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Random Point Fields Associated with Fermion, Boson and Other Statistics

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Abstract.

We show that the grand canonical ensembles of ideal gas under Fermi, Boson and other statistics give simple examples of the random point fields studied in the previous papers [13, 14, 15]. Also we present two classes of nonsymmetric integral operators for which such random point fields do exist.

§1. Introduction

In the present paper we are concerned with the nonnegativity problem of certain generalization of determinants and permanents denoted by $\det_\alpha$ in our previous paper [14]. The problem is almost equivalent to the existence problem of those random point fields or point processes whose Laplace transforms are given as the Fredholm determinants to the power $-1/\alpha$ of certain integral operators. They are also closely related to the Fermi, Boson and other statistics in quantum statistical mechanics of the ideal gas. Indeed, the grand canonical ensembles under these statistics (if any) are special examples of our random point fields.

Definition 1.1. Let α be a real number and $A = (a_{ij})$ be a square matrix of size n. Given a permutation σ, denote the number of cycles by $\nu(\sigma)$. The following quantity is called the α-permanent of A in [18]:

$$\det_\alpha(A) = \sum_{\sigma \in S_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.1}$$

where S_n is the symmetric group.

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For instance,

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \alpha a_{12}a_{21},$$

$$\begin{aligned} \det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} \\ &\quad +\alpha(a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22} + a_{23}a_{32}a_{11}) \\ &\quad +\alpha^2(a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}). \end{aligned}$$

It is immediate to see

$$\det_{-1}(A) = \det(A), \quad \det_1(A) = \mathrm{per}(A) \tag{1.2}$$

and

$$\det_0(A) = a_{11}a_{22}\dots a_{nn}. \tag{1.3}$$

In [14] we proved the following (though not stated directly there):

Theorem 1.2. *There holds the inequality* $\det_\alpha(A) \geq 0$ *if* α *and* A *satisfy one of the following three conditions* (A^-), (A) *and* (B).
(A^-) $\alpha \in \{-1/m \mid m = 1, 2, \dots\}$ *and* A *is nonnegative definite.*
(A) $\alpha \in \{2/m \mid m = 1, 2, \dots\}$ *and* A *is nonnegative definite.*
(B) $\alpha \in (0, \infty)$ *and* A *is a nonnegative matrix.*

The sufficiency of (B) is obvious from Definition 1.1. We give an alternative proof of the case (A) below in Section 3. In [14] we also proposed the following.

Conjecture 1.3. *If* $0 \leq \alpha \leq 2$, $\det_\alpha(A) \geq 0$ *for nonnegative definite matrix* A *of any size.*

The random point fields mentioned above are defined as follows. For simplicity, let R be a locally compact separable metric space and fix a nonnegative Radon measure λ on R. We define the locally finite configuration space Q over R as the set of nonnegative integer-valued Radon measures ξ on R and say that a function f is a test function if it is nonnegative and its support is compact. For a test function f and a (locally finite) configuration $\xi = \sum \delta_{x_i} \in Q$ we denote

$$e_f(\xi) = \exp\left(- \int_R f(x)\xi(dx) \right).$$

Now let α be a real number and K be a locally trace class integral operator on $L^2(R, \lambda)$, i.e., the restriction K_Λ of K to any compact set Λ

is of trace class. The Fredholm determinant $\mathrm{Det}(I+T)$ for a trace class operator T is defined as $\prod_{i=1}^{\infty}(1+\lambda_i)$, where $\lambda_i, i \geq 1$ are the eigenvalues (counting the multiplicity) of T.

Definition 1.4. A probability measure μ on Q will be called a random point field associated with (α, K) if it satisfies for any test function f

$$\int_Q \mu(d\xi)e_f(\xi) = \mathrm{Det}(I + \alpha\varphi K)^{-1/\alpha},$$

where $\varphi = 1 - e^{-f}$. In particular, μ is called a fermion point process and a boson point process in [8, 9] according as $\alpha = -1$ and $\alpha = 1$. Some people use the terminology "determinantal processes" for fermion processes (cf. [7, 16]).

In [13, 14] we essentially proved the following:

Theorem 1.5. *Assume that the kernel $K(x,y)$ is continuous and, in addition, that the operator norm K is so small that $\|\alpha K\| < 1$ when $\alpha < 0$. Set $J = (I + \alpha K)^{-1}K$.*
(i) The random point field μ associated with (α, K) exists and is unique if $\det_\alpha(J(x_i, x_j))_{i,j=1}^n$ is nonnegative for any n and any $x_1, \ldots, x_n$.
(ii) If the random point field μ associated with (α, K) exists, then both $\det_\alpha(J(x_i, x_j))_{i,j=1}^n$ and $\det_\alpha(K(x_i, x_j))_{i,j=1}^n$ are nonnegative for any n and any $x_1, \ldots, x_n$.

Combining Theorems 1.2 and 1.5 we showed

Theorem 1.6. ([14]) *The random point field μ associated with (α, K) exists and is unique if (α, K) satisfies one of the following conditions:*
(A^-) *$\alpha \in \{-1/m \mid m = 1, 2, \ldots\}$, $\|\alpha K\| < 1$ and K is nonnegative definite.*
(A) *$\alpha \in \{2/m \mid m = 1, 2, \ldots\}$ and K is nonnegative definite.*
(B) *$\alpha \in (0, \infty)$ and the kernel $J(x,y)$ defined by $J = (I + \alpha K)^{-1}K$ is nonnegative.*
Here $\|\cdot\|$ stands for the operator norm.

The case where $\|\alpha K\| = 1$ with $\alpha < 0$ can also be treated in [14, 15] although the operator J becomes unbounded.

In [13, 14, 15] we did not study the case where K is a nonsymmetric operator. But we can show the following:

Theorem 1.7. *Let $R = \mathbb{R}^1$, λ be the Lebesgue measure and $T_t, t \geq 0$ be the transition semigroup of a one dimensional diffusion process or* *let $R = \mathbb{N}$, λ be a counting measure and $T_t, t \geq 0$ be the transition semigroup of a birth and death process. Then the random point field μ associated with $(-1, T_t)$ exists and is unique.*

We would like to emphasize that T_t can be nonsymmetric in the above Theorem 1.7 (cf. [3, 12]).

We had better to mention here on a rather classical theorem of Karlin and McGregor [4, 5] from which Theorem 1.7 above follows immediately by using Theorem 1.5. Recall that a matrix A is called totally positive if all of its square minors are nonnegative and that a kernel function $K(x, y)$ is called totally positive if $\det(K(x_i, y_j))_{i,j=1}^n$ is nonnegative for any n and any $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Theorem 1.8 (Karlin-McGregor). *Let $p(t, x, y)$ be the transition probability density of a one dimensional diffusion process or a birth and death process. Then for each $t > 0$ the kernel function $p(t, x, y)$ is totally positive.*

The proof of Karlin and McGregor is simple and is based only on the two facts: the strong Markov property and the one dimensionality. So it may also be applied to discrete time nearest neighbor random walks on $\mathbb{Z}^1$ under suitable settings (cf., [6]). Notice that $p(t, x, y)$ is not necessarily symmetric in x and y.

§2. Fermi statistics, Boson statistics and other statistics

Consider quantum statistical mechanics of ideal gas. If the energy levels are E_i, then the grand canonical partition function Z is given in text books, for instance [2] as

$$Z = \prod(1 + ze^{-\beta E_i}) \text{ and } Z = \prod(1 - ze^{-\beta E_i})^{-1}$$

under fermi statistics and boson statistics, respectively. If we introduce the trace class diagonal operator

$$J = \mathrm{diag}(ze^{-\beta E_1}, ze^{-\beta E_2}, \ldots)$$

on $\ell^2(\{1, 2, \ldots\})$ and a parameter α, then they can be written as

$$Z = \mathrm{Det}(I - \alpha J)^{-1/\alpha}$$

with $\alpha = -1$ for fermi statistics and $\alpha = 1$ for boson statistics.

For general values of α we might consider the α-statistics. If the underlying space R consists of a single point, then such statistics exist for

$\alpha \in \{1/m \mid m = 1, 2, \dots\}$ or $\alpha \in (0, \infty)$ and the corresponding distributions are called negative binomial or generalized binomial, respectively. There are some attempts to generalize these distributions to spaces consisting of two or more points [1]. Anyway we need some restriction on α in order to consider α-statistics.

The grand canonical ensemble, say μ, under α-statistics is, if any, described in terms of its Laplace transform as

$$\int_Q \mu(d\xi) e_f(\xi) = \frac{\mathrm{Det}(I - \alpha e^{-f} J)^{-1/\alpha}}{\mathrm{Det}(I - \alpha J)^{-1/\alpha}}. \tag{2.1}$$

For fermion and boson cases it is immediate to see, by setting f to be a linear combination of indicator functions of intervals and then by expanding the (infinite) product into the (infinite) sum, that one can obtain the micro canonical ensembles.

Thus, if we introduce an operator $K = (I - \alpha J)^{-1} J$, then we obtain

$$\int_Q \mu(d\xi) e_f(\xi) = \mathrm{Det}(I + \alpha \varphi K)^{-1/\alpha}.$$

Consequently, the grand canonical ensemble μ is the random point field over the set $\{1, 2, \dots\}$ associated with (α, K) in our terminology. By Theorem 1.6 we may consider the α-statisitcs as a real object at least for $\alpha \in \{-1/m \mid m = 1, 2, \dots\} \cup [0, \infty)$ since J is now nonnegative definite and has nonnegative entries provided that $\|\alpha J\| < 1$ when α is positive.

Moreover, the operator J may not be of trace class nor diagonal. Indeed, J can be a locally trace class operator and one can consider the infinite volume limit of grand canonical ensembles as is shown by the following theorem which is a restatement of results in [14]:

Theorem 2.1. *Let α be a real number and J be a locally trace class operator. The random point field μ satisfying (2.1) exists if (α, J) satisfies one of the following conditions.*
(A^-) $\alpha \in \{-1/m \mid m = 1, 2, \dots\}$ *and J is nonnegative definite.*
(A) $\alpha \in \{2/m \mid m = 1, 2, \dots\}$ *and J is nonnegative definite with* $\|\alpha J\| < 1$.
(B) $\alpha \in (0, \infty)$ *and the kernel $J(x, y)$ is nonnegative with* $\|\alpha J\| < 1$.

The points of the proof of Theorem 2.1 are to introduce restrictions J_Λ of J to compact subsets and to show that the operators $(I - \alpha J_\Lambda)^{-1} J_\Lambda$ converge to $K := (I - \alpha J)^{-1} J$ as $\Lambda \to R$. Then it turns out that the grand canonical ensemble over the compact subsets Λ converges to the limiting grand canonical ensemble which is nothing but our random point field associated with (α, K).

Roughly to say, the random point fields associated with powers of Fredholm determinants are Gibbs random fields as the argument above suggests. If R is discrete, say d-dimensional square lattice $\mathbb{Z}^d$, then we obtained the following rigorous result in the fermion case($\alpha = -1$). Since the fermion point fields have no multiple points, we may safely identify the configuration space Q with the power set of R or $\{0,1\}^R$.

Theorem 2.2. ([15]) *Let $R = \mathbb{Z}^d$ and λ be the counting measure. Assume the the operator $K : \ell^2(R) \to \ell^2(R)$ is positive definite with $\|K\| < 1$. Set $J = (I-K)^{-1}K$ and write its restriction to a subset $\Lambda_1 \times \Lambda_2$ by J_{Λ_1,Λ_2}. Then the fermion point field μ associated with K exists and is the unique Gibbs measure for the potential*

$$U(x_0|\xi) = J(x_0,x_0) - J_{\{x_0\},\xi}(J_{\xi,\xi})^{-1}J_{\xi,\{x_0\}}, (x_0 \in R, \xi \in Q).$$

Here the potential $U(x_0|\xi)$ is defined by

$$U(x_0|\xi) = -\log \frac{\mu(\xi\{x_0\} = 1|\mathcal{B}_{\{x_0\}^c})(\xi)}{\mu(\xi\{x_0\} = 0|\mathcal{B}_{\{x_0\}^c})(\xi)}, \tag{2.2}$$

where $\mathcal{B}_{\{x_0\}^c}$ is the σ-algebra generated by $\xi(x), x \neq x_0$.

To conclude this section we want to point out an analogy. The following formula for Schur functions is well known:

$$\prod_{i,j=1}^{n} (1 - x_i y_j)^{-1} = \sum_p S_p(x)S_p(y),$$

where the summation is taken over all partition $p = (p_1, \dots, p_n), p_1 \geq \cdots \geq p_k \geq 1$, of the number $|p| := p_1 + \cdots + p_n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and $S_p(x)$ is the Schur function. Note that the left hand side is the reciprocal of a certain determinant. Similarly, $\prod_{i,j=1}^n (1 - 2x_i y_j)^{-1/2}$ is expanded in terms of zonal functions $Z_p(x)$ and $Z_p(y)$ with suitable coefficients. Moreover, the case of general α can be expanded in terms of the Jack polynomials which is a new face in the study of symmetric functions. (cf. [10]). The zonal function has been introduced and studied by mathematical statisticians mainly to apply the noncentered Wishart distributions (cf., for instance [11, 17]).

§3. Logarithmic derivatives of Fredholm determinant: Alternative proof of the nonnegativity under Condition (A)

The quantity $\det_\alpha$ can be characterized as follows:

Lemma 3.1. *Let α be a real number and A be a square matrix of size n. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ write $X = \operatorname{diag}(x_1, \dots, x_n)$. Then,*

$$\det_\alpha(A) = \frac{\partial^n}{\partial x_1 \dots \partial x_n}\Big|_{x=0} \det(I - \alpha AX)^{-1/\alpha}.$$

Proof. Denote by E_i the diagonal matrix with 1 on the ith entry and 0 elsewhere. For a while we write $G = (I - \alpha AX)^{-1}$ for simplicity of notations. Then,

$$\frac{\partial}{\partial x_i} \det(I - \alpha AX)^{-1/\alpha} = \operatorname{Tr}(GAE_i) \det(I - \alpha AX)^{-1/\alpha}.$$

Moreover, since $\frac{\partial}{\partial x_j} G = \alpha GAE_j G$, we obtain

$$\begin{aligned} & \frac{\partial}{\partial x_j} \operatorname{Tr}(GAE_{i_1} \dots GAE_{i_k}) \\ = \; & \alpha\{\operatorname{Tr}(GAE_j GAE_{i_1} \dots GAE_{i_k}) + \operatorname{Tr}(GAE_{i_1} GAE_j GAE_{i_2} \dots GAE_{i_k}) \\ & + \dots + \operatorname{Tr}(GAE_{i_1} \dots GAE_{i_k} GAE_j)\} \end{aligned}$$

for any $j, i_1, \dots, i_k \in \{1, 2, \dots, n\}$ and $k \geq 1$. From these two algorithms we obtain the above formula. Q.E.D.

The above proof also shows the following:

Lemma 3.2. *For all $k, i_1, \dots, i_k \in \{1, 2, \dots, n\}$,*

$$\det_\alpha(A_{i_1,\dots,i_k}) = \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}}\Big|_{x=0} \det(I - \alpha AX)^{-1/\alpha},$$

where $A_{i_1,\dots,i_k}$ stands for the square matrix of size k whose (j,k)-element is the (i_j, i_k)-element of A.

Example 3.3. *Let $\alpha = 2$, $X = \operatorname{diag}(x_1, \dots, x_n)$ and C be a positive definite matrix of size n. Assume $\max|x_i|$ is sufficiently small. Then,*

$$\begin{aligned} & \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sqrt{\det C}} \int_{\mathbb{R}^n} \exp(-\sum_{i=1}^n x_i u_i^2) \exp(-\frac{1}{2}\langle C^{-1}u, u\rangle) du \\ = \; & (\det(I + 2CX))^{-1/2}. \end{aligned}$$

Hence,

$$\det_2(C) = \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sqrt{\det C}} \int_{\mathbb{R}^n} u_1^2 \cdots u_n^2 \exp(-\frac{1}{2}\langle C^{-1}u, u\rangle) du.$$

In other words, if $Z = (Z_1, \dots, Z_n)$ is a Gaussian random variable with mean 0 and covariance matrix C, then

$$\det_2(C) = E[Z_1^2 \cdots Z_n^2].$$

Proposition 3.4. *Let α be a real number and T be a trace class integral operator on $L^2(R, \lambda)$ with kernel $T(x, y)$. Then,*

$$\begin{aligned} &\mathrm{Det}(I - \alpha z T)^{-1/\alpha} \\ = \;& 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_R \cdots \int_R \det_\alpha(T(x_i, x_j))_{i,j=1}^n \lambda(dx_1) \dots \lambda(dx_n) \end{aligned}$$

if $z \in \mathbb{C}$ and $|z|$ is sufficiently small(so that $|\alpha z| \|T\| < 1$).

Proof. If T is a finite dimensional operator, the assertion follows immediately from the Taylor expansion of $\det(I - \alpha z T)^{-1/\alpha}$ in z based on Lemmas 3.1 and 3.2 where λ is the counting measure. The generalization to the trace class operators is obtained by a routine approximation procedure. Q.E.D.

As an application of Proposition 3.4 one can give a proof to the following well-known formula.

Proposition 3.5. *Let $Z(x), x \in R$ be a Gaussian random field with mean 0 and covariance $K(x, y)$ in the sense that*

$$E[\{\int_R Z(x)\varphi(x)\lambda(dx)\}^2] = \int_R \int_R K(x, y)\varphi(x)\varphi(y)\lambda(dx)\lambda(dy).$$

Then,

$$E[\exp(-\int_R Z(x)^2 \varphi(x)\lambda(dx))] = \mathrm{Det}(I + 2\varphi K)^{-1/2}.$$

Proof. Expand the exponential in the right hand side. Then the expectation of each term is expressed as

$$\begin{aligned} & \int_R \cdots \int_R \lambda(dx_1) \dots \lambda(dx_n) E[Z(x_1)^2 \dots Z(x_n)^2] \\ = \;& \int_R \cdots \int_R \lambda(dx_1) \dots \lambda(dx_n) \det_2(K(x_i, x_j))_{i,j=1}^n. \end{aligned}$$

Consequently, we obtain the desired formula from the previous Proposition 3.4. Q.E.D.

Theorem 3.6. *Let A be a nonnegative definite symmetric matrix and $\alpha \in \{2/m \mid m = 1, 2, \dots\}$. Then,*

$$\det_\alpha(A) \geq 0.$$

Moreover, if $Z = (Z_1, \dots, Z_n)$ be a Gaussian random variable with mean 0 and covariance matrix $(1/m)A$ and $Z^{(1)}, \dots, Z^{(m)}$ be m independent copies of Z, then

$$\det_{2/m}(A) = E[\prod_{i=1}^{n}(\sum_{j=1}^{m}(Z_i^{(j)})^2)]. \tag{3.1}$$

Proof. Consider

$$E[\exp(-\sum_{i=1}^{n} x_i \sum_{j=1}^{m}(Z_i^{(j)})^2)] = \det(I + (2/m)XA)^{-m/2}$$

and differentiate it in $x_1, \dots, x_n$ successively. Then we obtain the formula (3.1) and so the desired nonnegativity. Q.E.D.

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