{\eta_0\}.$$

Here $L_{\{0,1\}} = L_{01} + L_{10}$. The explicit form of the currents are

$$\begin{aligned} w_{01}^P &= c_{\text{ex}}(\eta_0)(1-\xi_1) - c_{\text{ex}}(\eta_1)(1-\xi_0) \\ w_{01}^E &= c_{\text{ex}}(\eta_0)(1-\xi_1)\eta_0 + c_{\text{zr}}(\eta_0)\xi_1 - c_{\text{ex}}(\eta_1)(1-\xi_0)\eta_1 - c_{\text{zr}}(\eta_1)\xi_0. \end{aligned}$$

We can show that

$$\begin{pmatrix} w_{01}^P \\ w_{01}^E \end{pmatrix} - D(p,\rho)\begin{pmatrix} \nabla^-\xi \\ \nabla^-\eta \end{pmatrix} \in \overline{\left\{ \begin{pmatrix} Lf_1 \\ Lf_2 \end{pmatrix} : f_1, f_2 \in \mathcal{F}_c^K \text{ for some } K \in \mathbf{N} \right\}}^{p,\rho},$$

where $\overline{\{\cdots\}}^{p,\rho}$ is the closure relative to the central limit theorem variance $V^{p,\rho}$ (see Section 3). This would lead one to expect that the hydrodynamic equation for the limit densities $p = p(t,\theta)$ and $\rho = \rho(t,\theta)$ should be

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ \rho \end{pmatrix} = \frac{\partial}{\partial \theta} D(p,\rho) \frac{\partial}{\partial \theta}\begin{pmatrix} p \\ \rho \end{pmatrix}.$$

Unfortunately in deriving this equation there arises serious difficulty due to the unboundedness of the spin values. While the marginal of our grandcanonical measure is roughly Poisson, the energy current w_{01}^E involves the term $c_{\text{ex}}(\eta_0)\eta_0$ that is bounded below by $\delta\eta_0^2$ ($\delta > 0$) and cannot be controlled by the grandcanonical measure as in the case of Ginzburg-Landau model, the logarithm of the Poisson density function being of the order $O(\eta_0 \log \eta_0)$. Nagahata [3] studies a similar model and derives a system of diffusion equations of the same form as above: his model is the same as the present one except that the energy values are bounded by a constant.

In the rest of this article we shall state some results on the equilibrium fluctuations and the central limit theorem variances without proof, and give certain asymptotic estimates for the density-density correlation coefficients and for the least upper bound of the spectrum of an operator of the form $V_N + L$ as consequences of these results. In the last part of the paper some upper and lower bounds of the diffusion matrix will be given.

§2. Density-Density Correlation Function

Consider an infinite particle system on the whole lattice $\mathbf{Z}$ whose formal generator is $L = \sum c_b \Gamma_b$. It is well defined on $\mathcal{F}_c$:

$$Lf(\eta) = \sum_{b \in \mathbf{Z}^*} c_b(\eta)\Gamma_b f(\eta), \qquad f \in \mathcal{F}_c.$$

Let $\mathcal{F}_c^\circ$ be the set of all $f \in \mathcal{F}_c$ such that both f and Lf are in $L^2(\nu_{p,\rho}, \mathcal{X})$. Then the operator L with the domain $\mathcal{F}_c^\circ$ is a symmetric and non-negative transformation in $L^2(\nu_{p,\rho}, \mathcal{X})$. Clearly $\mathcal{F}_c^\circ$ is dense in $L^2(\nu_{p,\rho}, \mathcal{X})$. Hence L has the Friedrichs extension, which we denote by $\mathcal{L}$: namely $\mathcal{L}$ is the smallest self-adjoint extension of L. The following theorem is a consequence from the standard theory on the semigroup of operators. Let Λ_K be the finite interval $\{-K, \dots, K\}$ and $L_{\Lambda(K)}$ the generator of the lattice gas on Λ_K, namely

$$L_{\Lambda(K)} = \sum_{b \in \Lambda^*(K)} L_b;$$

also put $\mathcal{X}_{\Lambda(K)} = \mathbf{Z}_+^{\Lambda(K)}$. Here $\Lambda(K)$ is used in stead of Λ_K in sub- or superscripts and $\Lambda^*(K) = (\Lambda(K))^*$ (the set of all oriented bonds in Λ).

Theorem 1. *The operator $\mathcal{L}$ generates a strongly continuous Markov semigroup on $L^2(\nu_{p,\rho}, \mathcal{X})$. Denote by $S(t)$, $t \ge 0$ this semigroup, and by $S_K(t)$ the semigroup on $L^2(\mathcal{X}_{\Lambda(K)})$ generated by $L_{\Lambda(K)}$. Then*

$$\lim_{K \to \infty} S_K(t) f(\eta|_{\Lambda(K)}) = S(t) f(\eta), \qquad f \in \mathcal{F}_c^\circ,$$

strongly in $L^2(\nu_{p,\rho}, \mathcal{X})$. The convergence is locally uniform in t.

Fix $0 < p < 1$ and $\rho > p$. Let $\eta(t)$ be a Markov process on $\mathcal{X}$ whose infinitesimal generator and initial distribution are $\mathcal{L}$ and $\nu_{p,\rho}$, respectively. Denote the probability law of the process $\eta(t)$ by $P_{\text{eq}} = P_{\text{eq}(p,\rho)}$ and the expectation relative to it by $E_{\text{eq}(p,\rho)}$. Define the fluctuation processes $Y_{t,N}^P$ and $Y_{t,N}^E$ by

$$Y_{t,N}^P(J) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbf{Z}} J(x/N)(\xi_x(N^2 t) - p), \qquad J \in C_0^\infty(\mathbf{R}),$$

$$Y_{t,N}^E(J) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbf{Z}} J(x/N)(\eta_x(N^2 t) - \rho), \qquad J \in C_0^\infty(\mathbf{R})$$

respectively. ($C_0^\infty(\mathbf{R})$ is the set of smooth functions with compact supports.) Under the equilibrium measure $P_{\text{eq}(p,\rho)}$ the process $Y_{t,N} =$

$(Y^P_{t,N}, Y^E_{t,N})$ converges in the sense of finite dimensional distributions, namely for each set of $J_1, \dots, J_k \in C_0^\infty(\mathbf{R})$ and $t_1, \dots, t_k \in [0, \infty)$, the joint distribution of $Y_{t_1,N}(J_1), \dots, Y_{t_k,N}(J_k)$ converges ([6]). The limit process $Y_t = (Y_t^P, Y_t^E)$ is an infinite dimensional Ornstein-Uhlenbeck process. The distribution of Y_t is described as follows.

Let K_D denote the fundamental solution for the heat equation

$$\frac{\partial}{\partial t}\underline{u} = D^T \frac{\partial^2}{\partial \theta^2}\underline{u}$$

and U_t a matrix of corresponding convolution operators:

$$U_t\underline{J}(\theta) = \int_{-\infty}^{\infty} K_D(t, \theta - \theta')\underline{J}(\theta')d\theta',$$

where $\underline{J} = (J^1, J^2)^T \in C_0^\infty(\mathbf{R}) \times C_0^\infty(\mathbf{R})$. Let $\underline{J}_1$ and $\underline{J}_2$ be vector functions of the same kind. Then the distribution of the limit process Y_t is given by

$$E\Big[e^{i(Y_0, \underline{J}_1)}e^{i(Y_t, \underline{J}_2)}\Big] = \exp\Big[-\frac{1}{2}\int_0^t Q\{U_r\underline{J}_2\}dr - \frac{1}{2}\sigma^2\{U_t\underline{J}_2 + \underline{J}_1\}\Big];$$

in particular

$$E[(Y_0, \underline{J}_1)(Y_t, \underline{J}_2)] = \sigma^2(U_t\underline{J}_2, \underline{J}_1) = (\chi(p, \rho)U_t\underline{J}_2, \underline{J}_1)_{L^2(\mathbf{R})}. \tag{4}$$

Here E denotes the expectation by the probability law of the limit process and

$$Q\{\underline{J}\} = 2(\underline{J}', \hat{c}\underline{J}')_{L^2(\mathbf{R})}, \quad \sigma^2\{\underline{J}\} = (\underline{J}, \chi\underline{J})_{L^2(\mathbf{R})}.$$

(Also $(Y_t, \underline{J}) = Y_t^P(J_1) + Y_t^E(J_2)$, $(\underline{J}_1, \underline{J}_2)_{L^2(\mathbf{R})} = \int_{\mathbf{R}}(J_1^1 J_2^1 + J_1^2 J_2^2)d\theta$; $\hat{c} = \hat{c}(p, \rho)$ is the matrix appearing in the definition of $D = D(p, \rho)$; $\underline{J}'$ is the (component-wise) derivative of $\underline{J}$; $\sigma^2(\cdot, \cdot)$ is the bilinear form associated with the quadratic form $\sigma^2\{\cdot\}$.) The kernel K_D may be explicitly written down in the form

$$\begin{aligned} K_D(t, \theta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-t\lambda^2 D^T\}e^{-i\lambda\theta}d\lambda \\ &= \sqrt{4\pi t D^T}^{\,-1} \exp\{-\theta^2(4tD^T)^{-1}\}. \end{aligned}$$

Here D^T is the transpose of D; for a 2×2 real matrix A whose eigenvalues are positive,

$$\sqrt{A} := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\{-\theta^2 A^{-1}\}d\theta,$$

which is a real matrix having positive eigenvalues such that $A = (\sqrt{A})^2$.

Define the symmetric matrix $\Sigma(x,t)$ with parameters $(x,t) \in \mathbf{Z} \times [0,\infty)$ by

$$\underline{\alpha} \cdot \Sigma(x,t)\underline{\alpha} = E_{eq(p,\rho)}[u_{\underline{\alpha}}(0,0)u_{\underline{\alpha}}(x,t)]$$

$$\text{where} \quad u_{\underline{\alpha}}(x,t) = \alpha(\xi_x(t) - p) + \beta(\eta_x(t) - \rho).$$

Since $P_{\mathrm{eq}(p,\rho)}$ is invariant under the translation, $\Sigma(x,t)$ is the covariance matrix of $(\xi_x(s), \eta_x(s))$ and its space-time translation $(\xi_{x+y}(s+t), \eta_{x+y}(s+t))$. Hence if we define

$$R(x,t) := \Sigma(x,t)\chi^{-1}(p,\rho),$$

then $R(x-y, t-s)$ is the space-time correlation coefficient of $(\xi_x(t), \eta_x(t))$. The next theorem states that $R(x,t)$ behaves like $R(x,t) \approx K_D(t,x)$ as $x, t \to \infty$, as being expected ([5]).

Theorem 2. *For* $\underline{J} = (J^1, J^2)^T \in C_0^\infty(\mathbf{R}) \times C_0^\infty(\mathbf{R})$

$$\lim_{N\to\infty} \sum_{x\in\mathbf{Z}} \mathbf{R}(x, N^2t)\underline{J}(x/N) = \int_{-\infty}^{\infty} K_D(t,\theta)\underline{J}(\theta)d\theta.$$

Theorem 2 is deduced from (4). Indeed by (4),

$$\begin{aligned} \text{(5)} \qquad & \lim_{N\to\infty} \frac{1}{N} \sum_x \sum_y \underline{J}_1(y/N) \cdot R(x-y, N^2t)\underline{J}_2(x/N) \\ & = \int_{-\infty}^{\infty} \underline{J}_1(\theta) \cdot U_t\underline{J}_2(\theta)d\theta \end{aligned}$$

because the formula under the limit on the left side equals $E[(Y_{0,N}, \underline{J}_1)(Y_{t,N}, \underline{J}_2)]$. If the delta function could be taken for $\underline{J}_1$, the relation of Theorem 2 would come out. For justification we take Fourier transform in (5). To this end let $\hat{R}$ be the Fourier series with coefficients R:

$$\begin{aligned} \hat{R}(\lambda,t) &= \hat{\Sigma}(\lambda,t)\chi^{-1}, \qquad \lambda \in \mathbf{R} \\ \hat{\Sigma}(\lambda,t) &= \sum_{x\in\mathbf{Z}} e^{i\lambda x}\Sigma(x,t). \end{aligned}$$

Lemma 3.

$$0 \le \hat{\Sigma}(\lambda,t) \le \hat{\Sigma}(\lambda,0) = \chi.$$

Proof. If $a_x = e^{i\lambda x}\Sigma(x,t)$, then

$$\sum_{x=-k}^{k-1}\sum_{y=-k}^{k-1} a_{y-x} = \sum_{u=-2k}^{2k}(2k-|u|)a_u.$$

The right-hand side divided by $2k$ converges, as $k\to\infty$, to $\hat\Sigma(\lambda,t)$. Since $S(t)$ is a symmetric operator, the first diagonal component of a_{y-x} may be expressed in the form

$$a_{y-x}^{11} = E^{\nu_{p,\rho}}\left[e^{i\lambda y}S(t/2)\{\xi_y - p\}e^{-i\lambda x}S(t/2)\{\xi_x - p\}\right],$$

and similarly for the other components; hence

$$\underline{\alpha}\cdot\hat\Sigma(\lambda,t)\underline{\alpha} = \lim_{k\to\infty}\frac{1}{2k}E^{\nu_{p,\rho}}\left|S(t/2)\left\{\sum_{x=-k}^{k-1}e^{i\lambda x}[\alpha(\xi_x-p)+\beta(\eta_x-\rho)]\right\}\right|^2.$$

The inequalities of the lemma now follow from the fact that $S(t)$ is contraction in $L^2(\nu_{p,\rho})$. Q.E.D.

Proof of Theorem 2. Rewriting the relation (5) by means of $\hat R$, we have

$$\begin{aligned}&\lim_{N\to\infty}\int_{-N\pi}^{N\pi}\hat{\underline{J}}_1^N(\lambda)\cdot\hat R(\lambda/N,N^2t)\hat{\underline{J}}_2^N(-\lambda)d\lambda\\ (6)\qquad &=\int_{-\infty}^{\infty}\hat{\underline{J}}_1(\lambda)\cdot e^{-t\lambda^2D^T}\hat{\underline{J}}_2(-\lambda)d\lambda.\end{aligned}$$

Here

$$\hat{\underline{J}}^N(\lambda) = \frac{1}{N}\sum \underline{J}(x/N)e^{i\lambda x/N},\quad \hat{\underline{J}}(\lambda) = \int_{-\infty}^{\infty}\underline{J}(\theta)e^{i\lambda\theta}d\theta.$$

By the Poisson summation formula, $\hat{\underline{J}}^N(\lambda) = \sum_{x\in\mathbf{Z}}\hat{\underline{J}}(\lambda+2\pi Nx)$. The class of J_1^i $(i=1,2)$ in (6) may be extended to the set of rapidly decreasing functions. Let $\delta>0$, $g_\delta(\theta) = (4\pi\delta)^{-1/2}e^{-\theta^2/(4\delta)}$ and $\underline{J}_1(\theta) = g_\delta(\theta)\underline{\alpha}$. Then, $\hat g_\delta(\lambda) = e^{-\delta\lambda^2}$ and

$$e^{-\delta\lambda^2}\le \hat g_\delta^N(\lambda)\le e^{-\delta\lambda^2}+\frac{2e^{-\delta(\pi N)^2}}{1-e^{-\delta(\pi N)^2}}\quad(|\lambda|\le N\pi);$$

and writing $\underline{J}$ for $\underline{J}_2$ in (6), we infer that

$$\begin{aligned}&\lim_{N\to\infty}\int_{-N\pi}^{N\pi}e^{-\delta\lambda^2}\underline{\alpha}\cdot\hat R(\lambda/N,N^2t)\hat{\underline{J}}^N(-\lambda)d\lambda\\ &=\int_{-\infty}^{\infty}e^{-\delta\lambda^2}\underline{\alpha}\cdot e^{-t\lambda^2D^T}\hat{\underline{J}}(-\lambda)d\lambda.\end{aligned}$$

On taking the limit as $\delta \downarrow 0$ this relation is also valid for $\delta = 0$. The proof is complete. Q.E.D.

§3. Central Limit Theorem Variance

The canonical measure for the configurations on Λ_n with the number of particles m and the total energy E is the conditional law

$$P_{n,m,E}[\,\cdot\,] = \frac{\nu_{p,\rho}(\,\cdot\, \cap \{|\xi|_{\Lambda(n)} = m,\ |\eta|_{\Lambda(n)} = E\} \,|\, \mathcal{F}_{\mathbf{Z}\setminus\Lambda(n)}\,)}{\nu_{p,\rho}(|\xi|_{\Lambda(n)} = m,\ |\eta|_{\Lambda(n)} = E\,)}.$$

Here for $\Lambda \subset \mathbf{Z}$, $|\xi|_\Lambda = \sum_{x\in\Lambda} \xi_x$ and $|\eta|_\Lambda = \sum_{x\in\Lambda} \eta_x$; $\mathcal{F}_\Lambda$ stands for the σ-field in $\mathcal{X}$ generated by $\eta_y, y \in \Lambda$. From the reversibility relation it follows that for any functions f and g of η and any bond $b \in \Lambda_n^*$,

$$E_{n,m,E}[c_b(\eta)f(S^b\eta)g(\eta)] = E_{n,m,E}[c_{b'}(\eta)f(\eta)g(S^{b'}\eta)],$$

where b' is the bond obtained from b by reversing its direction. The Dirichlet form for $L_{\Lambda(n)}$ accordingly is given by

$$\begin{aligned} \mathcal{D}_{n,m,E}\{f\} &:= -E_{n,m,E}[fL_{\Lambda(n)}f] \\ &= \sum_{b\in\Lambda^*(n)} \mathcal{D}^b_{n,m,E}\{f\} \end{aligned}$$

where $\mathcal{D}^b_{n,m,E}\{f\} = \frac{1}{2}E_{n,m,E}[(\Gamma_b f)^2 c_b],$; the corresponding bilinear form is given by

$$\mathcal{D}^{01}_{n,m,E}(f,g) = -\frac{1}{2}E_{n,m,E}[f\cdot(L_{01}+L_{10})g] = \frac{1}{2}E_{n,m,E}[(\Gamma_{01}f)(\Gamma_{01}g)c_{01}].$$

We introduce a function space on which the central limit theorem variance is well defined. The numbers p and ρ are fixed so that $0 < p < 1$ and $\rho \geq p$ unless otherwise specified. They will be dropped from the notations if used as sub- or superscripts.

Definition 4. *Let $\mathcal{G}$ denote the linear space of all functions $h \in \mathcal{F}_c$ of the form*

$$L_I H := \sum_{b\in I^*} L_b H = h, \tag{7}$$

where I is an interval of $\mathbf{Z}$ and H is a local function such that for some positive integer K,

$$\sum_{b\in I^*} (\Gamma_b H(\eta))^2 \leq K \sum_{x\in I} (\eta_x)^K, \quad \eta \in \mathcal{X}. \tag{8}$$

(This bound, which may be replaced by a weaker one, is adopted only for convenience sake. We may take I as the minimal of intervals Λ such that $h \in \mathcal{F}_\Lambda$.)

If $h \in \mathcal{F}_c$ satisfies

$$E^\nu[h \mid \mathcal{F}_{\mathbf{Z}\setminus I} \vee \sigma\{|\xi|_I, |\eta|_I\}] = 0 \text{ a.s.},$$

then it admits a representation (7) but the condition (8) may fail to hold. The functions w_{01}^P, w_{01}^E are in $\mathcal{G}$: the requirements are satisfied with $I = \{0,1\}$ and $H = -\xi_0$ and $H = -\eta_0$, respectively. For each positive integer K put

$$\mathcal{F}_c^K = \{f \in \mathcal{F}_c : |f(\eta)| \leq K \sum_{|x| \leq K} (\eta_x)^K\}$$

Then the linear space $L\mathcal{F}_c^K$ is obviously included in $\mathcal{G}$.

Let $L_{n,m,E}$ denote the restriction of $L_{\Lambda(n)}$ to the space of functions on $\mathcal{X}_{n,m,E} := \{\eta \in \mathcal{X}_{\Lambda(n)} : |\xi|_{\Lambda(n)} = m, |\eta|_{\Lambda(n)} = E\}$, and for $h,\ g \in \mathcal{G}$, define

$$V_{n,m,E}(h,g) = \frac{1}{2n} E_{n,m,E} \left[\sum_{|x|<n'} \tau_x h \cdot (-L_{n,m,E})^{-1} \sum_{|x|<n'} \tau_x g \right],$$

where n' is the maximal integer among those for which both sums in the brackets are $\mathcal{F}_{\Lambda(n)}$-measurable.

Theorem 5. *For every $h, g \in \mathcal{G}$ and for every $p > 0, \rho \geq p$, there exists a following limit*

$$\lim_{m/2n \to p, E/2n \to \rho} V_{n,m,E}(h,g),$$

where the limit is taken in such a way that n, m and E are sent to infinity so that $m/2n \to p$ and $E/2n \to \rho$. The functional defined by this limit makes a bilinear form on $\mathcal{G}$. If it is denoted by

$$V(h,g) = V^{p,\rho}(h,g),$$

then the subspace

$$\mathcal{G}_\circ := \{\alpha w_{01}^P + \beta w_{01}^E - Lf : \alpha, \beta \in \mathbf{R}, f \in \mathcal{F}_c^K \text{ for some } K\}$$

is dense in $\mathcal{G}$ with respect to the quadratic form $V^{p,\rho}\{h\} := V^{p,\rho}(h,h)$.

Theorem 5 says that every $h \in \mathcal{G}$ can be approximated by an element of $\mathcal{G}_\circ$ in the metric $\sqrt{V^{p,\rho}}$ as accurately as one needs. To apply this to the gradients $\nabla^- \xi := \xi_0 - \xi_1$ and $\nabla^- \eta := \eta_0 - \eta_1$, we need the following lemma (cf. [6]).

Lemma 6. *Suppose that (1) and (2) are satisfied. Then both* $\nabla^-\xi$ *and* $\nabla^-\eta$ *are in* $\mathcal{G}$. *Let* H^P *and* H^E *stand for the corresponding* H*'s (with* $I(h) = \{0,1\}$ *). Then*

$$\Gamma_{01} H^P = \xi_0 / c_{\mathrm{ex}}(\eta_0) \quad \textit{and} \quad \Gamma_{01} H^E = \eta_0 / c_{\mathrm{ex}}(\eta_0) \quad \textit{if} \quad \xi_0(1-\xi_1) = 1$$

and $\Gamma_{01} H^P = 0$ *if* $\xi_0(1-\xi_1) = 0$*; moreover there exists a constant* $\delta > 0$ *such that* $\delta \le \Gamma_{01} H^E \le 1/\delta$ *whenever* $\mathbf{1}(\eta_0 \ge 2)\xi_1 = 1$.

The proof of Theorem 5 may be carried out along the same lines as in [7] or [8].

§4. The Least Upper Bound of Spectrum

In this section we are concerned with the Markov process whose infinitesimal generator is $\mathcal{L}$, a self-adjoint operator on $L^2(\nu_{p,\rho})$ (see Theorem 1). Let $\mathcal{P}(\mathcal{X})$ be the set of all probability measures on $\mathcal{X}$. Define a functional $\mathcal{I}(\mu)$ of $\mu \in \mathcal{P}(\mathcal{X})$ by

$$\mathcal{I}(\mu) = E^\nu[\varphi(-\mathcal{L})\varphi], \text{ where } \varphi = \sqrt{d\mu/d\nu}$$

if μ is absolutely continuous relative to $\nu = \nu_{p,\rho}$ and φ is in the domain of $\sqrt{-\mathcal{L}}$; and $\mathcal{I}(\mu) = \infty$ otherwise. For a local function G on $\mathcal{X}$ let $\Omega_\circ\{G + \mathcal{L}\}$ denote the least upper bound of the spectrum of the operator $G + \mathcal{L}$. It has the variational representation

$$\Omega_\circ\{G + \mathcal{L}\} = \sup_{\mu \in \mathcal{P}(\mathcal{X})} \Big(E^\mu[G] - \mathcal{I}(\mu) \Big).$$

Given a positive integer n and $h \in \mathcal{G}$, let n' be the maximal integer such that $\tau_y h \in \mathcal{F}_{\Lambda(n)}$ if $|y| < n'$, and define a function $G_n = G_n^h$ by

$$G_n = \frac{1}{2n} \sum_{y:|y|<n'} \tau_y h.$$

Theorem 7. *Let* $h \in \mathcal{G}$. *Let the interval* $I = I(h)$ *and the function* H *be chosen so that*

$$\sum_{b \in I^*} (\Gamma_b H)^2 c_b \le A \sum_{x \in I} \eta_x^K \tag{9}$$

where $\eta_x^K = (\eta_x)^K$, and A and K are positive constants with $K \geq 1$. Let $G_n = G_n^h$ be defined as above. Also define a function $\zeta_n^l(\eta)$ for $l \geq 1$ by

$$\zeta_n^l(\eta) = \frac{1}{2n} \sum_{x:|x|\leq n} \eta_x^K \mathbf{1}(\eta_x > l).$$

Then, if $\lambda \in (-1,1)$, $J \in C_0^2(\mathbf{R})$, and C is a positive constant such that $A|I|^2(1-2^{-K})^{-1} \leq C$, it holds that for all $n, l \in \mathbf{N}$,

$$\begin{aligned} &\varlimsup_{N\to\infty} \Omega_\circ \left\{ \sum_{x\in\mathbf{Z}} \left[N^\lambda J(x/N)\tau_x G_n - \frac{C}{N} J^2(x/N)\tau_x \zeta_n^l \right] + N^{1+2\lambda}\mathcal{L} \right\} \\ &\leq ||J||_{L^2}^2 \sup_{m,E:E/m\leq 2l} V_{n,m,E}\{h\}. \end{aligned}$$

where $||J||_{L^2}^2 = \int_{\mathbf{R}} J^2 d\theta$ and the supremum is taken over all couples of positive integers m and E such that $m \leq E \leq 2lm$.

Proof. The proof is divided into three steps.

Step 1. This step is quite similar to a corresponding argument in [7], so we provide only an outline. The supremum of the spectrum $\Omega_\circ$ that is to be estimated may be given by the variational formula

$$\Omega^N = \sup_{\mu\in\mathcal{P}(\mathcal{X})} E^\mu \left[\sum_{x\in\mathbf{Z}} \left[N^\lambda j_x \tau_x G_n - \frac{C}{N} j_x^2 \tau_x \zeta_n^l \right] - N^{1+2\lambda}\mathcal{I}(\mu) \right].$$

where we put $j_x = J(x/N)$.

Let $\varphi = \sqrt{d\mu/d\nu}$ and $\mathcal{D}^\Lambda = \sum_{b\in\Lambda^*} \mathcal{D}^b$, then $\mathcal{I}(\mu) = \sum_{b\in\mathbf{Z}^*} \mathcal{D}^b\{\varphi\} = \frac{1}{2n}\sum_{x\in\mathbf{Z}} \mathcal{D}^{\Lambda(n)}\{\tau_x\varphi\}$. We substitute this into the variational expression given above. To compute the expectation appearing in it we first take the conditional expectation conditioned on $\omega = \eta|_{\Lambda_n^c}$. If $\mu(\,\cdot\,|\omega)$ stands for this conditional law, then $E^\mu[G_n]$ is expressed as an integral of $F(\omega) = E^{\mu(\,\cdot\,|\omega)}[G_n]$ by μ. We have a similar expression for the form $\mathcal{D}^{\Lambda(n)}\{\varphi\}$, which may be naturally restricted to the space $L^2(\nu^{\Lambda(n)}, \mathcal{X}_{\Lambda(n)})$ (ν^Λ is the product measure on $\mathcal{X}_\Lambda$ with the same common one-site marginal as that of $\nu = \nu_{p,\rho}$). Rewriting μ for $\mu(\,\cdot\,|\omega) \in \mathcal{P}(\mathcal{X}_{\Lambda(n)})$ and taking the supremum in μ, we see that Ω^N is not greater than

$$\frac{N^{1+2\lambda}}{2n} \sum_{x\in\mathbf{Z}} \sup_{\mu\in\mathcal{P}(\mathcal{X}_{\Lambda(n)})} \left\{ \frac{2n}{N^{1+2\lambda}} E^\mu \left[N^\lambda j_x G_n - \frac{C}{N} j_x^2 \zeta_n^l \right] - \mathcal{D}^{\Lambda(n)}\{\varphi\} \right\}.$$

Decomposing $\mathcal{X}_{\Lambda(n)}$ into the ergodic classes $\mathcal{X}_{n,m,E}$ we may express $\mathcal{D}^{\Lambda(n)}\{\varphi\}$ in the form $\mathcal{D}^{\Lambda(n)}\{\varphi\} = \sum_m \sum_E p_{m,E} \mathcal{D}_{n,m,E}\{\varphi_{m,E}\}$, where

$p_{m,E} = \mu(\mathcal{X}_{n,m,E})$ and $\varphi_{m,E}$ is the square root of a probability density on $\mathcal{X}_{n,m,E}$. As a consequence we see that if

$$\Omega^N_{n,m,E,x} = \sup_{\mu\in\mathcal{P}(\mathcal{X}_{n,m,E})} \left\{ \frac{2nj_x}{N^{1+\lambda}} E^\mu[G_n] - \frac{2nCj_x^2}{N^{2+2\lambda}} E^\mu[\zeta_n^l] - \mathcal{D}_{n,m,E}\{\varphi\} \right\},$$

then

$$\Omega^N \le \frac{N^{1+2\lambda}}{2n} \sum_{x=1}^{N} \sup_{m,E} \Omega^N_{n,m,E,x}. \tag{10}$$

Step 2. Let $\langle \cdot \rangle_{n,m,E}$ stand for the expectation by $P_{n,m,E}$. For H introduced in Definition 4 and for any $\mathcal{F}_{\Lambda(n)}$-measurable function u, we have the following identity

$$\langle u\tau_x h\rangle_{n,m,E} = -\frac{1}{2} \sum_{b\in I^*(h)} \Big\langle \Gamma_{b+x} u \cdot \tau_x(c_b\Gamma_b H)\Big\rangle_{n,m,E} \tag{11}$$

or in terms of the Dirichlet form

$$\langle u\tau_x h\rangle_{n,m,E} = -\sum_{b\in I^*(h)} \mathcal{D}^{b+x}_{n,m,E}(u, \tau_x H). \tag{12}$$

(Here $b+x$ is the oriented bond obtained by translating b by x.) From this it follows that

$$E^\mu[G_n] = -\frac{1}{2n} \sum_{|x|<n'} \sum_{b\in I^*(h)} \mathcal{D}^{b+x}_{n,m,E}(\tau_x H, \varphi^2).$$

A simple computation verifies that the terms $|\mathcal{D}^b_{n,m,E}(F,\varphi^2)|$, where $F \in C(\mathcal{X}_{n,m,E})$, are bounded by

$$\sqrt{\tfrac{1}{2}\Big\langle \Big[(\Gamma_b F)^2 c_b + (\Gamma_{b'} F)^2 c_{b'}\Big]\varphi^2 \Big\rangle_{n,m,E}} \sqrt{\mathcal{D}^b_{n,m,E}\{\varphi\}}.$$

where b' is the bond b but reversely oriented. By employing Schwarz inequality and the assumption (9) on H it therefore follows that $|E^\mu[G_n]|$ is at most

$$\begin{aligned} &\frac{1}{2n}\sqrt{\sum_{|x|<n'} \sum_{b\in I^*(h)} \Big\langle (\Gamma_{b+x}\tau_x H)^2 c_{b+x}\varphi^2 \Big\rangle_{n,m,E}} \sqrt{|I^*|\mathcal{D}_{n,m,E}\{\varphi\}} \\ &\le \frac{|I|}{n}\sqrt{A\sum_{|x|\le n} \Big\langle \eta_x^K \varphi^2 \Big\rangle_{n,m,E}} \sqrt{\mathcal{D}_{n,m,E}\{\varphi\}}. \end{aligned}$$

By the inequality $2ab - a^2 \le b^2$ this shows that

$$(13) \qquad \frac{2nj_x}{N^{1+\lambda}} E^{\mu}[G_n] - \mathcal{D}_{n,m,E}\{\varphi\} \le \frac{A|I|^2 j_x^2}{N^{2+2\lambda}} \sum_{|x|\le n} \left\langle \eta_x^K \varphi^2 \right\rangle_{n,m,E}.$$

Since $(m^{-1}\sum \eta_x)^K \le m^{-1}\sum \eta_x^K$, the condition $E = \sum \eta_x > 2lm$ implies the inequality $2^{-K}\sum \eta_x^K \ge l^K m$, which in turn implies that

$$2n\zeta_n^l = \sum \eta_x^K \mathbf{1}(\eta_x > l) \ge \sum \eta_x^K - l^K m \ge (1 - 2^{-K}) \sum \eta_x^K.$$

This combined with (13) shows that if the constant C is chosen so that $A|I|^2 \le (1-2^{-K})C$, then

$$\Omega^N_{n,m,E,x} \le 0 \quad \text{whenever } E/m > 2l,$$

and accordingly that the supremum over the pairs of m and E in (10) may be restricted to those satisfying $E/m \le 2l$. Consequently

$$(14) \qquad \Omega^N \le \frac{N^{1+2\lambda}}{2n} \sum_{x\in\mathbf{Z}} \sup_{m,E:E/m\le 2l} \Omega^N_{n,m,E,x}.$$

Step 3. Now we apply the following estimate for the spectrum of the Schrödinger type operator $L_{n,m,E} + F$ with $F \in C(\mathcal{X}_{n,m,E})$ satisfying $\langle F \rangle_{n,m,E} = 0$:

$$(15) \qquad \Omega_\circ\{F + L_{n,m,E}\} \le \langle F(-L_{n,m,E})^{-1} F \rangle_{n,m,E} + \frac{4}{\kappa_n^2} \|F\|_\infty^3,$$

where $\kappa_n = \kappa_{n,m,E}$ is the second eigenvalue of $-L_{n,m,E}$ (cf. [7],[1] etc.). Taking $F = (2nj_x/N^{1+\lambda})G_{n,m,E}$ in (15), where $G_{n,m,E} = G_n|_{\mathcal{X}_{n,m,E}}$,

$$\begin{aligned} \Omega^N_{n,m,E,x} &\le \Omega_\circ\{(2nj_x/N^{1+\lambda})G_{n,m,E} + L_{n,m,E}\} \\ &\le (2n)V_{n,m,E}\left\{\frac{j_x}{N^{1+\lambda}} h\right\} + \frac{4}{\kappa_n^2} \cdot \left[\frac{2nj_x \|G_{n,m,E}\|_\infty}{N^{1+\lambda}}\right]^3 \\ &= \frac{2nj_x^2}{N^{2+2\lambda}} V_{n,m,E}\{h\} + O\left(\frac{1}{N^{3+3\lambda}}\right). \end{aligned}$$

From (14) we thus obtain $\overline{\lim}_{N\to\infty} \Omega^N \le \|J\|_{L^2}^2 \sup_{m,E:E/m\le 2l} V_{n,m,E}\{h\}$, the required bound. Q.E.D.

The next theorem is essentially a corollary of Theorem 7.

Theorem 8. *Let $h \in \mathcal{G}$ and put*

$$F^N(\eta) = \sqrt{N} \sum_{x \in \mathbf{Z}} J(x/N) \tau_x h(\eta).$$

Then there exists a constant C such that for all positive constants β and l,

$$\overline{\lim_{N\to\infty}} E_{\text{eq}} \left| \int_0^T F^N(\eta(N^2 t)) dt \right| \leq \beta T ||J||_{L^2}^2 \sup_{p_o, \rho_o : \rho_o/p_o \leq l} V^{p_o, \rho_o}\{h\} + (\log 2)/\beta + (C\beta)/l.$$

Proof. We may replace F^N by

$$F_n^N := \sqrt{N} \sum_{x \in \mathbf{Z}} J(x/N) \frac{1}{2n} \sum_{y : |y-x| < n'} \tau_y h.$$

In fact if

$$a_{N,n}^x = \frac{N}{2n^2} \sum_{y:\, |y-x|<n'} [J(x/N) - J(y/N)],$$

then $|a_{N,n}^x| \leq \int_{-n/N}^{n/N} |J''(s + N^{-1}x)| ds$ and the difference

$$F^N - F_n^N = \frac{n}{\sqrt{N}} \sum_{x \in \mathbf{Z}}^{N} a_{N,n}^x \tau_x h$$

is obviously negligible under the equilibrium measure.

Introducing the random variable $X^N = \int_0^T F_n^N(\eta(N^2 t)) dt$, we may write $E_{\text{eq}}|X^N|$ for what to estimate. Let $K \geq 1$ be a constant for which the condition (9) is satisfied. Let ζ_n^l be a function defined in Theorem 7 and put

$$Y^N = \int_0^T \frac{C}{N} \sum_{x \in \mathbf{Z}} J^2(x/N) \tau_x \zeta_n^l(\eta(N^2 t)) dt.$$

Then by Jensen's inequality and the Feynman-Kac formula

$$\begin{aligned} &E_{\text{eq}}[|X^N| - \beta Y^N] \\ &\leq \frac{1}{\beta} \log \max_{+,-} E_{\text{eq}}[e^{\pm\beta X^N - \beta^2 Y^N}] + \frac{\log 2}{\beta} \\ &\leq \frac{T}{\beta} \max_{+,-} \Omega_o \Big\{ \pm \beta F^N - \frac{C}{N} \sum_{x \in \mathbf{Z}} |\beta J(x/N)|^2 \tau_x \zeta_n^l + N^2 L \Big\} + \frac{\log 2}{\beta}. \end{aligned}$$

According to Theorems 7 and 5, if C is chosen suitably large, then

$$\varlimsup_{N\to\infty} E_{\rm eq}[\ |X^N| - \beta Y^N\ ] \le \beta T||J||^2_{L^2} \sup_{p_o,\rho_o : \rho_o/p_o \le l} V^{p_o,\rho_o}\{h\} + \frac{\log 2}{\beta}.$$

This gives the required inequality since $E_{\rm eq}[\beta Y^N] \le C_1\beta/l$. Q.E.D.

§5. Upper and Lower Bounds For $D(p,\rho)$

Let $\underline{\kappa} = \underline{\kappa}(p,\rho)$ and $\bar{\kappa} = \bar{\kappa}(p,\rho)$ stand for the eigen-values of $D(p,\rho)$ such that $\underline{\kappa} \le \bar{\kappa}$. We here prove that for some positive constants m and M,

$$\frac{m}{p + (1+\lambda)^{-1}} \le \underline{\kappa} \le \bar{\kappa} \le M(1+\lambda) \qquad (\rho \ge p > 0),$$

where $\lambda = \lambda(p,\rho)$ is the parameter appearing in the definition of $\nu_{p,\rho}$.

Proof of the upper bound. We shall apply the fact that if $\hat{c}_o$ is a symmetric 2×2 matrix and $\hat{c}_o \ge \hat{c}$, then $\mathrm{Tr}(\hat{c}_o\chi^{-1}) \ge \mathrm{Tr}(\hat{c}\chi^{-1})$. Let $\langle\cdot\rangle$ indicate the expectation under $\nu_{p,\rho}$. Then

$$\begin{aligned}\underline{\alpha}\cdot\hat{c}(p,\rho)\underline{\alpha} &\le \Big\langle\Big(\Gamma_{01}\{\alpha\xi_0+\beta\eta_0\}\Big)^2 c_{01}\Big\rangle \\ &= \Big\langle\{\alpha\xi_0+\beta\eta_0\}^2(1-\xi_1)c_{\rm ex}(\eta_0)\Big\rangle + \beta^2\langle\xi_0\xi_1c_{\rm zr}(\eta_0)\rangle\end{aligned}$$

In view of the conditions (2) and (3), $c_{\rm ex}(\eta_0) \le C[c_{\rm zr}(\eta_0)+\mathbf{1}(\eta_0=1)]$. By combining this with the relations $\langle c_{\rm zr}(\eta_0)\rangle = p\lambda$, $\langle\eta_0 c_{\rm zr}(\eta_0)\rangle = (\rho+p)\lambda$ and $\langle\eta_0^2 c_{\rm zr}(\eta_0)\rangle = (\langle\eta_0^2\rangle + 2\rho + p)\lambda$, the last line above is dominated by $\beta^2p^2\lambda$ plus a constant multiple of

$$(1-p)[\alpha^2p\lambda + 2\alpha\beta(\rho+p)\lambda + \beta^2(\langle\eta_0^2\rangle + 2\rho + p)\lambda + (\alpha+\beta)^2\langle\mathbf{1}(\eta_0=1)\rangle].$$

Recalling what is remarked at the beginning of this proof, noticing $\det\chi = (p\langle\eta_0^2\rangle - \rho^2)(1-p)$ so that

$$\chi^{-1}(p,\rho) = \frac{1}{(p\langle\eta_0^2\rangle - \rho^2)(1-p)}\begin{pmatrix}\langle\eta_0^2\rangle - \rho^2 & -(1-p)\rho \\ -(1-p)\rho & (1-p)p\end{pmatrix}$$

and carrying out simple computations, we see that

$$\mathrm{Tr}(\hat{c}\chi^{-1}) \le C_1[\lambda + p^2(\lambda^2)(p\langle\eta_0^2\rangle - \rho^2)^{-1} + \lambda].$$

Since $\bar{\kappa} + \underline{\kappa} = \mathrm{Tr}(\hat{c}\chi^{-1})$, these yield the required upper bound, if we can find a positive constant δ so that

$$p\langle\eta_0^2\rangle - \rho^2 \ge \delta p^2\lambda. \tag{16}$$

(This is certainly true for $\lambda \le 1$.) To this end set $\ell = \ell(\lambda) = \max\{k : c_{\rm zr}(k) \le \lambda\}$ and $p_k = \nu_{p,\rho}\{\eta : \eta_0 = k\}/p$. Noticing that $p_{k+1}/p_k = \lambda/c_{\rm zr}(k+1)$, we infer from $|c_{\rm zr}(k) - c_{\rm zr}(\ell)| \le a_1|k-\ell|$ that for all sufficiently large λ,

$$p_k \ge p_\ell \exp\{-a_1(k-\ell)^2/\lambda\} \quad \text{if} \quad |k-\ell| \le 2\sqrt{\lambda},$$

or, what we are about to apply, $\min\{\sum_{k<\ell-\sqrt{\lambda}} p_k, \sum_{k>\ell+\sqrt{\lambda}} p_k\} \ge \delta$ with some constant $\delta > 0$ independent of λ. Hence

$$\begin{aligned}\langle \eta_0^2\rangle/p - (\rho/p)^2 &= E^{\nu_{p,\rho}}[|\eta_0 - \rho/p|^2 \mid \eta_0 > 0] \\ &\ge \lambda P^{\nu_{p,\rho}}[|\eta_0 - \rho/p| \ge \sqrt{\lambda} \mid \eta_0 > 0] \ge \delta\lambda.\end{aligned}$$

Thus we have shown (16).

Proof of the lower bound. Let $A = A(p,\rho)$ be a 2×2 symmetric matrix whose quadratic form is

$$\underline{\alpha}\cdot A\underline{\alpha} = V\{\alpha\nabla^-\xi + \beta\nabla^-\eta\}.$$

Then $D(p,\rho) = \chi(p,\rho)A^{-1}(p,\rho)$ and it holds that $V\{\alpha\nabla^-\xi + \beta\nabla^-\eta\} \le \langle(\Gamma_{01}\{\alpha H^P + \beta H^E\})^2 c_{01}\rangle$ (cf. [6]), where H^P and H^E are functions introduced in Lemma 6. We shall apply the inequality

$$\underline{\kappa} \ge \frac{\det(\chi A^{-1})}{\mathrm{Tr}(\chi A^{-1})} = \frac{1}{\mathrm{Tr}(\chi^{-1}A)}. \tag{17}$$

By employing Lemma 6 as well as the conditions (1) through (3) we see that for some constant C,

$$\begin{aligned}\underline{\alpha}\cdot A\underline{\alpha} &\le \langle(\Gamma_{01}\{\alpha H^P + \beta H^E\})^2 c_{01}\rangle \\ &\le C\left\langle \frac{\xi_0(1-\xi_1)}{c_{\rm zr}(\eta_0+1)}(\alpha\xi_0 + \beta\eta_0)^2\right\rangle + C\beta^2\langle \xi_1 c_{\rm zr}(\eta_0)\rangle.\end{aligned}$$

One observes that the right-hand side equals C times

$$\alpha^2(1-p)\frac{p}{\lambda}\left(1 - \frac{1}{Z_\lambda}\right) + 2\alpha\beta(1-p)\frac{\rho - p}{\lambda}$$
$$+\beta^2\left(\frac{1-p}{\lambda}\langle(\eta_0 - \xi_0)^2\rangle + p^2\lambda\right).$$

Noticing that $Z_\lambda = 1 + \lambda/c_{\rm zr}(2) + O(\lambda^2)$ as $\lambda \downarrow 0$ and $\nu_{p,\rho}\{\eta_0 = 2\} = p\lambda/c_{\rm zr}(2)Z_\lambda$, and applying the inequality used in the preceding proof, we infer that

$$\det(\chi)\mathrm{Tr}(\chi^{-1}A) \le C'p^2(1-p)\lambda \quad \text{for} \quad 0 < \lambda < 1. \tag{18}$$

For large values of λ we make an elementary computation (as we did for the upper bound) to see that $\det(\chi)\mathrm{Tr}(\chi^{-1}A)$ is at most C times

$$\frac{1-p}{\lambda}(2-p)(p\langle\eta_0^2\rangle-\rho^2)+\frac{(1-p)^2p^2}{\lambda}-\frac{(1-p)p}{\lambda Z_\lambda}(\langle\eta_0^2\rangle-\rho^2)+(1-p)p^3\lambda.$$

Hence, in view of (16),

$$\mathrm{Tr}(\chi^{-1}A)\le C'\left[\frac{1}{\lambda}+p\right]\ (\lambda\ge 1).$$

This together with (17) and (18) concludes the asserted lower bound of $\underline{\kappa}$.

References

[1] C. Kipnis and C. Landim, Scaling limits of particle systems, Springer, 1999.

[2] C. Landim, S. Sethuraman and S. Varadhan, Spectral Gap for Zero-Range Dynamics, Ann. Probab. 24 (1996), pp. 1871-1902.

[3] Y. Nagahata, Fluctuation dissipation equation for lattice gas with energy, to appear in Jour. Stat. Phys., Vol. 110 Nos.1/2 (2003) 219-246.

[4] Y. Nagahata and K. Uchiyama: Spectral gap for zerorange-exclusion dynamics, preprint

[5] H. Spohn, Large scale dynamics of interacting particles, Text and Monographs in Physics, Springer, 1991.

[6] K. Uchiyama, Equilibrium fluctuations for zero-range-exclusion processes, preprint

[7] S.R.S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions - II, Asymptotic problems in probability theory: stochastic models and diffusions on fractals (eds. K.D. Elworthy and N. Ikeda), Longman (1993), pp. 75-128.

[8] S.R.S. Varadhan and H.T. Yau., Diffusive limit of lattice gases with mixing condition, Asian J.Math vol. 1 (1997), 623-678.

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro
Tokyo 152-8551
Japan
E-mail address: uchiyama@math.titech.ac.jp