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The Dobrushin-Hryniv Theory for the Two-Dimensional Lattice Widom-Rowlinson Model

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Abstract.

We consider the fluctuation of the phase boundary separating two phases of the Widom-Rowlinson model in the plane square lattice. The phase boundary is conditioned to have specified values of the area underneath and the height difference of two end points. Dobrushin and Hryniv studied the phase boundary of the Solid-on-Solid model [DH1] and of the Ising model [DH2], and obtained the central limit theorem for the fluctuation of the phase boundary from the Wulff profile. The phase boundary of the Ising model is well approximated by that of the Solid-on-Solid model with the aid of the cluster expansion. Their argument seems to be applicable to the general models which have polymer representation. We apply their theory to the Widom-Rowlinson model.

§1. Introduction

Let $\mathbf{Z}^2$ be the square lattice and let $\Lambda_{L,M}$ be the rectangle $[1, L-1] \times [-M, M]$ in $\mathbf{Z}^2$. We consider a system of particles in $\Lambda_{L,M}$. These particles are of two types, either A or B. There is strong repulsive interaction between particles of different types. Namely, a B particle can not occupy a site within distance $\sqrt{2}$ from a site where an A particle sits, and vice versa.

A *configuration* ω is a function from $\Lambda_{L,M}$ to $\{-1, 0, +1\}$. $\omega(x) = +1$ denotes that the site x is occupied by an A particle, $\omega(x) = -1$ denotes that x is occupied by a B particle and $\omega(x) = 0$ denotes that there is no particle at x. We say that a configuration ω is *feasible* if

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$\omega(x)\omega(y) \geq 0$ for all pairs x, y with $|x-y| \leq \sqrt{2}$, where $|\cdot|$ denotes the Euclidean distance.

Let $\Omega_{L,M}$ denote the set of all feasible configurations in $\Lambda_{L,M}$. The Hamiltonian of our system is a function on $\Omega_{L,M}$ given by

$$H(\omega) = \sum_{x \in \Lambda_{L,M}} \mu\big(1-\omega(x)^2\big) \tag{1.1}$$

for every $\omega \in \Omega_{L,M}$. Here, μ denotes the chemical potential.

Let $h>0$ be fixed and assume that $M > Lh$. Then we can put the following boundary condition:

$$\eta^h(x) = \begin{cases} +1, & \text{if } x^2 > \lceil hx^1 \rceil, \\ 0, & \text{if } x^2 = \lceil hx^1 \rceil, \\ -1, & \text{otherwise}, \end{cases}$$

for every $x = (x^1, x^2) \in \partial\Lambda_{L,M} = [0, L] \times [-M-1, M+1] \setminus \Lambda_{L,M}$. Let $\Omega^h_{L,M}$ denote the set of all configurations ω in $\Omega_{L,M}$ such that $\omega \circ \eta^h$ is feasible, where $\omega \circ \eta$ is given by

$$\omega \circ \eta(x) = \begin{cases} \omega(x), & \text{if } x \in \Lambda_{L,M}; \\ \eta(x), & \text{if } x \in \partial\Lambda_{L,M}. \end{cases}$$

The conditional Gibbs distribution on $\Omega^h_{L,M}$ with the boundary condition η^h is given by

$$P^h_{L,M}(\omega) = \big(Z^h_{L,M}\big)^{-1} \exp\{-\mu|S^0(\omega)|\}, \tag{1.2}$$

where $S^0(\omega)$ is the set of points in $\Lambda_{L,M}$ such that ω takes 0 value, $|S|$ denotes the cardinality of a set S, and $Z^h_{L,M}$ is the normalizing constant, which we call the *partition function.*

For a feasible configuration ω, we call a connected component of $S^0(\omega)$ a *contour.* Among contours we can find a unique contour which connects $(0,0)$ with $(L, \lceil hL \rceil)$. We call this the *separating contour* with the starting point $(0,0)$ and the end point $(L, \lceil hL \rceil)$, and denote it by $\Gamma(\omega)$. Let $S^h_{L,M}$ denote the collection

$$\{\Gamma(\omega); \omega \in \Omega^h_{L,M} \text{ is feasible}\}.$$

The aim of this paper is to investigate the fluctuation of the separating contour via Dobrushin-Hryniv theory.

the backbone

We say that a set $C \subset \mathbf{Z}^2$ is $*$*connected* if for every $x, y \in C$, there exist a sequence $z_0 = x, z_1, \ldots, z_m = y$ in C such that $|z_i - z_{i-1}| \leq \sqrt{2}$ for every $1 \leq i \leq n$. A *hole* of a connected set $C \subset \mathbf{Z}^2$ is a finite $*$connected component of $C^c = \mathbf{Z}^2 \setminus C$. Let $C_1, C_2, \dots, C_n$ be connected subsets of $\Lambda_{L,M}$. We say that contours $\{C_j\}$ are *compatible* if they are connected components of the set $\cup_{1 \leq j \leq n} C_j$. We also say that $\{C_j\}$ are *compatible* with a connected set D if $\{D, C_j\}$ are compatible for every j. Then the partition function $Z^h_{L,M}$ can be rewritten as

$$Z^h_{L,M} = \sum_{\Gamma \in \mathcal{S}^h_{L,M}} \sum_{\{C_j\}} 2^{N(\Gamma)} \exp\{-\mu|\Gamma|\} \prod_j (2^{N(C_j)} \exp\{-\mu|C_j|\}),$$

where the second summation is taken over compatible families $\{C_j\}$, which are compatible with Γ, $|\Gamma|$ is the number of points in Γ and $N(C)$ is the number of holes in C. Therefore, we can find μ_0 sufficiently large so that we have a cluster expansion (see [KP])

$$(1.3) \qquad Z^h_{L,M} = \sum_{\Gamma \in \mathcal{S}^h_{L,M}} \exp\{-\mu|\Gamma| + N(\Gamma)\ln 2 + \sum_{\substack{\Lambda \subset \Lambda_{L,M} \\ \Lambda\, c\, \Gamma}} \Phi(\Lambda)\}$$

for $\mu > \mu_0$, where $\Lambda\, c\, \Gamma$ denotes that Λ is compatible with Γ. Moreover, the function $\Phi(\Lambda)$ satisfies the estimate

$$(1.4) \qquad \sum_{\Lambda \ni 0} |\Phi(\Lambda)| e^{(\mu-\mu_0)|\Lambda|} < 1,$$

and $\Phi(\Lambda) = 0$ unless Λ is connected. Let

$$Z^+_{L,M} = \exp\{ \sum_{\Lambda \subset \Lambda_{L,M}} \Phi(\Lambda)\}.$$

Dividing both sides of (1.3) by $Z^+_{L,M}$, we have

$$(1.5) \qquad \frac{Z^h_{L,M}}{Z^+_{L,M}} = \sum_{\Gamma \in \mathcal{S}^h_{L,M}} \exp\{-\mu|\Gamma| + N(\Gamma)\ln 2 - \sum_{\substack{\Lambda \subset \Lambda_{L,M} \\ \Lambda\, i\, \Gamma}} \Phi(\Lambda)\},$$

where $\Lambda\, i\, \Gamma$ denotes that Λ is incompatible with Γ. We use the summand in the right hand side of (1.5) as a statistical weight of the separating contour Γ. Let $\Gamma \in \mathcal{S}^h_{L,M}$. We extract a self-avoiding path from Γ in the following way.

First we define an order of preference among four directions;
up > down> right > left.

This order naturally defines an order among self-avoiding paths connecting $(0,0)$ with $(L, \lceil hL \rceil)$. To be more precise, let $\pi = \{x_1, x_2, \dots, x_n\}$

and $\pi' = \{y_1, y_2, \dots, y_k\}$ be two self-avoiding paths connecting $(0,0)$ with $(L, \lceil hL \rceil)$. Let j_0 be the first number j such that $x_j \neq y_j$. We define that $\pi > \pi'$ if the direction of the ordered edge $\{x_{j_0-1}, x_{j_0}\}$ is preferred to the direction of the ordered edge $\{y_{j_0-1}, y_{j_0}\}$. Now, let

$$\Pi_\Gamma := \{\pi : \text{ self-avoiding path in } \Gamma \text{ connecting } (0,0) \text{ with } (L, \lceil hL \rceil)\}.$$

Let $\pi(\Gamma)$ be the unique maximal element of Π_Γ with respect to this order. We call $\pi(\Gamma)$ the *backbone* of Γ. This backbone will play the role of the phase separation line of the 2D Ising model.

For $\Gamma \in \mathcal{S}^h_{L,M}$, $\pi(\Gamma)$ separates $[0, L] \times [-M-1, M+1]$ into two $*$connected components. One is above $\pi(\Gamma)$ and the other is below $\pi(\Gamma)$. Let $a^-(\pi(\Gamma))$ and $a^+(\pi(\Gamma))$ be the number of points in $\mathbf{Z}^{2^*} \cap [0, L] \times [-M-1, M+1]$, which are below $\pi(\Gamma)$ and above $\pi(\Gamma)$, respectively. Here, $\mathbf{Z}^{2^*} = \mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. The area $a(\pi(\Gamma))$ is defined by

$$a(\pi(\Gamma)) := a^-(\pi(\Gamma)) - a^+(\pi(\Gamma)). \tag{1.6}$$

This value is independent of M if M is sufficiently large.

free energy

If μ is sufficiently large, (1.5) has a limit as $M \to \infty$:

$$\lim_{M\to\infty} \frac{Z^h_{L,M}}{Z^+_{L,M}} = \sum_{\Gamma \in \mathcal{S}^h_L} \exp\left\{ -\mu|\Gamma| + N(\Gamma)\ln 2 - \sum_{\substack{\Lambda \subset \Lambda_{L,\infty} \\ \Lambda\, i\, \Gamma}} \Phi(\Lambda) \right\}, \tag{1.7}$$

where $\mathcal{S}^h_L := \cup_{M>0} \mathcal{S}^h_{L,M}$, $\Lambda_{L,\infty} := [1, L-1] \times (-\infty, \infty) \cap \mathbf{Z}^2$.

Let $W(\Gamma)$ be the weight in the right hand side of (1.7);

$$W(\Gamma) := \exp\left\{ -\mu|\Gamma| + N(\Gamma)\ln 2 - \sum_{\substack{\Lambda \subset \Lambda_{L,\infty} \\ \Lambda\, i\, \Gamma}} \Phi(\Lambda) \right\}$$

for $\Gamma \in \cup_{h\in\mathbf{R}} \mathcal{S}^h_L = \mathcal{S}_L$. For $\Gamma \in \mathcal{S}_L$, we denote by $A(\Gamma) = (0,0)$ and $B(\Gamma) = (L, k(\Gamma))$ the starting point and endpoint of Γ, respectively.

For $\zeta \in \mathbf{C}$, we define

$$\varphi(\zeta) := \lim_{L\to\infty} \frac{1}{L} \ln \sum_{\Gamma \in \mathcal{S}_L} e^{\mu\zeta k(\Gamma)} W(\Gamma). \tag{1.8}$$

if the limit exists. This is the free energy of the height of the last point of Γ. For $\Gamma \in \mathcal{S}_L$, we define $\{X_L(t);\ t \in [0,1]\} = \{X_L(t;\Gamma);\ t \in [0,1]\}$

by

$$\begin{cases} X_L\left(\frac{j}{L}\right) = \max\{k \in \mathbf{Z};\ (j,k) \in \pi(\Gamma)\}, \\ X_L(t) \ \ = (j+1-Lt)X_L\left(\frac{j}{L}\right) + (Lt-j)X_L\left(\frac{j+1}{L}\right) \quad (j \le Lt \le j+1) \end{cases}$$

Let P_L be the probability measure on $\mathcal{S}_L$ defined by

$$P_L(\Gamma) = \left[\sum_{\Gamma' \in \mathcal{S}_L} W(\Gamma')\right]^{-1} W(\Gamma). \tag{1.9}$$

Theorem There exists $\mu_1 > \mu_0$ such that for $\mu > \mu_1$, (1.9) is well defined on $\mathcal{S}_L$ and the followings hold.

Assume that for $h > 0$ and $a \ge \frac{h}{2}$ there exist a $\delta > 0$ and a pair $(\zeta_0, \zeta_1) \in \mathbf{R}^2$ with $\max\{|\zeta_0 + \zeta_1|, |\zeta_1|\} \le 1 - \frac{\delta}{\mu}$ such that

$$\frac{1}{\mu}\int_0^1 \nabla_{(\zeta_0,\zeta_1)}\varphi(\zeta_0(1-x)+\zeta_1)dx = (a,h).$$

Then the process

$$Y_L(t) := \frac{1}{\sqrt{L}}\left\{X_L(t) - \frac{L}{\mu}\int_0^t \varphi'(\zeta_0(1-x)+\zeta_1)dx\right\}$$

under $P_L(\cdot \mid a(\pi(\Gamma)) = \lceil aL^2 \rceil, k(\Gamma) = \lceil hL \rceil)$ converges

$$Y(t) = \frac{1}{\mu}\int_0^t \sqrt{\varphi''(\zeta_0(1-x)+\zeta_1)}dB(x)$$

conditioned that

$$\int_0^1 Y(t)dt = 0, \quad Y(1) = 0.$$

Here, $\{B(t)\}_{t \ge 0}$ is the one dimensional standard Brownian motion.

Remark Although $X_L(t)$ is defined by the backbone $\pi(\Gamma)$, the width (in the x^2 direction) of the separating contour Γ is negligible and, hence, the limiting process $Y(t)$ depends only on Γ. So, the choice of the backbone is for technical reasons only.

The proof of the theorem goes along the line of [DH1,2], and we regard our model as a perturbation of Solid-on-Solid(SOS) model. This SOS model corresponds to the ensemble of (site) self avoiding paths in $[0, L] \times \mathbf{Z}$ starting from $(0, 0)$ and ending at a site in $\{x^1 = L\}$, which do not go back in the horizontal direction. Let us call such a path an *SOS path*. There are no $\{\Lambda_\alpha\}$'s for the SOS model.

An SOS path will be cut into simple polymers. A simple polymer is obtained from intersection of an SOS path with a vertical line $\{x^1 = j\}$ for some $1 \le j \le L$, shifted so that its starting point is at height zero. So, it has a form $\{(j,0),(j,1),\dots,(j,k)\}$ for some $k \ge 0$ or $\{(j,0),(j,-1),\dots,(j,k)\}$ for some $k < 0$.

Let

$$Q(\zeta) = \sum_{\substack{\xi: simple\ polymer \\ starting\ from\ (0,0)}} e^{\mu\zeta k(\xi) - \mu|\xi|}$$

where $k(\xi)$ and $|\xi|$ are the height of the endopoint of ξ and number of sites in ξ, respectively. Then

$$\sum_{\Gamma: SOS\ path\ in\ [0,L]\times\mathbf{Z}} e^{\mu}\zeta k(\Gamma) W(\Gamma) = Q(\zeta)^L.$$

We would like to show that

$$Q(\zeta)^{-L} \sum_{\Gamma\in\mathcal{S}_L} e^{\mu\zeta k(\Gamma)} W(\Gamma)$$

has a form;

$$\sum_{\substack{I_1,\dots,I_r\subset[0,L];\\ disjoint\ intervals}} \prod_{j=1}^{r} X(I_j) \tag{1.10}$$

which admits a cluster expansion, and is equal to $e^{L\varphi_L(\zeta)}$ for some function $\hat{\varphi}_L$ analytic in ζ. Further, we need that the second derivative in ζ of $\hat{\varphi}_L$ is sufficiently small in absolute value compared to the second derivative (in ζ) of $\ln Q$ in order to show the non-degeneracy of the covariance of the limit process $Y(t)$.

These two points, i.e., a) existence and analyticity of the free energy and b) non-degeneracy of the limiting covariance are to be checked depending on our model. Remaining arguments are the same as in [DH1,2], and we present them for the sake of completeness.

Finally, recent progress of understanding the fluctuation of interfaces provides us a beautiful and systematic approach using the renewal theory ([Ioffe], [KH]). For our problem, it seems also possible to follow this new line. However, what we have to check are the same, and at this stage we are not able to present our result in a compact form following this general approach.

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is possible for the continuum Widom-Rowlinson model, which should be true, but we have not completed the whole story, yet.

§2. Local limit theorem

We will first show the existence of the limit (1.8) and its analyticity. Let $\Gamma \in \mathcal{S}_L$, $A(\Gamma) = (0,0)$, $B(\Gamma) = (L, k(\Gamma))$ be its starting and ending points. Let $\pi(\Gamma)$ be the backbone of Γ connecting $A(\Gamma)$ with $B(\Gamma)$. We decompose $\Gamma \setminus \pi(\Gamma)$ into connected components $\{C_j\}_{j=1}^{s}$. As in [DH2] we expand

$$\exp\left\{-\sum_{\substack{\Lambda\subset\Lambda_{L,\infty}\\ \Lambda\, i\, \Gamma}} \Phi(\Lambda)\right\} = \sum_{n=0}^{\infty} \sum_{\substack{\Lambda_1,\dots,\Lambda_n\subset\Lambda_{L,\infty}\\ \Lambda_\nu\, i\, \Gamma}} \prod_{\nu=1}^{n} \left(e^{-\Phi(\Lambda_\nu)} - 1\right).$$

Then

$$(2.1)\qquad \sum_{\Gamma\in\mathcal{S}_L} e^{\mu\zeta k(\Gamma)} W(\Gamma)$$
$$= \sum_{k=-\infty}^{\infty} \sum_{\substack{\pi;(0,0)\to(L,k)\\ self-avoiding\\ \pi\subset\Lambda_{L,\infty}}} \sum_{\substack{C_1,\dots,C_s;compatible\\ C_\nu\, i\, \pi,\ C_\nu\cap\pi=\emptyset\\ \pi;backbone\ of\ \pi\cup C_1\cup\ldots\cup C_s}} \sum_{\substack{\Lambda_1,\ldots\Lambda_t;\\ \Lambda_\alpha is connected\\ \Lambda_\alpha\, i\, \pi\cup C_1\cup\ldots\cup C_s}}$$
$$e^{\mu\zeta k} e^{-\mu|\pi| + N(\pi,C_1,\dots,C_s)\ln 2 - \mu\sum_{\nu=1}^{s}|C_\nu|} \prod_{\alpha=1}^{t} \left(e^{-\Phi(\Lambda_\alpha)} - 1\right),$$

where $N(\pi, C_1, \dots, C_s)$ denotes the number of holes of $\pi \cup \cup_{\nu=1}^{s} C_\nu$.

polymers

Defining polymers is to cut the separating contour Γ into elementary pieces according to the additional information of $\{\Lambda_\alpha\}$. A simplest way to do it would be to cut γ at lines $\{x^1 = \ell + \frac{1}{2}\}$ of dual lattice such that they intersect only one edge of Γ and intersection with edges of Λ_α's is empty. But the resulting pieces, say polymers, do interact. Even a "simple polymer" can interact with some polymers.

For example, a part of Γ like Fig 1 will be separated into two parts: one having $\sqsubset$ shape and one point to the right of it. If instead of one point, there comes a simple polymer of height three to the right of $\sqsubset$, then they are put together and there is no natural way to cut them (Fig. 2).

Thus, in a natural way of cutting procedure, Γ will be cut into interacting polymers. This causes us to introduce a polymer chain below, working with which we can use usual cluster expansion. The idea is to

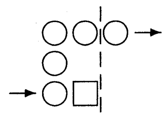

Fig. 1

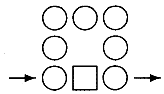

Fig. 2

treat a cluster of polymers interacting each other possibly through simple polymers which are at neiboring sites of such 'active' polymers.

Let $\hat{l} \leq \hat{r}$ be positive integers. A polymer ξ with base $[\hat{l}, \hat{r}]$ is a collection $\xi = (\gamma, C_1, \dots, C_s, \Lambda_1, \dots, \Lambda_t)$ such that

(a) γ is a self-avoiding path in $\{\hat{l} \leq x^1 \leq \hat{r}\}$ starting from $(\hat{l}, 0)$ and ending at a point $(\hat{r}, k)$ in $\{x^1 = \hat{r}\}$. Here, we understand γ as an edge set.

(b) $\{C_\nu\}_{\nu=1}^s$ is a compatible family of connected subsets of $\{x \in \Lambda_{L,\infty}; \hat{l} \leq x^1 \leq \hat{r}\}$ such that
  (b-1) $C_\nu \cap V(\gamma) = \emptyset$, where $V(\gamma)$ is the set of vertices in γ.
  (b-2) $C_\nu \cup V(\gamma)$ is connected.
  (b-3) γ is the backbone of $\gamma \cup C_1 \cup \dots \cup C_s$ with starting point $(\hat{l}, 0)$ and endpoint $(\hat{r}, k)$.

(c) $\{\Lambda_\alpha\}_{\alpha=1}^t$ is a collection of connected subsets of $\{x \in \Lambda_{L,\infty}; \hat{l} \leq x^1 \leq \hat{r}\}$ such that

$$\Lambda_\alpha \, i \, V(\gamma) \cup \cup_{\nu=1}^s C_\nu .$$

Besides these conditions, we need a technical condition for a polymer. This condition is to subtract 'simple polymers' from the phase separating contour Γ as much as possible.

An edge e is called an *edge of* ξ if

$$e \in \gamma \cup \mathcal{E}(\cup_{\nu=1}^s C_\nu \cup \cup_{\alpha=1}^t \Lambda_\alpha) \cup \mathcal{E}(\gamma, \cup_{\nu=1}^s C_\nu \cup \cup_{\alpha=1}^t \Lambda_\alpha),$$

where $\mathcal{E}(\cup_{\nu=1}^s C_\nu \cup \cup_{\alpha=1}^t \Lambda_\alpha)$ is the set of nearest neighbor edges in $\cup_{\nu=1}^s C_\nu \cup \cup_{\alpha=1}^t \Lambda_\alpha$, and $\mathcal{E}(\gamma, \cup_{\nu=1}^s C_\nu \cup \cup_{\alpha=1}^t \Lambda_\alpha)$ is the set of edges that connect γ with $\cup_{\nu=1}^{s} C_\nu \cup \cup_{\alpha=1}^{t} \Lambda_\alpha$. An edge $e = \{x, y\}$ of ξ is *not admissible* if it is a horizontal edge in $\mathcal{E}(\gamma, \cup_{\nu=1}^{s} C_\nu \cup \cup_{\alpha=1}^{t} \Lambda_\alpha)$, such that

(1) The left vertex x is in a connected component D of $\cup_{\nu=1}^{s} C_\nu \cup \cup_{\alpha=1}^{t} \Lambda_\alpha$ and the right vertex y is in $V(\gamma)$,

(2) further, there exists a horizontal edge $e' = \{x', y'\}$ of ξ such that $x' \in V(\gamma)$ and $y' \in D$, where x' is the left vertex of e'.

Other edges of ξ are *admissible.* Also, we identify an edge $\{x, y\}$ of $\mathbf{Z}^2$ with the line segment connecting x and y. Now we introduce the remaining condition (d) for a polymer ξ.

(d) If $\hat{l} < \hat{r}$, then for $\hat{l} \le j < \hat{r}$, $j \in \mathbf{N}$, the line $\ell_j = \{x^1 = j + \frac{1}{2}\}$ intersects at least two admissible edges of ξ.

We call γ the *backbone* of ξ. For two disjoint self-avoiding paths γ_1, γ_2 such that the starting point of γ_2 is nearest neighbor of the endpoint of γ_1, we can define the concatenation $\gamma_1 \circ \gamma_2$ of these paths by simply connecting them.

Let $\xi = (\gamma, C_1, \dots, C_u, \Lambda_1, \dots, \Lambda_v)$ and $\xi' = (\gamma', C_1', \dots, C_w', \Lambda_1', \dots, \Lambda_z')$ be two polymers with bases $[\hat{l}, \hat{r}]$ and $[\hat{l}', \hat{r}']$ $(\hat{l} \le \hat{l}')$, respectively. We say that ξ and ξ' are *compatible* if either of the following conditions holds;

(1) $\hat{r} + 1 < \hat{l}'$,

(2) $\hat{l}' = \hat{r} + 1$, the backbone of

$$\tilde{\Gamma} := \gamma \cup C_1 \cup \dots \cup C_u \cup (\gamma' + (0, k(\gamma))) \cup (C_1' + (0, k(\gamma))) \cup \dots \cup (C_w' + (0, k(\gamma)))$$

is the concatenation $\gamma \circ (\gamma' + (0, k(\gamma)))$, and connected components of the set $\tilde{\Gamma} \setminus \gamma \circ (\gamma' + (0, k(\gamma)))$ are $\{C_1, \dots, C_u, C_1' + (0, k(\gamma)), \dots, C_w' + (0, k(\gamma))\}$. Here, $k(\gamma)$ is the hight of the endpoint of γ.

The family $\{\xi_p\}_{p=0}^{n+1}$ is compatible if ξ_p and $\xi_{p'}$ $(p \ne p')$ are compatible.

Let π be a self-avoiding path in $\Lambda_{L,\infty}$ connecting $(0,0)$ with $(L, k(\pi))$, $\{C_\nu\}_{\nu=1}^{s}$ be a compatible family of connected subsets of $\Lambda_{L,\infty}$ such that

(1) $C_\nu \, i \, \pi$ and $C_\nu \cap \pi = \emptyset$,

(2) π is the backbone of $V(\pi) \cup \cup_{\nu=1}^{s} C_\nu$.

Let also $\{\Lambda_\alpha\}_{\alpha=1}^{t}$ be a collection of connected subsets of $\Lambda_{L,\infty}$ such that $\Lambda_\alpha \, i \, \pi \cup \cup_{\nu=1}^{s} C_\nu$ for each α. We say that the line $\ell_j = \{x^1 = j + \frac{1}{2}\}$ $(0 \le j \le L-1)$ is a *cutting line* of $(\pi, \{C_\nu\}_{\nu=1}^{s}, \{\Lambda_\alpha\}_{\alpha=1}^{t})$ if ℓ_j intersects only one admissible edge of $(\pi, \{C_\nu\}_{\nu=1}^{s}, \{\Lambda_\alpha\}_{\alpha=1}^{t})$.

Let $\ell_0 < \ell_{j_1} < \dots < \ell_{j_n} < \ell_{j_{n+1}} = \ell_{L-1}$ be all the cutting lines of $(\pi, \{C_\nu\}_{\nu=1}^{s}, \{\Lambda_\alpha\}_{\alpha=1}^{t})$. For each $m \in \{0, 1, \dots, n+1\}$, there is a unique edge $e_m = \{B_m, A_{m+1}\}$ of π which intersects ℓ_{j_m}. Let γ_m be the portion of π starting from A_m and ending at B_m. Also let $\{C_\nu^{(m)}\}_{\nu=1}^{s(m)}$ and $\{\Lambda_\alpha^{(m)}\}_{\alpha=1}^{t(m)}$ be the set of elements of $\{C_\nu\}_{\nu=1}^{s}$ and $\{\Lambda_\alpha\}_{\alpha=1}^{t}$ such that they are subsets of $[j_{m-1}+1, j_m] \times (-\infty, \infty) \cap \mathbf{Z}^2$. Then $A_m = (j_{m-1}+1, p)$ for some $p \in \mathbf{Z}$. Thus we obtain the m-th polymer ξ_m by setting

$$\xi_m = (\gamma_m - (0,p), \{C_\nu^{(m)} - (0,p)\}_{\nu=1}^{s(m)}, \{\Lambda_\alpha^{(m)} - (0,p)\}_{\alpha=1}^{t(m)}).$$

By definition, $\{\xi_0, \xi_1, \dots, \xi_{n+1}\}$ are compatible.

For a polymer $\xi_m = (\gamma_m, \{C_\nu^{(m)}\}, \{\Lambda_\alpha^{(m)}\})$, let $k_m = k(\xi_m) = k(\gamma_m)$ be the hight of the endpoint of the self-avoiding path γ_m. Then the hight $k(\pi)$ of the endpoint of the original path π is given by

$$k(\pi) = \sum_{m=0}^{n+1} k(\gamma_m).$$

For a polymer $\xi = (\gamma, \{C_\nu\}_{\nu=1}^{u}, \{\Lambda_\alpha\}_{\alpha=1}^{v})$, set

$$\text{(2.2)}\quad \Psi(\xi) = e^{-\mu|\gamma| + N^*(\gamma, C_1, \dots, C_u)\ln 2 - \mu\sum_{\nu=1}^{u}|C_\nu|} \times \prod_{\alpha=1}^{v}(e^{-\Phi(\Lambda_\alpha)} - 1),$$

Where

$$\begin{aligned} N^*(\gamma, C_1, \dots, C_s) &= N(\gamma, C_1, \dots, C_s) \\ &+ N_l(\gamma, C_1, \dots, C_s) + N_r(\gamma, C_1, \dots, C_s) \end{aligned}$$

and $N_l(\gamma, C_1, \dots, C_s)$ is the number of new holes created by $V(\gamma) \cup \cup_{\nu=1}^{s} C_\nu$ and the line $\{x^1 = \hat{l} - 1\}$, where $base(\xi) = [\hat{l}, \hat{r}]$. Similarly, $N_r(\gamma, C_1, \dots, C_s)$ is the number of new holes created by $V(\gamma) \cup \cup_{\nu=1}^{s} C_\nu$ and the line $\{x^1 = \hat{r} + 1\}$.

A polymer ξ is called *simple* if $base(\xi)$ is one point and $\xi = (\gamma, \emptyset, \emptyset)$. Thus, the weight $\Psi(\xi)$ is given by $\Psi(\xi) = e^{-\mu|\gamma|}$. A polymer ξ is called *decorated* if it is not simple.

A decorated polymer $\xi = (\gamma, \{C_\nu\}, \{\Lambda_\alpha\})$ with $base(\xi) = [\hat{l}, \hat{r}]$ is said *r-active* if there exists a simple polymer $\xi_1 = (\gamma_1, \emptyset, \emptyset)$ with $base(\xi_1) = \{\hat{r}+1\}$ such that ξ_1 is incompatible with ξ or the concatenation of γ and γ_1 together with $\cup_\nu C_\nu$ produces a new hole. ξ is said *l-active* if there exists a simple polymer $\xi_2 = (\gamma_2, \emptyset, \emptyset)$ with $base(\xi_2) = \{\hat{l} - 1\}$ such that ξ_2 is incompatible with ξ or the concatenation of γ_2 and γ together with $\cup_\nu C_\nu$ produces a new hole. If ξ is both r-active and l-active, we call it *bi-active*. A *polymer chain* is a family of decorated polymers $\mathcal{C} = \{\xi_1, \dots, \xi_m\}$ such that

(1) $\{\xi_1, \ldots, \xi_n\}$ are compatible.
(2) If $base(\xi_u) = [\hat{l}_u, \hat{r}_u]$, $1 \le u \le n$, then $\hat{l}_{u+1} = \hat{r}_u + 1$ or $\hat{r}_u + 2$ for every u.
(3) If $\hat{l}_{u+1} = \hat{r}_u + 2$ for some u, then ξ_u is r-active and ξ_{u+1} is l-active.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two polymer chains. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *compatible* if $\mathcal{C}_1 \cup \mathcal{C}_2$ is a compatible family of polymers, but it is not a polymer chain.

For a polymer chain $\mathcal{C} = \{\xi_1, \ldots, \xi_m\}$, let

$$base(\mathcal{C}) = base(\xi_1) \cup \ldots \cup base(\xi_m).$$

For a polymer ξ, we define

$$\hat{\Psi}(\xi;\zeta) := e^{\mu\zeta k(\xi)}\Psi(\xi)Q(\zeta)^{-|base(\xi)|},$$

where $|base(\xi)| = \hat{r} - \hat{l} + 1$ when $base(\xi) = [\hat{l}, \hat{r}]$, and $Q(\zeta)$ is the generating function of the hight of the endpoint of a simple polymer ;

$$Q(\zeta) = e^{-\mu} \sum_{k=-\infty}^{\infty} e^{\mu\zeta k} e^{-|k|\mu}.$$

Also, for a polymer chain $\mathcal{C} = \{\xi_1, \ldots, \xi_m\}$, we put

$$\mathbf{F}_{\hat{\Psi}}(\mathcal{C};\zeta) := \prod_{u=1}^{m} \hat{\Psi}(\xi_u;\zeta) \times \mathcal{J}_l(\xi_1)\mathcal{J}_r(\xi_m) \prod_{u=1}^{m-1} \mathcal{J}(\xi_u, \xi_{u+1}),$$

where for $base(\xi) = [\hat{l}, \hat{r}]$ and $base(\xi_1) = [c, d]$ with $c > \hat{r}$, $\mathcal{J}_l, \mathcal{J}_r, \mathcal{J}$ are defined in the following way.

$$\mathcal{J}_l(\xi) = \begin{cases} \sum\limits_{\xi' c \xi}^{\hat{l}-1} \hat{\Psi}(\xi';\zeta) 2^{N(\xi',\xi) - N_l(\xi, C_1, \ldots, C_s)} & \text{if } \xi \text{ is l-active} \\ 1, & \text{otherwise,} \end{cases}$$

where $\sum\limits_{\xi' c \xi}^{\hat{l}-1}$ means over simple polymers $\xi' = (\gamma', \emptyset, \emptyset)$ with base $\{\hat{l} - 1\}$ compatible with ξ, and $N(\xi', \xi)$ is the number of new holes created by the concatenation of γ' and γ together with $\cup_\nu C_\nu$, which is not larger than $N_l(\gamma, C_1, \ldots, C_s)$. Similarly,

$$\mathcal{J}_r(\xi) = \begin{cases} \sum\limits_{\xi' c \xi}^{\hat{r}+1} \hat{\Psi}(\xi';\zeta) 2^{N(\xi,\xi') - N_r(\gamma, C_1, \ldots, C_s)}, & \text{if } \xi \text{ is r-active} \\ 1, & \text{otherwise,} \end{cases}$$

and $\mathcal{J}(\xi,\xi_1)$ is defined in two cases.

(i) If $c=\hat{r}+2$, ξ is r-active and ξ_1 is l-active, then

$$\mathcal{J}(\xi,\xi_1)=\sum_{\xi' c\,\xi,\xi_1}^{\hat{r}+1}\hat{\Psi}(\xi';\zeta)2^{N(\xi,\xi')+N(\xi',\xi_1)-N_r(\gamma,C_1,\dots,C_s)-N_l(\gamma_1,\tilde{C}_1,\dots,\tilde{C}_{s_1})},$$

(ii) If $c=\hat{r}+1$, and ξ and ξ' are compatible, then

$$\mathcal{J}(\xi,\xi_1)=2^{N(\xi,\xi_1)-N_r(\gamma,C_1,\dots,C_s)-N_l(\gamma_1,\tilde{C}_1,\dots,\tilde{C}_{s_1})}.$$

Let $\mathcal{K}_L$ be the set of all decorated polymers with base in $[0,L]$, and $\mathcal{CP}_L$ be the set of polymer chains with base in $[0,L]$. Then we have

$$\text{(2.3)}\qquad \frac{1}{Q(\zeta)^L}\sum_{\Gamma\in\mathcal{S}_L}e^{\mu\zeta k(\Gamma)}W(\Gamma)=\sum_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_r\in\mathcal{CP}_L;\\ compatible}}\prod_{i=1}^{r}\mathbf{F}_{\hat{\Psi}}(\mathcal{C}_i;\zeta).$$

Lemma 2.1 Let $\delta>0$ be given. Then there exists $\mu_4>\mu_0$ such that for $\mu>\mu_4$, the free energy $\varphi(\zeta)$ in (1.8) exists and is analytic in ζ if $Re\zeta<1-\frac{\delta}{\mu}$.

Proof. It is sufficient to show that

$$\frac{1}{L}\ln\sum_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_r\in\mathcal{CP}_L;\\ compatible}}\prod_{i=1}^{r}\mathbf{F}_{\hat{\Psi}}(\mathcal{C}_i;\zeta)$$

converges as $L\to\infty$ and its limit $\hat{\varphi}(\zeta)$ is analytic for $Re\zeta<1-\frac{\delta}{\mu}$. Then we have

$$\varphi(\zeta)=\hat{\varphi}(\zeta)+\ln Q(\zeta),$$

which is analytic in this region.

In order to verify the convergence and analyticity, we have to check that there exist functions $c^*,d^*:\mathcal{CP}=\{\mathcal{C};\text{ polymer chain}\}\to[0,\infty)$ such that

$$\text{(2.4)}\qquad \sum_{\mathcal{C}\in\mathcal{CP};\mathcal{C}\,i\,\mathcal{C}_0}e^{c^*(\mathcal{C})+d^*(\mathcal{C})}|\mathbf{F}_{\hat{\Psi}}(\mathcal{C};\zeta)|\le c^*(\mathcal{C}_0)$$

for any polymer chain $\mathcal{C}_0$ and for any $\zeta\in\mathbf{C}$ with $Re\zeta<1-\frac{\delta}{\mu}$ (see e.g. [KP]). For a decorated polymer $\xi=(\gamma,\{C_\nu\},\{\Lambda_\alpha\})$, we put $c(\xi)=3|base(\xi)|$ and

$$d(\xi)=\begin{cases}(\mu-\mu_4)|base(\xi)|+\frac{\delta}{6}|\gamma|-(\mu-\mu_2-1), & \text{if } |base(\xi)|\ge 2,\\ (\mu-\mu_4)|base(\xi)|+\frac{\delta}{6}|\gamma|, & \text{if } |base(\xi)|=1.\end{cases}$$

Then we set

$$c^*(\mathcal{C}) = \sum_{\xi \in \mathcal{C}} c(\xi), \quad d^*(\mathcal{C}) = \sum_{\xi \in \mathcal{C}} d(\xi)$$

The constant μ_4 is specified later. We will first show that

$$\sum_{\xi \in \mathcal{K}_L; \xi i\, \xi_0} e^{c(\xi)+d(\xi)} |\hat{\Psi}(\xi; \zeta)| \leq c(\xi_0) \tag{2.5}$$

for every polymer ξ_0. Note first that

$$|\gamma| = N_v(\gamma) + N_h(\gamma) + 1, \tag{2.6}$$

where $N_v(\gamma)$ is the number of vertical edges in γ, and $N_h(\gamma)$ is the number of horizontal edges in γ. Also, by definition of decorated polymers, if $base(\xi)$ is one point, then

$$N_h(\gamma) + \sum_{\nu=1}^{s} |C_\nu| + \sum_{\alpha=1}^{t} |\Lambda_\alpha| \geq 1, \tag{2.7a}$$

since either $\{C_\nu\}$ or $\{\Lambda_\alpha\}$ is non-empty if $base(\xi)$ is one point. If $|base(\xi)| \geq 2$, then we have

$$N_h(\gamma) + \sum_{\nu=1}^{s} |C_\nu| + \sum_{\alpha=1}^{t} |\Lambda_\alpha| \geq 2(|base(\xi)| - 1). \tag{2.7b}$$

Let γ be a self-avoiding path such that it is the backbone of some decorated polymer with base $I = [\hat{l}, \hat{r}]$. We estimate the following sum.

$$G(\gamma) := \sum_{\xi; \gamma \textit{ is the backbone of } \xi} |\Psi(\xi) e^{\mu k(\gamma)\zeta}|.$$

From (1.4), $|\Phi(\Lambda)| \leq e^{-(\mu-\mu_0)|\Lambda|} < 1$ and therefore we have

$$|e^{-\Phi(\Lambda)} - 1| \leq e^{-(\mu-\mu_0-1)|\Lambda|}.$$

Using this, if $\hat{l} = \hat{r}$, i.e., $|I| = 1$, then we have $N^*(\gamma, C_1, \ldots, C_s) = 0$ and

$$
\begin{aligned}
(2.8) \qquad G(\gamma) &\le e^{-\mu|\gamma|} e^{\mu k(\gamma) Re\zeta} \sum_{\{C_\nu\}; C_\nu\, i\, \gamma} e^{-\mu \sum_\nu |C_\nu|} \\
&\quad \times \sum_{\{\Lambda_\alpha\}; \Lambda_\alpha\, i\, \gamma \cup C_1 \cup \cdots \cup C_s} e^{-(\mu-\mu_0-1)\sum_\alpha |\Lambda_\alpha|} \\
&\le e^{-\mu|\gamma| + \mu k(\gamma) Re\zeta - (\mu-\mu_2-1)} \\
&\quad \times \sum_{\{C_\nu\}; C_\nu\, i\, \gamma} e^{-\mu_2 \sum_\nu |C_\nu|} \\
&\quad \times \sum_{\{\Lambda_\alpha\}; \Lambda_\alpha\, i\, \gamma \cup C_1 \cup \cdots \cup C_s} e^{-(\mu_2-\mu_0)\sum_\alpha |\Lambda_\alpha|}
\end{aligned}
$$

The summation over $\{\Lambda_\alpha\}$ is estimated as follows.

$$
\begin{aligned}
&\sum_{\{\Lambda_\alpha\}; \Lambda_\alpha\, i\, \gamma \cup C_1 \cup \cdots \cup C_s} e^{-(\mu_2-\mu_0)\sum_\alpha |\Lambda_\alpha|} \\
&\le \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{\Lambda_1\, i\, \gamma \cup C_1 \cup \cdots \cup C_s} \cdots \sum_{\Lambda_t\, i\, \gamma \cup C_1 \cup \cdots \cup C_s} e^{-(\mu_2-\mu_0)\sum_\alpha |\Lambda_\alpha|} \\
&\le \exp\left\{ 4|\gamma \cup C_1 \cup \cdots \cup C_s| \sum_{\Lambda \ni 0; connected} e^{-(\mu_2-\mu_0)|\Lambda|} \right\} \\
&= \exp\left\{ (|\gamma| + \sum_\nu |C_\nu|) g_1(\mu_2, \mu_0) \right\}.
\end{aligned}
$$

Since there exist constants $K_1, \kappa > 0$ such that the number N_n of connected sets of n points in $\mathbf{Z}^2$ which contain the origin is bounded as

$$N_n \le K_1 \kappa^n \quad (n \ge 1),$$

we know that $g_1(\mu_2, \mu_0) = 4\sum_{\Lambda \ni 0; connected} e^{-(\mu_2-\mu_0)|\Lambda|}$ goes to zero exponentially fast as $\mu_2 \to \infty$. Thus, summing up the RHS of (2.8) over $\{\Lambda_\alpha\}$'s we obtain

$$
\begin{aligned}
G(\gamma) &\le e^{-(\mu - g_1(\mu_2,\mu_0))|\gamma| + \mu k(\gamma) Re\zeta} \\
&\quad \times e^{-(\mu-\mu_2-1)} \sum_{\{C_\nu\}; C_\nu\, i\, \gamma} e^{-(\mu_2 - g_1(\mu_2,\mu_0))\sum_\nu |C_\nu|} \\
&\le e^{-(\mu - g_1(\mu_2,\mu_0) - g_2(\mu_2,\mu_0))|\gamma| + \mu k(\gamma) Re\zeta - (\mu-\mu_2-1)},
\end{aligned}
$$

where $g_2(\mu_2,\mu_0) = 4\sum_{C\ni 0;connected} e^{-(\mu_2-g_1(\mu_2,\mu_0))|C|}$. If $\hat{r} > \hat{l}$, i.e., $|I|\geq 2$, then since $N^*(\gamma, C_1,\dots,C_s) \leq N_h(\gamma)+\sum_\nu |C_\nu|$, we have from (2.7b) as in (2.8),

$$\begin{aligned}(2.9)\qquad G(\gamma) \leq & e^{-\mu|\gamma|+\mu k(\gamma)Re\zeta} \sum_{\{C_\nu\};C_\nu\, i\, \gamma} e^{-\mu\sum_\nu |C_\nu|} 2^{N^*(\gamma,C_1,\dots,C_s)} \\ & \times \sum_{\{\Lambda_\alpha\};\Lambda_\alpha\, i\, \gamma\cup C_1\cup\dots C_s} e^{-(\mu-\mu_0-1)\sum_\alpha |\Lambda_\alpha|} \\ \leq & e^{-\mu|\gamma|+\mu k(\gamma)Re\zeta-(\mu-\mu_2-1)(2|I|-N_h(\gamma)-2)} 2^{N_h(\gamma)} \\ & \times \sum_{\{C_\nu\};\, C_\nu\, i\, \gamma} e^{-(\mu_2-\ln 2)\sum_\nu |C_\nu|} \\ & \times \sum_{\{\Lambda_\alpha\};\Lambda_\alpha\, i\, \gamma\cup C_1\cup\dots\cup C_s} e^{-(\mu_2-\mu_0)\sum_\alpha |\Lambda_\alpha|} \\ \leq & e^{-(\mu-g_1(\mu_2,\mu_0))|\gamma|+\mu k(\gamma)Re\zeta-(\mu-\mu_2-1)(2|I|-N_h(\gamma)-2)} \\ & \times \sum_{\{C_\nu\}} e^{-(\mu_2-g_1(\mu_2,\mu_0)-\ln 2)\sum_\nu |C_\nu|} e^{N_h(\gamma)\ln 2} \\ \leq & e^{-(\mu-g_1(\mu_2,\mu_0)-g_3(\mu_2,\mu_0))|\gamma|+\mu k(\gamma)Re\zeta} \\ & \times e^{-(\mu-\mu_2-1)(2|I|-N_h(\gamma)-2)+N_h(\gamma)\ln 2},\end{aligned}$$

where $g_3(\mu_2,\mu_0) = 4\sum_{C\ni 0;connected} e^{-(\mu_2-g_1(\mu_2,\mu_0)-\ln 2)}$. We take μ_2 sufficiently large so that $g_1(\mu_2,\mu_0)$, $g_2(\mu_2,\mu_0)$ and $g_3(\mu_2,\mu_0)$ are all smaller than $\frac{\delta}{4}$.

Assume that $Re\zeta < 1-\frac{\delta}{\mu}$. Then since $N_v(\gamma)\geq |k(\gamma)|$, from (2.6) we have

$$(2.10)\qquad G(\gamma) \leq e^{-\frac{\delta}{2}N_v(\gamma)-(\mu_2-\frac{\delta}{2})(N_h(\gamma)+1)-(\mu-\mu_2-1)(2|I|-1)},$$

if $|I|\geq 2$, and

$$(2.11)\qquad G(\gamma) \leq e^{-\frac{\delta}{2}N_v(\gamma)-(\mu_2-\frac{\delta}{2})-2(\mu-\mu_2-1)}$$

if $|I|=1$. Since $c(\xi)$ and $d(\xi)$ depend only on the backbone γ, we write them $c(\gamma)$ and $d(\gamma)$. Then

$$\begin{aligned}(2.12)\qquad & \sum_{\xi;\gamma\ is\ the\ backbone\ of\ \xi} \left|\Psi(\xi)e^{\mu k(\gamma)\zeta}\right| e^{c(\xi)+d(\xi)} \\ & = G(\gamma)e^{c(\gamma)+d(\gamma)} \\ & \leq e^{-\frac{\delta}{3}N_v(\gamma)-(\mu_2-\frac{2\delta}{3})(N_h(\gamma)+1)} e^{-(\mu+\mu_4-2\mu_2-5)|base(\gamma)|},\end{aligned}$$

where $base(\gamma) = base(\xi)$ for any ξ such that γ is the backbone of ξ. Therefore we have for a fixed interval I,

$$\sum_{base(\gamma)=I} G(\gamma)e^{c(\gamma)+d(\gamma)} \leq e^{-(\mu+\mu_4-2\mu_2-5)|I|} \sum_{base(\gamma)=I} e^{-\frac{\delta}{3}N_v(\gamma)-(\mu_2-\frac{2\delta}{3})(N_h(\gamma)+1)}. \tag{2.13}$$

To estimate the RHS of (2.13) we separate γ into fragments following the idea of [DKS]. Let $\gamma = \{x_0, x_1, \dots, x_n\}$ be a self-avoiding path with $base(\gamma) = I$. Let $j_0 = 0$, and for $i \geq 1$, let

$$j_i := \min\{j > j_{i-1};\ \{x_{j-1}, x_j\} \text{ is a horizontal edge}\}.$$

Each vertical part $\{x_{j_{i-1}}, x_{j_{i-1}+1}, \dots, x_{j_i-1}\}$ of γ with the direction of the exit vector $\{x_{j_i-1}, x_{j_i}\}$ is called a fragment. For a fragment $f = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_p\}$ with exit direction $e(f)$, we define

$$W(f) := e^{-\frac{\delta}{3}N_v(f)-(\mu_2-\frac{2\delta}{3})} = e^{-\frac{p\delta}{3}-(\mu_2-\frac{2\delta}{3})}.$$

Then the decomposition of γ into fragments $\{f_1, \dots, f_r\}$ leads to the identity

$$e^{-\frac{\delta}{3}N_v(\gamma)-(\mu_2-\frac{2\delta}{3})(N_h(\gamma)+1)} = \prod_{j=1}^{r} W(f_j).$$

Therefore we have

$$\begin{aligned}\sum_{\gamma;base(\gamma)=I} e^{-\frac{\delta}{3}N_v(\gamma)-(\mu_2-\frac{2\delta}{3})(N_h(\gamma)+1)} &= \sum_{r=|I|}^{\infty} \sum_{f_1,\dots,f_r} \prod_{j=1}^{r} W(f_j) \\ &\leq \sum_{r=|I|}^{\infty} \left(2 \sum_{k=-\infty}^{\infty} e^{-\frac{\delta}{3}|k|}\right)^r \times e^{-(\mu_2-\frac{2\delta}{3})r} \\ &= \frac{R(\mu_2,\delta)^{|I|}}{1-R(\mu_2,\delta)},\end{aligned}$$

if μ_2 is sufficiently large. Thus, if $Re\zeta < 1 - \frac{\delta}{\mu}$ and $\mu > \mu_2$, where μ_2 is sufficiently large, we have

$$\sum_{base(\gamma)=I} G(\gamma)e^{c(\gamma)+d(\gamma)} \leq e^{-(\mu+\mu_4-2\mu_2-5)|I|} \frac{R(\mu_2,\delta)^{|I|}}{1-R(\mu_2,\delta)}.$$

Since

$$
|Q(\zeta)| = e^{-\mu}\left|\frac{\sinh\mu}{\cosh\mu - \cosh\mu\zeta}\right| \geq \frac{e^{-\mu}\tanh\mu_2}{1+e^{-\delta}} := e^{-\mu-\mu_3} \tag{2.14}
$$

if $Re\zeta < 1 - \frac{\delta}{\mu}$ and $\mu > \mu_2$, we have

$$
\sum_{base(\xi)=I} \left|\hat{\Psi}(\xi;\zeta)\right| e^{c(\xi)+d(\xi)} \leq e^{-(\mu_4-2\mu_2-\mu_3-5)|I|}\frac{R(\mu_2,\delta)^{|I|}}{1-R(\mu_2,\delta)}. \tag{2.15}
$$

Let $\mu_4 > 2\mu_2 + \mu_3 + 5$. For $\mu > \mu_4$ we will estimate the RHS of (2.5). Fix ξ_0 and write $base(\xi_0) = [\hat{l}, \hat{r}]$. Then we have

$$
\begin{aligned}
\sum_{\xi\, i\, \xi_0} \left|\hat{\Psi}(\xi;\zeta)\right| e^{c(\xi)+d(\xi)} &\leq \sum_{x\in[\hat{l}-1,\hat{r}+1]}\sum_{I\ni x}\frac{R(\mu_2,\delta)^{|I|}}{1-R(\mu_2,\delta)} \\
&= \frac{(\hat{r}-\hat{l}+3)}{1-R(\mu_2,\delta)}\sum_{k=1}^{\infty} kR(\mu_2,\delta)^k \\
&\leq 3|base(\xi_0)|\frac{R(\mu_2,\delta)}{\left(1-R(\mu_2,\delta)\right)^3} \leq c(\xi_0).
\end{aligned}
$$

if μ_2 is large. Thus, (2.5) is proved. From (2.5) to (2.4), we argue in the following way. We call a family of intervals $I_1 = [\hat{l}_1, \hat{r}_1], \ldots, I_n = [\hat{l}_n, \hat{r}_n]$ *linked intervals* if for each $1 \leq u \leq n$, $\hat{r}_u < \hat{l}_{u+1} \leq \hat{r}_u + 2$ holds. The base of a polymer chain forms linked intervals. For a fixed polymer chain $\mathcal{C}_0$, let $[base(\mathcal{C}_0)] = [\hat{l}_0, \hat{r}_0]$ be the smallest interval including $base(\mathcal{C}_0)$. Then noting that the distance of $base(\mathcal{C}_0)$ and $base(\mathcal{C})$ is less than 2 if $\mathcal{C}_0$ and $\mathcal{C}$ are incompatible, we have

$$
\begin{aligned}
&\sum_{\mathcal{C}\, i\, \mathcal{C}_0} |\mathbf{F}_{\hat{\Psi}}(\mathcal{C};\zeta)| e^{c^*(\mathcal{C})+d^*(\mathcal{C})} \\
&\leq \sum_{x\in[\hat{l}_0-2,\hat{r}_0+2]}\sum_{n=1}^{\infty}\sum_{\substack{I_1,\ldots,I_n\subset[0,L];\cup I_u\ni x\\ linked\, intervals,}}\sum_{\substack{\xi_1,\ldots,\xi_n\in\mathcal{K}_L;\\ base(\xi_u)=I_u, 1\leq u\leq n}} \\
&\prod_{u=1}^{n}\left[\hat{\Psi}(\xi_u;\zeta)e^{c(\xi_u)+d(\xi_u)}\right]\mathcal{J}_l(\xi_1)\mathcal{J}_r(\xi_n)\prod_{u=1}^{n-1}\mathcal{J}(\xi_u,\xi_{u+1})
\end{aligned}
$$

By definition and (2.14), there exists $\mu_3^* > 0 = \mu_3^*(\delta)$ such that $|\mathcal{J}_r|$, $|\mathcal{J}_l|$, $|\mathcal{J}|$ are all bounded by $e^{\mu_3^*}$ from above if $Re(\zeta) < 1 - \frac{\delta}{\mu}$. Therefore from the estimate (2.15), we have

$$\sum_{\substack{\xi_1,\dots,\xi_n \in \mathcal{K}_L;\\ base(\xi_u)=I_u, 1\le u\le n}} \prod_{u=1}^{n} \left[\hat{\Psi}(\xi_u;\zeta)e^{c(\xi_u)+d(\xi_u)}\right] \mathcal{J}_l(\xi_1)\mathcal{J}_r(\xi_n) \prod_{u=1}^{n-1} \mathcal{J}(\xi_u,\xi_{u+1})$$
$$\le \prod_{u=1}^{n} e^{-(\mu_4-2\mu_2-\mu_3-2\mu_3^*-5)|I_u|} \frac{R(\mu_2,\delta)^{|I_u|}}{1-R(\mu_2,\delta)}.$$

Assuming that $\mu_4 > 2\mu_2 + \mu_3 + 2\mu_3^* + 5$, we have

$$\sum_{\mathcal{C}\, i\, \mathcal{C}_0} |\mathbf{F}_{\hat{\Psi}}(\mathcal{C};\zeta)| e^{c^*(\mathcal{C})+d^*(\mathcal{C})}$$
$$\le (\hat{r}_0 - \hat{l}_0 + 4) \sum_{n=1}^{\infty} \sum_{u=1}^{n} \sum_{\substack{I_1,\dots,I_n\subset[0,L];I_u\ni x,\\ linked\, intervals}} \prod_{u=1}^{n} \frac{R(\mu_2,\delta)^{|I_u|}}{1-R(\mu_2,\delta)}$$
$$\le (\hat{r}_0 - \hat{l}_0 + 4) \frac{R(\mu_2,\delta)}{(1-R(\mu_2,\delta))^3} \sum_{n=1}^{\infty} n\left(\frac{2R(\mu_2,\delta)}{(1-R(\mu_2,\delta))^2}\right)^{n-1}$$
$$\le \frac{(\hat{r}_0 - \hat{l}_0 + 4)}{2}$$

if μ_2 is large. Since $\sum_{\xi\in\mathcal{C}_0} |base(\xi)| \ge \max\left\{\frac{2}{3}[base(\mathcal{C}_0)], 1\right\}$, the RHS of the above inequality is not larger than $c^*(\mathcal{C}_0)$.

This allows us to apply general theory of cluster expansion so that there exists a function

$$\mathbf{F}_{\hat{\Psi}}^T : \mathcal{P}_f(\mathcal{CP}) \times \mathbf{C} \to \mathbf{C}$$

such that $\mathbf{F}_{\hat{\Psi}}^T$ is analytic for $Re\zeta < 1 - \frac{\delta}{\mu}$ and it satisfies

$$\sum_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_r\in\mathcal{CP}_L;\\ compatible}} \prod \mathbf{F}_{\hat{\Psi}}(\mathcal{C}_i;\zeta) = \exp\left\{ \sum_{\Delta\in\mathcal{P}_f(\mathcal{CP}_L)} \mathbf{F}_{\hat{\Psi}}^T(\Delta;\zeta) \right\} \tag{2.16}$$

and

$$\sum_{\Delta\, i\, \mathcal{C}_0} |\mathbf{F}_{\hat{\Psi}}^T(\Delta;\zeta)| e^{d^*(\Delta)} \le c^*(\mathcal{C}_0), \tag{2.17}$$

where $\mathcal{P}_f(\mathcal{CP}_L)$ is the collection of all finite subsets of $\mathcal{CP}_L$ and $d^*(\Delta) = \sum_{\mathcal{C}\in\Delta} d^*(\mathcal{C})$. If Δ is decomposed into two disjoint subsets Δ_1 and Δ_2 such that $\{\mathcal{C}_1, \mathcal{C}_2\}$ are compatible for every pair $\mathcal{C}_1 \in \Delta_1$, $\mathcal{C}_2 \in \Delta_2$, then $\mathbf{F}_{\hat{\Psi}}^T(\Delta;\zeta) = 0$. We call $\Delta \in \mathcal{P}_f(\mathcal{CP}_L)$ a *cluster* if there are no such decomposition $\Delta = \Delta_1 \cup \Delta_2$. Also, we note that $\mathbf{F}^T_{\hat{\Psi}}(\Delta;\zeta)$ is invariant under horizontal translation of Δ. For $\Delta \in \mathcal{P}_f(\mathcal{CP})$, put $base(\Delta) = \cup_{\mathcal{C}\in\Delta} base(\mathcal{C})$. Then (2.16) and (2.17) implies that the limit

$$\begin{aligned}\hat{\varphi}(\zeta) &= \lim_{L\to\infty} \frac{1}{L} \ln \sum_{\mathcal{C}_1,\dots,\mathcal{C}_r\in\mathcal{CP}_L} \prod_{u=1}^{r} \mathbf{F}_{\hat{\Psi}}(\mathcal{C}_u;\zeta) \\ &= \sum_{\substack{\Delta\in\mathcal{P}_f(\mathcal{CP});[base(\Delta)]=[0,k] \\ for\ some\ k\geq 0}} \mathbf{F}^T_{\hat{\Psi}}(\Delta;\zeta)\end{aligned}$$

exists and analytic for $\zeta < 1 - \frac{\delta}{\mu}$ if $\mu > \mu_4$.

free energy for a joint distribution

Let $q \geq 1$, and let $0 < t_1 < \cdots < t_{q+1} = 1$. For $\underline{\zeta} = (\zeta_0, \zeta_1, \dots, \zeta_{q+1}) \in \mathbf{C}^{q+1}$, let

$$\varphi^{(q)}(\underline{\zeta}; t_1, \dots, t_{q+1}) = \lim_{L\to\infty} \frac{1}{L} \ln \sum_{\Gamma\in\mathcal{S}_L} e^{\mu\underline{\zeta}\cdot\hat{X}^{(q)}_L(t_1,\dots,t_{q+1})} W(\Gamma) \tag{2.18}$$

if the limit exists. Here, the random vector $\hat{X}^{(q)}_L(t_1,\dots,t_{q+1})$ is defined by

(2.19)
$$\hat{X}^{(q)}_L(t_1,\dots,t_{q+1}) = \left(\frac{a(\pi(\Gamma))}{L}, X_L(\frac{\lfloor Lt_1\rfloor}{L}), \dots, X_L(\frac{\lfloor Lt_q\rfloor}{L}), X_L(1)\right).$$

With a slight change of the proof of Lemma 2.1, we can prove existence and analyticity of the limit $\varphi^{(q)}(\underline{\zeta}; t_1, \dots, t_{q+1})$. To be more precise, we decompose $a(\pi(\Gamma))$ into terms corresponding to polymers appearing in the decomposition of Γ. Let $\xi = (\gamma, \{C_\nu\}, \{\Lambda_\alpha\})$ be a polymer with base $[a,b]$. The area $area(\xi)$ is then defined by

$$\begin{aligned}area(\xi) =& \#\{x \in [\hat{l},\hat{r}] \times [-M,M] \cap \mathbf{Z}^{2*};\ x \text{ is below } \gamma\} \\ &- \#\{x \in [\hat{l},\hat{r}] \times [-M,M] \cap \mathbf{Z}^{2*};\ x \text{ is above } \gamma\}.\end{aligned}$$

This is independent of large M. For a $\Gamma \in \mathcal{S}_L$, denote $\mathcal{D}(\Gamma)$ all polymers, which obtained through any triple $(\pi(\Gamma), \{C_\nu\}, \{\Lambda_\alpha\})$ with its cutting lines, where $\{\Lambda_\alpha\}$ is taken over all families of connected sets such that $\Lambda_\alpha i \Gamma$ for each α. We have

$$a(\pi(\Gamma)) = \sum_{\xi\in\mathcal{D}(\Gamma)} \{area(\xi) + k(\gamma)(L - \hat{r}(\xi))\}, \tag{2.20}$$

where $base(\xi) = [\hat{l}(\xi), \hat{r}(\xi)]$ for $\xi \in \mathcal{D}(\Gamma)$. Therefore,

$$\begin{aligned}\underline{\zeta} \cdot \hat{X}_L^{(q)}(t_1, \dots, t_{q+1}) = \zeta_0 \sum_{\xi \in \mathcal{D}(\Gamma)} \left\{ \frac{area(\xi)}{L} + k(\gamma)\left(1 - \frac{\hat{r}(\xi)}{L}\right) \right\} \\ + \sum_{i=1}^{q+1} \zeta_i \sum_{\xi \in \mathcal{D}(\Gamma)} 1_{[\hat{r}(\xi) < Lt_i]} k(\gamma) \\ + \sum_{i=1}^{q+1} \zeta_i \sum_{\xi \in \mathcal{D}(\Gamma)} 1_{[\hat{l}(\xi) \le Lt_i \le \hat{r}(\xi)]} k(\gamma; t_i L),\end{aligned}$$

where $k(\gamma; t_i L)$ is the maximal hight of the intersection of polygonal line γ and the vertical line $\{x^1 = t_i L\}$.

Proposition 2.2. Let $\mu > \mu_4$. If $\underline{\zeta}$ satisfies

$$(2.21) \quad \begin{cases} \max\{|Re(\zeta_0 + \zeta_{q+1})|, |Re\zeta_{q+1}|\} \le 1 - \frac{2\delta}{\mu}, \\ |Re\zeta_i| \le \frac{\delta}{4(q+1)\mu}, \qquad i = 1, 2, \dots, q, \end{cases}$$

then the limit $\varphi^{(q)}(\underline{\zeta}; t_1, \dots, t_{q+1})$ exists and is analytic in $\underline{\zeta}$.

Proof. Let ξ be a polymer with base $[\hat{l}(\xi), \hat{r}(\xi)] \subset [0, L]$. We decompose ξ into fragments $\{f_p\}_{p=1}^P$. The hight of a fragment $f = \{x_1, \dots, x_u\}$ is defined by

$$h(f) = x_u^2 - x_1^2$$

and the position of f is given by

$$pos(f) = x_1^1 = x_u^1.$$

Then we have as in [DH2],

$$area(\xi) = \sum_{p=1}^P h(f_p)(\hat{r}(\xi) - pos(f_p)).$$

Since $k(\gamma) = \sum_{p=1}^P h(f_p)$, we have

$$\frac{area(\xi)}{L} + k(\gamma)\left(1 - \frac{\hat{r}(\xi)}{L}\right) = \sum_{p=1}^P h(f_p)\left(1 - \frac{pos(f_p)}{L}\right).$$

Thus, we have

$$\Big|Re\Big[\zeta_0\big(\frac{area(\xi)}{L}+k(\gamma)(1-\frac{\hat{r}(\xi)}{L})\big)+\sum_{i=1}^{q+1}\zeta_i 1_{[\hat{r}(\xi)<Lt_i]}k(\gamma)$$

$$+\sum_{i=1}^{q+1}\zeta_i 1_{[\hat{l}(\xi)\le Lt_i\le\hat{r}(\xi)]}k(\gamma;Lt_i)\Big]\Big|$$

$$\le\Big|Re(\zeta_0+\zeta_{q+1})\sum_{p=1}^{P}h(f_p)\big(1-\frac{pos(f_p)}{L}\big)+Re\zeta_{q+1}\sum_{p=1}^{P}h(f_p)\frac{pos(f_p)}{L}\Big|$$

$$+\sum_{i=1}^{q}|Re\zeta_i|N_v(\gamma)$$

$$\le|Re(\zeta_0+\zeta_{q+1})|\sum_{p=1}^{P}|h(f_p)|\big(1-\frac{pos(f_p)}{L}\big)+|Re\zeta_{q+1}|\sum_{p=1}^{P}|h(f_p)|\frac{pos(f_p)}{L}$$

$$+\sum_{i=1}^{q}|Re\zeta_i|N_v(\gamma)$$

$$\le\Big[\max\{|Re(\zeta_0+\zeta_{q+1})|,|Re\zeta_{q+1}|\}+\sum_{i=1}^{q}|Re\zeta_i|\Big]N_v(\gamma).$$

Set

$$X^{(L)}(\underline{\zeta};\xi)=X^{(L)}_{t_1,\dots,t_{q+1}}(\underline{\zeta};\xi)$$

$$=\zeta_0\big(\frac{area(\xi)}{L}+k(\gamma)(1-\frac{\hat{r}(\xi)}{L})\big)+\sum_{i=1}^{q+1}\zeta_i 1_{[\hat{r}(\xi)<Lt_i]}k(\gamma)$$

$$+\sum_{i=1}^{q+1}\zeta_i 1_{[\hat{l}(\xi)\le Lt_i\le\hat{r}(\xi)]}k(\gamma;Lt_i).$$

As before, let

$$\hat{\Psi}(\xi;\ \underline{\zeta},t_1,\dots,t_{q+1})=\Psi(\xi)e^{\mu X^{(L)}(\underline{\zeta};\xi)}\prod_{\ell=\hat{l}(\xi)}^{\hat{r}(\xi)}Q^{-1}(\zeta_L(\ell)),\tag{2.22}$$

where $\zeta_L(\ell)=\zeta_0(1-\frac{\ell}{L})+\sum_{i=1}^{q+1}\zeta_i 1_{[\ell\le Lt_i]}$. For simplicity we write $\hat{\Psi}(\xi;\underline{\zeta})$ for $\hat{\Psi}(\xi;\underline{\zeta};t_1,\dots,t_{q+1})$ for the moment. Then for a polymer chain $\mathcal{C}=$

$\{\xi_1, \dots, \xi_m\}$, we define $\mathbf{F}_{\hat{\Psi}}(\mathcal{C}; \underline{\zeta}) = \mathbf{F}_{\hat{\Psi}}(\mathcal{C}; \underline{\zeta}, t_1, \dots, t_{q+1})$ analogously to $\mathbf{F}_{\hat{\Psi}}(\mathcal{C}; \zeta)$. Namely,

$$\mathbf{F}_{\hat{\Psi}}(\mathcal{C}; \underline{\zeta}) = \prod_{u=1}^{m} \hat{\Psi}(\xi_u; \underline{\zeta}) \mathcal{J}_l^{(q)}(\xi_1) \mathcal{J}_r^{(q)}(\xi_m) \prod_{u=1}^{m-1} \mathcal{J}^{(q)}(\xi_u, \xi_{u+1}),$$

where $\mathcal{J}_l^{(q)}$, $\mathcal{J}_r^{(q)}$ and $\mathcal{J}^{(q)}$ are defined as $\mathcal{J}_l$, $\mathcal{J}_r$ and $\mathcal{J}$ by replacing $\hat{\Psi}(\xi; \zeta)$ with $\hat{\Psi}(\xi; \underline{\zeta})$. If $\underline{\zeta}$ satisfies (2.21), then $Q(\zeta_L(\ell))$ is analytic in $\underline{\zeta}$ and satisfies the estimate

$$|Q(\zeta_L(\ell))^{-1}| \le e^{\mu + \mu_3} \quad \ell = 0, 1, \dots, L$$

if $\mu > \mu_2$. Therefore as in the proof of Lemma 2.1, for $\mu > \mu_4$ we have convergent cluster expansion:

(2.23)
$$\frac{1}{L} \ln \sum_{\substack{\mathcal{C}_1, \dots, \mathcal{C}_n \in \mathcal{CP}_L;\\ compatible}} \prod_{j=1}^{n} \mathbf{F}_{\hat{\Psi}}(\mathcal{C}_j; \underline{\zeta}) = \frac{1}{L} \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}_L)} \mathbf{F}^T_{\hat{\Psi}}(\Delta; \underline{\zeta}, t_1, \dots, t_{q+1})$$

such that $\mathbf{F}^T_{\hat{\Psi}}(\Delta; \underline{\zeta}, t_1, \dots, t_{q+1}) = 0$ unless Δ is a cluster, and (2.17) holds uniformly in $\underline{\zeta}$ satisfying (2.21). So, if (2.23) converges uniformly in $\underline{\zeta}$ satisfying (2.21), then the limit is analytic in this region.

For an interval $I \subset [0, L]$, set

$$\hat{\Xi}(I; \underline{\zeta}, t_1, \dots, t_{q+1}) := \sum_{\substack{\mathcal{C}_1, \dots, \mathcal{C}_m;\ compatible\\ base(\mathcal{C}_i) \subset I,\ 1 \le i \le m}} \prod_{i=1}^{m} \mathbf{F}_{\hat{\Psi}}(\mathcal{C}_i; \underline{\zeta}, t_1, \dots, t_{q+1}).$$

Then by cluster expansion we have

$$\ln \hat{\Xi}(I; \underline{\zeta}, t_1, \dots, t_{q+1}) = \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}); base(\Delta) \subset I} \mathbf{F}^T_{\hat{\Psi}}(\Delta; \underline{\zeta}, t_1, \dots, t_{q+1})$$

if $\underline{\zeta}$ satisfies (2.21), where $base(\Delta) = \cup_{\mathcal{C} \in \Delta} base(\mathcal{C})$. Writing

(2.24)
$$\Phi(J; \underline{\zeta}) := \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}); [base(\Delta)] = J} \mathbf{F}^T_{\hat{\Psi}}(\Delta; \underline{\zeta}, t_1, \dots, t_{q+1})$$

for an interval $J \subset I$, we obtain

$$\ln \hat{\Xi}(I; \underline{\zeta}, t_1, \dots, t_{q+1}) = \sum_{J \subset I} \Phi(J; \underline{\zeta}).$$

From Möbius' inversion formula, we also have

$$\Phi(J;\underline{\zeta}) = \sum_{\tilde{I}\subset J}(-1)^{|J|-|\tilde{I}|}\ln\hat{\Xi}(\tilde{I};\underline{\zeta},t_1,\dots,t_{q+1}). \tag{2.25}$$

Let us also define

$$\Phi_0(J;\zeta) := \sum_{\Delta\in\mathcal{P}_f(\mathcal{CP});[base(\Delta)]=J} \mathbf{F}^T_{\hat{\Psi}}(\Delta;\zeta),$$

where $\mathbf{F}^T_{\hat{\Psi}}(\Delta;\zeta)$ is given in (2.16) through cluster expansion. Then by (2.17) and the definition of $d^*(\Delta)$, $\Phi(J;\underline{\zeta})$ and $\Phi_0(J;\zeta)$ satisfy the following estimate.

$$\max\{|\Phi(J;\underline{\zeta})|,|\Phi_0(J;\zeta)|\} \le 3e^{-(\mu-2\mu_4+\mu_2+1)\lceil\frac{|J|}{3}\rceil} \tag{2.26}$$

if $\mu > 2\mu_4-\mu_2-1$, $|Re\zeta| \le 1-\frac{\delta}{\mu}$ and $\underline{\zeta}$ satisfies (2.21).

Lemma 2.3. Let $\mu > 2\mu_4-\mu_2-1$. If $\underline{\zeta}$ satisfies (2.21), then

$$\lim_{L\to\infty}\frac{1}{L}\sum_{J=[\hat{l},\hat{r}]\subset[0,L]}|\Phi(J;\underline{\zeta})-\Phi_0(J;\zeta_L(\hat{r}))| = 0, \tag{2.27}$$

where $\zeta_L(\hat{r}) = \zeta_L(\hat{r};\underline{\zeta}) := \zeta_0(1-\frac{\hat{r}}{L})+\sum_{i=1}^{q+1}\zeta_i 1_{[0,Lt_i]}(\hat{r})$.

Lemma 2.3 implies that

$$\lim_{L\to\infty}\frac{1}{L}\sum_{J=[\hat{l},\hat{r}]\subset[0,L]}\Phi(J;\underline{\zeta}) = \lim_{L\to\infty}\frac{1}{L}\sum_{J=[\hat{l},\hat{r}]\subset[0,L]}\Phi_0(J;\zeta_L(\hat{r})).$$

Note that for ζ satisfying $|Re\zeta| < 1-\frac{\delta}{\mu}$,

$$\hat{\varphi}(\zeta) = \sum_{\substack{J=[-k,0]\\ for\ some\ k\ge 0}}\Phi_0(J;\zeta),$$

which implies that

(2.28)

$$\lim_{L\to\infty}\frac{1}{L}\sum_{J=[\hat{l},\hat{r}]\subset[0,L]}\Phi_0(J;\zeta_L(\hat{r})) = \int_0^1\hat{\varphi}(\zeta_0(1-x)+\sum_{i=1}^{q+1}\zeta_i 1_{[0,t_i]}(x))dx$$

uniformly in $\underline{\zeta}$ satisfying (2.21). As a result of Proposition 2.2 and Lemma 2.3, we obtain

Corollary 2.4 For $\mu > \mu_4$,

(2.29)

$$\varphi^{(q)}(\underline{\zeta}; t_1, \dots, t_{q+1}) = \int_0^1 (\hat{\varphi} + \ln Q)\big(\zeta_0(1-x) + \sum_{i=1}^{q+1} \zeta_i 1_{[0,t_i]}(x)\big)dx$$

if $\underline{\zeta}$ satisfies (2.21). This function is analytic in $\underline{\zeta}$ in this region.

Proof of Lemma 2.3. We first introduce an intermediate weight $\tilde{\Psi}(\xi; \underline{\zeta})$ by

$$\begin{aligned}&\tilde{\Psi}(\xi;\underline{\zeta})\\ &:= \Psi(\xi)\exp\left[\mu\{\zeta_0(\frac{area(\xi)}{L} + (1 - \frac{\hat{r}(\xi)}{L})k(\gamma)) + \sum_{i=1}^{q+1}\zeta_i 1_{[\hat{r}(\xi)<Lt_i]}k(\gamma)\}\right]\\ &\quad\times \prod_{\ell=a(\xi)}^{b(\xi)} Q^{-1}(\zeta_L(\ell)).\end{aligned}$$

It is easy to verify that $\tilde{\Psi}(\xi;\underline{\zeta})$ also satisfies (2.5) if $\underline{\zeta}$ satisfies (2.21), and therefore we have corresponding $\tilde{\Phi}$ by

$$\ln \sum_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_m;\, compatible\\ base(\mathcal{C}_p)\subset I,\, 1\le p\le m}} \prod_{p=1}^{m} \mathbf{F}_{\tilde{\Psi}}(\mathcal{C}_p;\underline{\zeta}) = \sum_{J\subset I; interval} \tilde{\Phi}(J;\underline{\zeta})$$

for every interval $I \subset [0, L]$. $\tilde{\Phi}$ also satisfies the estimate (2.26). By the Möbius inversion formula $\Phi(I;\underline{\zeta}) = \tilde{\Phi}(I;\underline{\zeta})$ if I contains none of $\{Lt_i\}_{j=1}^{q+1}$. This means by (2.26) that

$$\lim_{L\to\infty}\frac{1}{L}\sum_{J\subset[0,L]} |\Phi(J;\underline{\zeta}) - \tilde{\Phi}(J;\underline{\zeta})| = 0. \tag{2.30}$$

For $s \in [0,1]$, let us define

$$\tilde{\Psi}_s(\xi;\underline{\zeta}) := s\tilde{\Psi}(\xi;\underline{\zeta}) + (1-s)\hat{\Psi}(\xi;\zeta_L(\hat{r})),$$

and let $\tilde{\Phi}_s$ be the corresponding function defined through cluster expansion. Then we have

$$\begin{aligned}&|\tilde{\Phi}(J;\underline{\zeta}) - \Phi_0(J;\zeta_L(\hat{r}))| \\ &\le \sum_{\xi\in\mathcal{K}(J)} \sup_{s,\underline{\zeta}} \left|\frac{\partial\tilde{\Phi}_s(J;\underline{\zeta})}{\partial\tilde{\Psi}_s(\xi;\underline{\zeta})}\right| |\tilde{\Psi}(\xi;\underline{\zeta}) - \hat{\Psi}(\xi;\zeta_L(b))|.\end{aligned} \tag{2.31}$$

Like (2.25) we have

$$\tilde{\Phi}_s(J;\underline{\zeta}) = \sum_{I'\subset J; interval} (-1)^{|J|-|I'|} \ln \tilde{\Xi}(I'),$$

where

$$\tilde{\Xi}(I') := \sum_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_m \in \mathcal{CP} \\ base(\mathcal{C}_p)\subset I',\, 1\le p\le m}} \prod_{p=1}^{m} \mathbf{F}_{\tilde{\Psi}_s}(\mathcal{C}_p;\underline{\zeta}).$$

For a polymer chain $\mathcal{C}$ with $base(\mathcal{C}) \subset I'$, we have

$$\left|\frac{\partial \ln \tilde{\Xi}(I')}{\partial \mathbf{F}_{\tilde{\Psi}_s}(\mathcal{C})}\right| \le \exp\Big\{ \sum_{\substack{\Delta\, i\, \mathcal{C}; \\ base(\Delta)\subset I'}} \left|\mathbf{F}^T_{\tilde{\Psi}_s}(\Delta;\underline{\zeta})\right| \Big\}$$

$$\le \exp\big\{c^*(\mathcal{C}) e^{-(\mu-2\mu_4+\mu_2+1)}\big\}.$$

Therefore for a polymer ξ with $base(\xi) \subset I'$, we have

$$\left|\frac{\partial \ln \tilde{\Xi}(I')}{\partial \tilde{\Psi}_s(\xi)}\right|$$

$$\le \sum_{\substack{\mathcal{C}\in\mathcal{CP},\mathcal{C}\ni\xi, \\ base(\mathcal{C})\subset I'}} \left|\frac{\partial \mathbf{F}_{\tilde{\Psi}_s}(\mathcal{C};\underline{\zeta})}{\partial \tilde{\Psi}_s(\xi;\underline{\zeta})}\right| \exp\big\{c^*(\mathcal{C}) e^{-(\mu-2\mu_4+\mu_2+1)}\big\}$$

$$\le \sum_{n,m\ge 0} \sum_{\substack{\{I_1,\dots,I_n\}; \\ I_1,\dots,I_n, base(\xi)\ form \\ linked\ intervals}} \sum_{\substack{\{I_{n+1},\dots,I_{n+m}\}; \\ base(\xi),I_{n+1},\dots,I_{n+m}\ form \\ linked\ intervals}}$$

$$\times \exp\big\{c(\xi) e^{-(\mu-2\mu_4+\mu_2+1)}\big\} e^{(n+m+2)\mu_3^*}$$

$$\times \prod_{p=1}^{n+m} \Big(\sum_{base(\xi_p)=I_p} |\tilde{\Psi}_s(\xi_p;\underline{\zeta})| e^{d(\xi_p)+c(\xi_p)} \Big)$$

$$\le \sum_{n,m\ge 0} \left\{ \frac{2R(\mu_2,\delta)e^{2\mu_3^*}}{(1-R(\mu_2,\delta))^2} \right\}^{n+m} \exp\big\{c(\xi) e^{-(\mu-2\mu_4+\mu_2+1)}\big\}$$

$$\le 4 \exp\big\{c(\xi) e^{-(\mu-2\mu_4+\mu_2+1)}\big\},$$

if μ_2 is sufficiently large. This implies the uniform bound

$$\left|\frac{\partial \tilde{\Phi}_s(J;\underline{\zeta})}{\partial \tilde{\Psi}_s(\xi;\underline{\zeta})}\right| \le 4|J|^2 \exp\big\{3|base(\xi)| e^{-(\mu-2\mu_4+\mu_2+1)}\big\} \tag{2.32}$$

for $s \in [0,1]$, $\xi \in \mathcal{K}(J)$ and $\underline{\zeta}$ satisfying (2.21). Let $J = [\hat{l}, \hat{r}]$ be an interval in $[0, L]$ with $|J| \le (\ln L)^2$ and $Lt_i \notin J$ for any $i = 1, \dots, q+1$, and let $\xi \in \mathcal{K}(J)$ be such that $N_v(\xi) \le (\ln L)^2$. Let $K > 0$ be an arbitrary positive number and we fix it. We assume that $\underline{\zeta}$ satisfies (2.21) with $|Im\zeta_0| \le K$. By analyticity, for $\hat{l} \le \ell \le \hat{r}$ we have

$$\log Q(\zeta_L(\hat{r})) - \log Q(\zeta_L(\ell)) \le Const. \frac{(\ln L)^2}{L}.$$

uniformly in $\underline{\zeta}$ in this region. From this and the fact that

$$\mu \frac{area(\xi)}{L} + \mu(\frac{\hat{r}}{L} - \frac{\hat{r}(\xi)}{L})k(\gamma) \le \frac{\mu}{L}\sum_f |h(f)|(\hat{r} - pos(\xi))$$
$$\le \mu \frac{(\ln L)^2}{L} N_v(\xi) \le \mu(\ln L)^4/L,$$

using the inequality $|e^z - 1| \le |z|e^{|z|}$ we have

$$(2.33) \quad \frac{|\tilde{\Psi}(\xi;\underline{\zeta}) - \hat{\Psi}(\xi;\zeta_L(\hat{r}))|}{|\hat{\Psi}(\xi;\underline{\zeta})|}$$
$$= \left| \frac{Q(\zeta_L(\hat{r}))^{|base(\xi)|}}{\prod_{\ell=\hat{l}(\xi)}^{\hat{r}(\xi)} Q(\zeta_L(\ell))} \exp\left[\mu\zeta_0 \frac{area(\xi)}{L} + \mu\zeta_0(\frac{\hat{r}}{L} - \frac{\hat{r}(\xi)}{L})k(\gamma)\right] - 1\right|$$
$$\le Const. \frac{(\ln L)^4}{L}.$$

The constant does not depend on L or $\underline{\zeta}$ satisfying $|Im\zeta_0| \le K$ and (2.23). Hence we have

$$|\tilde{\Phi}(J;\underline{\zeta}) - \Phi_0(J;\zeta_L(\hat{r}))|$$
$$\le Const. \sum_{\substack{\xi;\, base(\xi)\subset J \\ N_v(\xi)\le(\ln L)^2}} |J|^2 e^{3|base(\xi)|e^{-(\mu-2\mu_4+\mu_2+1)}} |\hat{\Psi}(\xi;\underline{\zeta})| \frac{(\ln L)^4}{L}$$
$$+ \sum_{\substack{\xi;base(\xi)\subset J \\ N_v(\xi)\ge(\ln L)^2}} |J|^2 e^{3|base(\xi)|e^{-(\mu-2\mu_4+\mu_2+1)}} \left(|\tilde{\Psi}(\xi;\underline{\zeta})| + |\hat{\Psi}(\xi;\zeta_L(\hat{r}))|\right)$$
$$:= I + II.$$

Since $|J| \le (\ln L)^2$ and $\underline{\zeta}$ satisfies (2.21), we can bound I and II in the following way.

$$I \le Const.|J|^3 \Big\{ \sum_{\substack{\xi;\, base(\xi)=[0,k] \\ for\ some\ k\ge 0}} |\hat{\Psi}(\xi;\underline{\zeta})| e^{c(\xi)+d(\xi)} \Big\} \frac{(\ln L)^4}{L}$$

$$= O\Big(\frac{(\ln L)^{10}}{L}\Big),$$

$$II \le |J|^2 e^{-\frac{\delta}{6}(\ln L)^2} \sum_{\xi;\, base(\xi)\subset J} \big[|\tilde{\Psi}(\xi;\underline{\zeta})| + |\hat{\Psi}(\xi;\zeta_L(\hat{r}))|\big] e^{c(\xi)+d(\xi)}$$

$$\le 6(\ln L)^6 e^{-\frac{\delta}{6}(\ln L)^2}.$$

Using this and (2.26), we have

$$\frac{1}{L} \sum_{J=[\hat{l},\hat{r}]\subset[0,L]} \big|\tilde{\Phi}(J;\underline{\zeta}) - \Phi_0(J;\zeta_L(\hat{r}))\big|$$

$$\le \frac{6}{L} \sum_{\substack{J\subset[0,L]; \\ |J|>(\ln L)^2}} e^{-(\mu-2\mu_4+\mu_2+1)\lceil\frac{|J|}{3}\rceil} + \frac{6}{L} \sum_{\substack{J\subset[0,L]; \\ |J|\le(\ln L)^2, \\ Lt_i\in J \text{ for some } i}} e^{-(\mu-2\mu_4+\mu_2+1)\lceil\frac{|J|}{3}\rceil}$$

$$+ \frac{1}{L} \sum_{\substack{J=[\hat{l},\hat{r}]\subset[0,L]; \\ |J|\le(\ln L)^2, \\ Lt_i\notin J \text{ for any } i=1,\dots,q+1}} \big|\tilde{\Phi}(J;\underline{\zeta}) - \Phi_0(J;\zeta_L(\hat{r}))\big|$$

$$= O\Big(\frac{(\ln L)^{10}}{L}\Big)$$

uniformly in $\underline{\zeta}$ satisfying (2.21) with $Im\zeta_0 \le K$. Since we can take $K > 0$ in an arbitrary way, we proved (2.27).

the limiting quadratic form

Let $\underline{\zeta}$ satisfy (2.21). We introduce a $(q+1)\times(q+1)$ matrix $V_L(\underline{\zeta})$ by

$$V_L(\underline{\zeta}) = \frac{1}{\mu^2 L} \text{Hess } \ln \sum_{\Gamma\in\mathcal{S}_L} e^{\mu\underline{\zeta}\cdot\hat{X}_L^{(q)}(t_1,\dots,t_{q+1})} W(\Gamma).$$

This is analytic in $\underline{\zeta}$ satisfying (2.21).

Lemma 2.5. Assume that $\mu > 2\mu_4 - \mu_2 - 1$ and that $\underline{\zeta} \in \mathbf{R}^{q+2}$ and $\underline{\zeta}$ satisfies (2.21). Then uniformly in $\underline{\zeta}$ and $\underline{\eta} = (\eta_0,\dots,\eta_{q+1}) \in \mathbf{R}^{q+2}$

such that $|\underline{\eta}| = 1$,

$$\underline{\eta} \cdot V_L(\underline{\zeta})\underline{\eta} \longrightarrow \underline{\eta} \cdot V(\underline{\zeta})\underline{\eta}$$

as $L \to \infty$, where

$$V(\underline{\zeta}) = \frac{1}{\mu^2}\mathrm{Hess} \int_0^1 (\ln Q + \hat{\varphi})(\zeta(x))dx, \tag{2.34}$$

and

$$\zeta(x) = \zeta_0(1-x) + \sum_{i=1}^{q+1} \zeta_i 1_{[0,t_i]}(x). \tag{2.35}$$

Further, there exists $\mu_5 > 2\mu_4 - \mu_2 - 1$ such that $V(\underline{\zeta})$ is uniformly positive definite for $\mu > \mu_5$.

Proof. Let $\mu_5 > \mu_4 + 1$ be fixed and let $\mu > \mu_5$. It is easy to see that $\ln Q(\zeta(x))$ is analytic in $\underline{\zeta}$ for every $x \in [0,1]$, and

$$\underline{\eta} \cdot V(\underline{\zeta})\underline{\eta} = \frac{1}{\mu^2} \int_0^1 \Big(\eta_0(1-x) + \sum_{i=1}^{q+1} \eta_i 1_{[0,t_i]}(x)\Big)^2 \big(\ln Q + \hat{\varphi}\big)''(\zeta(x))dx.$$

The uniform convergence of

$$\frac{1}{L}\ln \sum_{\Gamma \in \mathcal{S}_L} e^{\mu\underline{\zeta}\cdot X_L^{(q)}(t_1,\dots,t_{q+1})} W(\Gamma)$$

to

$$\int_0^1 (\ln Q + \hat{\varphi})(\zeta(x))dx$$

assures the convergence $V_L(\underline{\zeta}) \to V(\underline{\zeta})$ by Cauchy's formula. What remains to prove is the non-degeneracy of $V(\underline{\zeta})$. First, note that for any $\zeta \in \mathbf{R}$ with $|\zeta| < 1$,

$$\frac{1}{\mu^2}(\ln Q)''(\zeta) = \frac{\cosh\mu\cosh\mu\zeta - 1}{(\cosh\mu - \cosh\mu\zeta)^2} \geq e^{-\mu}\frac{\cosh\mu_5 - 1}{\cosh\mu_5} \tag{2.36}$$

holds if $\mu > \mu_5$.

We prove the lemma in two different cases depending on whether $|\zeta_0 + \zeta_{q+1}|$ and $|\zeta_{q+1}|$ are both small or not.

Case 1) $|\zeta_0 + \zeta_{q+1}| < 1/5, \ |\zeta_{q+1}| < 1/5.$

In this case, we have

$$|\zeta(x)| \leq (1-x)|\zeta_0 + \zeta_{q+1}| + x|\zeta_{q+1}| + \sum_{i=1}^{q} |\zeta_i|$$
$$\leq \frac{1}{5} + \frac{\delta}{4\mu}$$

for every $x \in [0,1]$. By Cauchy's formula, we have

$$\hat{\varphi}''(\zeta(x)) = \frac{1}{\pi i} \int_{|z-\zeta(x)|=\frac{1}{5}} \frac{\hat{\varphi}(z)}{(z-\zeta(x))^3} dz$$

If $|z-\zeta(x)| = \frac{1}{5}$, then $|Rez| < \frac{3}{5} < 1-\frac{\delta}{\mu}$. Therefore by (2.26) and (2.28) we have

$$|\hat{\varphi}(z)| \leq 9 \sum_{n=1}^{\infty} e^{-(\mu - 2\mu_4 + \mu_2 + 1)n}.$$

Therefore as $\mu \to \infty$

$$\left|\frac{1}{\mu^2}\hat{\varphi}''(\zeta(x))\right| \leq \frac{18 \cdot 5^2}{\mu^2} e^{-\mu}(1+o(1)) \tag{2.37}$$

uniformly in $x \in [0,1]$. Taking μ_5 sufficiently large, we have

$$\frac{1}{\mu^2}(\ln Q + \hat{\varphi})''(\zeta(x)) \geq \frac{e^{-\mu}}{2} > 0$$

for $\mu > \mu_5$.

Case 2) $|\zeta_{q+1}| > \frac{1}{5}$ or $|\zeta_0 + \zeta_{q+1}| > \frac{1}{5}$.

We assume that $|\zeta_0 + \zeta_{q+1}| > \frac{1}{5}$. The argument for the case where $|\zeta_{q+1}| > \frac{1}{5}$ is the same. For $x \in [0, \frac{1}{16}]$ we have

$$|\zeta(x)| \geq (1-x)|\zeta_0 + \zeta_{q+1}| - x|\zeta_{q+1}| - \sum_{i=1}^{q} |\zeta_i|$$
$$\geq \frac{1}{8} - \frac{\delta}{4\mu} > \frac{1}{10}$$

for $\mu > \mu_5$, if μ_5 is sufficiently large. This means that

$$\frac{1}{\mu^2}(\ln Q)''(\zeta(x)) \geq \frac{e^{-\frac{9}{10}\mu}}{4} \frac{\cosh \mu_5 - 1}{\cosh \mu_5}$$

for $x \in [0, \frac{1}{16}]$ and $\mu > \mu_5$. Therefore by (2.36)

$$\begin{aligned}(2.38)\qquad &\int_0^{\frac{1}{16}} \frac{1}{\mu^2}(\ln Q)''(\zeta(x))\big(\eta_0(1-x)+\sum_{i=1}^{q+1}\eta_i 1_{[0,t_i]}(x)\big)^2 dx \\ &\geq \frac{e^{-\frac{9}{10}\mu}}{16\cdot 4}\frac{\cosh\mu_5 - 1}{\cosh\mu_5}\int_0^1 \big(\eta_0(1-x)+\sum_{i=1}^{q+1}\eta_i 1_{[0,t_i]}(x)\big)^2 dx.\end{aligned}$$

Since $\underline{\zeta} \in \mathbf{R}^{q+2}$ satisfies (2.21), $|\zeta(x)| < 1 - \frac{7\delta}{4\mu}$ for every $x \in [0,1]$. By Cauchy's formula,

$$\hat{\varphi}''(\zeta(x)) = \frac{1}{\pi i}\int_{|z-\zeta(x)|=\frac{\delta}{2\mu}} \frac{\hat{\varphi}(z)}{(z-\zeta(x))^3} d\zeta.$$

Since the circle $\{|z-\zeta(x)| = \frac{\delta}{2\mu}\}$ lies entirely in the region $\{Rez < 1-\frac{\delta}{\mu}\}$, by (2.26) and (2.28) we have

$$(2.39)\qquad \Big|\int_0^1 \frac{1}{\mu^2}\hat{\varphi}''(\zeta(x))dx\Big| \leq \frac{12}{\delta^2}e^{-\mu}(1+o(1)).$$

Thus, by (2.38) and (2.39) $V(\underline{\zeta})$ is uniformly positive definite.

Let $\hat{P}_L^{(q)}$ be the distribution of $\hat{X}_L^{(q)}(t_1, \dots, t_{q+1})$ under P_L, and $\hat{P}_{L,\underline{\zeta}}^{(q)}$ be given by

$$\hat{P}_{L,\underline{\zeta}}^{(q)}(\underline{\eta}) = E_L\left[e^{\mu\underline{\zeta}\cdot\hat{X}_L^{(q)}(t_1,\dots,t_{q+1})}\right]^{-1} e^{\mu\underline{\zeta}\cdot\underline{\eta}}\hat{P}_L^{(q)}(\underline{\eta})$$

for $\mu > \mu_5$, $\underline{\zeta} \in \mathbf{R}^{q+2}$ satisfying (2.21).

Lemma 2.6. Let $\delta > 0$ be small and $\mu > \mu_5$. Assume that $\underline{\zeta}_L, \underline{\zeta} \in \mathbf{R}^{q+2}$ satisfy (2.21) and $\underline{\zeta}_L \to \underline{\zeta}$ as $L \to \infty$. Then, under $\hat{P}_{L,\underline{\zeta}_L}^{(q)}$ the centralized random vector

$$\hat{Y}_L^{(q)}(t_1,\dots,t_{q+1}) = \frac{1}{\sqrt{L}}\big(\hat{X}_L^{(q)}(t_1,\dots,t_{q+1}) - \hat{E}_{L,\underline{\zeta}_L}^{(q)}\hat{X}_L^{(q)}(t_1,\dots,t_{q+1})\big)$$

converges weakly to a centered Gaussian random vector $\hat{Y}^{(q)}(t_1, \dots, t_{q+1})$ of which covariance matrix is given by $V(\underline{\zeta})$.

Proof. Let

$$g_L(\underline{\eta}) = \hat{E}_{L,\underline{\zeta}_L}^{(q)}\left[e^{i\underline{\eta}\cdot\hat{Y}_L^{(q)}(t_1,\dots,t_{q+1})}\right].$$

Then

$$\ln g_L(\underline{\eta}) = L\varphi_L(\underline{\zeta}_L + \frac{i}{\sqrt{L}\mu}\underline{\eta}) - L\varphi_L(\underline{\zeta}) - \frac{i\underline{\eta}}{\sqrt{L}}\cdot\hat{E}_{L,\underline{\zeta}_L}^{(q)}\left[\hat{X}_L^{(q)}(t_1,\dots,t_{q+1})\right],$$

where $\varphi_L(\underline{\zeta})$ is given by

$$\varphi_L(\underline{\zeta}) = \frac{1}{L} \ln \sum_{\Gamma \in \mathcal{S}_L} e^{\mu \underline{\zeta} \cdot \hat{X}_L^{(q)}(t_1, \dots, t_{q+1})} W(\Gamma).$$

Since $\underline{\zeta}_L$ satisfies (2.21), so does $\underline{\zeta}_L + \frac{i}{\mu\sqrt{L}}\underline{\eta}$, and we have

$$\begin{aligned} &\varphi_L(\underline{\zeta}_L + \frac{i}{\mu\sqrt{L}}\underline{\eta}) - \varphi_L(\underline{\zeta}_L) \\ =& \frac{i}{\mu L\sqrt{L}} E^{(q)}_{L,\underline{\zeta}_L}\left[\hat{X}_L^{(q)}(t_1, \dots, t_{q+1})\right] - \frac{1}{2\mu^2 L^2} \sum_{j,k=1}^{q+1} \eta_j \eta_k \frac{\partial^2 \varphi_L}{\partial \zeta_j \partial \zeta_k}\bigg|_{\underline{\zeta}=\underline{\zeta}_L} + R_L. \end{aligned}$$

Since

$$\frac{1}{\mu^2 L} \sum_{j,k=1}^{q+1} \eta_j \eta_k \frac{\partial^2 \varphi_L}{\partial \zeta_j \partial \zeta_k}\bigg|_{\underline{\zeta}=\underline{\zeta}_L} = \sum_{j,k=1}^{q+1} \eta_j \eta_k V_L(\underline{\zeta}_L)_{j,k},$$

this term converges to $-\frac{1}{2}\eta \cdot V(\underline{\zeta})\underline{\eta}$. So it remains to show that $LR_L \to 0$ as $L \to \infty$. Formally, R_L has the following integral representation.

$$\text{(2.40)} \qquad R_L = \Big(\frac{i}{\mu\sqrt{L}}\Big)^3 \sum_{1 \le j \le k \le m \le n} R_{j,k,m},$$

where for $j < k < m$,

$$R_{j,j,j} = \frac{\eta_j^3}{2\pi i}\int_{C_j} \frac{\varphi_L\big(\underline{\zeta}_L + (\xi_j - \zeta_{L,j})\mathbf{e}_j + \sum_{\nu=j+1}^n (\frac{i}{\mu\sqrt{L}})\eta_\nu \mathbf{e}_\nu\big)}{(\xi_j - \zeta_L, j)^3(\xi_j - \zeta_{L,j} - (\frac{i}{\mu\sqrt{L}})\eta_j)} d\xi_j,$$

$$R_{j,j,k} = \frac{\eta_j^2\eta_k}{(2\pi i)^2}\int_{C_j} \frac{d\xi_j}{(\xi_j - \zeta_{L,j})^3}\int_{C_k} d\xi_k \times \frac{\varphi_L\big(\underline{\zeta}_L + \sum_{\alpha=j,k}(\xi_\alpha - \zeta_{L,j})\mathbf{e}_\alpha + \sum_{\beta=k+1}^n (\frac{i}{\mu\sqrt{L}})\eta_\beta \mathbf{e}_\beta\big)}{(\xi_k - \zeta_{L,k})(\xi_k - \zeta_{L,k} - (\frac{i}{\mu\sqrt{L}})\eta_k)},$$

$$R_{j,k,k} = \frac{\eta_j\eta_k^2}{(2\pi i)^2}\int_{C_j} \frac{d\xi_j}{(\xi_j - \zeta_{L,j})^2}\int_{C_k} d\xi_k \times \frac{\varphi_L\big(\underline{\zeta}_L + \sum_{\alpha=j,k}(\xi_\alpha - \zeta_{L,\alpha})\mathbf{e}_\alpha + \sum_{\beta=k+1} (\frac{i}{\mu\sqrt{L}})\eta_\beta \mathbf{e}_\beta\big)}{(\xi_j - \zeta_{L,j})(\xi_k - \zeta_{L,k} - (\frac{i}{\mu\sqrt{L}})\eta_k)},$$

$$R_{j,k,m} = \frac{\eta_j\eta_k\eta_m}{(2\pi i)^3}\int_{C_j} \frac{d\xi_j}{(\xi_j - \xi_{L,j})^2}\int_{C_k} \frac{d\xi_k}{(\xi_k - \zeta_{L,k})^2}\int_{C_m} d\xi_m \times \frac{\varphi_L\big(\underline{\zeta}_L + \sum_{\alpha=j,k,m}(\xi_\alpha - \zeta_{L,\alpha})\mathbf{e}_\alpha + \sum_{\beta=m+1}^n (\frac{i}{\mu\sqrt{L}})\eta_\beta \mathbf{e}_\beta\big)}{(\xi_m - \zeta_{L,m})(\xi_m - \zeta_{L,m} - (\frac{i}{\mu\sqrt{L}})\eta_m)}.$$

Here, C_p is a curve composed of the lower half of the circle $\{|\xi_p - \zeta_{L,p}| = \rho\}$, upper half of the circle $\{|\xi_p - \zeta_{L,p} - (\frac{i}{\mu\sqrt{L}})\eta_p| = \rho\}$, and vertical line segments connecting them, and ρ is a small positive

number. Let us estimate $|R_{j,j,j}|$. Other terms can be estimated similarly. Set

$$\underline{w}_L(j) := \underline{\zeta}_L + (\xi_j - \zeta_{L,j})\mathbf{e}_j + \sum_{\nu=j+1}^n \frac{i}{\mu\sqrt{L}}\eta_\nu \mathbf{e}_\nu.$$

Then it is easy to see that

$$\max\{|Re(w_L(j)_0 + w_L(j)_{q+1})|, |Re(w_L(j)_{q+1})|\} \le 1 - \frac{2\delta}{\mu} + \rho(\delta_{0,j} + \delta_{q+1,j})$$

and $|Re(w_L(j)_\alpha)| \le \dfrac{\delta}{4(q+1)\mu} + \rho\delta_{\alpha,j}$, where $\delta_{j,k} = 1$ if $j = k$ and $= 0$ if $j \ne k$. Note that

$$\varphi_L(w_L(j)) = \hat{\varphi}(w_L(j)) + \frac{1}{L}\sum_{\ell=0}^L \ln Q(\underline{\tilde{w}}_L(j;\ell)),$$

where

$$\underline{\tilde{w}}_L(j;\ell) := w_L(j)_0(1-\frac{\ell}{L}) + \sum_{p=1}^{q+1} w_L(j)_p 1_{[\ell \le Lt_p]},$$

and that

$$|Re\underline{\tilde{w}}_L(j;\ell)| \le \max\{|Re(w_L(j)_0 + w_L(j)_{q+1})|, |Re(w_l(j)_{q+1})|\} + \sum_{p=1}^{q} |Re(w_L(j)_p)| < 1 - \frac{7\delta}{4\mu} + \rho.$$

If $\rho < \delta/4\mu$, then we have analyticity of the integrand in the expression of $R_{j,j,j}$ as in the proof of Proposition 2.2. This is true when $\underline{\zeta}_L$ satisfies (2.21) and $\xi_j \in C_j$. Thus, we can assume that $|\varphi_L(\underline{w}_L(j))|$. From this we easily obtain that

$$|R_{j,j,j}| \le 2M \frac{|\eta_j|^3}{\rho^3}$$

for some $M > 0$, which is independent of L. This means that $LR_L = O(L^{-1/2})$ uniformly in $\underline{\eta}$.

Let $g_{\underline{\zeta}}$ be the density function of the Gaussian vector $\hat{Y}^{(q)}(t_1, \dots, t_{q+1})$ given in Lemma 2.6.

Proposition 2.7. Let $\mathcal{X}_L^{(q)} = (L^{-1}\mathbf{Z}) \times \mathbf{Z}^{q+1}$. For each $\underline{x}_L \in \mathcal{X}_L^{(q)}$ and $\underline{\zeta}_L \in \mathbf{R}^{q+2}$ satisfying (2.21), let

$$\underline{y}_L := \frac{1}{\sqrt{L}}(\underline{x}_L - \hat{E}^{(q)}_{L,\underline{\zeta}_L} \hat{X}_L^{(q)}(t_1, \dots, t_{q+1})).$$

Then we have

$$2L^{(q+4)/2}\hat{P}_L^{(q)}(\underline{x}_L) - g_{\underline{\zeta}_L}(\underline{y}_L) \to 0$$

uniformly in $\underline{x}_L \in \mathcal{X}_L$ and $\underline{\zeta}_L \in \mathbf{R}^{q+2}$ satisfying (2.21).

The proof is a complete repetition of the proof of Theorem 6.3 in [DH2], so we omit it. Let $h > 0$ and $a \ge \frac{h}{2}$ be such that

(2.41a) $$\frac{1}{\mu}\int_0^1 (1-x)\varphi'((1-x)\zeta_0^* + \zeta_1^*)dx = a$$

(2.41b) $$\frac{1}{\mu}\int_0^1 \varphi'((1-x)\zeta_0^* + \zeta_1^*)dx = h$$

hold for some $(\zeta_0^*, \zeta_1^*) \in \mathbf{R}^2$ with

$$\max\{|\zeta_0^* + \zeta_1^*|, |\zeta_1^*|\} \le 1 - \frac{2\delta}{\mu}, \tag{2.42}$$

where $\varphi = \ln Q + \hat{\varphi}$. Let also $a_L > 0$ and $h_L > 0$ satisfy

$$\frac{1}{\mu}\frac{\partial \varphi_L}{\partial \zeta_0}(\zeta_{L,0}, 0, \dots, 0, \zeta_{L,1}) = \frac{a_L}{L^2} \tag{2.43a}$$

$$\frac{1}{\mu}\frac{\partial \varphi_L}{\partial \zeta_1}(\zeta_{L,0}, 0, \dots, 0, \zeta_{L,1}) = \frac{h_L}{L} \tag{2.43b}$$

for some $(\zeta_{L,0}, \zeta_{L,1})$ satisfying (2.42), and

$$(\frac{a_L}{L^2}, \frac{h_L}{L}) \to (a, h).$$

For simplicity, we write $\varphi_L(\zeta_0, \zeta_1)$ for $\varphi_L(\zeta_0, 0, \dots, 0, \zeta_1)$. By the argument in the proof of Lemma 2.5, for a sufficiently small $\rho > 0$, $\varphi_L(\zeta_{L,0}, \zeta_{L,1})$ and

$$\mathcal{L}(\zeta_0, \zeta_1) := \int_0^1 \varphi(\zeta_0(1-x) + \zeta_1)dx$$

are analytic in $(\zeta_0, \zeta_1) \in \mathcal{D}_\rho$, where

$$\mathcal{D}_\rho := \{(\zeta_0, \zeta_1) \in \mathbf{C}^2; \max\{|\zeta_0 - \zeta_0^*|, |\zeta_1 - \zeta_1^*|\} \le \rho\}.$$

Also, $\varphi_L(\zeta_0, \zeta_1)$ converges to $\mathcal{L}(\zeta_0, \zeta_1)$ uniformly in $\mathcal{D}_\rho$. Therefore we also have the convergence;

$$(\nabla_{(\zeta_0,\zeta_1)}\varphi_L)(\zeta_0^*, \zeta_1^*) \to (\nabla_{(\zeta_0,\zeta_1)}\mathcal{L})(\zeta_0^*, \zeta_1^*). \tag{2.44}$$

This convergence is uniform in (ζ_0^*, ζ_1^*) satisfying (2.42). By Lemma 2.5 for $q = 0$, there exist $L_0 \ge 1$ and $\varepsilon = \varepsilon(\rho, \mu, \delta, \zeta_0^*, \zeta_1^*) > 0$ such that

$$\sum_{j,k=0}^{1} [Hess_{(\zeta_0,\zeta_1)}\varphi_L(\zeta_0, \zeta_1)]_{j,k}\eta_j\eta_k \ge \varepsilon(|\eta_0|^2 + |\eta_1|^2),$$

$$\sum_{j,k=0}^{1} [Hess_{(\zeta_0,\zeta_1)}\mathcal{L}(\zeta_0, \zeta_1)]_{j,k}\eta_j\eta_k \ge \varepsilon(|\eta_0|^2 + |\eta_1|^2),$$

for $(\zeta_0, \zeta_1) \in \mathcal{D}_\rho \cap \mathbf{R}^2$, $L \geq L_0$ and $\eta_0, \eta_1 \in \mathbf{R}$. This implies that $\nabla_{(\zeta_0,\zeta_1)}\varphi_L$ and $\nabla_{(\zeta_0,\zeta_1)}\mathcal{L}$ are one-to-one bicontinuous maps on $\mathcal{D}_\rho \cap \mathbf{R}^2$ for every $L \geq L_0$. In particular, we have

(2.45a)
$$\| (\nabla_{(\zeta_0,\zeta_1)}\varphi_L)(\zeta_0, \zeta_1) - (\nabla_{(\zeta_0,\zeta_1)}\varphi_L)(\zeta_0^*, \zeta_1^*) \| \geq \frac{\varepsilon}{2} \| (\zeta_0, \zeta_1) - (\zeta_0^*, \zeta_1^*) \|$$

and

(2.45b)
$$\| (\nabla_{(\zeta_0,\zeta_1)}\mathcal{L})(\zeta_0, \zeta_1) - (\nabla_{(\zeta_0,\zeta_1)}\mathcal{L})(\zeta_0^*, \zeta_1^*) \| \geq \frac{\varepsilon}{2} \| (\zeta_0, \zeta_1) - (\zeta_0^*, \zeta_1^*) \|$$

for every $(\zeta_0, \zeta_1) \in \mathcal{D}_\rho \cap \mathbf{R}^2$. By (2.44) and the definition of (a, h) and (a_L, h_L), we have

$$\| \frac{1}{\mu}\nabla_{(\zeta_{L,0},\zeta_{L,1})}\varphi_L(\zeta_0^*, \zeta_1^*) - (\frac{a_L}{L^2}, \frac{h_L}{L}) \| \to 0.$$

This means that we can find $(\zeta_{L,0}, \zeta_{L,1}) \in \mathcal{D}_\rho$ which solves (2.43a, 2.43b) and by (2.45a, 2.45b) it converges to (ζ_0^*, ζ_1^*).

In order to discuss convergence of $X_L(t)$ from Proposition 2.7, except tightness we need one more estimate which assures that the separating contour itself neither fluctuates a lot nor is fat. To do this, let us define

$$vol(\xi) := |\gamma| + \sum_{\alpha+1}^{u} |C_\alpha| \tag{2.46}$$

for a polymer $\xi = (\gamma, \{C_\alpha\}_{\alpha=1}^u, \{\Lambda_\beta\}_{\beta=1}^v)$.

Lemma 2.8. Let $\mu > \mu_5$, $h > 0$, $a \geq \frac{h}{2}$ and a, h, a_L, h_L be given as above such that $(\frac{a_L}{L^2}, \frac{h_L}{L}) \to (a, h)$ as $L \to \infty$. Then for every $k \in \mathbf{N}$, there exists a constant $L_0 \geq 1$ such that for $L \geq L_0$,

(2.47)
$$P_L\left(\max\{vol(\xi); \xi \in \Delta(\Gamma)\} \geq \frac{6}{\delta}\ln L + k \;\middle|\; a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\right)$$
$$\leq 1 - \exp\left(-4e^{-\frac{\delta}{6}k}\right).$$

Proof. Let (ζ_0^*, ζ_1^*) solves (2.41a, 2.41b) satisfying (2.42) and $(\zeta_{L,0}, \zeta_{L,1})$ be a solution of (2.43a, 2.43b) satisfying (2.42) such that $(\zeta_{L,0}, \zeta_{L,1})$

converges to (ζ_0^*, ζ_1^*) as $L \to \infty$. Put

$$\hat{X}_L^{(0)}(\Gamma) := (\frac{a(\pi(\Gamma))}{L}, k(\Gamma)) = \sum_{\xi \in \Delta(\Gamma)} \Big(\frac{area(\xi)}{L} + k(\gamma)\big(1 - \frac{\hat{r}(\xi)}{L}\big), k(\gamma) \Big).$$

Then for $N := \frac{6}{\delta} \ln L + k$,

$$(2.48) \quad P_L\big(\max\{vol(\xi); \xi \in \Delta(\Gamma)\} \leq N \mid a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\big)$$
$$= \Big[\sum_{\Gamma \in \mathcal{S}_L; a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L} e^{\mu(\zeta_{L,0}, \zeta_{L,1}) \cdot \hat{X}_L^{(0)}(\Gamma)} W(\Gamma) \Big]^{-1}$$
$$\times \sum_{\substack{\Gamma \in \mathcal{S}_L, a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L \\ vol(\xi) \leq N \ for \ every \ \xi \in \Delta(\Gamma)}} e^{\mu(\zeta_{L,0}, \zeta_{L,1}) \cdot \hat{X}_L^{(0)}(\Gamma)} W(\Gamma).$$

By Proposition 2.7 we have

$$(2.49) \quad \sum_{\Gamma \in \mathcal{S}_L; a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L} e^{\mu(\zeta_{L,0}, \zeta_{L,1}) \cdot \hat{X}_L^{(0)}(\Gamma)} W(\Gamma)$$
$$= e^{L \varphi_L(\zeta_{L,0}, \zeta_{L,1})} \hat{P}^{(0)}_{L, (\zeta_{L,0}, \zeta_{L,1})} (\frac{a_L}{L}, h_L)$$
$$= e^{L \varphi_L(\zeta_{L,0}, \zeta_{L,1})} \frac{g_{(\zeta_{L,0}, \zeta_{L,1})}(0,0)}{2L^2} \{1 + o(1)\}$$

as $L \to \infty$.

Let (ζ_0, ζ_1) satisfy (2.42) and

$$\varphi_L^{(N)}(\zeta_0, \zeta_1) := \frac{1}{L} \ln \sum_{\substack{\Gamma \in \mathcal{S}_L, a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L \\ vol(\xi) \leq N \ for \ every \ \xi \in \Delta(\Gamma)}} e^{\mu(\zeta_0, \zeta_1) \cdot \hat{X}_L^{(0)}(\Gamma)} W(\Gamma).$$

It is straightforward to check that the estimate (2.4) is still valid when we replace $d(\xi)$ with

$$d_1(\xi) := d(\xi) - \frac{\delta}{6}|\gamma| + \frac{\delta}{6} vol(\xi).$$

The only change is that we introduce

$$G_1(\gamma) := \sum_{\xi;\ \gamma \ is \ the \ backbone \ of \ \xi} \Big| \Psi(\xi) e^{\mu(\zeta_0, \zeta_1) \cdot \hat{X}_L^{(0)}} e^{\frac{\delta}{6} \sum_\alpha |C_\alpha|} \Big|,$$

in place of $G(\gamma)$, and in estimating $G_1(\gamma)$, we have to put

$$g_2(\mu_2, \mu_0) = 4 \sum_{C \ni 0;\ connected} e^{-(\mu_2 - g_1(\mu_2,\mu_0) - \ln 2 - \delta/6)|C|}.$$

Therefore we have convergent cluster expansion;

$$\varphi_L^{(N)}(\zeta_0, \zeta_1) = \frac{1}{L} \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}_L(N))} \mathbf{F}^T_{\hat{\Psi}}(\Delta; \zeta_0, \zeta_1),$$

where $\mathcal{CP}_L(N) := \{\mathcal{C} \in \mathcal{CP}_L;\ vol(\mathcal{C}) \leq N\}$ and

$$\text{(2.50)} \qquad \sum_{\Delta\, i\, \mathcal{C}_0,\ \Delta \in \mathcal{P}_f(\mathcal{CP})} \left|\mathbf{F}^T_{\hat{\Psi}}(\Delta; \zeta_0, \zeta_1)\right| e^{d_1^*(\Delta)} \leq c^*(\mathcal{C}_0).$$

Therefore we have

(2.51)

$$\left|\varphi_L(\zeta_0, \zeta_1) - \varphi_L^{(N)}(\zeta_0, \zeta_1)\right| \leq \frac{1}{L} \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}_L) \setminus \mathcal{P}_f(\mathcal{CP}_L(N))} \left|\mathbf{F}^T_{\hat{\Psi}}(\Delta; \zeta_0, \zeta_1)\right|.$$

If $\Delta \in \mathcal{P}_f(\mathcal{CP}_L(N))$, then Δ contains at least one $\xi \in \mathcal{K}_L$ such that $vol(\xi) \geq N$. Therefore by (2.50) the RHS of (2.51) is bounded by

$$\frac{e^{-\frac{\delta}{6}N}}{L} \sum_{\Delta \in \mathcal{P}_f(\mathcal{CP}_L)} \left|\mathbf{F}^T_{\hat{\Psi}}(\Delta; \zeta_0, \zeta_1)\right| e^{-d_1(\Delta)} \leq 3e^{-\frac{\delta}{6}N}.$$

This estimate is uniform for (ζ_0, ζ_1) satisfying (2.42). By analyticity of φ_L and $\varphi_L^{(N)}$, we have for $\alpha, \beta \in \{0, 1\}$,

$$\text{(2.52a)} \qquad \left|\frac{1}{\mu} \frac{\partial}{\partial \zeta_\alpha} [\varphi_L - \varphi_L^{(N)}](\zeta_{L,0}, \zeta_{L,1})\right| \leq \frac{3}{\rho} e^{-\frac{\delta}{6}N}$$

and

$$\text{(2.52b)} \qquad \left|\frac{1}{\mu^2} \frac{\partial^2}{\partial \zeta_\alpha \partial \zeta_\beta} \left[\varphi_L - \varphi_L^{(N)}\right](\zeta_{L,0}, \zeta_{L,1})\right| \leq \frac{3}{\rho^2} e^{-\frac{\delta}{6}N}$$

where $0 < \rho < \frac{\delta}{4\mu}$. Since $N \to \infty$ as $L \to \infty$,

$$Hess_{(\zeta_0,\zeta_1)} \varphi_L^{(N)}(\zeta_{L,0}, \zeta_{L,1}) \to Hess_{(\zeta_0,\zeta_1)} \mathcal{L}(\zeta_0^*, \zeta_1^*)$$

as $L \to \infty$. Let $\hat{P}_{L,(\zeta_{L,0},\zeta_{L,1})}(\Gamma)$ be the probability weight which is proportional to $e^{\mu(\zeta_{L,0},\zeta_{L,1}) \cdot \hat{X}^{(0)}(\Gamma)} W(\Gamma)$ restricted to the ensemble

$$\{\Gamma \in \mathcal{S}_L;\ vol(\xi) \leq N \ \text{ for every } \ \xi \in \Delta(\Gamma)\}.$$

Then by (2.52a, 2.52b) as in the proof of Proposition 2.7, we see that

$$\frac{1}{\sqrt{L}}\Big(\big(\frac{a(\pi(\Gamma))}{L}, k(\Gamma)\big) - \frac{1}{\mu}E^{(N)}_{L,(\zeta_{L,0},\zeta_{L,1})}\big(\frac{a(\pi(\Gamma))}{L}, k(\Gamma)\big)\Big)$$

converges to a centered Gaussian vector with covariance matrix

$$\frac{1}{\mu^2}Hess_{(\zeta_0,\zeta_1)}\mathcal{L}(\zeta_0^*,\zeta_1^*)$$

as far as $N \to \infty$ as $L \to \infty$. Further, since $N - \frac{3}{\delta}\ln L \to \infty$,

$$\frac{1}{\mu}|\nabla_{(\zeta_0,\zeta_1)}\varphi_L^{(N)}(\zeta_{L,0},\zeta_{L,1}) - (\frac{a_L}{L^2}, \frac{h_L}{L})| = o(\frac{1}{\sqrt{L}})$$

as $L \to \infty$ and by this we have

$$\hat{P}^{(N)}_{L,(\zeta_{L,0},\zeta_{L,1})}\Big(\big(\frac{a(\pi(\Gamma))}{L}, k(\Gamma)\big) = \big(\frac{a_L}{L}, h_L\big)\Big) = \frac{g_{(\zeta_{L,0},\zeta_{L,1})}(0,0)}{2L^2}\{1+o(1)\}$$

as in the proof of Proposition 2.7. Combining this with (2.48) and (2.49), we see that there exists an $L_0 \geq 1$ such that for $L \geq L_0$ and $N = \frac{6}{\delta}\ln L + k$,

$$\begin{aligned}
&P_L(vol(\xi) \leq N \text{ for every } \xi \in \Delta(\Gamma) \mid a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L)\\
&\geq \exp\{-L|\varphi_L(\zeta_{L,0},\zeta_{L,1}) - \varphi_L^{(N)}(\zeta_{L,0},\zeta_{L,1})|\}\exp\{-e^{-\frac{\delta}{6}k}\}\\
&\geq \exp\{-4e^{-\frac{\delta}{6}k}\}.
\end{aligned}$$

Theorem 2.9. Let $\mu > \mu_5$, $h > 0, a \geq \frac{h}{2}$ and a_L, h_L be given as above. Further, we assume that $aL^2 - a_L = o(\sqrt{L^3})$ and $hL - h_L = o(\sqrt{L})$ as $L \to \infty$. Then the process $Y_L(t)$ under $P_L(\cdot \mid a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L)$ converges in finite dimensional distribution to the process

$$Y(t) = \frac{1}{\mu}\int_0^t \sqrt{\varphi''(\zeta_1 + (1-x)\zeta_0)}dB(x)$$

conditioned that

$$\int_0^1 Y(t)dt = 0, \quad Y(1) = 0.$$

Proof. Let $q \geq 1$, and $0 < t_1 < \cdots < t_{q+1} = 1$ be given arbitrarily. We take $(\zeta_0,\zeta_1) \in \mathbf{R}^2$ which satisfies (2.42) and solves (2.41a, 2.41b). Also, we take $(\zeta_{L,0},\zeta_{L,1})$ as a solution of (2.43a, 2.43b) which satisfies

(2.42). Then by the above argument $(\zeta_{L,0}, \zeta_{L,1}) \to (\zeta_0, \zeta_1)$ as $L \to \infty$. Let $\underline{\zeta}_L^\circ, \underline{\zeta}^\circ \in \mathbf{R}^{q+2}$ be

$$\underline{\zeta}_L^\circ = (\zeta_{L,0}, 0, \dots, 0, \zeta_{L,1})$$
$$\underline{\zeta}^\circ = (\zeta_0, 0, \dots, 0, \zeta_1).$$

From the assumption of the theorem and (2.45a) and the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_L$, we have

$$\begin{aligned}
&\hat{E}^{(q)}_{L,\underline{\zeta}_L^\circ} \hat{X}_L^{(q)}(t_1, \dots, t_{q+1}) \\
=&(\frac{a_L}{L}, \hat{E}^{(q)}_{L,\underline{\zeta}_L^\circ} X_L(\frac{\lfloor Lt_1 \rfloor}{L}), \cdots, \hat{E}^{(q)}_{L,\underline{\zeta}_L^\circ} X_L(\frac{\lfloor Lt_q \rfloor}{L}), h_L) \\
=&\frac{L}{\mu}(\nabla_{\underline{\zeta}}\varphi_L)(\underline{\zeta}_L^\circ) \\
=&\frac{L}{\mu}(\nabla_{\underline{\zeta}}\varphi^{(q)})(\underline{\zeta}^\circ; t_1, \dots, t_{q+1}) + o(\sqrt{L}).
\end{aligned}$$

By proposition 2.7 we have for $-\infty < \hat{l}_j < \hat{r}_j < \infty, 1 \le j \le q$,

$$\begin{aligned}
&\lim_{L\to\infty} \hat{P}_L^{(q)}\big(y_j \in [\hat{l}_j, \hat{r}_j] \quad 1 \le j \le q \mid x_0 = \frac{a_L}{L}, x_{q+1} = h_L\big) \\
=&\lim_{L\to\infty} \hat{P}^{(q)}_{L,\underline{\zeta}_L^\circ}\big(y_j \in [\hat{l}_j, \hat{r}_j] \quad 1 \le j \le q \mid x_0 = \frac{a_L}{L}, x_{q+1} = h_L\big) \\
=&\frac{\int_{[\hat{l}_1,\hat{r}_1]\times\cdots\times[\hat{l}_q,\hat{r}_q]} g_{\underline{\zeta}^\circ}(0, y_1, \dots, y_q, 0)dy_1 \cdots dy_q}{\int_{\mathbf{R}^q} g_{\underline{\zeta}^\circ}(0, y_1, \dots, y_q, 0)dy_1 \cdots dy_q}.
\end{aligned}$$

Let

$$\hat{Y}^{(q)}(t_1, \dots, t_{q+1}) = (Y_0, Y(t_1), Y(t_2), \dots, Y(t_{q+1}))$$

be a Gaussian random vector with distribution density $g_{\underline{\zeta}}(y_0, \dots, y_{q+1})$. Then its covariance matrix is given by

$$E\left[Y(t_j)Y(t_k)\right] = \frac{1}{\mu^2}\int_0^{t_j \wedge t_k} \varphi''(\zeta_0(1-x)+\zeta_1)dx$$

for $j, k = 1, \dots, q+1$, where $a \wedge b = \min\{a, b\}$, and

$$\begin{aligned}
E\left[Y_0 Y(t_j)\right] =& \frac{1}{\mu^2}\int_0^{t_j} \varphi''(\zeta_0(1-x)+\zeta_1)dx \\
E\left[Y_0^2\right] =& \frac{1}{\mu^2}\int_0^{1} \varphi''(\zeta_0(1-x)+\zeta_1)dx
\end{aligned}$$

for $j = 1, 2, \ldots, q+1$. This means that $\{Y_0, \{Y(t)\}_{t\in[0,1]}\}$ is a Gaussian system with covariance given above for every $0 < t_1 < \ldots < t_{q+1} = 1$, $q \geq 1$. Finally, by Lemma 2.8 we can replace $\hat{E}^{(q)}_{L,\underline{\zeta}^\circ_L} X_L(t_j)$ with $\hat{E}^{(q)}_{L,\underline{\zeta}^\circ_L} X_L(\frac{\lfloor Lt_j \rfloor}{L})$ for every $1 \leq j \leq q$ in the above argument.

§3. Tihgtness

As usual, we will estimate the fourth moment of $Y_L(t) - Y_L(s)$ for every $s, t \in [0,1]$. First, we show the following one polymer estimate. For an integer $x \in [0, L]$ and $\Gamma \in \mathcal{S}_L$, let $\xi(x) = \xi(x, \Gamma)$ be the unique element of $\mathcal{D}(\Gamma)$ whose base contains x.

Lemma 3.1 Let $\mu > \mu_5, h > 0, a \geq \frac{h}{2}$ and a_L, h_L be given as in Lemma 2.8. Then there exist constants $C > 0$ and $L_1 \geq 1$ such that for $L \geq L_1$,

$$E_L\left[e^{\frac{1}{2}d(\xi(x))}\,\middle|\, a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\right] \leq C.$$

Proof. Let (ζ_0^*, ζ_1^*) satisfy (2.41a, 2.41b) and (2.42), and $(\zeta_{L,0}, \zeta_{L,1})$ satisfy (2.42) and (2.43a, 2.43b) such that $(\zeta_{L,0}, \zeta_{L,1}) \to (\zeta_0^*, \zeta_1^*)$ as $L \to \infty$. For $\Gamma \in \mathcal{S}_L$ such that $\mathcal{D}(\Gamma) \ni \xi$, let $\Gamma'(\xi)$ denote the set of elements of $\mathcal{S}_L$ such that $\mathcal{D}(\Gamma'(\xi)) = \mathcal{D}(\Gamma) \setminus \{\xi\}$. Also we put for a polymer ξ,

$$\hat{X}_L^{(0)}(\xi) = \Big(\frac{area(\gamma)}{L} + k(\gamma)(1 - \frac{\hat{r}(\xi)}{L}), k(\gamma)\Big),$$

and $\Psi(\xi; \zeta_0, \zeta_1) := \Psi(\xi) \exp\{\mu \hat{X}_L^{(0)}(\xi) \cdot (\zeta_0, \zeta_1)\}$, where γ stands for the backbone of ξ. Then

$$\begin{aligned}
&P_{L,(\zeta_{L,0},\zeta_{L,1})}\big[\{\mathcal{D}(\Gamma) \ni \xi\} \cap \{\hat{X}_L^{(0)}(\Gamma) = (\tfrac{a_L}{L}, h_L)\}\big] \\
=&e^{-L\varphi_L(\zeta_{L,0},\zeta_{L,1})}\Psi(\xi; \zeta_{L,0}, \zeta_{L,1}) \\
&\times \sum_{\substack{\Gamma\in\mathcal{S}_L;\ \mathcal{D}(\Gamma)\ni\xi,\\ a(\pi(\Gamma))=a_L, k(\Gamma)=h_L}} e^{\mu(\zeta_{L,0},\zeta_{L,1})\cdot\hat{X}_L^{(0)}(\Gamma'(\xi))} W(\Gamma'(\xi)).
\end{aligned}$$

By the cluster expansion we have

$$\begin{aligned}
(3.1)\quad &P_{L,(\zeta_{L,0},\zeta_{L,1})}\big[\{\mathcal{D}(\Gamma) \ni \xi\} \cap \{\hat{X}_L^{(0)}(\Gamma) = (\tfrac{a_L}{L}, h_L)\}\big] \\
&= \sum_{\substack{C\in\mathcal{CP}_L;\\ C\ni\xi}} \mathbf{F}_{\hat{\Psi}}(C; \zeta_{L,0}, \zeta_{L,1}) \exp\{-\sum_{\Delta\in\mathcal{P}_f(\mathcal{CP}_L);\Delta i C} \mathbf{F}^T_{\hat{\Psi}}(\Delta; \zeta_{L,0}, \zeta_{L,1})\} \\
&\quad\times P_{L,(\zeta_{L,0},\zeta_{L,1})}\big[a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L \mid \mathcal{D}(\Gamma) \ni \xi\big],
\end{aligned}$$

where

$$\hat{\Psi}(\xi;\zeta_{L,0},\zeta_{L,1}) := \Psi(\xi;\zeta_{L,0},\zeta_{L,1})\prod_{\ell=0}^{L} Q^{-1}(\zeta_{L,0}(1-\frac{\ell}{L})+\zeta_{L,1}).$$

Since the final term in the RHS of (3.1) is not larger than 1, by the same argument to derive (2.32) we have for $C>0$,

$$\begin{aligned}&\sum_{\substack{\xi;base(\xi)\ni x,\\ |\gamma|\geq C\ln L}} e^{\frac{1}{2}d(\xi)} P_{L,(\zeta_{L,0},\zeta_{L,1})}\left[\{\mathcal{D}(\Gamma)\ni\xi\}\cap\{\hat{X}_L^{(0)}(\Gamma)=(\frac{a_L}{L},h_L)\}\right]\\ \leq&4\sum_{\xi;base(\xi)\ni x,|\gamma|\geq C\ln L} e^{c(\xi)+\frac{1}{2}d(\xi)}\hat{\Psi}(\xi;\zeta_{L,0},\zeta_{L,1}).\end{aligned}$$

As in the proof of Lemma 2.1,

$$\begin{aligned}(3.2)\qquad &\sum_{\xi;\ base(\xi)\ni x,\ |\gamma|\geq C\ln L} e^{c(\xi)+\frac{1}{2}d(\xi)}|\hat{\Psi}(\xi;\zeta_{L,0},\zeta_{L,1})|\\ \leq&e^{-\frac{\delta}{12}C\ln L}\sum_{\xi;\ base(\xi)\ni x} e^{c(\xi)+d(\xi)}|\hat{\Psi}(\xi;\zeta_{L,0},\zeta_{L,1})|\\ \leq&3e^{-\frac{\delta}{12}C\ln L}.\end{aligned}$$

By (2.49), we have for a constant $C_1>0$ and a sufficiently large L,

$$\begin{aligned}&E_{L,(\zeta_{L,0},\zeta_{L,1})}\left[e^{\frac{1}{2}d(\xi(x))}1_{\{|\gamma(x)|\geq C\ln L\}}\;\middle|\;\hat{X}_L^{(0)}(\Gamma)=(\frac{a_L}{L},h_L)\right]\\ \leq&C_1L^2e^{-\frac{\delta}{12}C\ln L},\end{aligned}$$

which goes to zero as $L\to\infty$. Here, $\gamma(x)$ stands for the backbone of $\xi(x)$.

Assume that $|\gamma|\leq C\ln L$ for the backbone γ of ξ. Then since

$$\begin{aligned}\varphi_L(\zeta_{L,0},\zeta_{L,1}\mid\xi) :=& \frac{1}{L}\ln\sum_{\Gamma\in\mathcal{S}_L;\mathcal{D}(\Gamma)\ni\xi} e^{\mu(\zeta_{L,0},\zeta_{L,1})\cdot\hat{X}_L^{(0)}(\Gamma)}W(\Gamma)\\ =&\varphi_L(\zeta_{L,0},\zeta_{L,1})-\frac{1}{L}\sum_{\Delta\in\mathcal{K}_L;\ \Delta i\xi}\mathbf{F}_{\hat{\Psi}}^T(\Delta;\zeta_{L,0},\zeta_{L,1}),\end{aligned}$$

$[Hess_{(\zeta_0,\zeta_1)}\varphi_L(\cdot\mid\xi)](\zeta_{L,0},\zeta_{L,1})$ converges to $[Hess_{(\zeta_0,\zeta_1)}\mathcal{L}](\zeta_0^*,\zeta_1^*)$ uniformly in ξ with $|\gamma|\leq C\ln L$. Therefore there exist constants $C_2>0$

and $L_0 \geq 1$ such that for $L \geq L_0$,

$$P_{L,(\zeta_{L,0},\zeta_{L,1})}\big(a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L \mid \mathcal{D}(\Gamma) \ni \xi\big) \leq \frac{C_2}{L^2} \tag{3.3}$$

uniformly in ξ such that $|\gamma| \leq C \ln L$. Combining (2.49) with (3.3), we can find L_1 such that for $L \geq L_1$,

$$\begin{aligned} &E_{L,(\zeta_{L,0},\zeta_{L,1})}\left[e^{\frac{1}{2}d(\xi(x))}1_{\{|\gamma|\leq C\ln L\}}\,\middle|\, \hat{X}_L^{(0)}(\Gamma) = \big(\frac{a_L}{L}, h_L\big)\right] \\ \leq &C_1C_2 \sum_{base(\xi)\ni x, |\gamma|\leq C\ln L} |\hat{\Psi}(\xi)|e^{\frac{1}{2}d(\xi)+c(\xi)} \leq 3C_1C_2. \end{aligned} \tag{3.4}$$

This together with (3.3) proves Lemma 3.1.

Now let us turn to the estimate of the fourth moment of $Y_L(t) - Y_L(s)$. It is sufficient to consider the case where Ls, Lt are integers and $s < t$.

Lemma 3.2 There exist constants $C_3 > 0$ and $L_2 \geq 1$ such that for $L \geq L_2$, if $|t-s| \leq L^{-\frac{4}{5}}$, then

$$E_L\big(|Y_L(t) - Y_L(s)|^4 | a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\big) \leq C_3|t-s|^{\frac{3}{2}}. \tag{3.5}$$

Proof. Since

$$Y_L(t) - Y_L(s) = \frac{1}{\sqrt{L}}\big[X_L(t) - X_L(s) - \frac{L}{\mu}\int_s^t \varphi'(\zeta_0^*(1-x) + \zeta_1^*)dx\big],$$

we estimate

$$E_L(|X_L(t) - X_L(s)|^4 \mid a(\pi(\gamma)) = a_L, k(\gamma) = h_L)$$

and

$$E_L(|L\int_s^t \varphi'(\zeta_0^*(1-x) + \zeta_1^*)dx|^4 \mid a(\pi(\gamma)) = a_L, k(\gamma) = h_L)$$

separately, where (ζ_0^*, ζ_1^*) solves (2.41a), (2.41b) and satisfies (2.42). By analyticity the latter is bounded by $C(L|t-s|)^4$ for some positive constant C. Also, by Lemma 3.1, the former is bounded by

$$C'(L|t-s|)^4$$

for some positive constant C'. It remains to check that

$$L^2|t-s|^4 \leq |t-s|^{\frac{3}{2}},$$

which is true when $|t-s| \le L^{-\frac{4}{5}}$.

To handle the case where $|t-s| \ge L^{-\frac{4}{5}}$, we introduce a moment generating function $\varphi_L^{(s,t)}$ by

$$\varphi_L^{(s,t)}(\zeta_0,\zeta_1,\zeta_2) := \frac{1}{L}\ln \sum_{\Gamma\in\mathcal{S}_L} e^{\mu \hat{X}_L^{(s,t)}(\Gamma)\cdot\underline{\zeta}} W(\Gamma),$$

where

$$\hat{X}_L^{(s,t)}(\Gamma) := \Big(\frac{a(\pi(\Gamma))}{L}, k(\Gamma), \frac{X_L(t)-X_L(s)}{\sqrt{t-s}}\Big)$$

and $\underline{\zeta} = (\zeta_0,\zeta_1,\zeta_2) \in \mathbf{R}^3$ such that (ζ_0,ζ_1) satisfies (2.42) and

$$|\zeta_2| \le \frac{\delta}{2\mu}\sqrt{t-s}. \tag{3.6}$$

To complete the proof of the tightness of $\{Y_L(t),\ 0 \le t \le 1\}$, it is sufficient to show that there exists a constant ε_0 independent of L such that (3.5) holds for all $s,t \in [0,1]$ with $|t-s| \le \varepsilon_0$.

Let $a, h, a_L, h_L, (\zeta_0^*,\zeta_1^*), (\zeta_{L,0},\zeta_{L,1})$ be taken as before; i.e.,

1. (ζ_0^*,ζ_1^*) and $(\zeta_{L,0},\zeta_{L,1})$ satisfy (2.42),
2. (ζ_0^*,ζ_1^*) solves (2.41a), (2.41b), and
3. $(\zeta_{L,0},\zeta_{L,1})$ solves (2.43a), (2.43b).

Put

$$v_L := \frac{1}{\mu}\frac{\partial \varphi_L^{(s,t)}}{\partial \zeta_2}(\zeta_{L,0},\zeta_{L,1},0). \tag{3.7}$$

Then as in the proof of Lemma 2.3, we can show that

$$v_L - \frac{1}{\mu\sqrt{t-s}}\int_s^t \varphi'(\zeta_0^*(1-x)+\zeta_1^*)dx = O\big(L^{-\frac{3}{5}}(\ln L)^{10}\big). \tag{3.8}$$

Therefore

$$\frac{Y_L(t)-Y_L(s)}{\sqrt{t-s}} = \frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} - \sqrt{L}v_L + o(1).$$

So, we will show that for some $\varepsilon_0 > 0$ and for all $s,t \in [0,1]$ such that $|t-s| < \varepsilon_0$,

$$\sum_{k=0}^{\infty}(k+1)^4 P_L\left(\frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} - v_L\sqrt{L} \ge k \,\middle|\, \begin{matrix} a(\pi(\gamma)) = a_L, \\ k(\Gamma) = h_L \end{matrix}\right)$$

converges and is bounded from above by a constant independent of L, s, t. For $k \in \mathbf{N}$, let $\underline{\zeta}_L^{(k)} = (\zeta_{L,0}^{(k)}, \zeta_{L,1}^{(k)}, \zeta_{L,2}^{(k)})$ be the solution of

$$\frac{1}{\mu}[\nabla_{(\zeta_0,\zeta_1,\zeta_2)}\varphi_L^{(s,t)}](\underline{\zeta}_L^{(k)}) = (\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + \frac{k}{\sqrt{L}})$$

and $\zeta_L^{(0)} = (\zeta_{L,0}, \zeta_{L,1}, 0)$. For $\eta = (\eta_0, \eta_1, \eta_2)$, let $\varphi_L^{*(s,t)}(\underline{\eta})$ be the Legendre transform of $\frac{1}{\mu}\varphi_L^{(s,t)}$. Then by duality,

$$[\nabla_{\underline{\eta}}\varphi_L^{*(s,t)}](\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + \frac{k}{\sqrt{L}}) = (\zeta_{L,0}^{(k)}, \zeta_{L,1}^{(k)}, \zeta_{L,2}^{(k)})$$

and

$$\zeta_{L,2}^{(k)} = \int_0^{\frac{k}{\sqrt{L}}} \frac{\partial^2 \varphi_L^{*(s,t)}}{\partial \eta_2^2}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + u)du \geq 0.$$

Therefore

(3.9)

$$\begin{aligned}
&P_L\left(\frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} \geq v_L\sqrt{L}+k \,\middle|\, \begin{array}{l} a(\pi(\Gamma)) = a_L, \\ k(\Gamma) = h_L \end{array}\right) \\
&= \sum_{j \geq Lv_L\sqrt{t-s}+k\sqrt{L(t-s)}} \frac{e^{L\varphi_L^{(s,t)}(\underline{\zeta}_L^{(k)})-\mu(\frac{a_L}{L},h_L,j)\cdot\zeta_L^{(k)}}}{e^{L\varphi_L^{(s,t)}(\underline{\zeta}_L^{(0)})-\mu(\frac{a_L}{L},h_L,Lv_L)\cdot\zeta_L^{(0)}}} \\
&\quad\times \frac{P_{L,\underline{\zeta}_L^{(k)}}(X_L(t)-X_L(s)=j, a(\pi(\Gamma))=a_L, k(\Gamma)=h_L)}{P_{L,\underline{\zeta}_L^{(0)}}(a(\pi(\Gamma))=a_L, k(\Gamma)=h_L)} \\
&\leq \exp\left\{-L\mu[\varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L+\frac{k}{\sqrt{L}}) - \varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L)]\right\} \\
&\quad\times \frac{P_{L,\underline{\zeta}_L^{(k)}}\left(\frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} \geq v_L\sqrt{L}+k, a(\pi(\Gamma))=a_L, k(\Gamma)=h_L\right)}{P_{L,\underline{\zeta}_L^{(0)}}(a(\pi(\Gamma))=a_L, k(\Gamma)=h_L)}.
\end{aligned}$$

From Proposition 2.7, the RHS of (3.9) is bounded by

$$\begin{aligned}
&\exp\{-L\mu[\varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L+\frac{k}{\sqrt{L}}) - \varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L)]\} \\
&\times Const.L^2.
\end{aligned}$$

as $L \to \infty$.

Lemma 3.3. There exist positive constants α_1, α_2, L_0 such that every eigenvalue of

$$\frac{1}{\mu^2} Hess_{(\zeta_0,\zeta_1,\zeta_2)} [\varphi_L^{(s,t)}(\zeta_0, \zeta_1, \frac{\zeta_2}{\sqrt{t-s}})]$$

is in the interval $[\alpha_1, \alpha_2]$ if $L \geq L_0$ and

$$(3.10) \qquad \begin{cases} |\zeta_2| & \leq \frac{\delta}{3\mu}\sqrt{|t-s|} \\ \max\{|\zeta_0+\zeta_1|, |\zeta_1|\} & \leq 1 - \frac{3\delta}{2\mu} \end{cases}$$

For the moment we take it for granted that Lemma 3.3 is true. Then, since $(\zeta_{L,0}, \zeta_{L,1})$ satisfies (2.42), by Lemma 3.3 and the continuity, we can find $\varepsilon > 0$ such that if $\frac{k}{\sqrt{L}} < \varepsilon\sqrt{t-s}$, then $|\zeta_{L,0}^{(k)} - \zeta_{L,0}|$, $|\zeta_{L,1}^{(k)} - \zeta_{L,1}|$, $|\zeta_{L,2}^{(k)}|$ are all bounded by $\frac{\delta}{4\mu}$ and every eigenvalue of

$$\left[Hess_{\underline{\eta}}\varphi_L^{*(s,t)}\right](\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + \frac{k}{\sqrt{L}})$$

is in the interval $[\alpha_2^{-1}, \alpha_1^{-1}]$. Thus, we have

$$(3.11) \qquad \varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + \frac{k}{\sqrt{L}}) - \varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L)$$
$$= \int_0^{\frac{k}{L}} (\frac{k}{\sqrt{L}} - u)\frac{\partial^2 \varphi_L^{*(s,t)}}{\partial \eta_2^2}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + u)du \geq \alpha_2^{-1}\frac{k^2}{2L}$$

if $k \leq \varepsilon\sqrt{L(t-s)}$. By convexity, this means that the LHS of (3.9) is not less than

(3.12)
$$\frac{k}{\sqrt{L}} \frac{\varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L + \varepsilon\sqrt{t-s}) - \varphi_L^{*(s,t)}(\frac{a_L}{L^2}, \frac{h_L}{L}, v_L)}{\varepsilon\sqrt{t-s}} \geq \frac{\alpha_2^{-1}}{2}\varepsilon L^{-\frac{9}{10}} k$$

(3.12) proves that

$$\sum_{k \geq \varepsilon\sqrt{L(t-s)}} (k+1)^4 P_L\left(\frac{X_L(t) - X_L(s)}{\sqrt{L(t-s)}} \geq v_L\sqrt{L} + k \middle| \begin{array}{l} a(\pi(\Gamma)) = a_L, \\ k(\Gamma) = h_L \end{array}\right)$$

$$= O(L^4 \exp\{-\mu\frac{\alpha_2^{-1}\varepsilon}{2}L^{\frac{1}{5}}\})$$

for large L. Also, for $k \le \varepsilon\sqrt{L(t-s)}$, $Hess_{(\zeta_0,\zeta_1,\zeta_2)}[\varphi_L^{(s,t)}(\zeta_0,\zeta_1,\frac{\zeta_2}{\sqrt{t-s}})]$ is uniformly positive definite and by Lemma 3.3,

$$\begin{aligned}
&P_{L,\underline{\zeta}_L^{(k)}}\left(\frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} \ge v_L\sqrt{L}+k, a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\right)\\
\le &P_{L,\underline{\zeta}_L^{(k)}}\left(a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\right)\\
\le &\frac{Const.}{L^2}.
\end{aligned}$$

This and (3.9) together with (3.8) prove

$$\begin{aligned}
&\sum_{k\le\varepsilon\sqrt{L(t-s)}} (k+1)^4\\
&\quad\times P_L\left(\frac{X_L(t)-X_L(s)}{\sqrt{L(t-s)}} \ge v_L\sqrt{L}+k \,\middle|\, a(\pi(\Gamma)) = a_L, k(\Gamma) = h_L\right)\\
\le &Const.\sum_{k=0}^{\infty}(k+1)^4 e^{-\frac{k^2}{2\alpha_2}} < \infty
\end{aligned}$$

Proof of Lemma 3.3. Put

$$\begin{aligned}
&\Psi_L^{(s,t)}(\xi;\zeta_0,\zeta_1,\frac{\zeta_2}{\sqrt{t-s}})\\
=&\Psi(\xi)\exp\{\zeta_0(\frac{\hat{l}(\xi)}{L}+(1-\frac{\hat{r}(\xi)}{L}k(\gamma))+\zeta_1 k(\gamma)+\frac{\zeta_2}{\sqrt{t-s}}k(\gamma;Ls,Lt)\}
\end{aligned}$$

where

(3.13)

$$k(\gamma;Ls,Lt) = \begin{cases} k(\gamma) & \text{if } base(\xi)\subset[Ls,Lt],\\ k(\gamma)-k(\gamma;Ls) & \text{if } \hat{l}(\xi)<Ls\le\hat{r}(\xi)\le Lt\\ k(\gamma;Lt) & \text{if } Ls\le\hat{l}(\xi)\le Lt<\hat{r}(\xi)\\ k(\gamma;Lt)-k(\gamma;Ls) & \text{if } \hat{l}(\xi)<Ls<Lt<\hat{r}(\xi).\end{cases}$$

Then as in the proof of Proposition 2.2, we have a convergent cluster expansion

$$\begin{aligned}
&\varphi_L(\zeta_0, \zeta_1, \frac{\zeta_2}{\sqrt{t-s}}) \\
&= \frac{1}{L} \sum_{J=[a,b]\subset[0,L]} \Phi^{(\Delta)}(J; \zeta_0, \zeta_1, \frac{\zeta_2}{\sqrt{t-s}}) \\
&= \frac{1}{L} \sum_{J=[\hat{l},\hat{r}]\subset[0,L]\setminus[s,t]} \Phi(J; \zeta_L(\hat{r})) \\
&\quad + \frac{1}{L} \sum_{\substack{J=[\hat{l},\hat{r}]\subset[0,L] \\ Ls\leq\hat{r}\leq Lt}} \Phi(J; \zeta_L(\hat{r}) + \frac{\zeta_2}{\sqrt{t-s}}) + O(\frac{(\ln L)^{10}}{L}) \\
&= \int_0^1 \varphi(\zeta_0(1-x) + \zeta_1 + \frac{\zeta_2}{\sqrt{t-s}} 1_{[s,t]}(x))dx + O(\frac{(\ln L)^{10}}{L}).
\end{aligned}$$

Note that

$$\frac{\partial}{\partial\zeta_2} \Phi^{(s,t)}(J; \zeta_0, \zeta_1, \frac{\zeta_2}{\sqrt{t-s}}) = 0$$

if $J \cap [s,t] = \emptyset$. By analitycity, this means that for $\underline{\eta} \in \mathbf{R}^3$

$$\begin{aligned}
(3.14) \qquad &\underline{\eta} \cdot \left[Hess_{(\zeta_0,\zeta_1,\zeta_2)} \varphi_L^{(s,t)}(\zeta_0, \zeta_1, \frac{\zeta_2}{\sqrt{t-s}})\right]\underline{\eta} \\
&= \int_0^1 \left\{(1-x)\eta_0 + \eta_1 + \frac{\eta_2}{\sqrt{t-s}} 1_{[s,t]}(x)\right\}^2 \\
&\quad \times \varphi''(\zeta_0(1-x) + \zeta_1 + \frac{\zeta_2}{\sqrt{t-s}} 1_{[s,t]}(x))dx \\
&\quad + |\eta|^2 O\big(L^{-\frac{1}{5}} (\ln L)^{10}\big)
\end{aligned}$$

as long as $t - s > L^{-\frac{4}{5}}$. If $(\zeta_0, \zeta_1, \zeta_2)$ satisfies (3.10), then as in the proof of Lemma 2.5, we have some $\alpha_1^0 > 0$ depending only on μ and δ such that

$$\alpha_1^0 \leq \varphi''(\zeta_0(1-x) + \zeta_1 + \frac{\zeta_2}{\sqrt{t-s}} 1_{[s,t]}(x))$$

for every $x \in [0,1]$. Also, by analyticity, there exists $\alpha_2^0 > 0$ depending only on μ and δ such that

$$\varphi''(\zeta_0(1-x) + \zeta_1 + \frac{\zeta_2}{\sqrt{t-s}} 1_{[s,t]}(x)) \leq \alpha_2^0$$

for every $x \in [0,1]$. Therefore we have

(3.15)

$$\begin{aligned}
&\alpha_1^0 \int_0^1 \Big\{\eta_0(1-x) + \eta_1 + \frac{\eta_2}{\sqrt{t-s}} 1_{[s,t]}(x)\Big\}^2 dx + O\big(L^{-\frac{1}{5}}(\ln L)^{10}\big) \cdot |\underline{\eta}|^2 \\
&\leq \text{the RHS of (3.14)} \\
&\leq \alpha_2^0 \int_0^1 \Big\{\eta_0(1-x) + \eta_1 + \frac{\eta_2}{\sqrt{t-s}} 1_{[s,t]}(x)\Big\}^2 dx + O\big(L^{-\frac{1}{5}}(\ln L)^{10}\big) \cdot |\underline{\eta}|^2.
\end{aligned}$$

Further, since

$$\begin{aligned}
&\int_0^1 \Big\{\eta_0(1-x) + \eta_1 + \frac{\eta_2}{\sqrt{t-s}} 1_{[s,t]}(x)\Big\}^2 dx \\
&= \int_0^1 \Big\{\eta_0(1-x) + \eta_1\Big\}^2 dx + \eta_2^2 + \frac{2\eta_2}{\sqrt{t-s}} \int_s^t \Big\{\eta_0(1-x) + \eta_1\Big\} dx
\end{aligned}$$

Since we know that the first term in the RHS of the above equality is bounded from below by $\alpha_1^0(\eta_0^2 + \eta_1^2)$, the RHS is bounded from below by

$$\begin{aligned}
&\alpha_1^0(\eta_0^2 + \eta_1^2) - 2\sqrt{t-s}(|\eta_0\eta_2| + |\eta_1\eta_2|) + \eta_2^2 \\
\geq &(\alpha_1^0 - \sqrt{t-s})(\eta_0^2 + \eta_1^2) + (1 - 2\sqrt{t-s})\eta_2^2.
\end{aligned}$$

Set $2\alpha_1 := \min\{\frac{1}{2}\alpha_0^1, \frac{1}{3}\}$. It is obvious that the RHS of the above inequality is larger than $\alpha_1|\underline{\eta}|^2$ if $\sqrt{t-s} < 2\alpha_1$. The existence of α_2 is obvious from (3.14).

References

[DH1] R.L. Dobrushin and O. Hryniv, Fluctuations of shapes of boundaries large areas under paths of random walks, Probab. Theo. Rel. Fields, **105**, 423–458(1996)

[DH2] R.L. Dobrushin and O. Hryniv, Fluctuations of the phase boundary in the 2D Ising Ferromagnet, Commun. Math. Phys. **189**, 395–445(1997)

[Ioffe] D. Ioffe, Ornstein-Zernike behaviour and analyticity of shapes for self-avoiding walks on $\mathbf{Z}^d$, Markov Process. Related Fields **4**, 323–350 (1998)

[KH] R. Kotecký and O. Hryniv Surface tension and the Ornstein-Zernike behaviour for the 2D Blume-Capel model, J. Stat. Phys. **106** (3-4), 431–476(2002)

[KP] R. Kotecký and D. Preis, Cluster expansions for abstract polymer models, Commun. Math. Phys. **103**, 491–498(1986)

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