Advanced Studies in Pure Mathematics 39, 2004
Stochastic Analysis on Large Scale Interacting Systems
pp. 355–376

Lévy Processes Conditioned to Stay Positive and Diffusions in Random Environments

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Abstract.

Some general properties of Lévy processes conditioned to stay positive are studied. As an application, which is our main concern, a result of localization is obtained for diffusion processes in Lévy environments.

§1. Introduction

We discuss a problem of localization of diffusion processes in Lévy random environments. For this we must first prepare some general properties of Lévy processes conditioned to stay positive, which were studied intensively by Bertoin [1, 2] and Chaumont [4, 5]. Some of our results in § **2** may also be found in [1] and [5] but our method is more or less analytical and different from theirs.

Let $\mathbf{W}$ denote the space of real valued right continuous functions on $[0,\infty)$ with left limits and vanishing at 0. For an element w of $\mathbf{W}$ we write $w = (w(t),\, t \geq 0)$ in § **2** and $w = (w(x),\, x \geq 0)$ in § **3**.

Given a one-dimensional Lévy process $W = \{w(t),\, t \geq 0,\, P\}$, we define a function h by $h(x) = \mu([0,x))$, $x > 0$, where μ is the measure in $[0,\infty)$ determined by (2.9). According to Silverstein [20] the function h is sub-invariant for the absorbing process W^- in $(0,\infty)$; it is invariant for W^- if $\sup w(t) = \infty$ a.s. Therefore $H(t,x,dy) = h(x)^{-1}P^-(t,x,dy)h(y)$ is a sub-Markov transition function in $(0,\infty)$ where $P^-(t,x,dy)$ denotes the transition function of W^-. Defining the transition $H(t,0,dy)$ from 0 in a suitable way, we will have a Markov process with state space $[0,\infty)$, called the h-transform of W^- and denoted by W^h. When $\sup w(t) = \infty$ a.s., the process W^h is what we call the Lévy process W conditioned to stay positive. This definition is the same as that of Bertoin and Chaumont. When $\sup w(t) < \infty$ a.s., Hirano [10] showed that there are two

Received December 17, 2002.
Revised March 27, 2003.

different ways of defining Lévy processes conditioned to stay positive (under some additional condition) and so, simply to avoid confusion in this case we do not call W^h the process conditioned to stay positive in this paper, though it seems so (if, in addition, W has no positive jumps, W^h is not the same as the one considered in [2]). By an analytic method we prove that, if W enters immediately into $(0,\infty)$ a.s. and if W enters $(-\infty,0)$ within a finite time with positive probability, then W^h has a Feller semigroup H_t strongly continuous on $C.[0,\infty)$, the space of continuous functions on $[0,\infty)$ vanishing at infinity. This fact was also noticed by Chaumont [5] as a consequence of more probabilistic arguments. For our application in § **3** we must also prepare some convergence theorems on the reversed pre-minimum and the post-minimum processes $\hat{V}_\lambda$ and V_λ defined in (2.43) and (2.44). Similar results were already obtained by Bertoin [1] and Chaumont [5] but there is a delicate difference and we need extra arguments.

In § **3** we are concerned with diffusion processes in Lévy random environments. Suppose that we are given a Lévy process $W = \{w(x),\, x \geq 0,\, P\}$. Let $\Omega = C[0,\infty)$ and for $\omega \in \Omega$ set $X(t) = X(t,\omega) = \omega(t)$ (the value of ω at time t). For each $w \in \mathbf{W}$ we denote by P^w the probability measure on Ω such that $\{X(t),\, t \geq 0,\, P^w\}$ is a reflecting diffusion process on $[0,\infty)$ with generator $\frac{1}{2}e^{w(x)}\frac{d}{dx}\left(e^{-w(x)}\frac{d}{dx}\right)$ and starting at 0. The reflecting barrier at $x = 0$ is not essential; it was considered just to simplify the situation; we may (but do not) consider the case where the Lévy environment $w(x)$ is given in the whole of $\mathbf{R}$. We set $\mathbb{P}(dwd\omega) = P(dw)P^w(d\omega)$, which is a probability measure on $\mathbf{W}\times\Omega$. We then regard $\{X(t),\, t \geq 0,\, \mathbb{P}\}$ as a process defined on the probability space $(\mathbf{W}\times\Omega, \mathbb{P})$ and call it the (reflecting) diffusion process in the Lévy environment W.

When W is a Brownian environment, Brox [3] and Schumacher [19] proved that $\{X(t),\, t \geq 0,\, \mathbb{P}\}$ has the same limiting behavior (or the same localization property) as Sinai's random walk in a random environment ([21]). A result of refinement, which corresponds to that of Golosov [8] for Sinai's random walk, was then obtained by Tanaka [22, 25] and some extension by Kawazu-Tamura-Tanaka [13]. In this paper for a certain class of Lévy environments we obtain such a results of localization, which is similar to those of [8],[22, 25],[13]. To be precise let $w \in \mathbf{W}, \lambda > 0$ and set

(1.1) $N(x) = N(x,w) = \inf\{w(y) : 0 \leq y \leq x\},\ w^{\#}(x) = w(x) - N(x),$

(1.2) $a_\lambda = a_\lambda(w) = \inf\{x > 0 : w^{\#}(x) > \lambda\},$

(1.3) $b_\lambda = b_\lambda(w) =$ the unique x such that $w(x)$ is equal to $N(a_\lambda)$.

In general there may be many x with $w(x) = N(a_\lambda)$ but, in the case we actually discuss, such an x is unique a.s. (see Lemma 7). It will be proved that, under a certain condition on W, the distribution of $X(e^\lambda) - b_\lambda$ under $\mathbb{P}$ tends to some nondegenerate distribution $\bar{\nu}$ as $\lambda \to \infty$. It can happen that the limit distribution of $X(e^\lambda)-b_\lambda$ under $\mathbb{P}$ exists even when the limit distribution of a suitably scaled $X(t)$ (without centering, under $\mathbb{P}$) does not; the latter exists if, in addition, b_λ has a limit distribution under a suitable scaling (without centering). We are also interested in the form of the limit distribution $\bar{\nu}$. Under a certain condition on W our result is that $\bar{\nu}$ can be expressed in terms of two independent Lévy processes conditioned to stay positive starting at 0, the one is related to W and the other to $-W$.

§2. Lévy processes conditioned to stay positive

We use the notation in § **1** and so $W = \{w(t),\ t \geq 0,\ P\}$ is a Lévy process starting at 0. Throughout the paper we exclude the trivial case where $w(t) = 0\,(t \geq 0)$ a.s. The infimum process $N(t)$ and the reflecting process $w^\#(t)$ are defined by (1.1) with x replaced by t. The hitting (entrance) times $\sigma(x)$ and $\tau(x)$ are defined by

$$\sigma(x) = \inf\{t > 0 : x + w(t) \leq 0\}, \quad x \geq 0,$$

$$\tau(x) = \inf\{t > 0 : x + w(t) < 0\}, \quad x \geq 0,$$

$$\sigma = \sigma(0), \quad \tau = \tau(0).$$

We often consider $\hat{W} = \{\hat{w}(t), t \geq 0\}$, the dual of W, where $\hat{w}(t) = -w(t)$ and define $\hat{N}(t), \hat{\sigma}, \hat{\tau}$, etc., similarly in terms of $\hat{W}$. The absorbing Lévy process W^- in $(0,\infty)$ is defined as the Markov process $\{x + w(t), 0 \leq t < \sigma(x), x > 0\}$ and its transition function, semigroup and Green operator are denoted by $P^-(t,x,dy)$, T_t^- and G_λ^-, respectively. We set $G^- = G_0^-$. Another absorbing Lévy process $W^=$ on $[0,\infty)$, which does not much differ from W^-, is the Markov process $\{x + w(t), 0 \leq t < \tau(x), x \geq 0\}$; its Green operator of order 0 is denoted by $G^=$. We also define the reflecting Lévy process $W^\#$ on $[0,\infty)$ associated with W as the Markov process $\{w^\#(t;x), t \geq 0, x \geq 0\}$, where

$$\begin{aligned} w^\#(t;x) &= \sup_{0 \leq s \leq t} \{w(t;x) - w(s;x)\} \vee w(t;x) \\ &= \begin{cases} x + w(t) & \text{if } x + N(t) > 0, \\ w(t) - N(t) & \text{if } x + N(t) \leq 0, \end{cases} \end{aligned}$$

wherein $w(t;x) = x + w(t)$ and $a \vee b = \max\{a, b\}$, and we denote by $P^{\#}(t, x, dy)$ the transition function of $W^{\#}$. The reflecting dual Lévy process $\hat{W}^{\#}$ on $[0, \infty)$ and its transition function $\hat{P}^{\#}(t, x, dy)$ are defined in a similar manner from $\hat{W}$. Throughout the paper T denotes an exponential random time with mean $1/\lambda$ and independent of W.

2.1 Preliminaries. In this subsection we present in an elementary way some preliminary and known facts concerning the measure μ such as the (sub-)invariance for $\hat{W}^{\#}$.

Lemma 1. (Silverstein [20, p.556]) (i) *For any fixed* $t > 0, x > 0$ *and* $y \geq 0$ *we have* $\hat{P}^{\#}(t, y, [0, x)) = P^{-}(t, x, (y, \infty))$.
(ii) *For any fixed* $t \geq 0$, $-N(t) \overset{d}{=} \hat{w}^{\#}(t)$ *and* $-\hat{N}(t) \overset{d}{=} w^{\#}(t)$, *where* $\overset{d}{=}$ *is the equality in distribution. In particular,* $-N(T) \overset{d}{=} \hat{w}^{\#}(T)$ *and* $-\hat{N}(T) \overset{d}{=} w^{\#}(T)$.

Let ν_λ be the distribution of $-N(T)$, or equivalently of $\hat{w}^{\#}(T)$, and $\hat{\nu}_\lambda$ be the distribution of $-\hat{N}(T)$, or equivalently of $w^{\#}(T)$. The fluctuation identity

$$\log E\{e^{\xi N(T)+\eta(w(T)-N(T))}\}$$
$$= \int_0^\infty \frac{e^{-\lambda t}}{t} \left\{E(e^{\xi w(t)} - 1; w(t) < 0) + E(e^{\eta w(t)} - 1; w(t) > 0)\right\} dt$$

due to Pecherskii-Rogozin [15] (see also Sato [18], Bertoin [2], Doney [6]) implies that $N(T)$ and $w^{\#}(T)$ are independent and for $\xi \geq 0$

$$\text{(2.1)} \quad \mathcal{L}(\xi, \nu_\lambda) = \exp \int_0^\infty t^{-1} e^{-\lambda t} E\{e^{\xi w(t)} - 1; w(t) < 0\} dt,$$

$$\text{(2.2)} \quad \mathcal{L}(\xi, \hat{\nu}_\lambda) = \exp \int_0^\infty t^{-1} e^{-\lambda t} E\{e^{-\xi w(t)} - 1; w(t) > 0\} dt,$$

where $\mathcal{L}(\xi, \nu)$ denotes the Laplace transform $\int_{[0,\infty)} e^{-\xi x} \nu(dx)$ of a measure ν. The equations (2.1) and (2.2) imply that there exist finite measures μ_λ and $\hat{\mu}_\lambda$ on $[0, \infty)$ such that

$$\text{(2.3)} \qquad \mathcal{L}(\xi, \mu_\lambda) = \exp \int_0^\infty t^{-1} e^{-\lambda t} E\{e^{\xi w(t)} - e^{-t}; w(t) < 0\} dt,$$

$$\text{(2.4)} \qquad \mathcal{L}(\xi, \hat{\mu}_\lambda) = \exp \int_0^\infty t^{-1} e^{-\lambda t} E\{e^{-\xi w(t)} - e^{-t}; w(t) > 0\} dt,$$

$$\text{(2.5)} \qquad \nu_\lambda = c_\lambda \, \mu_\lambda, \qquad \hat{\nu}_\lambda = \hat{c}_\lambda \, \hat{\mu}_\lambda,$$

$$\text{(2.6)} \qquad c_\lambda = \exp\left\{- \int_0^\infty t^{-1} e^{-\lambda t} (1 - e^{-t}) P(w(t) < 0) dt\right\},$$

and $\hat{c}_\lambda$ is defined with the replacement of "$w(t) < 0$" by "$w(t) > 0$" in the equation (2.6).

For our elementary and straightforward method the following simple lemma, probably known, is useful.

Lemma 2. *If W is not the zero process, then for any $\xi > 0$*

(i) $$E\{e^{-\xi|w(t)|}\} \le \text{const.} t^{-1/4} \qquad (t \ge 0),$$

(ii) $$E\{1 - e^{-\xi|w(t)|}\} \le \text{const.} t^{1/2} \qquad (0 \le t \le 1),$$

where const. may depend on ξ.

Proof. Assume that W is not deterministic. Then we can write $w(t) = w_0(t) + w_1(t)$ where $w_0(t)$ and $w_1(t)$ are independent Lévy processes and $E\{w_0(t)\} = 0$, $E\{|w_0(t)|^2\} = \sigma^2 t$, $(\sigma > 0)$, $E\{|w_0(t)|^3\} < \infty$. In fact, the decomposition can be obtained by noting the fact that any Lévy process having Lévy measure with bounded support admits finite absolute moments of all positive orders (e.g. see Sato [18, p.161]). We now make use of the Berry-Esseen theorem (e.g. see Feller [7, p.542]):

$$\sup_{x \in \mathbf{R}} \left| P\{(\sigma\sqrt{t})^{-1} w_0(t) \le x\} - \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2) dy \right| = O(t^{-1/2}),$$

as $t \to \infty$. Setting $Y(t) = (\sigma\sqrt{t})^{-1}|w_0(t) + x|$ we have

$$\begin{aligned}
&E\{e^{-\xi|w_0(t)+x|}\} \\
&= E\{e^{-\xi\sigma\sqrt{t}Y(t)}; Y(t) < t^{-1/4}\} + E\{e^{-\xi\sigma\sqrt{t}Y(t)}; Y(t) \ge t^{-1/4}\} \\
&\le P\{Y(t) < t^{-1/4}\} + \exp(-\xi\sigma t^{1/4}) \\
&\le \text{const.} t^{-1/2} + \int_{\{|y + (\sigma\sqrt{t})^{-1}x| < t^{-1/4}\}} (2\pi)^{-1/2} e^{-y^2/2} dy + e^{-\xi\sigma t^{1/4}} \\
&\le \text{const.} t^{-1/4} \quad (\text{for large } t),
\end{aligned}$$

where const. may depend on ξ. Therefore

$$E\{e^{-\xi|w(t)|}\} = \int_{-\infty}^{\infty} E\{e^{-\xi|w_0(t)+x|}\} P\{w_1(t) \in dx\} \le \text{const.} t^{-1/4}.$$

The proof of (ii) is omitted.

2.1.1 *A formula on Green operators of absorbing Lévy processes.* For $\lambda > 0$, $x > 0$ and $f \in C_0[0, \infty)$, the space of continuous functions with compact supports, we have

$$(2.7)\quad \begin{aligned} G_\lambda^- f(x) &= \int_0^\infty e^{-\lambda t} E\{f(x+w(t));\ \sigma(x)>t\}dt \\ &= \lambda^{-1} E\{f(x+N(T)+w^{\#}(T));\ -N(T)<x\} \\ &= \lambda^{-1}\int_{[0,x)} \nu_\lambda(du)\int_{[0,\infty)} \hat{\nu}_\lambda(dv) f(x-u+v), \end{aligned}$$

and similarly

$$(2.8)\quad G_\lambda^= f(x) = \lambda^{-1}\int_{[0,x]} \nu_\lambda(du)\int_{[0,\infty)} \hat{\nu}_\lambda(dv) f(x-u+v), \quad x\ge 0.$$

By Lemma 2 the integrals on the right hand sides of (2.3) and (2.4) are convergent for $\lambda=0$ and so the measures μ_λ and $\hat{\mu}_\lambda$ converge vaguely as $\lambda\downarrow 0$ to the measures μ and $\hat{\mu}$ in $[0,\infty)$, respectively, which are defined by

$$(2.9)\quad \mathcal{L}(\xi,\mu) = \exp\int_0^\infty t^{-1}E\{e^{\xi w(t)} - e^{-t};\ w(t)<0\}dt,$$

$$(2.10)\quad \mathcal{L}(\xi,\hat{\mu}) = \exp\int_0^\infty t^{-1}E\{e^{-\xi w(t)} - e^{-t};\ w(t)>0\}dt,$$

where $\xi>0$. Moreover, using the definition of c_λ and $\hat{c}_\lambda$ we see that $\lambda^{-1}c_\lambda\,\hat{c}_\lambda \to c^0$ as $\lambda\downarrow 0$ where

$$(2.11)\quad c^0 = \exp\int_0^\infty t^{-1}(1-e^{-t})P\{w(t)=0\}dt,$$

which is finite by Lemma 2. It is also known that (e.g. see Sato [18, p.372])

$$(2.12)\quad c^0 = 1 \quad \text{if } W \text{ is not a compound Poisson process.}$$

Thus $\lambda^{-1}\nu_\lambda\otimes\hat{\nu}^\lambda = \lambda^{-1}c_\lambda\hat{c}_\lambda\mu_\lambda\otimes\hat{\mu}_\lambda \to c^0\mu\otimes\hat{\mu}$ vaguely as $\lambda\downarrow 0$ and hence letting $\lambda\downarrow 0$ in (2.7) and (2.8) we obtain the following theorem.

Theorem 1. *If W is not the zero process, then for $f\in C_0[0,\infty)$*

$$(2.13)\quad G^- f(x) = c^0\int_{[0,x)} \mu(du)\int_{[0,\infty)} \hat{\mu}(dv) f(x-u+v), \quad x>0,$$

$$(2.14)\quad G^= f(x) = c^0\int_{[0,x]} \mu(du)\int_{[0,\infty)} \hat{\mu}(dv) f(x-u+v), \quad x\ge 0.$$

This theorem was obtained by Ray [16] for symmetric stable processes and by Silverstein [20] for general Lévy processes; the present derivation of (2.13) and (2.14) was taken from Tanaka [23, 24] with a slight improvement.

2.1.2 *The measure μ is a sub-invariant measure of the Markov process $\hat{W}^{\#}$.*

Theorem 2. (Silverstein [20]) (i) *If the Markov process $\hat{W}^{\#}$ is recurrent, then μ is an invariant measure of $\hat{W}^{\#}$.*
(ii) *If $\hat{W}^{\#}$ is transient, then μ is a sub-invariant measure of $\hat{W}^{\#}$; more precisely, for any $A \in \mathcal{B}[0,\infty)$*

$$\mu(A) = \bar{c}\,E\left\{\int_0^\infty \mathbf{1}_A(\hat{w}^{\#}(t))dt\right\}, \tag{2.15}$$

$$\int_{[0,\infty)} \mu(dx)\hat{P}^{\#}(t,x,A) = \mu(A) - \bar{c}\,E\left\{\int_0^t \mathbf{1}_A(\hat{w}^{\#}(s))ds\right\}, \tag{2.16}$$

$$\bar{c} = \exp\left\{-\int_0^\infty t^{-1}(1-e^{-t})P(w(t)\geq 0)dt\right\}. \tag{2.17}$$

Proof. (i) We assume that $\hat{W}^{\#}$ is recurrent and that W is not an increasing process. Take $a > 0$, let $T^{\#}$ be the time of first return of $\hat{w}^{\#}(t)$ to 0 after visiting (a,∞) and define the measures $\mu_\lambda^{\#}, \lambda \geq 0$, in $[0,\infty)$ by

$$\int f d\mu_\lambda^{\#} = E\left\{\int_0^{T^{\#}} e^{-\lambda t} f\big(\hat{w}^{\#}(t)\big)dt\right\}. \tag{2.18}$$

Then

$$\mu_\lambda^{\#} = \lambda^{-1}\left\{1 - E\big(e^{-\lambda T^{\#}}\big)\right\}\nu_\lambda = \lambda^{-1}\left\{1 - E\big(e^{-\lambda T^{\#}}\big)\right\}c_\lambda\,\mu_\lambda.$$

Since $\mu_\lambda^{\#} \to \mu_0^{\#}$ and $\mu_\lambda \to \mu$ as $\lambda \downarrow 0$, the above identity implies that the measure $\mu_0^{\#}$ is a constant multiple of μ. On the other hand it is easy to see that $\mu_0^{\#}$ is an invariant measure of the recurrent process $\hat{W}^{\#}$ and so is μ. (ii) If $\hat{W}^{\#}$ is transient, then $\bar{c} > 0$ and the assertion follows from

$$\int f d\mu = \bar{c}\,E\left\{\int_0^\infty f\big(\hat{w}^{\#}(t)\big)dt\right\}.$$

We now introduce a function $h(x), x \geq 0$, by

$$h(x) = \begin{cases} \mu([0,x)) & \text{for } x \geq 0, \\ \mu(\{0\}) & \text{for } x = 0. \end{cases} \tag{2.19}$$

Then using Lemma 1 we can rephrase Theorem 2 as follows.

Theorem 3. (Silverstein [20]) (i) *If $\hat{W}^{\#}$ is recurrent, then*

$$\int_{(0,\infty)} P^{-}(t,x,dy)h(y) = h(x), \quad x > 0, \tag{2.20}$$

$$\lambda G_{\lambda}^{-} h(x) = h(x), \quad x > 0. \tag{2.21}$$

(ii) *If $\hat{W}^{\#}$ is transient, then for any $x > 0$*

$$\int_{(0,\infty)} P^{-}(t,x,dy)h(y) = h(x) - \bar{c}\int_0^t P\{\hat{w}^{\#}(s) < x\}ds, \tag{2.22}$$

$$\lambda G_{\lambda}^{-} h(x) = h(x) - \lambda^{-1}\bar{c}\,\nu_{\lambda}([0,x)). \tag{2.23}$$

Remark. The following conditions are equivalent to each other (e.g. see Sato [18]).

(i) $\hat{W}^{\#}$ is recurrent. (ii) $\sup_{t\geq 0} w(t) = \infty$, a.s.
(iii) $\int_0^{\infty} t^{-1}(1-e^{-t})P\{w(t)>0\}dt = \infty$. (iv) $\bar{c} = 0$.
(v) $\int_1^{\infty} t^{-1}P\{w(t)>0\}dt = \infty$. (vi) $\hat{\mu}(\mathbf{R}) = \infty$.

2.2 The Feller property of the semigroup of W^h. We define the superharmonic transform $H(t,x,dy)$ of $P^{-}(t,x,dy)$ by

$$H(t,x,dy) = h(x)^{-1}P^{-}(t,x,dy)h(y). \tag{2.24}$$

We set $H_t f(x) = \int_{(0,\infty)} H(t,x,dy)f(y)$. Then $H_t f(x)$ is well-defined for $f \in C.[0,\infty)$ and for $x > 0$. We will prove that $H_t f$ can be extended to a function in $[0,\infty)$ so that H_t gives rise to a strongly continuous sub-Markov semigroup on $C.[0,\infty)$ provided that $\hat{\tau} = 0$ a.s. and $\tau < \infty$ with positive probability.

We prepare three lemmas.

Lemma 3. (i) If W is not a compound Poisson process, then $\mu(\{x\}) = \nu_{\lambda}(\{x\}) = 0$ for any $x > 0$ and $\lambda > 0$.
(ii) The condition $\tau = 0$ a.s. is equivalent to each of $\mu(\{0\}) = 0$ and $\nu_{\lambda}(\{0\}) = 0$.

The proof is easy; for instance, the equivalence of $\tau = 0$ (a.s.) and $\mu(\{0\}) = 0$ follows from the formula (2.14). The rest are omitted.

In what follows we often use the notation $\nu(f) = \int_{[0,\infty)} f d\nu$.

Lemma 4. *Suppose that W is not the zero process.*
(i) *For any $\lambda > 0$ and $x > 0$*

$$\text{(2.25)} \quad \lambda^{-1}\bar{c} + \int_{[0,\infty)} h(v)\hat{\nu}_\lambda(dv) \le \frac{h(x)}{\nu_\lambda([0,x))} \le \lambda^{-1}\bar{c} + \int_{[0,\infty)} h(x+v)\hat{\nu}_\lambda(dv).$$

(ii) *For any $\lambda > 0$*

$$\text{(2.26)} \qquad \lim_{x\downarrow 0} \frac{\nu_\lambda([0,x))}{h(x)} = \alpha_\lambda,$$

$$\text{(2.27)} \qquad \alpha_\lambda = \left\{\lambda^{-1}\bar{c} + \hat{\nu}_\lambda(h)\right\}^{-1} \in (0,\infty).$$

Proof. The function $\lambda G_\lambda^- h$ can be expresses in two ways:

$$\text{(2.28)} \qquad \lambda G_\lambda^- h(x) = \int_{[0,x)} \nu_\lambda(du) \int_{[0,\infty)} \hat{\nu}_\lambda(dv) h(x-u+v).$$

$$\text{(2.29)} \qquad \lambda G_\lambda^- h(x) = h(x) - \lambda^{-1}\bar{c}\,\nu_\lambda([0,x)).$$

Firstly we remark that the finiteness of $\hat{\nu}_\lambda(h)$ follows from (2.28); moreover, if $\hat{\nu}_\lambda(h) = 0$ then $\hat{\nu}_\lambda$ is the δ-distribution at 0 so W is decreasing and hence $\bar{c} > 0$. Thus $0 < \alpha_\lambda < \infty$ always. Secondly from (2.28) and (2.29) we have

$$h(x) = \lambda^{-1}\bar{c}\,\nu_\lambda([0,x)) + \int_{[0,x)} \nu_\lambda(du) \int_{[0,\infty)} \hat{\nu}_\lambda(dv) h(x-u+v),$$

and hence

$$\lambda^{-1}\bar{c} + \hat{\nu}_\lambda(h) \le \frac{h(x)}{\nu_\lambda([0,x))} \le \lambda^{-1}\bar{c} + \hat{\nu}_\lambda\left(h^x\right),$$

where $h^x(\cdot) = h(x+\cdot)$, which proves (2.25) and (2.26). The proof of the lemma is finished.

Let $\lambda > 0, f \in C_0[0,\infty)$ and set

$$\text{(2.30)} \qquad U_\lambda f(x) = \int_0^\infty e^{-\lambda t} H_t f(x) dt, \quad x > 0.$$

Then by (2.7) we have

$$(2.31)\quad U_\lambda f(x) = \lambda^{-1}h(x)^{-1}\int_{[0,x)}\nu_\lambda(du)\int_{[0,\infty)}\hat{\nu}_\lambda(dv)\,\tilde{f}(x-u+v),$$

where $\tilde{f} = fh$. If W is not a compound Poisson process, then $h(x)$ is continuous by Lemma 3 and hence $U_\lambda f(x)$ is also continuous in $x > 0$. Moreover, $U_\lambda f(x)$ tends to $\lambda^{-1}\alpha_\lambda\hat{\nu}_\lambda(\tilde{f})$ as $x \downarrow 0$ by (2.26) and (2.31). On the other hand it is clear that $U_\lambda f(x)$ tends to 0 as $x \to \infty$. Therefore $U_\lambda f(x), x > 0$, can be extended continuously to a function in $C.[0,\infty)$, which we denote by the same notation $U_\lambda f$. Since $\|U_\lambda f\|_\infty \le \lambda^{-1}\|f\|_\infty$, $U_\lambda f$ is well-defined also for $f \in C.[0,\infty)$. Thus we have a linear operator $U_\lambda : C.[0,\infty) \to C.[0,\infty)$, which clearly satisfies

$$(2.32)\qquad U_\lambda f \ge 0 \quad \text{if } f \ge 0,$$

$$(2.33)\qquad \|U_\lambda f\|_\infty \le \lambda^{-1}\|f\|_\infty,$$

$$(2.34)\qquad U_\lambda - U_{\lambda'} + (\lambda - \lambda')U_\lambda U_{\lambda'} = 0, \quad \lambda > 0,\ \lambda' > 0.$$

Now we introduce the following conditions.
(A) $\tau = \hat{\tau} = 0$, a.s.
(A′) $\hat{\tau} = 0$ a.s. and $0 < \tau < \infty$ with positive probability.

Lemma 5. *If either one of the conditions* (A) *and* (A′) *is satisfied, then*

$$(2.35)\qquad \lim_{\lambda\to\infty}\|\lambda U_\lambda f - f\|_\infty = 0 \quad \textit{for} \quad f \in C.[0,\infty).$$

Proof. Making use of (2.7) and (2.26) we have

$$(2.36)\quad \lambda U_\lambda f(0) = \lim_{x\downarrow 0}\lambda U_\lambda f(x) = \lambda\lim_{x\downarrow 0}h(x)^{-1}G_\lambda^{-}\tilde{f}(x) = \alpha_\lambda\hat{\nu}_\lambda(\tilde{f}).$$

If we set $\rho_\lambda(dx) = \alpha_\lambda h(x)\hat{\nu}_\lambda(dx)$, then ρ_λ is a measure in $[0,\infty)$ with total mass ≤ 1 and (2.36) yields

$$(2.37)\qquad \lambda U_\lambda f(0) = \rho_\lambda(f), \quad f \in C.[0,\infty).$$

We are going to prove that ρ_λ converges vaguely to δ_0 as $\lambda \to \infty$. To prove this, we assume that $f = U_\theta g$ with $g \in C_0[0,\infty)$ and $\theta > 0$. Then the equation (2.34) implies $\|\lambda U_\lambda f - f\|_\infty = \|\theta U_\lambda f - U_\lambda g\|_\infty \to 0$, as $\lambda \to \infty$. In particular,

$$(2.38)\qquad U_\theta g(0) = f(0) = \lim_{\lambda\to\infty}\lambda U_\lambda f(0) = \lim_{\lambda\to\infty}\rho_\lambda(f).$$

Let ρ be any vague limit of ρ_λ as $\lambda \to \infty$ via a sequence $\{\lambda_n\}$. Then (2.38) implies $U_\theta g(0) = \rho(f)$, which can be rewritten, again by making use of (2.7) and (2.26), as follows:

$$(2.39) \quad \theta^{-1}\alpha_\theta\, \hat{\nu}_\theta(\tilde{g}) = \rho(\{0\})\theta^{-1}\alpha_\theta\, \hat{\nu}_\theta(\tilde{g}) \\ + \theta^{-1}\int_{(0,\infty)} h(x)^{-1}\rho(dx)\int_{[0,x)} \nu_\theta(du)\int_{[0,\infty)} \tilde{g}(x-u+v)\hat{\nu}_\theta(dv).$$

We now prove that, under the assumption of the lemma, the equation (2.39) holds for $g(x) = h(x)^{-1}e^{-\xi x}$, $\xi > 0$. Since $\hat{\nu}_\theta(\{0\}) = 0$ by Lemma 3, the integration interval $[0,\infty)$ of $\hat{\nu}_\theta$ in (2.39) can be replaced by the open interval $(0,\infty)$. With such a replacement we take $g_n(x) = \min\{h(x)^{-1}e^{-\xi x}, n\}$ for $g(x)$ in (2.39) and then let $n \uparrow \infty$. The result is

$$\theta^{-1}\alpha_\theta \int_{(0,\infty)} e^{-\xi x}\hat{\nu}_\theta(dx) = \rho(\{0\})\theta^{-1}\alpha_\theta \int_{(0,\infty)} e^{-\xi x}\hat{\nu}_\theta(dx) \\ +\theta^{-1}\int_{(0,\infty)} h(x)^{-1}\rho(dx)\int_{[0,x)} \nu_\theta(du)\int_{(0,\infty)} e^{-\xi(x-u+v)}\hat{\nu}_\theta(dv),$$

or equivalently,

$$\alpha_\theta = \rho(\{0\})\alpha_\theta + \int_{(0,\infty)} h(x)^{-1}\rho(dx)\int_{[0,x)} e^{-\xi(x-u)}\nu_\theta(du).$$

Letting $\xi \uparrow \infty$ we obtain $\alpha_\theta = \rho(\{0\})\alpha_\theta$ so $\rho = \delta_0$. This proves that ρ_λ converges vaguely to δ_0 as $\lambda \to \infty$. Thus (2.37) implies

$$(2.40) \qquad \lim_{\lambda\to\infty} \lambda U_\lambda f(0) = f(0), \quad f \in C.[0,\infty).$$

On the other hand it is clear that, for any $x > 0$,

$$(2.41) \qquad \lim_{\lambda\to\infty} \lambda U_\lambda f(x) = f(x), \quad f \in C.[0,\infty).$$

From (2.40) and (2.41) we can easily derive (2.35). This completes the proof of Lemma 5.

As an immediate consequence of (2.32) $\sim$ (2.35) we obtain the following theorem.

Theorem 4. *If* $\hat{\tau} = 0$ a.s. *and* $\tau < \infty$ *with positive probability (namely, either one of the conditions* (A) *and* (A′) *is satisfied), then there exists a unique strongly continuous sub-Markov semigroup* H_t *on* $C.[0,\infty)$ *such that, for any* $t > 0, x > 0, f \in C.[0,\infty)$,

$$(2.42) \qquad H_t f(x) = h(x)^{-1}\int_{(0,\infty)} P^-(t,x,dy)f(y)h(y).$$

Denote by $C_\triangle$ the subspace of $C.[0,\infty)$ consisting of those functions f with $f(0) = E\{f\,(w(\hat{\tau}) - w(\hat{\tau}-))\,;\ \hat{\tau} < \infty\}$. We omit the proof of the following theorem since it is not used in our later arguments. Pictorial observation of the sample path of the reversed pre-minimum process of the next subsection suggests the result. .

Theorem 5. *If $\tau = 0, \hat{\tau} > 0$ a.s. and $\hat{\tau} < \infty$ with positive probability, then there exists a unique strongly continuous sub-Markov semigroup H_t on the subspace $C_\triangle$ such that* (2.42) *holds for $f \in C_\triangle$. H_t can not be strongly continuous at $t = 0$ on the whole space $C.[0,\infty)$.*

2.3 The reversed pre-minimum and the post-minimum processes. We assume that our Lévy process $W = \{w(t),\, t \geq 0,\, P\}$ satisfies the following conditions:
Condition (A). $\tau = \hat{\tau} = 0$ a.s.
Condition (B). $\sup\{w(t) : t > 0\} = -\inf\{w(t) : t > 0\} = \infty$ a.s.
So the process W^h is the process W conditioned to stay positive. We denote by W^+ such a process starting at 0. Similarly $\hat{W}^+$ denotes the process $\hat{W}$ conditioned to stay positive starting at 0. We consider the reversed pre-minimum and the post-minimum processes $\hat{V}_\lambda$ and V_λ defined by

$$\hat{V}_\lambda(t) = w((b_\lambda - t)-) - w(b_\lambda),\ 0 \leq t < b_\lambda, \tag{2.43}$$

$$V_\lambda(t) = w(b_\lambda + t) - w(b_\lambda),\ 0 \leq t < c_\lambda, \tag{2.44}$$

where a_λ and b_λ are defined by (1.2) and (1.3) and $c_\lambda = a_\lambda - b_\lambda$. It is known that $\hat{V}_\lambda$ and V_λ are independent for each fixed λ. We are interested in the convergence in law of $\hat{V}_\lambda$ to $\hat{W}^+$ and of V_λ to W^+ (as $\lambda \to \infty$). The proof of the former convergence is considerably easier but we can prove the latter convergence only under an additional condition (C) which is somewhat stronger. We have to omit the details of the latter part since our proof is too lengthy to be included here.

2.3.1 *The reversed pre-minimum process.* To prove the law convergence of $\hat{V}_\lambda$ first we express the sample functions of W and $\hat{V}_\lambda$, à la Itô [11, (6.6) of p.233], in terms of the Poisson point process (P.p.p. for short) of "W-excursions off the zeros of $W^\#$" which was first used by Greenwood-Pitman [9] (see also Bertoin [1]). So let $L(t)$ be the local time of the reflecting process $W^\#$ at 0, let $L^{-1}(s)$ be the right continuous inverse function of $L(t)$ and set

$$\triangle_s = w(L^{-1}(s)) - w(L^{-1}(s-)), \quad \zeta_s = L^{-1}(s) - L^{-1}(s-).$$

Some equations to follow hold under the phrase "a.s." but we shall often omit to write it. We have $L^{-1}(s) = \sum_{r \leq s} \zeta_r$ (the continuous part vanishes under the condition (A)). It can also be proved that the continuous part $N_c(t)$ of $N(t)$ is equal to $-cL(t)$ where c is the nonnegative constant determined by $E\{e^{-cL(T)}\} = E\{e^{-N_c(T)}\}$, T being an exponential random time with mean 1 and independent of W. Thus the decomposition of $N(t)$ to continuous and jump parts yields $N(t-) = -cs + \sum_{r<s} \triangle_r$, $t > 0$, where s is determined by $L^{-1}(s-) \le t \le L^{-1}(s)$. Now let

$$p_s(t) = w(L^{-1}(s-) + t) - w(L^{-1}(s-)) \quad \textit{for} \quad 0 \le t \le \zeta_s.$$

$p_s = \{p_s(t),\, 0 \le t \le \zeta_s\}$ is the W-excursion on $[L^{-1}(s-), L^{-1}(s)]$ that starts at 0 (this is a consequence of the condition (A)) and moves during $[0, \zeta_s]$ with the increments of w on $[L^{-1}(s-), L^{-1}(s)]$, ending at time ζ_s with final value $\triangle_s$. Then $\{p_s,\, s > 0\}$ is a P.p.p. and we have

$$w(t) = p_s(t - L^{-1}(s-)) - cs + \sum_{r<s} \triangle_r, \quad t > 0, \tag{2.45}$$

with s such that $L^{-1}(s-) \le t \le L^{-1}(s)$ where we use the convention that $p_s(\cdot) = 0$ whenever $L^{-1}(s-) = L^{-1}(s)$. Thus the process W is constructed from the P.p.p. $\{p_s,\, s > 0\}$. Moreover $b_\lambda = L^{-1}(s_\lambda -)$ where s_λ is the minimum of $s > 0$ such that the excursion p_s can cross the level λ.

Consider the reversed excursion $\hat{p}_s = \{\hat{p}_s(t),\, t \in \{0-\} \cup [0, \zeta_s)\}$ defined by

$$\hat{p}_s(0-) = \triangle_s \quad \textit{and} \quad \hat{p}_s(t) = p_s((\zeta_s - t)-) \quad \textit{for} \quad 0 \le t < \zeta_s.$$

Then $\{\hat{p}_s,\, s > 0\}$ is also a P.p.p., which we now modify as follows. Let $\lambda > 0$ be fixed, let s_λ be the same as before and define $\tilde{p}_s$ by

$$\tilde{p}_s = \begin{cases} \hat{p}_{s_\lambda - s} & \text{for } 0 < s < s_\lambda, \\ \hat{p}_s & \text{for } s \ge s_\lambda. \end{cases}$$

Then $\{\tilde{p}_s,\, s > 0\} \stackrel{d}{=} \{\hat{p}_s,\, s > 0\}$ and we can prove that, for $0 \le t < b_\lambda$,

$$\hat{V}_\lambda(t) = \tilde{p}_s(t - \tilde{L}^{-1}(s-)) + cs - \sum_{r \le s} \tilde{\triangle}_r, \tag{2.46}$$

where s is determined by $\tilde{L}^{-1}(s-) \le t \le \tilde{L}^{-1}(s)$. If we replace $\{\tilde{p}_s\}$ by $\{\hat{p}_s\}$, the right hand side of (2.46) has the form

$$\hat{p}_s(t - L^{-1}(s-)) + cs - \sum_{r \le s} \triangle_r$$

with s such that $L^{-1}(s-) \le t \le L^{-1}(s)$, $0 \le t < b_\lambda$ (it is to be noted that the inverse local time associated with $\{\hat{p}_s\}$ is still $L^{-1}(s)$). From these observations we see that a cadlag process $\{\hat{V}(t),\, t \ge 0\}$ is defined by

$$\hat{V}(t) = \hat{p}_s(t - L^{-1}(s-)) + cs - \sum\nolimits_{r \le s} \triangle_r, \tag{2.47}$$

with s such that $L^{-1}(s-) \le t \le L^{-1}(s)$, and for *each fixed* $\lambda > 0$

$$\{\hat{V}_\lambda(t),\, 0 \le t < b_\lambda\} \stackrel{d}{=} \{\hat{V}(t),\, 0 \le t < b_\lambda\}. \tag{2.48}$$

We are going to prove that $\{\hat{V}(t),\, t \ge 0\}$ is the $\hat{W}^+$-process. Let $\tilde{b}_\lambda$ be the unique t such that $w(t) = N(\lambda)$, define the reversed pre-minimum process $\{\tilde{V}_\lambda(t),\, 0 \le t < \tilde{b}_\lambda\}$ in a way similar to (2.43) and set $\bar{b}_\lambda = L^{-1}(\bar{s}_\lambda-)$ where $\bar{s}_\lambda$ is determined by $L^{-1}(\bar{s}_\lambda-) \le \lambda \le L^{-1}(\bar{s}_\lambda)$. Then as in the case of (2.48) we have

$$\{\tilde{V}_\lambda(t),\, 0 \le t < \tilde{b}_\lambda\} \stackrel{d}{=} \{\hat{V}(t),\, 0 \le t < \bar{b}_\lambda\}.$$

On the other hand $\{\tilde{V}_\lambda(t)\}$ converges in law to $\hat{W}^+$ as $\lambda \to \infty$ by Bertoin [1, Cor.3.2,Th.3.4]. See also Chaumont [5, Th.2]. ($\{\tilde{V}_\lambda(t)\}$ is identical in law to the post-minimum process for the dual process $\hat{W}$ while this will not be true for $\{\hat{V}_\lambda(t)\}$. We may also use Millar [14]; in this case it is better to define $\{\tilde{V}_\lambda(t)\}$ and the related quantities by replacing the constant time λ in $w(t) = N(\lambda)$ and $L^{-1}(\bar{s}_\lambda-) \le \lambda \le L^{-1}(\bar{s}_\lambda)$ with an exponential random time of mean λ and independent of W.) Therefore $\{\hat{V}(t),\, t \ge 0\}$ is $\hat{W}^+$ and we have the following:

Theorem 6. *Under the conditions* (A) *and* (B) $\{\hat{V}(t),\, t \ge 0\}$ *defined by* (2.47) *is the* $\hat{W}^+$*-process and* (2.48) *holds for each fixed* λ*. In particular,* $\hat{V}_\lambda$ *converges in law to* $\hat{W}^+$ *as* $\lambda \to \infty$*.*

2.3.2 *The post-minimum process.* Since we have no formula for V_λ like (2.48), we need extra arguments for the proof of the convergence in law of V_λ. And, assuming only the conditions (A) and (B) we did not succeed (the part (ii) of Theorem 7 in [24] lacked a complete proof); we had unexpected difficulty in proving the tightness concerning $\{V_\lambda\}$ and so we must assume the additional condition (C) below which is somewhat stronger. For $0 < x < \lambda$ let $h_\lambda(x)$ denote *the probability that* $x + w(t)$ *enters* (λ, ∞) *before it enters* $(-\infty, 0)$ and let us state the following lemma.

Lemma 6. *If W satisfies the condition* (A) *and* $\sup w(t) = \infty$ *a.s., then*

$$\lim_{\lambda\to\infty} h_\lambda(x)^{-1}h_\lambda(y) = h(x)^{-1}h(y), \quad x > 0,\ y > 0. \tag{2.49}$$

Outline of proof. We first prepare the result for a random walk analogous to (2.49), in which the right hand side is replaced by the ratio of certain renewal functions (e.g. see [13, Theorem 2.3, p.524]). This ratio is again replaced by the ratio of certain mean occupation measures of the reflecting dual random walk which are defined in a manner similar to (2.18). To go to the case of a Lévy process we make use of the uniform approximation of W by suitable step processes of semi-Markov type.

Remark. When $\sup w(t) < \infty$ a.s. contrary to the assumption of the lemma Hirano [10] proved, under some additional condition, that the limit in (2.49) exists but the equality does not hold so that there are two different processes conditioned to stay positive attached to the same W.

From Lemma 6 it follows that

$$\frac{h_\lambda(1)^{-1}h_\lambda(x)}{h(1)^{-1}h(x)} \to 1 \quad \text{as } \lambda \to \infty, \tag{2.50}$$

uniformly on any compact subset of $(0, \infty)$.

Condition (C). *The convergence in* (2.50) *is uniform on* $(0, a]$ *for any* $a > 0$.

Theorem 7. *Under* (A),(B) *and* (C), *the post-minimum process* V_λ *converges in law to* W^+ *as* $\lambda \to \infty$.

Key of proof. (i) By Lemma 6 the transition function of V_λ tends to that of W^+ as $\lambda \to \infty$, and (ii) the family of laws of $V_\lambda, \lambda > \lambda_0$, is tight; we used the condition (C) to check this.

§3. Diffusion processes in Lévy environments

In this section we write $w(x)$ instead of $w(t)$. Suppose, as stated in § **1**, that we are given the reflecting diffusion process $\{X(t),\ t \geq 0,\ \mathbb{P}\}$ in a Lévy environment $W = \{w, P\}$.

3.1 Let $w \in \mathbf{W}$ and $x > 0$. Then w is said to be locally right-oscillating (resp. locally left-oscillating) at x if $\sup\{w(y) : x < y < x + \epsilon\} > w(x)$ and $\inf\{w(y) : x < y < x + \epsilon\} < w(x)$ for any $\epsilon > 0$ (resp. if $\sup\{w(y) : x - \epsilon < y < x\} > w(x-)$ and $\inf\{w(y) : x - \epsilon < y < x\} < w(x-)$ for any $\epsilon > 0$). w is said to be locally oscillating at x if it is locally right- and left-oscillating at x. w is said to have a local maximum (resp. local minimum) at x if $\sup\{w(y),\ x-\epsilon<y<x+\epsilon\}=w(x)\vee w(x-)$ (resp. if $\inf\{w(y):\ x-\epsilon<y<x+\epsilon\}=w(x)\wedge w(x-)$) for some ϵ. An extreme point is a point of either local maximum or local minimum.

The following lemma can be proved in the same way as Lemma 3.1 of [13].

Lemma 7. *If the conditions* (A) and (B) *are satisfied, then there exists* $\mathbf{W}_0\subset\mathbf{W}$ *with* $P(\mathbf{W}_0)=1$ *such that any* $w\in\mathbf{W}_0$ *has the following properties:*
(i) $\tau=\hat{\tau}=0$.
(ii) $\sup\{w(x):\ x>0\}=-\inf\{w(x):\ x>0\}=\infty$.
(iii) *w can not have the same value at distinct extreme points.*
(iv) *w is locally oscillating at any point of discontinuity. In particular, w is continuous at any point of local minimum.*

We assume that W satisfies the conditions (A) and (B). We denote by μ_λ^w the distribution of $X(e^\lambda)-b_\lambda$ under P^w and by ν_λ^w the probability measure on $[-b_\lambda, a_\lambda-b_\lambda]$ defined by

$$\nu_\lambda^w(dx)=Z_{\lambda,w}^{-1}\exp\{-(w(x+b_\lambda)-w(b_\lambda))\}dx, \tag{3.1}$$

where $Z_{\lambda,w}=\int_{-b_\lambda}^{c_\lambda}\exp\,dx$ (normalization), a_λ and b_λ are defined by (1.2) and (1.3), and $c_\lambda=a_\lambda-b_\lambda$. In what follows $\|\cdot\|$ stands for the total variation. In computing $\|\mu_\lambda^w-\nu_\lambda^w\|$ we regard ν_λ^w as a probability measure in $(-\infty,\infty)$. Such a convention is often used. Note that $\|\mu_\lambda^w-\nu_\lambda^w\|$ is a random variable on the probability space $(\mathbf{W},P)$. For $\alpha>0, \lambda>0$ and $w\in\mathbf{W}$ we define $w_\lambda^\alpha\in\mathbf{W}$ by $w_\lambda^\alpha(x)=\lambda^{-1}w(\lambda^\alpha x)$, $x\geq 0$. Then $W_\lambda^\alpha=\{w_\lambda^\alpha(x),\ x\geq 0,\ P\}$ is a Lévy process.

Here are the conditions often used in the arguments to follow.

Condition (D_Λ). *Let $\Lambda=\{\lambda_n\}$ be a given positive sequence tending to ∞ and let it be fixed. There exists $\alpha>0$ such that W_λ^α converges in law, as $\lambda\to\infty$ along Λ, to some Lévy process $\tilde{W}=\{w(x),\ x\geq 0,\ \tilde{P}\}$ satisfying the conditions* (A) *and* (B).

Condition (D). *For any positive sequence $\{\lambda'_n\}$ tending to ∞ there exists a subsequence $\Lambda=\{\lambda_n\}$ of $\{\lambda'_n\}$ for which the condition* (D_Λ) *is satisfied.*

Most of strictly semi-stable Lévy processes satisfy the condition (D). A simple example of W satisfying (D) but is not strictly semi-stable is a Lévy process W with characteristic function

$$E\{e^{i\xi w(1)}\}=\exp\int_0^\infty(\cos i\xi x-1)x^{-\alpha-1}a(x)\,dx\,,$$

where $0<\alpha<2$ and $a(x)$ is a Borel function such that $a(e^t)$ is aperiodic in t and bounded from above and below by positive constants.

Theorem 8. *Suppose that W satisfies the conditions* (A),(B) *and* (D_Λ). *Then* $\|\mu_\lambda^w - \nu_\lambda^w\| \to 0$ *in probability with respect P as $\lambda \to \infty$ along Λ. If, in addition,* (D) *is satisfied, then the phrase "along Λ" is removed.*

3.2 This subsection is for preliminaries to the proof Theorem 8. Let $\Lambda = \{\lambda_n\}$, W_λ^α and $\tilde{W} = \{w(x),\, x \geq 0,\, \tilde{P}\}$ be the ones in the condition (D_Λ). Then by Lemma 7 there exists $\mathbf{W}_0 \subset \mathbf{W}$ with $P(\mathbf{W}_0) = \tilde{P}(\mathbf{W}_0) = 1$ such that any $w \in \mathbf{W}_0$ has the properties (i) $\sim$ (iv) of Lemma 7. Take an arbitrary $w \in \mathbf{W}_0$ and then let $\{w_n,\, n \geq 1\}$ be any sequence in $\mathbf{W}$ converging to w in the Skorohod topology. In the argument of this subsection $\{\lambda_n\}$, w and $\{w_n\}$ are all fixed.

We set $a = a_1$ and $b = b_1$ suppressing the suffix $\lambda = 1$. Then for any small $\epsilon > 0$ there exists a' with the following properties:

(i′) $a < a' < a + \epsilon$. (ii′) w is continuous at a'.

(iii′) $w(a) - \epsilon < w(x) < w(a')$ for any $x \in [a, a')$.

We set $d' = w(a') - w(b)$ and $e' = \sup\{w(y) - w(x) : 0 \leq x < y \leq b\}$. Then $d' > 1$ and $e' < 1$ (as for the latter we have to take $\mathbf{W}_0$ so that $e' < 1$ holds for any $w \in \mathbf{W}_0$ but this is certainly possible). We now employ the coupling method of Brox [3]. We use the notation $\omega(t)$ instead of $X(t)$ for the time being. Consider the product probability measure $P_n^{\otimes} = P^{\lambda_n w_n} \otimes \hat{P}_n$ on $\Omega \times \hat{\Omega}$ where $\hat{\Omega} = C([0,\infty) \to [0,a'])$ and $\hat{P}_n$ is the probability measure on $\hat{\Omega}$ with respect to which the coordinate process $\{\hat{\omega}(t),\, t \geq 0\}$ is a stationary reflecting diffusion process on $[0,a']$ with (local) generator

$$\frac{1}{2} e^{\lambda_n w_n} \frac{d}{dx}\left(e^{-\lambda_n w_n} \frac{d}{dx}\right).$$

Let $\hat{\nu}^{w_n}$ be the distribution (on $[0,a']$) of $\hat{\omega}(t)$ under $\hat{P}_n$; it is independent of t and has the density const.$\exp\{-\lambda_n w_n(x)\}$, $0 \leq x \leq a'$. We set

$$\tau' = \inf\{t > 0 : \omega(t) = a'\}, \quad \hat{\tau}' = \inf\{t > 0 : \hat{\omega}(t) = a'\},$$
$$\tilde{\sigma} = \inf\{t > 0,\, \omega(t) = \hat{\omega}(t)\}, \quad \tilde{\tau} = \inf\{t > \tilde{\sigma} : \hat{\omega}(t) = a'\}.$$
$$\tilde{\omega}(t) = \begin{cases} \omega(t) & \text{if } 0 \leq t < \tilde{\sigma}, \\ \hat{\omega}(t) & \text{if } t \geq \tilde{\sigma}. \end{cases}$$

Note that $\tilde{\sigma} \leq \tau'$ and $\hat{\tau}' \overset{d}{\leq} \tau'$. The following lemma can be proved as in Brox [3] (see also [12, p.179]).

Lemma 8. (i) *The process $\{\omega(t),\, 0 \leq t < \tau',\, P^{\lambda_n w_n}\}$ is equivalent in law to $\{\tilde{\omega}(t),\, 0 \leq t < \tilde{\tau},\, P_n^{\otimes}\}$.*

(ii) *Let $e' < r < d'$. Then as $n \to \infty$ we have, $P_n^{\otimes}\{\tilde{\sigma} < e^{\lambda_n r}\} \to 1$ and*

$$(3.2) \qquad P_n^{\otimes}\{\tilde{\tau} > e^{\lambda_n r}\} = P^{\lambda_n w_n}\{\tau' > e^{\lambda_n r}\} \geq \hat{P}_n\{\hat{\tau}' > e^{\lambda_n r}\} \to 1.$$

Denote by $E^{\lambda_n w_n}, \hat{E}_n$ and $E_n^{\otimes}$ the expectations with respect to $P^{\lambda_n w_n}, \hat{E}_n$ and $P_n^{\otimes}$, respectively. Then for any Borel function f in $[0, \infty)$ with $|f| \leq 1$ and for any positive sequence $\{r_n\}$ tending to 1 we have

$$\begin{aligned} E^{\lambda_n w_n}\{f(\omega(e^{\lambda_n r_n}))\} &= E_n^{\otimes}\{f(\tilde{\omega}(e^{\lambda_n r_n})); \tilde{\sigma} < e^{\lambda_n r_n} < \tilde{\tau}\} + \mathrm{o}(1) \\ = E_n^{\otimes}\{f(\hat{\omega}(e^{\lambda_n r_n}))\} + \mathrm{o}(1) &= \int_{[0,a']} f \, d\hat{\nu}^{w_n} + \mathrm{o}(1), \end{aligned}$$

where o(1), which may vary from place to place, denotes a term whose absolute value is dominated by some ϵ_n independent of f and tending to 0 as $n \to \infty$. Therefore, if μ^{w_n} denotes the distribution of $\omega(e^{\lambda_n r_n})$ under $P^{\lambda_n w_n}$, then $\|\mu^{w_n} - \hat{\nu}^{w_n}\| \to 0$ as $n \to \infty$. This can be rephrased as (3.3) below. Let ν^{w_n} be the probability measure on $[0, a]$ with density const.$\exp\{-\lambda_n w_n(x)\}$, $0 \leq x \leq a$. Since a' can be taken arbitrarily close to a and since $w_n \to w$ (the Skorohod convergence), we have $\|\hat{\nu}^{w_n} - \nu^{w_n}\| \to 0$ as $n \to \infty$ and hence

$$(3.3) \qquad \|\mu^{w_n} - \nu^{w_n}\| \to 0 \quad \text{as } n \to \infty.$$

3.3 We proceed to the proof of Theorem 8. From now on we use $X(t)$ for $\omega(t)$. As in [3] we have $\{\lambda^{-\alpha}X(\lambda^{2\alpha}t), \, t \geq 0, \, P^w\} \stackrel{d}{=} \{X(t), \, P^{\lambda w_\lambda^\alpha}\}$ and so

$$\{X(e^\lambda), \, P^w\} \stackrel{d}{=} \{\lambda^\alpha X(\lambda^{-2\alpha}e^\lambda), \, P^{\lambda w_\lambda^\alpha}\} = \{\lambda^\alpha X(e^{\lambda r(\lambda)}), \, P^{\lambda w_\lambda^\alpha}\},$$

where $r(\lambda) = 1 - 2\alpha\lambda^{-1}\log\lambda$ which tends to 1 as $\lambda \to \infty$. Now let us denote by $\tilde{\mu}_\lambda^w$ the distribution of $X(e^{\lambda r(\lambda)})$ under $P^{\lambda w_\lambda^\alpha}$ and by $\tilde{\nu}_\lambda^w$ the probability measure on $[0, a(w_\lambda^\alpha)]$ with density const.$\exp\{-\lambda w_\lambda^\alpha(x)\}, 0 \leq x \leq a(w_\lambda^\alpha)$. Noting that W_λ^α converges in law to $\tilde{W}$ as $\lambda \to \infty$ along $\{\lambda_n\}$, we first make use of Skorohod's realization theorem of almost sure convergence and then apply (3.3). As a result we have

$$(3.4) \qquad \|\tilde{\mu}_\lambda^w - \tilde{\nu}_\lambda^w\| \to 0 \quad \text{in probability as } \lambda \to \infty \text{ along } \{\lambda_n\}.$$

Since $a_\lambda(w) = \lambda^\alpha a(w_\lambda^\alpha), b_\lambda(w) = \lambda^\alpha b(w_\lambda^\alpha)$ and $\{X(e^\lambda) - b_\lambda, \, P^w\}$ is identical in law to $\{\lambda^\alpha(X(e^{\lambda r(\lambda)})) - b(w_\lambda^\alpha), \, P^{\lambda w_\lambda^\alpha}\}$, we have, for any

Borel function f in $\mathbf{R}$ with $|f| \leq 1$ and as $\lambda \to \infty$ along $\{\lambda_n\}$,

$$\int_{[-b_\lambda,\infty)} f\, d\mu_\lambda^w = \int_{[0,\infty)} f(\lambda^\alpha(x - b(w_\lambda^\alpha)))\tilde{\mu}_\lambda^w(dx) \tag{3.5}$$

$$= \int_{[0,a(w_\lambda^\alpha)]} f(\lambda^\alpha(x - b(w_\lambda^\alpha)))\tilde{\nu}_\lambda^w(dx) + \mathrm{o}(1) \tag{3.6}$$

$$= \text{const.} \int_{[0,a(w_\lambda^\alpha)]} f(\lambda^\alpha(x - b(w_\lambda^\alpha)))e^{-w(\lambda^\alpha x)}dx + \mathrm{o}(1) \tag{3.7}$$

$$= \text{const.}\lambda^{-\alpha} \int_{[0,a_\lambda]} f(x - b_\lambda)e^{-w(x)}dx + \mathrm{o}(1) \tag{3.8}$$

$$= \int_{[-b_\lambda,a_\lambda-b_\lambda]} f\, d\nu_\lambda^w + \mathrm{o}(1), \tag{3.9}$$

where we used (3.4) for (3.6), the definition of $\tilde{\nu}_\lambda^w$ for (3.7), change of variable for (3.8) and the definition (3.1) of ν_λ^w for (3.9). The proof of Theorem 8 is finished.

3.4 Let $\overline{\mathbf{W}}$ be the space of those nonnegative functions w in $\mathbf{R}$ which are right continuous and have left limits with $w(0) = w(0-) = 0$. Let $\bar{P}$ be the probability measure on $\overline{\mathbf{W}}$ such that $\bar{W}^- = \{w(-x-),\, x \geq 0\}$ is $\hat{W}^+$, $\bar{W}^+ = \{w(x),\, x \geq 0\}$ is W^+ and $\bar{W}^-$ and $\bar{W}^+$ are independent. The following lemma can be proved by making use of (2.13).

Lemma 9. *Under* (A) *and* (B), $\bar{E}\{\int_{-\infty}^\infty e^{-w(x)}\, dx\} < \infty$.

By this lemma we can define a probability measure $\bar{\nu}^w$ in $\mathbf{R}$, with superfix w outside some $\bar{P}$-negligible subset of $\overline{\mathbf{W}}$, and then $\bar{\nu}$ by

$$\bar{\nu}^w(dx) = Z_w^{-1}e^{-w(x)}dx \quad (Z_w = \int_{-\infty}^\infty e^{-w(x)}dx), \quad \bar{\nu} = \int \bar{\nu}^w \bar{P}(dw).$$

Of course ν_λ^w and $\bar{\nu}^w$ are random variables taking values of probability measures in $\mathbf{R}$; the former is governed by P and the latter by $\bar{P}$. From Theorem 6 and 7 it will be expected that ν_λ^w converges in law to $\bar{\nu}^w$ as $\lambda \to \infty$ but, to verify this, we still have to assume the following condition.

Condition (E). *There is a constant C such that, for any $x > 0$ and $y > 0$, the inequality $h_\lambda(x)^{-1}h_\lambda(y) \leq C\, h(x)^{-1}h(y)$ holds.*

In the following theorem μ_λ denotes the distribution of $X(e^\lambda) - b_\lambda$ under $\mathbb{P}$, namely, $\mu_\lambda = \int \mu_\lambda^w P(dw)$.

Theorem 9. *Under the conditions* (A)$\sim$ (E) ν_λ^w *and hence* μ_λ^w, *by* Theorem 8, *converge in law to* $\bar{\nu}^w$ *as* $\lambda \to \infty$. *In particular,* μ_λ *converges to* $\bar{\nu}$ *as* $\lambda \to \infty$.

For the proof we have to show the law convergence of $Z_{\lambda,w}$ (governed by P) to Z_w (governed by $\bar{P}$) as $\lambda \to \infty$. Set

$$Z^-_{\lambda,w} = \int_{-b_\lambda}^0 \exp\{-(w(x+b_\lambda) - w(b_\lambda))\}dx, \quad Z^+_{\lambda,w} = \int_0^{c_\lambda},$$

$$Z^-_w = \int_{-\infty}^0 e^{-w(x)}dx, \quad Z^+_w = \int_0^\infty e^{-w(x)}dx .$$

Then the law convergence of $Z^-_{\lambda,w}$ to Z^-_w follows immediately from Theorem 6 (in particular, from (2.48)). As for $Z^+_{\lambda,w}$, Theorem 7 alone is not enough; in fact, we have to show the uniform smallness (w.r.t. λ) of the tail $\int_{r\wedge c_\lambda}^{c_\lambda}$ for large r and this is done by using the inequality

$$P\{V_\lambda(t_k) \in \Gamma_k,\ 1 \le \forall k \le n\} \le C\bar{P}\{w(t_k),\ 1 \le \forall k \le n\}, \tag{3.10}$$

where $0 \le t_1 < t_2 < \cdots < t_n$ and Γ_k, $1 \le k \le n$, are Borel sets in $(0,\infty)$. The condition (E) is used for (3.10).

Remark. Our arguments remain valid when the conditions (D_Λ) and (D) are replaced by the following.
Condition (D'_Λ). *Let* $\Lambda = \{\lambda_n\}$ *be a given positive sequence tending to* ∞ *and let it be fixed. There exists a positive sequence* $\{\alpha_n\}$ *with* $\alpha_n = o(\lambda_n/\log\lambda_n)$ *and such that* $W^{\alpha_n}_{\lambda_n}$ *converges in law, as* $n \to \infty$, *to some Lévy process* $\tilde{W} = \{w(x),\ x \ge 0,\ \tilde{P}\}$ *satisfying the conditions* (A) *and* (B).
Condition (D′). *For any positive sequence* $\{\lambda'_n\}$ *tending to* ∞ *there exists a subsequence* $\Lambda = \{\lambda_n\}$ *of* $\{\lambda'_n\}$ *for which the condition* (D'_Λ) *is satisfied.*
Thus Theorem 9 still holds when the condition (D) is replaced by (D′). On the other hand the conditions (C) and (E) seem too strong and it is desirable to remove or relax these conditions.

Examples. (i) Let W be a strictly stable Lévy process with exponent $\alpha \in (0,2)$ such that $0 < \rho = P\{w(1) > 0\} < 1$. Then W satisfies all the conditions (A) $\sim$ (E) and $h(x) = \text{const.}x^{\alpha(1-\rho)}$. The verification of (C) and (E) is done by using, in detail, the explicit formula on $h_\lambda(x)$ obtained by Rogozin [17].
(ii) Spectrally negative Lévy processes satisfy the conditions (C) and (E) since $h_\lambda(x)^{-1}h_\lambda(y) = h(x)^{-1}h(y)$, $x,y \in (0,\lambda)$, for any λ. So any Lévy process W such that

$$E\{e^{i\xi w(1)}\} = \exp\int_{-\infty}^0 (e^{i\xi x} - 1 - i\xi x) \mid x \mid^{-\alpha-1} a(x)\, dx,$$

with $1 < \alpha < 2$ and $0 < c_1 \le a(x) \le c_2$, satisfies all the conditions (A)$\sim$(E).

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