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A PDE Approach for Motion of Phase-Boundaries by a Singular Interfacial Energy

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§1. Introduction

This is a review paper on geometric motions of phase boundaries like crystal surfaces when the interfacial energy is very singular. Such motions arise in nonequilibrium problem at low temperature. Our purpose is to review a macroscopic approach describing the phenomena by a partial differential equation (PDE) with singular diffusivity. Because of nonlocal effect of singular diffusivity the notion of solution itself is unclear. In this paper we focus the problem whether a solution of approximate parabolic problem converges to a 'solution' of PDE with the singular diffusivity. We do not intend to exhaust the references.

The equilibrium of a crystal shape is often explained as a solution of an anisotropic isoperimetric problem. The problem is described as follows. Let γ be a continuous function on $\mathbf{R}^n$ which is positively homogeneous of degree one, *i.e.*, $\gamma(\lambda p) = \lambda\gamma(p)$ for all $p \in \mathbf{R}^n$, $\lambda > 0$. Assume that $\gamma(p) > 0$ for $p \neq 0$. For an oriented hypersurface S with the orientation $\mathbf{n}$ (a unit normal vector field) in $\mathbf{R}^n$ let $I(S)$ be defined by

$$I(S) = \int_S \gamma(\mathbf{n})dS, \tag{1.1}$$

where dS denotes the surface element. The quantity $I(S)$ is called the interfacial energy and γ is called the *interfacial energy density* (depending upon the temperature τ through the structure of the crystal surface S).

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Anisotropic isoperimetric problem. Minimize $I(\partial D)$ among all open sets $D(\subset \mathbf{R}^n)$ with a fixed volume.

The answer is by now well-known. The interfacial energy $I(\partial D)$ is minimized if and only if D is a dilation of the Wulff shape $\mathcal{W}_\gamma$ defined by

$$\mathcal{W}_\gamma = \bigcap_{|\mathbf{m}|=1} \{x \in \mathbf{R}^n,\ x \cdot \mathbf{m} \leq \gamma(\mathbf{m})\}. \tag{1.2}$$

The reader is referred to [5], [34], [30] and references cited there for the development of the theory. This Wulff shape is considered as a shape of a crystal in an equilibrium state. The Wulff shape is always convex and closed, and its support function

$$\hat{\gamma}(p) = \sup\{p \cdot x; \quad x \in \mathcal{W}_\gamma\} \quad \text{for} \quad p \in \mathbf{R}^n$$

is the convex hull of γ. At low temperature the Wulff shape has a flat portion called a facet. In this case $\hat{\gamma}$ is not C^1 at the direction corresponding to the normal of the facet. We rather consider $\hat{\gamma}$ instead of γ, so we assume that γ is always convex, since $\mathcal{W}_\gamma = \mathcal{W}_{\hat{\gamma}}$.

The first variation of $I(\partial D)$ with respect to a variation of the volume of D is

$$H_\gamma = -\mathrm{div}_S((\nabla_p \gamma)(\mathbf{n})) \quad \text{with} \quad S = \partial D. \tag{1.3}$$

This is called the *weighted mean curvature* of S in the direction of $\mathbf{n}$, which is the unit outer normal vector field of ∂D. Here div_S denotes the surface divergence. If $\gamma(p) = |p|$, then H_γ is the usual mean curvature H. (We use the convention that H is the sum of all principal curvatures instead of its average.) The weighted mean curvature of $\partial \mathcal{W}_\gamma$ always equals $-(n-1)$ so $\mathcal{W}_\gamma$ substitutes the role of a unit sphere for usual mean curvature. If $H_\gamma = -(n-1)$, we expect that D is the Wulff shape but we do not know in general whether such a conjecture is settled except the case $\gamma(p) = |p|$ which is proposed by H. Hopf and solved affirmatively by [1]. If γ is not C^1 so that $\mathcal{W}_\gamma$ has a facet, we observe that H_γ should be a nonlocal quantity since otherwise $H_\gamma = 0$ on such a facet, since the second fundamental form equals zero on a facet.

In nonequilibrium state a phase-boundary such as a crystal surface moves. Its motion is often considered as a result of thermodynamical driving forces—variation of the free energy. A typical example is the mean curvature flow equation

$$V = H \quad \text{on} \quad S_t \tag{1.4}$$

proposed by Mullins [40] to describe the motion of the antiphase boundaries of grains in material sciences. Here, V denotes the normal velocity of evolving (embedded) (hyper) surface $\{S_t\}$ in the direction of $\mathbf{n}$; the parameter t denotes the time variable. The mean curvature is considered as the first variation of the area. To study a crystal growth problem anisotropic effect must be taken into account. For example one consider

$$V = M(\mathbf{n})(H_\gamma + C) \quad \text{on} \quad S_t \tag{1.5}$$

as proposed by [39]. Here C is a constant and $M(\mathbf{n})$ is a positive continuous function on the unit sphere S^{n-1}; H_γ is the weighted mean curvature defined in (1.3), which is considered as the first variation of I of (1.1). An axiomatic derivation of equations like (1.4) and (1.5) is found, for example, in [34]. Mathematical theory is well-developed for (1.4) and its generalization (1.5) if γ is smooth and convex. For example one is able to extend a solution beyond singularities (e.g. pinching) to a global-in-time solution by a level set method [43] (see also [46] and [42]) whose analytic foundation is established by [10], [13]; see [27], [30] and references cited there.

At low temperature τ the Frank diagram of $\gamma = \gamma^\tau$

$$\text{Frank}\gamma = \{p \in \mathbf{R}^n;\ \gamma(p) \le 1\}$$

may have a corner whose position vector directs to the normal of $\mathcal{W}_\gamma$. (Frank γ is a convex conjugate (or polar) of $\mathcal{W}_\gamma$.) There is a critical temperature $\tau_R(\mathbf{n})$ (depending on $\mathbf{n}$) called *roughening temperature* such that there is a facet of $\mathcal{W}_\gamma$ with the normal $\mathbf{n}$ if and only if $\tau < \tau_R(\mathbf{n})$. The evolution law also depends on temperature explicitly. When the latent heat is negligible, its general form [34] is

$$V = f(\mathbf{n},\ H_\gamma + C) \quad \text{on} \quad S_t \tag{1.6}$$

for a given smooth function $f = f^\tau$ on $S^{n-1} \times \mathbf{R}$, which is nondecreasing in the second variable. The theory of crystal growth [11] says that if $\tau \le \tau_R(\mathbf{n})$, then

$$\frac{\partial f^\tau}{\partial X}(\mathbf{n}, X) = 0 \quad \text{at} \quad X = 0$$

while $\tau > \tau_R(\mathbf{n})$,

$$\frac{\partial f^\tau}{\partial X}(\mathbf{n}, X) \ne 0 \quad \text{at} \quad X = 0.$$

So if $\tau > \tau_R(\mathbf{n})$, the equation (1.6) can be approximated by (1.5) at least for small $H_\gamma + C$. However, if $\tau \le \tau_R(\mathbf{n})$, we have to study (1.6) directly. Evolutions with facets are also discussed in surface sciences; see [9], [47] and papers cited there.

If Frank γ has a corner, the definition of solution is not clear even for (1.5). If Frank γ is a convex polyhedra, γ is called a crystalline energy (density). If $n = 2$ and S_t is a planar curve, a notion of solution is proposed by [2] and [48] by restricting S_t as a spcecial polygonal curve. This evolution is called a *crystalline motion.* Based on variational and order-preserving structure the notion of solution is extended by [16], [19], [21] for (1.5) and (1.6), when S_t is a general graph-like curve (§2.2 and Appendix). It applies for general graph-like curves with general γ not necessarily crystalline. Even the level set approach handling non graph-like curves is extended to this situation in [23], [24] ; see also [28] for a review. By now it is known that the problem for $n = 2$ is well-posed although the diffusion effect included in H_γ is not local. To see the difficulty of the problem we assume that $n = 2$, $S_t = \{(x, y); y = u(x, t)\}$ and $\gamma(p_1, p_2) = |p_1| + |p_2|$ and observe that (1.5) with $M \equiv 1, C \equiv 0$ equals

$$u_t(1 + u_x^2)^{-1/2} = (u_x/|u_x|)_x,$$

where subscripts t and x of u denote the partial derivatives. It formally equals (2.3) since $(1 + p^2)^{1/2}\delta(p) = \delta(p)$, where δ denotes the Dirac delta function; the notion of solution to (2.3) is unclear at all. Similar equation

$$u_t = (u_x/|u_x|)_x + u_{xx}$$

has been proposed by H. Spohn [47], where he proposed a notion of solution based on free boundary value problems.

In this paper we focus the problem whether our solution of (1.6) with singular γ can approximated by a solution of approximate equation (1.6) with regular γ, when S_t is given as the graph of a function. We discuss this problem in Section 2. Except the last convergence (2.11) the results are known (cf. [16], [21], [23], [24] and papers cited there). For evolving curves the notions (§2.2 and Appendix) of a solution are consistent with an ansatz that the flat portion called facet (whose normal corresponds to the corner of the Frank diagram) stays as flat during the evolution. We call this ansatz facet-stay-as-facet hypothesis. This hypothesis is invoked to define crystalline motion. Our convergence results in Section 2 assert that the facet-stay-as-facet hypothesis is fulfilled for a limit of solutions of approximate problems. This has a strong contrast to the problem for evolving surfaces where a facet may break (Remark 2.3 (i)). So far for three-dimensional problem even local-in-time solvability is unknown even when γ is crystalline. In Section 3 we claim that a solution proposed by H. Spohn [47] is a solution in our sense so it is obtained as a limit of approximate problems. For more examples of solutions see [36], [26]. There are several other applications of equations with singular diffusivity. The reader is referred to [45], [29], [25], [49] as well as [36], [26].

In the thermal grooving problem it is often more important to study evolution by surface diffusion [41]. This corresponds to the fourth order problem $V = -\Delta H_\gamma$ [8]. Although there are several analytic results when $H_\gamma = H$, for singular γ there are no analytic results; except [47] where several special solutions are proposed; however several numerical results are available as in [44]. This type of problem seems to be important to study thermal smoothing of surface [9].

Before we conclude this introduction we would like to point out several relations between microscopic approach and macroscopic approach. For equilibrium problems macroscopic model is justified as a limit of several microscopic models [5]. There is roughening transition in microscopic model [15]. At the low temperature macroscopic interfacial energy obtained from microscopic models may have singularities so that the Wulff shape has a facet for $n \geq 3$ (while it has no facet when $n = 2$). However, for nonequilibrium problem, the convergence from microscopic to macroscopic is only known above the roughening temperature [18], [35] mainly because of lack of estimates; see also a nice review by T. Funaki [17]. The authors are grateful to Professor Tadahisa Funaki and Professor Herbert Spohn for valuable informative remarks.

§2. General convergence results

We are interested in studying the convergence of a solution when singular interfacial energy is approximated by a smooth energy. So far there are two systematic ways to study this kind of problems. One is based on comparison priciples and is considered as an extension [19], [21] of the theory of viscosity solutions [12]. The other one is based on the theory of nonlinear semigroups initiated by [37] (see also [3]). It reflects the variational structure. The first method is so far restricted to one space dimensional problem but it applies to equations without divergence structure. The second method applies to general space dimension but it is restricted to a gradient system, which has a divergence structure. We first discuss the first method.

2.1. Viscosity approach

We consider a fully nonlinear evolution equation in one space dimension of the form

$$u_t + F(u_x, \Lambda_W(u)) = 0, \quad x \in \mathbf{R}, \quad t > 0 \tag{2.1}$$

with $\Lambda_W(u)$ formally written as $(W'(u_x))_x$. Here W is a given convex function on $\mathbf{R}$ and C^2 outside a discrete set P. Thus the derivative of W may have jumps. The function $F : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ is a given continuous function satisfying a monotonicity condition:

$$F(p,X) \le F(p,Y) \quad \text{for} \quad X \ge Y, \tag{2.2}$$

so that the equation (2.2) is (degenerate) parabolic. (The value Λ_W is actually unchanged by adding an affine function to W but we denote it by Λ_W rather than by $\Lambda_{W''}$:) If $F(p,X) = -X$, then (2.1) is the heat equation when $W(p) = p^2/2$. If $W(p) = |p|$, the equation (2.1) formally becomes

$$u_t = 2\delta(u_x)u_{xx}, \tag{2.3}$$

and the quantity $\delta(u_x)u_{xx}$ is not well-defined even in the distribution sense for smooth u. So we need to introduce a new notion of solution. (For this particular example both the first and the second methods provide a suitable notion of a solution.) In [19] (see also [21]) a notion of solution called a *viscosity solution* for initial value problem of (2.1) is proposed and its unique existence is proved under periodic boundary condition to avoid extra technicality; see [19] for other boundary conditions. We shall recall its definition as well as that of Λ_W briefly in the Appendix. Fortunately, in various settings we have the convergence principle.

CVP. Assume that $F_\varepsilon \to F$ and $W_\varepsilon \to W$ locally uniformly as $\varepsilon \to 0$. For $\varepsilon > 0$ let $u^\varepsilon \in C([0,T) \times \mathbf{T})$ be a solution of

(2.4)
$$u_t + F_\varepsilon(u_x, \Lambda_{W_\varepsilon}(u)) = 0 \quad \text{in} \quad (0,T)\times\mathbf{R} \quad \text{with} \quad u|_{t=0} = u_0^\varepsilon \quad \text{in} \quad \mathbf{R}$$

with $u_0^\varepsilon \in C(\mathbf{T}), \mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$, $\omega > 0$. If $u_0^\varepsilon \to u_0$ in $C(\mathbf{T})$, then u^ε convergences to some function u locally uniformly in $[0,T)\times\mathbf{T}$ (without taking a subsequence) and u is a unique solution of (2.1) with the initial data $u_0 \in C(\mathbf{T})$. (The constant T may be taken as $+\infty$.) (We should not assume uniform convergence of derivatives of W_ε so that W is allowed to be non -C^1.)

To state the convergence result rigorously we need to introduce a class of W and F.

> $\mathcal{E}$ = $\{W; W$ is convex in $\mathbf{R}$ and W is C^2 except some discrete set P. Moreover, $\sup_{K\setminus P} W'' = C_K < \infty$ for every compact set K in $\mathbf{R}\}$.

Any piecewise linear, convex function W belongs to $\mathcal{E}$. Also $W(p) = a|p|/2+bp^2/2$ for $a, b > 0$ belongs to $\mathcal{E}$. Let $\mathcal{F}$ be the set of all continuous function F satisfying the monotonicity condition (2.2). We shall state a special version of the convergence result in [21] where F is allowed to depend on the time explicitly.

Theorem 2.1 *(Convergence). Assume that $F_\varepsilon, F \in \mathcal{F}$ and that W_ε, $W \in \mathcal{E}$. Then (CVP) holds for viscosity solutions.*

Of course, if W_ε and F_ε are smooth and the problem (2.4) is strictly parabolic with smooth initial data u_0^ε, then the classical theory [38] of parabolic equations provides a unique smooth solution u^ε for (2.4). So Theorem 2.1 justifies the notion of solution when W' may have jumps in the sense that the solution is the limit of classical solutions of the strictly parabolic problems. On the other hand, if W_ε is piecewise linear, and W is smooth, Theorem 2.1 also provides the convergence of the crystalline algorithm (proposed by [2] and [48]). Theorem 2.1 extends some aspects of earliear convergnece results [16], [33] of the algorithm for a general equation. The reader is referred to [21], [22] for details and generalizations. As in [22] we also have the convergence of derivatives.

Theorem 2.2 *(Convergence of derivaties). Assume that $F_\varepsilon, F \in \mathcal{F}$ and that $W_\varepsilon, W \in \mathcal{E}$. Under the situation of (CVP) assume furthermore that $u_{0xx}^\varepsilon (\varepsilon > 0)$ is a finite Radon measure with $\limsup_{\varepsilon\to 0} \|u_{0xx}^\varepsilon\|_1 < \infty$. Then*

$$\lim_{\varepsilon\to 0} \sup_{0\le t<T'} \|u_x^\varepsilon - u_x\|_{L^r(\mathbf{T})}(t) = 0$$

for every $r \in [1,\infty)$ and $0 < T' < T$. Here $\|\cdot\|_1$ denotes the total variation of the measure and u_{0xx}^ε represents the distributional second derivative of u_0^ε.

Remark 2.3. (i) So far this method does not apply to a spatially inhomogeneous equation or higher dimensional problems because of complexity of nonlocal curvatures. Despite proposal of several notions of solutions [31], [4], [32], the local existence of a solution approximated by smoother problem is not yet known.
(ii) Theorem 2.1 applies to the interface equation (1.6) when S_t is the graph of a spatially periodic function of one variable. In fact, if $S_t = \{y = u(t,x)\}$, then (1.6) can be written in the form of (2.1) with

$$W(p) = \gamma(-p,1),$$
$$F(p,X) = -(1+p^2)^{1/2} f((-p(1+p^2)^{-1/2}, (1+p^2)^{-1/2}), X + C)$$

when $\mathbf{n}$ is taken upward, *i.e.*, $\mathbf{n} = (-u_x, 1)/(1+u_x^2)^{1/2}$. The weighted curvature H_γ of S_t in the direction of $\mathbf{n}$ at $(x_0, u(x_0))$ equals $\Lambda_W(u)(x_0)$. Thus **CVP** implies that the solution $\{S_t^\varepsilon\}$ of

$$V = f_\varepsilon(\mathbf{n}, H_{\gamma_\varepsilon}) \quad \text{on} \quad S_t^\varepsilon = \{y = u^\varepsilon(t,x)\}$$

converges to the solution of $\{S_t\}$ of (1.6) in the Hausdorff distance sense in $[0,T) \times \mathbf{T} \times \mathbf{R}$, provided that $f_\varepsilon \to f$, $\gamma_\varepsilon \to \gamma$ locally uniformly as $\varepsilon \to 0$ and that $S_0^\varepsilon \to S_0$ in $\mathbf{T} \times \mathbf{R}$ in the Hausdorff distance sense. Such a convergence result has been proved for closed curves in more general setting [24].

2.2. Variational approach

We consider a gradient system in a multi-dimensional space $\mathbf{R}^n$ under periodic boundary condition, *i.e.* in $\mathbf{T}^n = \Pi_{j=1}^n(\mathbf{R}/\omega_j\mathbf{Z}), \omega_j > 0 (j = 1, ..., n)$:

$$u_t - \operatorname{div}((DW)(\nabla u)) = 0 \quad \text{in} \quad \mathbf{T}^n \times (0, \infty). \tag{2.5}$$

Here $W : \mathbf{R}^n \to \mathbf{R}$ be a convex function and DW denotes the gradient of W. If initial data u_0 is Lipschitz continuous, the maximum principle yields a priori bound

$$\|\nabla u\|_\infty(t) \le \|\nabla u_0\|_\infty \quad \text{for all} \quad t \ge 0, \tag{2.6}$$

where $\|\cdot\|_\infty$ denotes the L^∞- norm in $L^\infty(\mathbf{T}^n)$. For example,

$$\|\nabla u\|_\infty(t) = \text{ess.sup}_{x \in \mathbf{T}^n} |\nabla u(x,t)|.$$

Let K be a number such that $\|\nabla u_0\|_\infty \le K$. We may modify $W(p)$ for $|p| \ge K+1$ so that $W(p)$ is coercive in the sense that

$$\lim_{|p| \to \infty} W(p)/|p| = \infty \tag{2.7}$$

without changing the notion of a solution with initial data u_0 since (2.6) holds. This modification simplifies the formulation. As in [16] we formulate the problem by using subdifferentials. If we define the energy for $v \in H = L^2(\mathbf{T}^n)$ by

(2.8)

$$\varphi(v) = \begin{cases} \displaystyle\int_{\mathbf{T}^n} W(\nabla v)dx & \text{if } \nabla v \in L^1(\mathbf{T}^n) \text{ and } W(\nabla v) \in L^1(\mathbf{T}^n), \\ \infty & \text{otherwise,} \end{cases}$$

then φ is convex and lower semicontinuous in H as in [6]. (The coercivity assumption (2.7) is important to conclude the lower semicontinuity.)

In [16] only one dimensional problem is treated but we follow their approach. Then the initial value problem for (2.5) with $\|\nabla u_0\|_\infty \leq K$ is formulated as an abstract ordinary differential equation for $u(t) = u(\cdot, t)$ in the Hilbert space $L^2(\mathbf{T}^n)$ with the standard inner product $< f, g >= \int_{\mathbf{T}^n} fg dx$:

$$\frac{du}{dt}(t) \in -\partial\varphi(u(t)) \quad \text{for} \quad t > 0, \quad u|_{t=0} = u_0, \tag{2.9}$$

where $\partial\varphi$ denotes the subdifferential of φ, *i.e.*,

$$\partial\varphi(v) = \{w \in H; \quad \varphi(v+h) - \varphi(v) \geq < w, h > \quad \text{holds for all} \quad h \in H\}.$$

A general theory guarantees that for $u_0 \in H$ satisfy $\|\nabla u_0\|_\infty \leq K$ there is a unique solution $u \in C([0, T); H)$ of (2.9) (with (2.8)) which is absolutely continuous with values in H on $[\delta, T]$ as a function of t for every $\delta > 0, T > 0$; see [3]. We refer this u as the *solution of* (2.5) (in the *variational sense*) with initial data u_0. As in [16] a general stability theorem due to J. Watanabe [50] (see also [26]) based on a result of H. Brezis and A. Pazy [7] implies the following convergence result.

Theorem 2.4. *Assume that W^ε and W are convex in $\mathbf{R}^n$. Assume that $W^\varepsilon(p) = W(p)$ for $|p| \geq K + 1$ and satisfies (2.7). Assume that $W^\varepsilon \to W$ (locally uniformly) as $\varepsilon \to 0$. Let u^ε be the solution of*

$$u_t - div((DW^\varepsilon)(\nabla u)) = 0 \quad in \quad \mathbf{T}^n \times (0, \infty), \quad u|_{t=0} = u_0^\varepsilon$$

with $\|\nabla u_0^\varepsilon\|_\infty \leq K$. Assume that $u_0^\varepsilon \to u_0$ in $L^2(\mathbf{T}^n)$ as $\varepsilon \to 0$. Then u^ε converges to a solution u of (2.5) with the initial data u_0 in the sense that for any $T > 0$

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \|u^\varepsilon - u\|_{L^2(\mathbf{T}^n)}(t) = 0. \tag{2.10}$$

Remark 2.5.(i) Since $\|\nabla u^\varepsilon\|_\infty(t) \leq K$ for all $t \geq 0$ (cf (2.6)), by Arzelà-Ascoli's theorem we also get the uniform convergence

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \|u^\varepsilon - u\|_\infty(t) = 0. \tag{2.11}$$

The proof of (2.11) admitting (2.10) and (2.6) is elementery; see [25, Lemma 4.3].

(ii) This convergence result also asserts that the solution with singular energy is approximated by that of smoother problem.

(iii) This approach applies to spatially inhomogeneous equation of the form

$$u_t - \frac{1}{b(x)}\{\mathrm{div}(a(x)(DW)(\nabla u)) + C(x)\} = 0$$

as described in [26] and [20].
(iv) By Theorem 2.1 and 2.4 if both variational and viscosity notion of solution is available it must agree each other, since both solutions are obtained as the same limiting procedure [20].

§3. Examples

Example 1. We consider (2.3) with ω-periodic boundary condition, *i.e.*, in $\mathbf{T} \times (0,\infty)$ with $\mathbf{T} = \mathbf{R}/(\omega\mathbf{Z})$ and $\omega > 0$. Assume that the initial data

$$u_0(x) = \begin{cases} A(x), & 0 \leq x \leq \alpha_0, \\ h_0, & \alpha_0 \leq x \leq \beta_0, \\ B(x), & \beta_0 \leq x \leq \omega/2 \end{cases} \tag{3.1}$$

with $\alpha_0 \leq \beta_0, A' > 0, B' < 0, A(0) = B(\omega/2) = 0, A(\alpha_0) = B(\beta_0) = h_0$. Here A and B are C^1 and $h_0 > 0$ is a constant. We also assume that $A' \leq K, -B' \leq K$ with some $K > 0$ so that u_0 is Lipschitz continuous. We extend u_0 to $[-\omega/2,0]$ as an odd function, and further extend u_0 as an ω-periodic function in $\mathbf{R}$, *i.e.*, a function on $\mathbf{T}$. The problem (2.3) with (3.1) is formulated as in (2.9) if we take φ in (2.8) with $W(p) = |p|$ for $p, |p| \leq K+1$ where $\|u_{0x}\|_\infty \leq K$. The solution is explicitly written as follows; see [19],[26]. Let $h(t)$ be a function defined by

$$h(t) = S^{-1}(2t),\ S(k) = \int_k^{h_0} (B^{-1}(\eta) - A^{-1}(\eta))d\eta,$$

where -1 represents the inverse of a function. This $h(t)$ satisfies $h(0) = h_0$ and is decreasing in time. Moreover, $h(T) = 0$ for $T = S(0)/2$. We set

$$u(x,t) = \begin{cases} \min(h(t), u_0(x)), & t \leq T, \quad x \in [0,\omega/2], \\ 0, & t > T \end{cases}$$

and extend $u(\cdot,t)$ to $[-\omega/2,0]$ as an odd function, and further extend $u(\cdot,t)$ as an ω-periodic function in $\mathbf{R}$. It turns out that $u(x,t)$ is the unique solution of (2.3) with (3.1) (in the variational sense). Indeed, if we set

$$\alpha(t) = A^{-1}(h(t)), \quad \beta(t) = B^{-1}(h(t)) \quad \text{with} \quad \alpha(0) = \alpha_0,\ \beta(0) = \beta_0,$$

then

$$h_t(t) = -\frac{2}{\beta(t) - \alpha(t)}. \tag{3.2}$$

The right hand side can be interpreted as $-(W'(+0) - W'(-0))/$ {length of flat portion} where $W(p) = |p|$. See Definition A.3 in the Appendix. Since

$$u(x,t) = \begin{cases} A(x), & 0 \le x \le \alpha(t), \\ h(t), & \alpha(t) \le x \le \beta(t), \\ B(x), & \beta(t) \le x \le \omega/2 \end{cases} \tag{3.3}$$

for $t \le T$, it is not difficult to derive

$$u_t(\cdot,t) \in -\partial\varphi(u(\cdot,t)) \quad \text{for all} \quad t > 0$$

from (3.2) [36], [26]. Indeed, we fix $t \in [0,T)$ and set

$$\zeta(x) = \begin{cases} -1, & 0 \le x \le \alpha, \\ \frac{2}{\beta-\alpha}(x-\alpha) - 1, & \alpha \le x \le \beta, \\ 1, & \beta \le x \le \omega/2; \end{cases}$$

here we suppress t-dependence of α, β and ζ. We extend ζ to $[-\omega/2, 0]$ as an even function, and further extend ζ as an ω-periodic function in $\mathbf{R}$. We then observe that $u_t(x,t) = -\zeta_x(x)$ for $x \in (0,\omega)$. To show $u_t \in -\partial\varphi(u)$ it suffices to prove

$$\zeta_x \in \partial\varphi(v) \quad \text{with} \quad v(x) = u(x,t).$$

We observe that

$$-\zeta(x) \in \partial W(v_x(x)), \quad 0 \le x \le \omega,$$

where ∂W is the subdifferential of a convex function W on $\mathbf{R}$. In other words, $W(v_x(x) + q) - W(v_x(x)) \ge -\zeta(x)q$ for all $q \in \mathbf{R}$, $x \in [0,\omega]$. Thus by definition of ∂W

$$\begin{aligned} \varphi(v+h) \quad - \quad \varphi(v) &= \int_0^\omega \{W(v_x(x) + h_x(x)) - W(v_x(x))\}dx \\ &\ge \int_0^\omega -\zeta(x)h_x(x)dx \end{aligned}$$

for all $h \in L^2(\mathbf{T})$ with $h_x \in L^2(\mathbf{T})$, $W(h_x) \in L^1(\mathbf{T})$. Integrating by parts yields

$$\varphi(v+h) - \varphi(v) \ge \int_0^\omega \zeta_x h dx$$

so we conclude that $\zeta_x \in \partial\varphi(v)$. Thus, we conclude that $u_t(\cdot,t) \in -\partial\varphi(u(\cdot,t))$ for each $t \in (0,T)$. It is clear that this relation holds for all $t \geq T$ since $0 \in -\partial\varphi(0)$.

Note that u_t is a constant on each flat portion of u and its quantity depends on the length of the flat portion so is determined nonlocally. We also note that the flat portion (facet) instantaneously (spontaneously) formed when $\alpha_0 = \beta_0$. The speed of $\alpha(t), \beta(t)$ at $t = 0$ is infinite in this case. By the way we note that the speed (3.2) at the facet can be formally obtained by integrating (2.3) on interval $(\alpha(t) - 0, \beta(t) + 0)$ if one assumes the facet-stay-as-facet hypothesis (see [2],[48]).

Our convergence theorems (Theorems 2.1, 2.4 and Remark 2.5 (i)) in partiular imply that such a solution u is obtained as a local uniform limit of the solution u^ε of

$$(A) \qquad u_t = a_\varepsilon(u_x)u_{xx}, \quad u|_{t=0} = u_0$$

with a smooth positive function a_ε such that $a_\varepsilon \to 2\delta$ as a weak convergence of measures in $(-K-1, K+1)$ as $\varepsilon \to 0$. (We may assume that $a_\varepsilon(p) = 1$ for p with $|p| \geq K+2$.) Moreover, u is the viscosity solution as shown in [19].

Example 2. We give another example of an equation that a facet is spontaneously formed. We consider

$$(3.4) \qquad u_t = 2c_0\delta(u_x)u_{xx} + u_{xx}$$

instead of (2.3) with $c_0 > 0$. For initial value u_0 we again consider ω-periodic function in $\mathbf{R}$ defined in Example 1. Our equation (3.4) is formulated as (2.9) by taking φ of (2.8) by setting

$$W(p) = c_0|p| + |p|^2/2 \quad \text{for} \quad p \quad \text{with} \quad |p| \leq K+1.$$

In [47] H. Spohn solves the initial value problem for (3.4) with (3.1) by reducing it to the Stefan problem studied by [14] under a symmetry assumption

$$(3.5) \qquad u_0(x - \omega/4) = u_0(-x - \omega/4).$$

Since his proposed solution is expressed by a different dependent variable, it is a priori not clear that it is the solution in our sense. We shall recall his solution. Assume that u is of the form

$$\begin{cases} u(x,t) = h(t), & \alpha(t) \leq x \leq \omega_1, \quad \omega_1 = \omega/4, \\ u(0,t) = 0, & \\ u_x(x,t) > 0, & 0 \leq x \leq \alpha(t) \end{cases}$$

with some free boundary $\alpha(t)$ at least for small $t > 0$. By our symmetry assumption (3.5) it is natural to assume that $u(x-\omega_1, t) = u(-x-\omega_1, t)$. Differentiate $u_t = (W'(u_x))_x$ with $W(p) = c_0|p| + |p|^2/2$ in x formally and set $w = u_x$ to get

$$w_t = g(w)_{xx} \tag{3.6}$$

with g defined by

$$g(w) = \begin{cases} c_0 + w & \text{for} \quad w > 0, \\ -c_0 + w & \text{for} \quad w < 0. \end{cases}$$

We set $v = g(w)$ and observe that $v(x,t) > c_0$ for $x \in [0, \alpha(t))$. As in [47] we postulate v and v_x is continuous accross $x = \alpha(t)$ and $v(\alpha(t), t) = c_0$ for (small) $t > 0$. Since $u_x = 0$ on $(\alpha(t), 2\omega_1 - \alpha(t))$, it is natural to postulate $0 < v(x,t) < c_0$ for $(\alpha(t), \omega_1)$ by symmetry. Here the case $\alpha_0 = \beta_0$ is also allowed. By (3.6) v satisfies

$$v_t = v_{xx} \quad \text{for} \quad x \in (0, \alpha(t)), \tag{3.7}$$

$$0 = v_{xx} \quad \text{for} \quad x \in (\alpha(t), \omega_1). \tag{3.8}$$

Since $v(\omega_1, t) = 0$ by symmetry, the equation (3.8) yields

$$v(x,t) = c_0(\omega_1 - x)/(\omega_1 - \alpha(t)), \quad x \in (\alpha(t), \omega_1).$$

By continuity assumption of v_x we have

$$v_x(\alpha(t) - 0, t) = -c_0/(\omega_1 - \alpha(t)) \quad \text{for} \quad \text{(small)} \quad t > 0. \tag{3.9}$$

Thus we obtain the Stefan type problem (3.7), (3.9) with $v(\alpha(t), t) = c_0$. The boundary condition $v_x(0,t) = 0$ is provided by the symmetry assumption of u_0. If (v, α) satisfies these equations, $u(x,t)$ for $0 < x < \alpha(t)$ must satisfiy the heat equation. According to [14] this problem is solvable until $\alpha(t)$ becomes zero provided that A in u_0 is C^3 in $[0, \alpha_0]$. The free boundary $\alpha = \alpha(t)$ is C^1 for $t > 0$ and continuous up to $t = 0$. Thus our u has the property that $u \in C([0,T), L^2(\mathbf{T}))$ and that u is absolutely continuous on $[\delta, T - \delta]$ for $\delta > 0$. To see that u is a solution of (3.4) in our variational sense it suffices to show that

$$\begin{aligned} u_t(x,t) &= \frac{-2c_0}{\beta(t) - \alpha(t)} \\ &= -(W'(+0) - W'(-0))/\{\text{the length of flat portion}\}, \end{aligned}$$

for $x \in (\alpha(t), \beta(t))$ and for $t \in (0, T)$ with

$$T = \sup\{t; \quad \alpha(\tau) > 0 \quad \text{for} \quad \tau \in [0, t)\},$$

where $\beta(t) = 2\omega_1 - \alpha(t)$. In fact, as in Example 1 this speed relation together with $u_t = u_{xx}$ for $0 \le x \le \alpha(t)$ yields $u_t \in -\partial\varphi(u)$ for a.e. $t \in (0, T)$ by observing that for each $t \in (0, T)$

$$\zeta(x) = \begin{cases} -u_x, & 0 \le x \le \alpha, \\ \frac{2c_0}{\beta - \alpha}(x - \alpha) - c_0, & \alpha \le x \le \beta, \\ -u_x, & \beta \le x \le \omega/2 \end{cases}$$

fulfills $u_t = -\zeta_x$ and $\zeta_x \in \partial\varphi(v)$ with $v(x) = u(x, t)$, where we suppress t-dependence of α, β and ζ. (As in Example 1, we extend ζ as an ω-periodic function in $\mathbf{R}$.)

Since for $t \ge T$ we have $u \equiv 0$ so (2.9) is clearly fulfilled for $t \ge T$. In other words it suffices to prove that

$$u_t(x, t) = -c_0/(\omega_1 - \alpha(t)), \quad \alpha(t) \le x \le \beta(t). \tag{3.10}$$

We integrate (3.7) with respect to $x \in (0, \alpha(s))$ and then the time variable $s \in (0, t)$. We observe that

$$\begin{aligned} \int_0^{\alpha(s)} v_{xx}(x, s)dx &= v_x(\alpha(s) - 0, s) - 0 \\ &= -c_0/(\omega_1 - \alpha(s)) \end{aligned}$$

by (3.9) and that

$$\begin{aligned} &\int_0^t \left(\int_0^{\alpha(s)} v_t(x, s)dx \right) ds = \int_0^t \left(\int_0^{\alpha(s)} (v - c_0)_t(x, s)dx \right) ds \\ &= \int_0^{\alpha(t)} (v(x, t) - c_0)dx - \int_0^{\alpha(0)} (v(x, 0) - c_0)dx, \end{aligned}$$

by changing the order of integration and $v(\alpha(s), s) = c_0$. Thus from (3.7) it follows that

$$\int_0^{\alpha(0)} (v(x, 0) - c_0)dx = \int_0^{\alpha(t)} (v(x, t) - c_0)dx + c_0 \int_0^t (\omega_1 - \alpha(s))^{-1} ds.$$

Since $u(\alpha(t), t) = \int_0^{\alpha(t)} (v(x, t) - c_0)dx$, we have

$$\frac{d}{dt}(u(\alpha(t), t)) = \frac{d}{dt} \int_0^{\alpha(t)} (v(x, t) - c_0)dx = -\frac{c_0}{\omega_1 - \alpha(t)}.$$

Since $v(\alpha(t), t) = c_0$ so that $u_x(\alpha(t), t) = 0$, this implies (3.10).

We thus conclude that Spohn's solution is the solution in our variational sense. Thus, again our convergence theorems (Theorem, 2.1, 2.4 and Remark 2.5(i)) in paticular implies that such a solution u can be obtained as a local uniform limit of the solution of the approximate equation (A) if $a_\varepsilon \to 2c_0\delta + 1$ as $\varepsilon \to 0$. Moreover, it is the viscosity solution. Thus as noted in [21], [22] it can be approximated numerically by a *crystalline algorithm.* A similar remark also applies to Example 1.

If u_0 is concave in $[0, \omega/2]$, $u(x,t)$ is also concave in $[0, \omega/2]$ for $t \in [0, T]$. This can be proved by above approximation and the standard maximum principle. In this case our solution u of (3.4) is a subsolution of (2.3) on $(0, \omega/2)$. Thus by a comparison theorem [19] $u \equiv 0$ for $t > T_0$ with some $T_0 > 0$ since the solution of (2.3) vanishes in finite time. In [47] this phenomena has been proved by a different method under the assmption that $A'' < 0$ in $[0, \omega/2]$. In his situation α is monotone decreasing as shown in [47].

Appendix. Definition of viscosity solution and its existence

We recall the definition of viscosity solution for (2.1) and the existence theorem for the reader's convenience [19], [21]. In the appendix we assume $W \in \mathcal{E}$ and $F \in \mathcal{F}$. Let Ω be an open interval.

Definition A.1 (P-faceted). A function $f \in C(\Omega)$ is called *faceted* at $x_0 \in \Omega$ with slope p in Ω if f fulfills the following properties : there is a closed nontrivial finite interval I $(\subset \Omega)$ (called a *faceted region*) containing x_0 such that f agrees with an affine function

$$l_p(x) = p(x - x_0) + f(x_0)$$

in I and $f(x) \neq l_p(x)$ for all $x \in J \setminus I$ with some neighborhood $J(\subset \Omega)$ of I. The length of I is denoted by $L(f, x_0)$. For a discrete set P in $\mathbf{R}$ a function f is called *P-faceted* at x_0 in Ω if f is faceted at x_0 in Ω with some slope $p \in P$.

Definition A.2 (Space of test functions). Let P be the set of jump discontinuities of W'. Let $C_P^2(\Omega)$ be the set of all $f \in C^2(\Omega)$ such that f is P-faceted at x_0 in Ω whenever $f'(x_0) \in P$. For $Q := (0, T) \times \Omega$ with $T > 0$ let $A_P(Q)$ be the set of all function on Q of the form

$$f(x) + g(t), \quad f \in C_P^2(\Omega), \quad g \in C^1(0, T).$$

Definition A.3 (Weighted curvature). Let P be the set of jump discontinuities of W'. Let x_0 be a point in Ω.

For $f \in C^2(\Omega)$ we set the value

$$\Lambda_W(f)(x_0) = W''(f'(x_0))f''(x_0)$$

if $f'(x_0) \notin P$, and

$$\Lambda_W(f)(x_0) = \frac{\chi(f, x_0)}{L(f, x_0)} \Delta(f'(x_0))$$

if $f'(x_0) \in P$ and f is P-faceted at x_0 in Ω. Here $\Delta(p) = W'(p+0) - W'(p-0)$ for $p \in P$, and $\chi(f, x_0)$ is the *transition number* defined by

$$\begin{cases} \chi = +1 & \text{if} \quad f \geq l_p \quad \text{in} \quad J, \\ \chi = -1 & \text{if} \quad f \leq l_p \quad \text{in} \quad J, \\ \chi = 0 & \text{otherwise} \end{cases}$$

for some neighborhood J of the faceted region I.

Definition A.4 (Viscosity solution). A real valued continuous function u on $Q = (0, T) \times \Omega$ is a *viscosity subsolution* of

(1) $$u_t + F(u_x, \Lambda_W(u)) = 0 \quad \text{in} \quad Q$$

if

(2) $$\psi_t(\hat{t}, \hat{x}) + F(\psi_x(\hat{t}, \hat{x}), \Lambda_W(\psi(\hat{t}, \cdot), \hat{x})) \leq 0$$

whenever $(\psi, (\hat{t}, \hat{x})) \in A_P(Q) \times Q$ fullfills

$$\max_Q (u - \psi) = (u - \psi)(\hat{t}, \hat{x}).$$

A *viscosity supersolution* is defined by replacing max by min, and the inequality in (2) by the reverse one. If u is a sub- and supersolution, u is called a *viscosity solution.*

Theorem A.5 (Existence [19]). *Suppose that* $u_0 \in C(\mathbf{R})$ *is periodic with period* $\omega > 0$. *Then there exists a unique function* $u \in C([0,T] \times \mathbf{R})$ *(for any* $T > 0$*) that satisfies*
(i) *u is a viscosity solution of (1) in* $(0, T) \times \mathbf{R}$*;*
(ii)$u(0, x) = u_0(x)$ *for* $x \in \mathbf{R}$*;*
(iii) $u(t, x + \omega) = u(t, x)$ *for* $(t, x) \in [0, T) \times \mathbf{R}$.

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