Advanced Studies in Pure Mathematics 39, 2004
Stochastic Analysis on Large Scale Interacting Systems
pp. 173–211

Large Deviations for $\nabla\varphi$ Interface Model and Derivation of Free Boundary Problems

Tadahisa Funaki and Hironobu Sakagawa

Abstract.

We consider the $\nabla\varphi$ interface model with weak self potential (one-body potential) under general Dirichlet boundary conditions on a large bounded domain and establish the large deviation principle for the macroscopically scaled interface height variables. As its application the law of large numbers is proved and the limit profile is characterized by a variational problem which was studied by Alt-Caffarelli [1], Alt-Caffarelli-Friedman [2] and others. The minimizers generate free boundaries inside the domain. We also discuss the $\nabla\varphi$ interface model with δ-pinning potential in one dimension.

§1. Introduction

Interfaces and variational problems.

It is one of the quite general and fundamental principles in physics that physically realizable phenomena may be characterized by variational problems. Such principle is expected to hold in the problem related to the phase coexistence and separation as well. Indeed, under the situation that two distinct pure phases like crystal/vapor coexist in space, hypersurfaces called interfaces are formed and separate these distinct phases at macroscopic level. The shape of the interface in equilibrium is assumed to minimize the anisotropic total surface energy. The corresponding solutions may be obtained by the so-called Wulff construction (see [5], [8] and references therein). The underlying variational problems change depending on the physical situations of interest.

In statistical mechanics, to derive the shape of the macroscopic interface, one need to determine its total surface energy based on statistical

---

Received January 24, 2003.
Revised May 28, 2003.
The first author is partially supported by JSPS, Grant-in-Aids for Scientific Research 14340029 and for Exploratory Research 13874015.

ensembles at microscopic level, which are formulated as Gibbs measures. This procedure can be accomplished by analyzing a proper scaling limit in the ensembles, which connects microscopic and macroscopic levels.

$\nabla\varphi$ interface model.

The basic microscopic model we study in this article is the $\nabla\varphi$ interface model, which is a continuous analogue of SOS type model. In this model, the interface is already considered as a microscopic object and described by height variables $\phi = \{\phi(x)\}$, the vertical distance of the surface measured from the points x on a fixed reference hyperplane located in the space (see [18], [19] for example). Assuming interfaces are formed in $d+1$ dimensional space, the variables ϕ are defined on a large bounded domain D_N in the d-dimensional square lattice $\mathbb{Z}^d$. Here D_N corresponds to the reference hyperplane which is discretized and $N \in \mathbb{Z}_+$ is the scaling parameter representing the ratio of the macroscopically typical length to the microscopic one.

Given strictly convex symmetric nearest neighbor interactions $V : \mathbb{R} \to \mathbb{R}$ and boundary conditions $\psi = \{\psi(x) \in \mathbb{R}; x \in \partial^+ D_N\}$, an interface energy $H_N^\psi(\phi)$ at microscopic level called Hamiltonian is assigned to each interface height variable $\phi = \{\phi(x) \in \mathbb{R}; x \in D_N\}$ on D_N as a sum of $V(\phi(x) - \phi(y))$ taken over all pairs of neighboring sites x and y in the domain $\overline{D_N}$. Here $\overline{D_N} = D_N \cup \partial^+ D_N$ is the closure of D_N, $\partial^+ D_N = \{x \notin D_N; |x-y| = 1 \text{ for some } y \in D_N\}$ is the outer boundary of D_N and $\phi(x) = \psi(x)$ for $x \in \partial^+ D_N$ in the sum; note that $x \notin D_N$ means $x \in \mathbb{Z}^d \setminus D_N$. We shall take $D_N = ND \cap \mathbb{Z}^d$ for a fixed bounded domain D in $\mathbb{R}^d$ having piecewise Lipschitz boundary ∂D, where $ND = \{N\theta \in \mathbb{R}^d; \theta \in D\}$; D is the macroscopic reference hyperplane while D_N is its microscopic correspondence.

Weak self potentials.

We further assume the space is filled by a media changing in the distances from D_N. Such situation can be realized by adding self potentials (one-body potentials) $U : D \times \mathbb{R} \to \mathbb{R}$ to the Hamiltonian which has therefore the following form:

$$(1.1)\quad H_N^{\psi,U}(\phi) = \sum_{x,y \in \overline{D_N}, |x-y|=1} V(\phi(x) - \phi(y)) + \sum_{x \in D_N} U\left(\frac{x}{N}, \phi(x)\right).$$

The first sum here is over all pairs of neighboring sites. Then the statistical ensemble for the height variables ϕ is defined by the finite volume

Gibbs measure on D_N

$$(1.2) \qquad \mu_N^{\psi,U}(d\phi) = \frac{1}{Z_N^{\psi,U}} \exp\{-H_N^{\psi,U}(\phi)\} \prod_{x\in D_N} d\phi(x),$$

where $Z_N^{\psi,U}$ is a normalization factor; note that $\mu_N^{\psi,U} \in \mathcal{P}(\mathbb{R}^{D_N})$, the family of all probability measures on $\mathbb{R}^{D_N}$. We shall sometimes regard $\mu_N^{\psi,U} \in \mathcal{P}(\mathbb{R}^{\overline{D_N}})$ by considering $\phi(x) = \psi(x)$ for $x \in \partial^+ D_N$ under $\mu_N^{\psi,U}$. We consider the case that U is represented as $U(\theta, r) = Q(\theta)W(r)$, where the function $Q : D \to [0,\infty)$ is bounded and the basic assumption on $W : \mathbb{R} \to \mathbb{R}$ is that the limits $\alpha = \lim_{r\to+\infty} W(r)$ and $\beta = \lim_{r\to-\infty} W(r)$ exist, and the values of W are always between α and β; see the conditions (Q1), (W1) and (W2) in Section 2. The self potential U is called weak since it is bounded. A typical example of W we have in mind throughout this paper is a function of the form

$$(1.3) \qquad W(r) = \beta 1_{\{r<0\}} + \alpha 1_{\{r\geq 0\}}, \quad r \in \mathbb{R}.$$

This potential describes the situation that the space is filled by two different media above and below the hyperplane D_N. If $\beta < \alpha$, the negative values are more favorable than the positive ones for the interface height variables ϕ under the Gibbs measures. In other words the interface is weakly attracted to the negative side, namely by the media below the hyperplane D_N.

Scaling limit and large deviations.

The aim of the present paper is to study the macroscopic behavior of the microscopic height variables ϕ under the Gibbs measures $\mu_N^{\psi,U}$ as $N \to \infty$. The scaling connecting microscopic and macroscopic levels is introduced by associating the macroscopic height variables $h^N = \{h^N(\theta); \theta \in D\}$ with ϕ as step functions (or their polilinear approximations (2.1)) on D, which satisfy

$$h^N(x/N) = N^{-1}\phi(x), \quad x \in D_N.$$

Note that both x- and ϕ-axis are rescaled by the same factor $1/N$, since the interface is located in the $d+1$ dimensional space. The boundary conditions ψ should be simultaneously scaled to have macroscopic limits $g(\theta), \theta \in \partial D$, see the conditions ($\psi$1), ($\psi$2) in Section 2. We shall prove that the law of large numbers holds for h^N distributed under $\mu_N^{\psi,U}$ as $N \to \infty$ and the limit $h = \{h(\theta); \theta \in D\}$ is characterized as the minimizer of the macroscopic total surface energy

$$(1.4) \qquad \int_D \sigma(\nabla h(\theta))\, d\theta - A \int_D Q(\theta) 1(h(\theta) \leq 0) d\theta$$

in the class of h having boundary condition g if the minimizer is unique, see Corollary 2.1. Here $\sigma = \sigma(u) \in \mathbb{R}$ is the so-called surface tension of the (macroscopic) surface with tilt $u \in \mathbb{R}^d$ (see (2.3) or [18]) and we assume $A = \alpha - \beta \geq 0$. When $A < 0$, the formula (1.4) should be slightly modified.

We shall actually establish the large deviation principle (LDP) for h^N under $\mu_N^{\psi,U}$, see Theorem 2.1. As its application, one can prove the law of large numbers. The variational problem characterizing the limit generates free boundaries inside D. Such variational problem was thoroughly studied by Alt and Caffarelli [1] for non-negative macroscopic boundary data g with $A > 0$ and by Alt, Caffarelli and Friedman [2] for general g especially when σ is quadratic: $\sigma(u) = |u|^2$, and by Weiss [26] for more general σ.

Bibliographical notes.

Our results are related to those obtained by Pfister and Velenik [24]. They considered the two dimensional Ising model at low temperature on a large box with attractive wall set at the bottom line. This line segment corresponds to our hyperplane D_N, although it has an effect of hard wall at the same time, since the interfaces separating $\pm$-phases can not go down beyond the bottom line in their setting. One of the motivations of [24] was to understand the so-called wetting or pinning/depinning transition.

The problem of the wetting transition is recently discussed for the $\nabla\varphi$ interface model as well by several authors. We shortly summarize the known results. The potential

$$U(\theta, r) = U(r) = -b1_{\{|r| \leq a\}}, \quad r \in \mathbb{R} \tag{1.5}$$

with $a, b > 0$ is called of square well type and yields a weak pinning effect to the interface near D_N, i.e. the level $\phi(x) = 0$. The limit as $a \downarrow 0$ keeping $s = 2a(e^b - 1)$ constant is called δ-pinning. Dunlop et al. [16] first proved the localization of the ϕ-field, namely the uniform boundedness in N of the expected height variables $E^{\mu_N^{0,U}}[|\phi(x)|]$ under the Gibbs measures $\mu_N^{0,U}$ with 0-boundary conditions or the existence of infinite volume limit of $\mu_N^{0,U}$ as $N \to \infty$, if the Hamiltonian contains arbitrarily weak pinning potentials U when $d = 2$ for quadratic V. This should be compared with the case without pinning (i.e. $U \equiv 0$) in which the localization occurs only when $d \geq 3$ and also compared with the case of strong pinning (or massive) potentials satisfying $\lim_{|r|\to\infty} U(r) = +\infty$ for which the localization occurs for all dimensions. The result of [16] is extended for general convex potential V by Deuschel and Velenik [15] later. In addition to the localization, the mass generation, namely the exponential decay of the correlations of the ϕ-field is shown by Ioffe and Velenik [20] for $d = 2$ with δ-pinning. Further precise estimates on the asymptotic behaviors of the mass and the degree of localization by means of the variances of the field as the pinning effect becomes smaller were established by Bolthausen and Velenik [9]. The basic assumption in our paper (W2) on the potential $W(r)$ unfortunately excludes the potential U of square well type given in (1.5).

When $U(r) = +\infty$ for $r < 0$, we say that the hard wall is settled at the level $\phi(x) = 0$ or at D_N. The ϕ-field can take only non-negative values. To discuss the wetting transition for the $\nabla\varphi$ interface model, the effects of the hard wall and the pinning near 0-level are introduced at the same time. Fisher [17] proved the existence of the wetting transition, namely the qualitative change in the localization/delocalization of the field depending on which of these two competitive effects dominate the other, when $d = 1$ for the SOS type discrete model. This result is extended by Caputo and Velenik [10] for $d = 2$. The precise path level behavior is discussed by Isozaki and Yoshida [21] when $d = 1$. Bolthausen et al. [7] showed that, contrarily when $d \geq 3$, no transition occurs and the field is always localized, i.e. only the phase of partial wetting appears. Note that the field on a hard wall is delocalized for all dimensions d if there is no pinning effect, i.e. $U \equiv 0$ for $r \geq 0$. The latter property is called entropic repulsion. Bolthausen and Ioffe [8] proved the law of large numbers in the partial wetting phase in 2-dimension (i.e. $d = 2$) under the Gibbs measures with 0-boundary conditions, hard wall, δ-pinning and quadratic V conditioned that the macroscopic total volume of the interfaces is kept constant. They derived the so-called Winterbottom shape in the limit and the variational problem characterizing it. The 1-dimensional case with general V was discussed by De Coninck et al. [11].

Our model only takes a special class of self potentials, in particular satisfying the condition (W2), into account and neglects the effect of the hard wall. Since the field can take negative values and the potential U has no strong singularity like hard wall, the situation becomes mild in a sense. On the other hand, this makes us possible to discuss the corresponding dynamics without making much effort, which will be discussed elsewhere; see also [23] for dynamics with general boundary conditions when $U \equiv 0$.

Organization of the paper.

In Section 2, the model is introduced in more precise way and the main results are stated. The proof of the large deviation principle is reduced to the case of $U \equiv 0$ in Section 3, since the potential U can be treated as a rather simple perturbation. The large deviation principle for general boundary conditions without the self potential U is proved in Sections 4 and 5. The case with 0-boundary conditions without U was discussed by Deuschel et al. [13]. Our main effort is therefore made for the treatment of the general boundary conditions. By a simple shift the problem can be reduced to the 0-boundary case, however with bond-depending interaction potentials. Finally, in Section 6, we prove the large deviation principle for δ-pinning case when $d = 1$ and Gaussian potential.

§2. Model and Results

Model and basic assumptions.

Recall that a bounded domain D in $\mathbb{R}^d$ with piecewise Lipschitz boundary is given and microscopic regions $D_N, \overline{D_N}$ and $\partial^+ D_N$, $N \in \mathbb{Z}_+$ in $\mathbb{Z}^d$ are defined from D. For a configuration $\phi = \{\phi(x); x \in D_N\} \in \mathbb{R}^{D_N}$ of the random interface on D_N and microscopic boundary condition $\psi = \{\psi(x); x \in \partial^+ D_N\} \in \mathbb{R}^{\partial^+ D_N}$, $\phi \vee \psi$ represents that on $\overline{D_N}$ which coincides with ϕ on D_N and ψ on $\partial^+ D_N$. For every $\Lambda \subset \mathbb{Z}^d$, Λ^* denotes the set of all directed bonds $b = \langle x, y \rangle$ in Λ, which are directed from y to x. We write $x_b = x$, $y_b = y$ for $b = \langle x, y \rangle$. For each $b \in (\mathbb{Z}^d)^*$ and $\phi = \{\phi(x); x \in \mathbb{Z}^d\}$, define $\nabla\phi(b) = \phi(x_b) - \phi(y_b)$. We also define $\nabla_j \phi(x) = \phi(x + e_j) - \phi(x)$, $1 \le j \le d$ for $x \in \mathbb{Z}^d$ where $e_j \in \mathbb{Z}^d$ is the j-th unit vector. $\nabla\phi(x) = \{\nabla_j\phi(x)\}_{1 \le j \le d}$ denotes vector field of height differences of ϕ.

The Hamiltonian on D_N with boundary condition ψ is defined by

$$H_N^{\psi}(\phi) = \frac{1}{2} \sum_{b \in \overline{D_N}^*} V(\nabla(\phi \vee \psi)(b)), \quad \phi \in \mathbb{R}^{D_N}.$$

Note that this coincides with the first term of (1.1). For the interaction potential V, we assume the following conditions:

(V1) $V \in C^2(\mathbb{R})$,
(V2) $V(\eta) = V(-\eta)$ for every $\eta \in \mathbb{R}$,
(V3) there exist $c_-, c_+ > 0$ such that $c_- \le V''(\eta) \le c_+$ for every $\eta \in \mathbb{R}$.

Next, let $U : D \times \mathbb{R} \to \mathbb{R}$ be a self potential which has an effect attracting the interface ϕ to the negative or positive side. We consider the case that U is decomposed as $U(\theta, r) = Q(\theta)W(r)$, where $Q : D \to [0, \infty)$, $W : \mathbb{R} \to \mathbb{R}$ and assume the following conditions:

(Q1) Q is non-negative, bounded and piecewise continuous,
(W1) W is measurable,
(W2) there exist $\alpha, \beta \in \mathbb{R}$ such that $\lim_{r\to+\infty} W(r) = \alpha$, $\lim_{r\to-\infty} W(r) = \beta$ and $\alpha \wedge \beta \le W(r) \le \alpha \vee \beta$ for every $r \in \mathbb{R}$ (in particular, W is bounded).

Then, $H_N^{\psi,U}(\phi) = H_N^{\psi}(\phi) + \sum_{x\in D_N} U(\frac{x}{N}, \phi(x))$ is the Hamiltonian (1.1) on D_N with boundary condition ψ and self potential U. The corresponding finite volume Gibbs measure $\mu_N^{\psi,U}$ on D_N is defined by (1.2). We shall denote $\mu_N^{\psi,0}$ by μ_N^{ψ}. In the Gaussian case i.e. $V(\eta) = \frac{1}{2}\eta^2$ and $U \equiv 0$, we shall denote it by $\mu_N^{\psi,*}$.

For $g \in C^\infty(\mathbb{R}^d)$, define $H_g^1(D) = \{h \in H^1(D); h - g\big|_D \in H_0^1(D)\}$. The function $g\big|_{\partial D}$ will be the macroscopic boundary condition. We assume the following conditions for the corresponding microscopic boundary condition $\psi \in \mathbb{R}^{\partial^+ D_N}$.

(ψ1) $\max_{x\in\partial^+ D_N} |\psi(x)| \le CN$,
(ψ2) $\sum_{x\in\partial^+ D_N} |\psi(x) - Ng(\frac{x}{N})|^{p_0} \le CN^d$ for some $C > 0$ and $p_0 > 2$.

Remark 2.1. *Since ∂D is piecewise Lipschitz and $g\big|_D \in C^\infty(\bar{D})$, by Theorem 8.7 and Theorem 8.9 of* [27]*, there exists a continuous linear trace operator $T_0 : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ such that $T_0 u = u\big|_{\partial D}$ for every $u \in C^\infty(\bar{D})$ and it holds that $H_g^1(D) = \{h \in H^1(D); T_0 h = g\big|_{\partial D}\}$.*

Scaling and polilinear interpolation.

Our scaled random interface $\{h^N(\theta); \theta \in D\}$ is defined by polilinear interpolation of the macroscopically scaled height variables i.e. $h^N(\theta) = \frac{1}{N}\phi(x)$ for $\theta = \frac{x}{N}$, $x \in \overline{D_N}$ and

$$(2.1)\qquad h^N(\theta) = \sum_{\lambda\in\{0,1\}^d} \Big[\prod_{i=1}^d \big(\lambda_i\{N\theta_i\} + (1-\lambda_i)(1-\{N\theta_i\})\big)\Big]\, h^N\Big(\frac{[N\theta]+\lambda}{N}\Big),$$

for general $\theta \in D$, where $[\cdot]$ and $\{\cdot\}$ denote the integral and the fractional parts, respectively, see (1.17) of [13]. We also define the scaled profile $\{\bar{h}^N(\theta); \theta \in D\}$ by step function i.e. $\bar{h}^N(\theta) = \frac{1}{N}\phi([N\theta])$ for $\theta \in D$. Similarly, for each scalar lattice field $\{u(\frac{x}{N}); x \in D_N\}$, we will define $\{u^N(\theta); \theta \in D\}$ by $u^N(\theta) = u(\frac{x}{N})$ for $\theta = \frac{x}{N}$, $x \in D_N$ and by (2.1) for general $\theta \in D$ and $\{\bar{u}^N(\theta); \theta \in D\}$ by $\bar{u}^N(\theta) = u(\frac{[N\theta]}{N})$ for $\theta \in D$. Also, given a continuous function $f(\theta)$ of $\theta \in D$, we will define $\{f^N(\theta); \theta \in D\}$ and $\{\bar{f}^N(\theta); \theta \in D\}$ from scalar lattice field $\{f(\frac{x}{N}); x \in D_N\}$ as above. Using Jensen's inequality and elementary estimates, we can see that for each $p > 1$, there exists a constant $C_0 = C_0(d,p) > 0$ such that

$$C_0 \|\bar{u}^N\|_{\mathbb{L}^p(D)} \leq \|u^N\|_{\mathbb{L}^p(D)} \leq \|\bar{u}^N\|_{\mathbb{L}^p(D)}, \tag{2.2}$$

for every scalar lattice field $\{u(\frac{x}{N}); x \in D_N\}$.

LDP in the case with weak self potentials.

Now we are in the position to state the main result of this paper. The (normalized) surface tension with tilt $u \in \mathbb{R}^d$ is defined by

$$\sigma(u) = -\lim_{N\to\infty} \frac{1}{N^d} \log \frac{Z_{\Lambda_N}^{\psi_u}}{Z_{\Lambda_N}^0}, \tag{2.3}$$

where $Z_{\Lambda_N}^{\psi}$ is a partition function for $\mu_{\Lambda_N}^{\psi} (= \mu_{\Lambda_N}^{\psi,0})$ on $\Lambda_N = [1, N-1]^d \cap \mathbb{Z}^d$ and $\psi_u(x) = u \cdot x$, $x \in \overline{\Lambda_N}$ represents the u-tilted boundary condition (cf. [13], [18]). For $h \in H^1(D)$, define surface free energy (integrated surface tension)

$$\Sigma(h) = \int_D \sigma(\nabla h(\theta)) d\theta.$$

Theorem 2.1. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\mu_N^{\psi,U}$ satisfies the large deviation principle (LDP) on $\mathbb{L}^2(D)$ with speed N^d and the rate functional $I^U(h)$, that is, for every closed set $\mathcal{C}$ and open set $\mathcal{O}$ of $\mathbb{L}^2(D)$ we have that*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mu_N^{\psi,U}(h^N \in \mathcal{C}) \leq -\inf_{h\in\mathcal{C}} I^U(h), \tag{2.4}$$

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mu_N^{\psi,U}(h^N \in \mathcal{O}) \geq -\inf_{h\in\mathcal{O}} I^U(h). \tag{2.5}$$

The functional $I^U(h)$ is given by

$$I^U(h) = \begin{cases} \Sigma^U(h) - \inf_{H_g^1(D)} \Sigma^U & \text{if } h \in H_g^1(D), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\inf_{H_g^1(D)} \Sigma^U = \inf\{\Sigma^U(h); h \in H_g^1(D)\}$ *and*

$$\Sigma^U(h) = \Sigma(h) + \alpha \int_D Q(\theta) 1(h(\theta) > 0) d\theta + \beta \int_D Q(\theta) 1(h(\theta) < 0) d\theta \\ + (\alpha \wedge \beta) \int_D Q(\theta) 1(h(\theta) = 0) d\theta.$$

Remark 2.2. *By the proof of Theorem 2.1* (*see* (3.8) *below*), *if* U *is given by* $U(\theta, r) = QW(r)$ *for some constant* $Q \geq 0$ *and* $W(r)$ *satisfies the condition* (W2) *with* $(\alpha, \beta) = (0, -A)$ *or* $(-A, 0)$ *for some* $A \geq 0$ *so that* $-A \leq W(r) \leq 0$ *for every* $r \in \mathbb{R}$, *then it holds that*

$$-AQ = -\lim_{N\to\infty} \frac{1}{N^d} \log \frac{Z_{\Lambda_N}^{0,U}}{Z_{\Lambda_N}^0}, \tag{2.6}$$

where the right hand side represents the difference of the free energies of the interface in the case with self potential and in the case without self potential. In this sense, $\Sigma^U(h)$ *above represents macroscopic total surface energy of the profile* h*; see also Remark 3.1 below.*

As a corollary of the upper bound (2.4) in Theorem 2.1, we obtain the following law of large numbers for $\{h^N(\theta); \theta \in D\}$ under $\mu_N^{\psi,U}$.

Corollary 2.1. *If* Σ^U *has a unique minimizer* $\bar{h}$ *in* $H_g^1(D)$, *then the law of large numbers holds under* $\mu_N^{\psi,U}$, *namely,*

$$\lim_{N\to\infty} \mu_N^{\psi,U}(\|h^N - \bar{h}\|_{\mathbb{L}^2(D)} > \delta) = 0,$$

for every $\delta > 0$.

Remark 2.3. **(Free boundary problems)** *If* $\sigma = \sigma(u)$ *is smooth enough* (*i.e.* $\sigma \in C^{2,\gamma}(\mathbb{R}^d), \gamma > 0$) *and if the free boundary* $\partial\{h > 0\}$ *of the minimizer* h *of* Σ^U *is locally* C^2, *then* h *satisfies the Euler equation* $\operatorname{div}\{\nabla\sigma(\nabla h)\} = 0$ *in* $D \setminus \partial\{h > 0\}$ *and the condition* $\Psi(\nabla h^+) - \Psi(\nabla h^-) = AQ$ *on the free boundary* $D \cap \partial\{h > 0\}$, *where* $\Psi(u) = u \cdot \nabla\sigma(u) - \sigma(u)$ *and* $A = (\alpha \vee \beta) - (\alpha \wedge \beta)$. *The Lipschitz continuity of the minimizer* h *and the regularity of its free boundary were studied by* [1], [2], [26] *and others. In our case, for the regularity of the surface tension,* $\sigma \in C^{1,1}(\mathbb{R}^d)$ *is only known in general, see* [18].

LDP for δ-pinning in one dimension.

The Gibbs measure with δ-pinning corresponds to the weak limit of the square-well pinning measure $\mu_N^{\psi,W}$ with $W(r) = -b1_{\{|r|\leq a\}}$ as

$a \downarrow 0, b \to \infty$ by keeping $2a(e^b - 1) = e^J$ for $J \in \mathbb{R}$ and has the following representation:

$$\mu_N^{\psi,J}(d\phi) = \frac{1}{Z_N^{\psi,J}} \exp\{-H_N^{\psi}(\phi)\} \prod_{x \in D_N} (e^J \delta_0(d\phi(x)) + d\phi(x)).$$

We regard $\mu_N^{\psi,J} \in \mathcal{P}(\mathbb{R}^{\overline{D_N}})$ by considering $\phi(x) = \psi(x)$ for $x \in \partial^+ D_N$ as before.

We study the large deviation principle for $\{h^N(\theta); \theta \in D\}$ under $\mu_N^{\psi,J}$ when $d = 1$ and with Gaussian potential i.e. $V(\eta) = \frac{1}{2}\eta^2$. Let $D = (0,1)$, $D_N = [1, N-1] \cap \mathbb{Z}$ and take the boundary condition $\psi(0) = aN$ and $\psi(N) = bN, a, b \in \mathbb{R}$. We shall denote $\mu_N^{\psi,J}$, $Z_N^{\psi,J}$, μ_N^{ψ} and Z_N^{ψ} as $\mu_N^{a,b,J}$, $Z_N^{a,b,J}$, $\mu_N^{a,b}$ and $Z_N^{a,b}$, respectively. Define

$$W_{a,b}(D) = \{h \in C([0,1]; \mathbb{R}); h(0) = a, h(1) = b\},$$
$$H^1_{a,b}(D) = \{h \in W_{a,b}(D); h \text{ is absolutely continuous and } h' \in \mathbb{L}^2(D)\}.$$

The space $W_{a,b}(D)$ is endowed with the topology determined by the sup-norm $\|\cdot\|_\infty$. Then, we have the following LDP.

Theorem 2.2. *Assume that $d = 1$ and $V(\eta) = \frac{1}{2}\eta^2$. Then the family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\mu_N^{a,b,J}$ satisfies the large deviation principle on $W_{a,b}(D)$ (i.e. the upper and lower bounds for closed and open subsets of $W_{a,b}(D)$, respectively) with speed N and the rate functional given by*

$$I^J(h) = \begin{cases} \Sigma^J(h) - \inf_{H^1_{a,b}(D)} \Sigma^J & \text{if } h \in H^1_{a,b}(D), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\Sigma^J(h) = \frac{1}{2}\int_0^1 (h')^2(\theta) d\theta + \tau(J) |\{\theta \in D; h(\theta) = 0\}|,$$

and

$$\tau(J) = -\lim_{N \to \infty} \frac{1}{N} \log \frac{Z_N^{0,0,J}}{Z_N^{0,0}}, \tag{2.7}$$

note that $|\cdot|$ stands for the Lebesgue measure.

Remark 2.4. *The function $\tau(J)$ is the so-called pinning free energy. By the proof of Theorem 2.2 and Remark 6.1 below, one can see that the limit exists and $\tau(J) < 0$ for every $J \in \mathbb{R}$.*

§3. Proof of Theorem 2.1: LDP with Self Potentials

LDP without self potentials.

This section reduces the proof of Theorem 2.1 to the LDP for $\mu_N^\psi (= \mu_N^{\psi,0})$, i.e. the Gibbs measure without self potential. The case where the boundary condition $\psi \equiv 0$ was studied in [13].

Proposition 3.1. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under μ_N^ψ satisfies the large deviation principle on $\mathbb{L}^2(D)$ with speed N^d and the rate functional given by*

$$I(h) = \begin{cases} \Sigma(h) - \inf_{H_g^1(D)} \Sigma & \text{if } h \in H_g^1(D), \\ +\infty & \text{otherwise.} \end{cases}$$

Treatment of boundary conditions.

One of the key observations for the proof of Proposition 3.1 is the following trivial identity:

$$\nabla(\phi \vee \psi)(b) = \nabla((\phi - \xi) \vee 0)(b) + \nabla(\xi \vee \psi)(b), \tag{3.1}$$

for every $\xi = \{\xi(x); x \in D_N\}$ and $b \in \overline{D_N}^*$. Now take ξ as $\xi(x) = Ng(\frac{x}{N})$ for $x \in D_N$ (and for $x \in \overline{D_N}$; recall $g \in C^\infty(\mathbb{R}^d)$) and define

$$\widetilde{H}_N^\psi(\phi) = \frac{1}{2} \sum_{b \in \overline{D_N}^*} V(\nabla(\phi \vee 0)(b) + \nabla(\xi \vee \psi)(b)).$$

Consider the finite volume Gibbs measure with Hamiltonian $\widetilde{H}_N^\psi(\phi)$ and 0-boundary condition:

$$\tilde{\mu}_N^\psi(d\phi) = \frac{1}{\widetilde{Z}_N^\psi} \exp\{-\widetilde{H}_N^\psi(\phi)\} \prod_{x \in D_N} d\phi(x).$$

Then the following LDP holds for $\tilde{\mu}_N^\psi$.

Proposition 3.2. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\tilde{\mu}_N^\psi$ satisfies the large deviation principle on $\mathbb{L}^2(D)$ with speed N^d and the rate functional given by*

$$\widetilde{I}(h) = \begin{cases} \widetilde{\Sigma}(h) - \inf_{H_0^1(D)} \widetilde{\Sigma} & \text{if } h \in H_0^1(D), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\widetilde{\Sigma}(h) = \int_D \sigma(\nabla h(\theta) + \nabla g(\theta))d\theta.$$

We shall prove this proposition in Sections 4 and 5.

Proof of Proposition 3.1. Consider the continuous map $\Phi_g : \mathbb{L}^2(D) \to \mathbb{L}^2(D)$ given by $\Phi_g(h) = h + g$. It is easy to see that

$$I(h) = \inf\{\tilde{I}(\tilde{h}); \tilde{h} \in \mathbb{L}^2(D), \Phi_g(\tilde{h}) = h\}.$$

Then by definitions of μ_N^ψ, $\tilde{\mu}_N^\psi$ and (3.1), Proposition 3.1 follows from the contraction principle (cf. [25], [14] and [12, Theorem 4.2.1]) and Proposition 3.2. Q.E.D.

Deduction of Theorem 2.1 from Proposition 3.1.

We shall prove Theorem 2.1 assuming that Proposition 3.2 and therefore Proposition 3.1 are shown. We only consider the case where $\alpha \geq \beta$. The case where $\alpha \leq \beta$ can be proved completely in an analogous manner or by turning the interfaces upside down by the map $\phi \mapsto -\phi$ and $\psi \mapsto -\psi$. The pinning potential $U(\theta, r) = Q(\theta)W(r)$ which satisfies the conditions (W1) and (W2) with $\alpha \geq \beta$ can be rewritten as $U(\theta, r) = Q(\theta)\alpha + Q(\theta)\widetilde{W}(r)$ and $\widetilde{W}(r)$ satisfies conditions (W1) and

(W2)′ there exists $A \geq 0$ such that $\lim_{r\to+\infty} W(r) = 0$, $\lim_{r\to-\infty} W(r) = -A$ and $-A \leq W(r) \leq 0$ for every $r \in \mathbb{R}$,

with $A = \alpha - \beta$. Since the contribution of the first term $Q(\theta)\alpha$ in $\exp\{-H_N^{\psi,U}(\phi)\}$ of $\mu_N^{\psi,U}$ cancels with the normalization factor, we only have to consider the case that W satisfies the conditions (W1) and (W2)′.

The following lemma allows us to replace the self potential part of the Hamiltonian by the integration of $-AQ$ on the domain where $g \in \mathbb{L}^2(D)$ is non-positive when the macroscopically scaled profile h^N is close enough to g. Note that g here represents a general function in $\mathbb{L}^2(D)$ and not the macroscopic boundary condition.

Lemma 3.1. *Assume the conditions* (Q1), (W1) *and* (W2)′ *on* $U(\theta, r) = Q(\theta)W(r)$. *Let* $g \in \mathbb{L}^2(D)$ *and* $0 < \delta < 1$ *be fixed. If* $h^N \in B_2(g, \delta) = \{h \in \mathbb{L}^2(D); \|h-g\|_{\mathbb{L}^2(D)} < \delta\}$ *for* N *large enough, then there exists some constant* $C > 0$ *such that*

$$\sum_{x\in D_N} U(\frac{x}{N}, \phi(x)) + N^d A \int_D Q(\theta)1(g(\theta) \leq -\delta^{\frac{1}{2}})d\theta \leq CN^d\delta,$$

for every N *large enough.*

Proof. There exists an approximating sequence $\{g_k\}_{k\geq 1} \subset C(D)$ of $g \in \mathbb{L}^2(D)$ such that $\|g_k - g\|_{\mathbb{L}^2(D)} \to 0$ as $k \to \infty$. Recall that one can define g_k^N (polilinear functions) and $\bar{g}_k^N$ (step functions) for $g_k \in C(D)$. Now, by (2.2), it holds that

$$\|\bar{h}^N - g\|_{\mathbb{L}^2(D)} \leq C\|h^N - g\|_{\mathbb{L}^2(D)} + a_{N,k},$$

for every $k \geq 1$, where

$$a_{N,k} = (C+1)\|g - g_k\|_{\mathbb{L}^2(D)} + C\|g_k - g_k^N\|_{\mathbb{L}^2(D)} + \|g_k - \bar{g}_k^N\|_{\mathbb{L}^2(D)},$$

which goes to 0 as $N \to \infty$ and $k \to \infty$. Hence,

$$\|\bar{h}^N - g\|_{\mathbb{L}^2(D)} < C\delta + a_{N,k}, \tag{3.2}$$

if $h^N \in B_2(g, \delta)$. The positive constants C in the estimates may change from line to line in the paper.

Now, for $\gamma > 0$, we rewrite

$$\begin{aligned}
&\sum_{x \in D_N} U\left(\frac{x}{N}, \phi(x)\right) + N^d A \int_D Q(\theta) 1(g(\theta) \leq -\gamma) d\theta \\
&\quad = N^d \int_D \big(W(N\bar{h}^N(\theta)) + A1(g(\theta) \leq -\gamma)\big) Q(\theta) d\theta \\
&\qquad + \Big\{ \sum_{x \in D_N} Q\left(\frac{x}{N}\right) W\left(N\bar{h}^N\left(\frac{x}{N}\right)\right) - N^d \int_D W(N\bar{h}^N(\theta)) Q(\theta) d\theta \Big\} \\
&\quad \equiv S_1 + S_2.
\end{aligned}$$

For S_1, we divide the integration on D into the sum of those on three domains $\{g > -\gamma\} (\equiv \{\theta \in D; g(\theta) > -\gamma\})$, $\{g \leq -\gamma\} \cap C_{N,\gamma}^c$ and $\{g \leq -\gamma\} \cap C_{N,\gamma}$, where $C_{N,\gamma} = \{|\bar{h}^N - g| < \gamma/2\}$ and $C_{N,\gamma}^c = D \setminus C_{N,\gamma}$. The integration on $\{g > -\gamma\}$ is non-positive, because $Q \geq 0$, $W \leq 0$ and $A1(g(\theta) \leq -\gamma) = 0$ on this domain. Next, since (3.2) implies $|C_{N,\gamma}^c| < \frac{4}{\gamma^2}(C\delta + a_{N,k})^2$, we obtain

$$\int_{\{g \leq -\gamma\} \cap C_{N,\gamma}^c} \big| W(N\bar{h}^N(\theta)) + A1(g(\theta) \leq 0) \big| d\theta \leq \frac{K}{\gamma^2}(C\delta + a_{N,k})^2,$$

where $K = 4(\|W\|_\infty + A)$. On $\{g \leq -\gamma\} \cap C_{N,\gamma}$, we have $\bar{h}^N(\theta) < -\gamma/2$. By this fact and the assumption (W2)$'$, $\big| W(N\bar{h}^N(\theta)) + A1(g(\theta) \leq -\gamma) \big| \leq \delta$ holds for N large enough and we see that

$$\int_{\{g \leq -\gamma\} \cap C_{N,\gamma}} \big| W(N\bar{h}^N(\theta)) + A1(g(\theta) \leq -\gamma) \big| d\theta \leq \delta |D|.$$

Therefore, we obtain

$$S_1 \le N^d \|Q\|_\infty \big(\frac{K}{\gamma^2}(C\delta + a_{N,k})^2 + \delta|D|\big),$$

for N large enough, every $k \ge 1$ and $\gamma > 0$. For S_2, we have

$$|S_2| \le N^d \|W\|_\infty \int_D \big| Q(\frac{[N\theta]}{N}) - Q(\theta) \big| \, d\theta + O(N^{d-1}),$$

where $O(N^{d-1})$ is the boundary term. Finally, taking $\gamma = \delta^{\frac{1}{2}}$ and N, k large enough, we complete the proof. Q.E.D.

Under the condition (W2)$'$, the rate functional $\Sigma^U(h)$ has the form

$$\Sigma^U(h) = \Sigma(h) - A \int_D Q(\theta) 1(h(\theta) \le 0) d\theta, \tag{3.3}$$

which coincides with (1.4), and enjoys the following properties.

Lemma 3.2. (1) *The functional $\Sigma^U(h)$ is lower semi-continuous on $\mathbb{L}^2(D)$.*
(2) *Let $\Sigma^U_-(h)$ be the functional defined by (3.3) with $1(h(\theta) \le 0)$ replaced by $1(h(\theta) < 0)$. Then, for every open set $\mathcal{O}$ of $\mathbb{L}^2(D)$, we have that*

$$\inf_{h \in \mathcal{O}} \Sigma^U(h) = \inf_{h \in \mathcal{O}} \Sigma^U_-(h).$$

Proof. (1) Decomposing D into two domains $C_\gamma = \{|h - g| < \gamma\}$ and C_γ^c, in a similar way to the proof of Lemma 3.1, one can prove that

$$\int_D Q(\theta) 1(h(\theta) \le 0) d\theta \le \int_D Q(\theta) 1(g(\theta) \le \gamma) d\theta + \|Q\|_\infty \frac{\delta^2}{\gamma^2},$$

for every $\gamma > 0$ if $h \in B_2(g, \delta)$. By this inequality and the property (strict convexity) of the surface tension (cf. [13, Lemma 3.6]):

$$\frac{1}{2} c_- |v - u|^2 \le \sigma(v) - \sigma(u) - (v - u) \cdot (\nabla\sigma)(u) \le \frac{1}{2} c_+ |v - u|^2, \tag{3.4}$$

for every $u, v \in \mathbb{R}^d$, it is easy to see the lower semi-continuity of $\Sigma^U(h)$ on $\mathbb{L}^2(D)$.
(2) Since $\Sigma^U(h) \le \Sigma^U_-(h)$ is obvious for every $h \in \mathbb{L}^2(D)$, the conclusion follows once we can show that

$$\inf_{h \in \mathcal{O}} \Sigma^U(h) \ge \inf_{h \in \mathcal{O}} \Sigma^U_-(h). \tag{3.5}$$

To this end, for every $\varepsilon > 0$, take $h \in \mathcal{O}$ such that $\Sigma^U(h) \le \inf_{\mathcal{O}} \Sigma^U + \varepsilon$. We approximate such h by a sequence $\{h^n\}_{n\ge 1}$ defined by $h^n(\theta) = h(\theta) - f^n(\theta)$, where $f^n \in C_0^\infty(D)$ are functions such that $f^n(\theta) \equiv \frac{1}{n}$ on $D_n = \{\theta \in D; \text{dist}\,(\theta, \partial D) \ge \frac{1}{n}\}$ and $|\nabla f^n(\theta)| \le C$ with $C > 0$. Note that h^n satisfy the same boundary condition as h. Then, since $\lim_{n\to\infty} \Sigma(h^n) = \Sigma(h)$ (recall $h \in H_g^1(D)$) and

$$
\begin{aligned}
-A\int_D Q(\theta)1(h^n(\theta) < 0)d\theta &\le -A\int_{D_n} Q(\theta)1(h(\theta) < \frac{1}{n})d\theta \\
&\le -A\int_D Q(\theta)1(h(\theta) \le 0)d\theta + A\|Q\|_\infty |D \setminus D_n|,
\end{aligned}
$$

we obtain $\limsup_{n\to\infty} \Sigma_-^U(h^n) \le \Sigma^U(h)$. However, $\mathcal{O}$ is an open set of $\mathbb{L}^2(D)$, so that $h^n \in \mathcal{O}$ for n large enough and thus (3.5) is shown. Q.E.D.

Proof of Theorem 2.1. *Step1 (lower bound).* Let $g \in \mathbb{L}^2(D)$ and $\delta > 0$. Then, by Lemma 3.1 and the LDP lower bound for μ_N^ψ (Proposition 3.1), we have

$$
\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,U}}{Z_N^\psi} \mu_N^{\psi,U}(h^N \in B_2(g,\delta)) \\
&\ge -\inf_{h\in B_2(g,\delta)} I(h) + A\int_D Q(\theta)1(g(\theta) \le -\delta^{\frac{1}{2}})d\theta - C\delta \\
&\ge -\{I(g) - A\int_D Q(\theta)1(g(\theta) \le -\delta^{\frac{1}{2}})d\theta\} - C\delta.
\end{aligned}
$$

Take now an arbitrary open set $\mathcal{O}$ of $\mathbb{L}^2(D)$. Then,

$$
\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,U}}{Z_N^\psi} \mu_N^{\psi,U}(h^N \in \mathcal{O}) \\
&\qquad \ge -\{I(h) - A\int_D Q(\theta)1(h(\theta) \le -\delta^{\frac{1}{2}})d\theta\} - C\delta
\end{aligned}
$$

for every $h \in \mathcal{O}$ and $\delta > 0$ such that $B_2(h,\delta) \subset \mathcal{O}$. Letting $\delta \downarrow 0$, since $h \in \mathcal{O}$ is arbitrary, we have

$$
\begin{aligned}
(3.6) \qquad &\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,U}}{Z_N^\psi} \mu_N^{\psi,U}(h^N \in \mathcal{O}) \\
&\qquad \ge -\inf_{h\in\mathcal{O}} \{I(h) - A\int_D Q(\theta)1(h(\theta) < 0)d\theta\}.
\end{aligned}
$$

However, by Lemma 3.2-(2), one can replace $1(h(\theta) < 0)$ with $1(h(\theta) \leq 0)$ in the right hand side of (3.6).
Step2 (*upper bound*). Let $g \in \mathbb{L}^2(D)$ and $\delta > 0$ be fixed. We define

$$\begin{aligned} L_N^+ &= N\{\theta \in D; g(\theta) > \delta^{\frac{1}{2}}\} \cap \mathbb{Z}^d, \\ L_N^- &= N\{\theta \in D; g(\theta) < -\delta^{\frac{1}{2}}\} \cap \mathbb{Z}^d, \\ I_N &= N\{\theta \in D; |g(\theta)| \leq \delta^{\frac{1}{2}}\} \cap \mathbb{Z}^d. \end{aligned}$$

By the assumption (W2)$'$ on W, for every $\varepsilon > 0$ there exists $K = K_\varepsilon > 0$ such that $W(r) \geq -(A-\varepsilon)1_{\{r \leq K\}} - \varepsilon$ for every $r \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} &\exp\{-\sum_{x\in D_N} U(\frac{x}{N}, \phi(x))\} \\ &\leq \exp\{(A-\varepsilon)\sum_{x\in D_N} Q(\frac{x}{N})1(\phi(x) \leq K) + \varepsilon \sum_{x\in D_N} Q(\frac{x}{N})\} \\ &= \exp\{\varepsilon \sum_{x\in D_N} Q(\frac{x}{N})\} \sum_{\Lambda \subset D_N} \prod_{x\in\Lambda} \big(e^{(A-\varepsilon)Q(\frac{x}{N})} - 1\big) 1(\phi(x) \leq K). \end{aligned}$$

Now, if $\phi(x) \leq K$ for $x \in L_N^+$, then $\frac{1}{N}\phi(x) - g(\frac{x}{N}) < -\frac{1}{2}\delta^{\frac{1}{2}}$ for N large enough. Thus, if $\phi(x) \leq K$ for every $x \in \Lambda \subset L_N^+$ on $\{h^N \in B_2(g,\delta)\}$, since $\|\bar{h}^N - \bar{g}^N\|_{\mathbb{L}^2(D)} < \frac{1}{C_0}(\delta + \|g - g^N\|_{\mathbb{L}^2(D)})$, we have for N large enough

$$\frac{2\delta^2}{C_0} > \frac{1}{N^d} \sum_{x\in D_N} (\frac{1}{N}\phi(x) - g(\frac{x}{N}))^2 > \frac{|\Lambda|\delta}{4N^d},$$

namely, $|\Lambda| < 8C_0^{-1}\delta N^d$, where $C_0 > 0$ is the constant appeared in (2.2). Combining these facts

$$\begin{aligned} &\exp\{-\varepsilon \sum_{x\in D_N} Q(\frac{x}{N})\} \frac{Z_N^{\psi,U}}{Z_N^{\psi}} \mu_N^{\psi,U}(h^N \in B_2(g,\delta)) \\ &\leq \sum_{\substack{\Lambda \subset L_N^+ \\ |\Lambda| < 8C_0^{-1}\delta N^d}} \prod_{x\in\Lambda} \big(e^{(A-\varepsilon)Q(\frac{x}{N})} - 1\big) \sum_{\Lambda' \subset I_N \cup L_N^-} \prod_{x\in\Lambda'} \big(e^{(A-\varepsilon)Q(\frac{x}{N})} - 1\big) \\ &\quad \times \frac{1}{Z_N^{\psi}} \int 1(h^N \in B_2(g,\delta)) 1(\phi(x) \leq K \text{ for every } x \in \Lambda \cup \Lambda') \\ &\qquad\qquad \times \exp\{-H_N^{\psi}(\phi)\} \prod_{x\in D_N} d\phi(x) \end{aligned}$$

$$\le \left(e^{(A-\varepsilon)\|Q\|_\infty}-1\right)^{8C_0^{-1}\delta N^d}\left|\{\Lambda\subset L_N^+;|\Lambda|<8C_0^{-1}\delta N^d\}\right|$$
$$\times\exp\{(A-\varepsilon)\sum_{x\in I_N\cup L_N^-}Q(\frac{x}{N})\}\mu_N^\psi(h^N\in B_2(g,\delta)).$$

By using Stirling's formula, we see that

$$\left|\{\Lambda\subset L_N^+;|\Lambda|<8C_0^{-1}\delta N^d\}\right|\le\frac{(CN^d)^{8C_0^{-1}\delta N^d}}{(8C_0^{-1}\delta N^d)!}$$
$$\le\frac{C}{\delta}N^d(\frac{C}{\delta})^{C\delta N^d}(1+o(1))$$

as $N\to\infty$, for some constant $C>0$ independent of N and δ. Hence, by the LDP upper bound for the measure μ_N^ψ (Proposition 3.1), we obtain

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,U}}{Z_N^\psi}\mu_N^{\psi,U}(h^N\in B_2(g,\delta))$$
$$\le(A-\varepsilon)\int_D Q(\theta)1(g(\theta)\le\delta^{\frac12})d\theta$$
$$-\inf_{h\in\bar B_2(g,\delta)}I(h)+C(\delta)+\varepsilon\int_D Q(\theta)d\theta,$$

where $C(\delta)$ is a constant independent of N and goes to 0 as $\delta\to0$. Then, by using the lower semi-continuity of $I(h)$ and the right-continuity of $\int_D Q(\theta)1(g(\theta)\le\delta^{\frac12})d\theta$ in δ, we see that for every $g\in\mathbb{L}^2(D)$ and $\varepsilon>0$, there exists $\delta>0$ small enough such that

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,U}}{Z_N^\psi}\mu_N^{\psi,U}(h^N\in B_2(g,\delta))$$
$$\le-\{I(g)-A\int_D Q(\theta)1(g(\theta)\le0)d\theta\}+\varepsilon.$$

Therefore, the standard argument in the theory of LDP yields

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,U}}{Z_N^\psi}\mu_N^{\psi,U}(h^N\in\mathcal{C})\tag{3.7}$$
$$\le-\inf_{h\in\mathcal{C}}\{I(h)-A\int_D Q(\theta)1(h(\theta)\le0)d\theta\},$$

for every compact set $\mathcal{C}$ of $\mathbb{L}^2(D)$. Since U is bounded, exponential tightness for $\mu_N^{\psi,U}$ can be proved in a similar way to those for μ_N^ψ which will be proved in Section 4 (see Remark 4.1 below). Thus, (3.7) holds for every closed set $\mathcal{C}$ of $\mathbb{L}^2(D)$.

Finally, taking $\mathcal{O} = \mathcal{C} = \mathbb{L}^2(D)$ in (3.6) (recall the remark subsequent to the estimate) and (3.7), we see that

$$\lim_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,U}}{Z_N^{\psi}} = - \inf_{H_g^1(D)} \Sigma^U + \inf_{H_g^1(D)} \Sigma, \tag{3.8}$$

and this concludes the proof. Q.E.D.

Remark 3.1. *As we mentioned in Remark 2.2, if U is given by $U(\theta, r) = QW(r)$ for some constant $Q \geq 0$ and $W(r)$ (or $W(-r)$) satisfying the condition* (W2)$'$*, then (3.8) with $D_N = \Lambda_N$ yields the difference of the free energies of the interface in the case with and without self potentials, see (2.6). This can also be proved in the following way under the condition* (W2)$'$*: for every $\varepsilon \in (0, A)$ there exists $K = K_\varepsilon > 0$ such that $W(r) \leq -(A-\varepsilon)1_{\{r \leq -K\}}$ for every $r \in \mathbb{R}$. Therefore, we have*

$$\begin{aligned}
\frac{Z^{0,U}_{\Lambda_N}}{Z^0_{\Lambda_N}} &= E^{\mu^0_{\Lambda_N}}\big[\exp\{-Q \sum_{x\in\Lambda_N} W(\phi(x))\}\big] \\
&\geq E^{\mu^0_{\Lambda_N}}\big[\exp\{(A-\varepsilon)Q \sum_{x\in\Lambda_{N,\varepsilon}} 1(\phi(x) \leq -K)\}\big] \\
&= E^{\mu^0_{\Lambda_N}}\big[\sum_{\Gamma\subset\Lambda_{N,\varepsilon}} (e^{(A-\varepsilon)Q}-1)^{|\Gamma|} 1(\phi(x) \leq -K \text{ for every } x \in \Gamma)\big] \\
&\geq e^{(A-\varepsilon)Q|\Lambda_{N,\varepsilon}|} \mu^0_{\Lambda_N}(\phi(x) \leq -K \text{ for every } x \in \Lambda_{N,\varepsilon}),
\end{aligned}$$

where $\Lambda_{N,\varepsilon} = \{x \in \Lambda_N; \operatorname{dist}(x, \Lambda_N^c) \geq \varepsilon N\}$. However, [6, Proposition 2.1] *shows that the probability in the last line is bounded below by*

$$\exp\{-CN^{d-2}\log N(1+o(1))\},$$

as $N \to \infty$ for some constant $C > 0$ independent of N. This implies

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z^{0,U}_{\Lambda_N}}{Z^0_{\Lambda_N}} \geq AQ.$$

The opposite inequality is obvious, since $W(r) \geq -A$.

§4. Proof of Proposition 3.2: LDP without Self Potentials

Convergence of average profiles.

In this section, the proof of Proposition 3.2 will be given assuming the convergence of average profiles (Lemma 4.1). We shall follow the strategy of [13]. The only difference is that the Dirichlet boundary data $g\big|_{\partial D}$ is given from $g \in C^\infty(\mathbb{R}^d)$ in our case, while [13] treated the case of $g \equiv 0$. For $f \in C_0^\infty(D)$, set

$$H_{N,f}^{\psi}(\phi) = H_N^{\psi}(\phi) - \frac{1}{N} \sum_{x \in D_N} f(\frac{x}{N})\phi(x),$$

$$\widetilde{H}_{N,f}^{\psi}(\phi) = \widetilde{H}_N^{\psi}(\phi) - \frac{1}{N} \sum_{x \in D_N} f(\frac{x}{N})\phi(x),$$

and consider the following two Gibbs probability measures:

$$\mu_{N,f}^{\psi}(d\phi) = \frac{1}{Z_{N,f}^{\psi}} \exp\{-H_{N,f}^{\psi}(\phi)\} \prod_{x \in D_N} d\phi(x),$$

$$\tilde{\mu}_{N,f}^{\psi}(d\phi) = \frac{1}{\widetilde{Z}_{N,f}^{\psi}} \exp\{-\widetilde{H}_{N,f}^{\psi}(\phi)\} \prod_{x \in D_N} d\phi(x),$$

having the different boundary conditions $\phi(x) = \psi(x)$ and $\phi(x) = 0$ for $x \in \partial^+ D_N$, respectively; recall that ψ and g satisfy the conditions (ψ1), (ψ2). We write the averages of the profile h^N defined by (2.1) under $\mu_{N,f}^{\psi}$ and $\tilde{\mu}_{N,f}^{\psi}$ as $\bar{h}_{N,f}^{\psi}(\theta) = E^{\mu_{N,f}^{\psi}}[h^N(\theta)]$ and $\widetilde{h}_{N,f}^{\psi}(\theta) = E^{\tilde{\mu}_{N,f}^{\psi}}[h^N(\theta)]$, respectively. For $f \in \mathbb{L}^2(D)$, h_f denotes the unique weak solution $h = h(\theta)$ in $H_0^1(D)$ of the following elliptic partial differential equation:

$$\operatorname{div}\{(\nabla\sigma)(\nabla h(\theta) + \nabla g(\theta))\} = -f(\theta), \quad \theta \in D.$$

The crucial step in the proof of Proposition 3.2 is the following lemma.

Lemma 4.1.

$$\widetilde{h}_{N,f}^{\psi} \to h_f \textit{ in } H_0^1(D) \textit{ as } N \to \infty.$$

We shall prove this lemma in Section 5. Next, define

$$\Xi_{N,f}^{\psi} \equiv \frac{\widetilde{Z}_{N,f}^{\psi}}{\widetilde{Z}_N^{\psi}} = E^{\tilde{\mu}_N^{\psi}}\Big[\exp\{\frac{1}{N} \sum_{x \in D_N} f(\frac{x}{N})\phi(x)\}\Big].$$

Then, in a similar way to the proof of Theorem 1.1 of [13] , by calculating the functional derivative of $\widetilde{\Sigma}(h)$ and using the differentiation-integration trick (i.e. computing $\frac{d}{dt}\log \widetilde{Z}_{N,tf}^{\psi}$ and integrating it in $t \in [0,1]$), Lemma 4.1 yields the following lemma. The proof is omitted.

Lemma 4.2. *The limit* $\Lambda(f) \equiv \lim_{N\to\infty} \frac{1}{N^d} \log \Xi^{\psi}_{N,f}$ *exists and it holds that*

$$\begin{aligned}\Lambda(f) &= \int_D \int_0^1 h_{tf}(\theta) f(\theta) dt d\theta, \\ &= \sup_{h\in H_0^1(D)} \{\langle h, f\rangle - \widetilde{\Sigma}(h)\} + \inf_{H_0^1(D)} \widetilde{\Sigma}, \\ &= \langle h_f, f\rangle - \widetilde{\Sigma}(h_f) + \inf_{H_0^1(D)} \widetilde{\Sigma},\end{aligned}$$

where $\langle h, f\rangle = \int_D h(\theta) f(\theta) d\theta$.

Exponential tightness.

For the proof of the LDP upper bound in Proposition 3.2, we prepare the following lemma.

Lemma 4.3. *There exists* $\varepsilon > 0$ *such that*

$$\sup_{N\geq 1} \frac{1}{N^d} \log E^{\tilde{\mu}^{\psi}_{N,f}} \Big[\exp\{\varepsilon \sum_{x\in \overline{D_N}} (|h^N(\frac{x}{N})|^2 + |\nabla^N h^N(\frac{x}{N})|^2)\}\Big] < \infty,$$

where for a scalar lattice field $\{u(\frac{x}{N}); x \in \overline{D_N}\}$, $\nabla^N u(\frac{x}{N}) = \{\nabla_j^N u(\frac{x}{N})\}_{1\leq j\leq d}$ *denotes a discrete gradient of u defined by* $\nabla_j^N u(\frac{x}{N}) = N\{u(\frac{x+e_j}{N}) - u(\frac{x}{N})\}, 1 \leq j \leq d$.

Proof. Since D is bounded, by discrete Poincaré's inequality and the definition of h^N, we only have to prove that there exists $\varepsilon > 0$ such that

$$\sup_{N\geq 1} \frac{1}{N^d} \log E^{\tilde{\mu}^{\psi}_{N,f}} \Big[\exp\{\varepsilon \sum_{b\in \overline{D_N}^*} |\nabla\phi(b)|^2\}\Big] < \infty. \tag{4.1}$$

However, this is shown by a simple direct computation. Indeed, by the strict convexity of V, it is easy to see that

$$\begin{aligned}&\frac{1}{2} c_- H_N^{0,*}(\phi) - \frac{1}{4} c_- \sum_{b\in \overline{D_N}^*} \big(\nabla(\xi\vee\psi)(b)\big)^2 \\ &\qquad \leq \widetilde{H}_N^{\psi}(\phi) \leq 2c_+ H_N^{0,*}(\phi) + \frac{1}{2} c_+ \sum_{b\in \overline{D_N}^*} \big(\nabla(\xi\vee\psi)(b)\big)^2,\end{aligned}$$

where $H_N^{0,*}(\phi) = \frac{1}{4} \sum_{b \in \overline{D_N}^*} \big(\nabla(\phi \vee 0)(b)\big)^2$. Therefore, the expectation in (4.1) is bounded above by

$$\exp\Big\{\Big(\frac{c_-}{4} + \frac{c_+}{2}\Big) \sum_{b \in \overline{D_N}^*} |\nabla(\xi \vee \psi)(b)|^2\Big\}$$
$$\times \frac{\int \exp\big\{\big(4\varepsilon - \frac{c_-}{2}\big)H_N^{0,*}(\phi) + \frac{1}{N}\sum_{x \in D_N} f(\frac{x}{N})\phi(x)\big\} \prod_{x \in D_N} d\phi(x)}{\int \exp\big\{-2c_+ H_N^{0,*}(\phi) + \frac{1}{N}\sum_{x \in D_N} f(\frac{x}{N})\phi(x)\big\} \prod_{x \in D_N} d\phi(x)}.$$

A simple Gaussian calculation yields

$$\int \exp\Big\{-\alpha H_N^{0,*}(\phi) + \frac{1}{N}\sum_{x \in D_N} f\Big(\frac{x}{N}\Big)\phi(x)\Big\} \prod_{x \in D_N} d\phi(x)$$
$$= \Big(\frac{2\pi}{\alpha}\Big)^{\frac{|D_N|}{2}} \sqrt{\det(-\Delta_{D_N})}\, \exp\Big\{\frac{1}{2\alpha N^2} V_{N,f}\Big\},$$

for every $\alpha > 0$, where Δ_{D_N} is a discrete Laplacian on D_N with 0-boundary condition,

$$V_{N,f} = \Big(f\Big(\frac{\cdot}{N}\Big),\ (-\Delta_{D_N})^{-1} f\Big(\frac{\cdot}{N}\Big)\Big)_{D_N} = \mathrm{Var}_{\mu_N^{0,*}}\Big(\sum_{x \in D_N} f\Big(\frac{x}{N}\Big)\phi(x)\Big),$$

and $(\ \cdot\ ,\ \cdot\ )_{D_N}$ denotes $l^2(D_N)$-scalar product. Therefore, for every $0 < \varepsilon < \frac{1}{8}c_-$, we obtain

$$\log E^{\tilde{\mu}^{\psi}_{N,f}}\Big[\exp\Big\{\varepsilon \sum_{b \in \overline{D_N}^*} |\nabla\phi(b)|^2\Big\}\Big]$$
$$\leq C|D_N| + C\frac{1}{N^2}V_{N,f} + C\sum_{b \in \overline{D_N}^*} |\nabla(\xi \vee \psi)(b)|^2,$$

for some $C = C_\varepsilon > 0$ independent of N. However, $V_{N,f} = O(N^{d+2})$ (cf. [13, Lemma 2.8]) and

$$\sum_{b \in \overline{D_N}^*} |\nabla(\xi \vee \psi)(b)|^2$$
$$\leq 2\sum_{b \in \overline{D_N}^*} |\nabla\xi(b)|^2 + 2\sum_{x \in \partial^+ D_N} |\xi(x) - \psi(x)|^2 = O(N^d),$$

as $N \to \infty$ by recalling the assumption on ψ and that $\xi(x) = Ng(\frac{x}{N})$ for $x \in \overline{D_N}$ with $g\big|_D \in C^\infty(\bar{D})$. This concludes the proof of (4.1). Q.E.D.

Proof of Proposition 3.2.

Proof of Proposition 3.2; upper bound. For every $f \in C_0^\infty(D)$ and measurable set $\mathcal{E}$ of $\mathbb{L}^2(D)$, Chebyshev's inequality shows

$$\tilde{\mu}_N^\psi(h^N \in \mathcal{E}) \le \exp\{-N^d \inf_{h\in\mathcal{E}} \langle h, f\rangle\} E^{\tilde{\mu}_N^\psi}\Big[\exp\{N^d\langle h^N, f\rangle\}\Big]. \tag{4.2}$$

Noting that

$$N^d\langle h^N, f\rangle \le \frac{1}{N}\sum_{x\in\overline{D_N}} f(\frac{x}{N})\phi(x) + \frac{1}{N^2}\|\nabla f\|_\infty \sum_{x\in\overline{D_N}} |\phi(x)|,$$

and using Hölder's inequality, the expectation in the right hand side of (4.2) is bounded above by

$$\begin{aligned} &E^{\tilde{\mu}_N^\psi}\Big[\exp\{\frac{p}{N}\sum_{x\in\overline{D_N}} f(\frac{x}{N})\phi(x)\}\Big]^{\frac{1}{p}} E^{\tilde{\mu}_N^\psi}\Big[\exp\{\frac{q}{N^2}\|\nabla f\|_\infty \sum_{x\in\overline{D_N}} |\phi(x)|\}\Big]^{\frac{1}{q}} \\ &\quad \equiv I_1^N \times I_2^N, \end{aligned}$$

for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. However, Lemmas 4.2 and 4.3 imply

$$\lim_{N\to\infty} \frac{1}{N^d} \log I_1^N = \frac{1}{p}\Lambda(pf),$$

and

$$\limsup_{N\to\infty} \frac{1}{N^d} \log I_2^N \le 0,$$

respectively. Hence, we have

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \tilde{\mu}_N^\psi(h^N \in \mathcal{E}) \le -\inf_{h\in\mathcal{E}} \langle h, f\rangle + \frac{1}{p}\Lambda(pf).$$

Now, by (3.4), we can prove the continuity of h_f in $H_0^1(D)$ with respect to $f \in \mathbb{L}^2(D)$ (cf. [13, Section 3.5]). Therefore, by taking the limit $p \downarrow 1$ and infimum with respect to $f \in C_0^\infty(D)$, we obtain

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \tilde{\mu}_N^\psi(h^N \in \mathcal{E}) \le -\sup_{f\in C_0^\infty(D)} \inf_{h\in\mathcal{E}} \big\{\langle h, f\rangle - \Lambda(f)\big\}.$$

Then by using Lemma 4.2, mini-max theorem (cf. [22, Appendix 2 Lemma 3.2]) and duality lemma (cf. [12, Lemma 4.5.8]), the standard argument yields the LDP upper bound for every compact set of $\mathbb{L}^2(D)$. This can be generalized for every closed set, since the exponential tightness of $\tilde{\mu}_{N,f}^\psi$ follows from Lemma 4.3. Q.E.D.

Remark 4.1. *Since the potential U is bounded, by recalling* (3.1) *and the assumption on ψ, we see that the estimate in Lemma 4.3 holds for $\mu_N^{\psi,U}$ in place of $\tilde{\mu}_{N,f}^{\psi}$ for some $\varepsilon_0 > 0$, which might be smaller than that in Lemma 4.3. In particular, the exponential tightness holds for $\mu_N^{\psi,U}$.*

Proof of Proposition 3.2; lower bound. By Lemmas 4.1 and 4.2, it is easy to see that

$$\lim_{N\to\infty} \frac{1}{N^d} H(\tilde{\mu}_{N,f}^{\psi}|\tilde{\mu}_N^{\psi}) = \widetilde{I}(h_f),$$

where $H(\tilde{\mu}_{N,f}^{\psi}|\tilde{\mu}_N^{\psi}) = E^{\tilde{\mu}_{N,f}^{\psi}}\big[\log \frac{d\tilde{\mu}_{N,f}^{\psi}}{d\tilde{\mu}_N^{\psi}}\big]$ is the relative entropy of $\tilde{\mu}_{N,f}^{\psi}$ with respect to $\tilde{\mu}_N^{\psi}$; see (5.4) in [13]. On the other hand, by Lemma 4.1, Brascamp-Lieb inequality (cf. [13, Lemma 2.8]) and the definition of $\tilde{h}_{N,f}^{\psi}$, one can prove that $\lim_{N\to\infty} E^{\tilde{\mu}_{N,f}^{\psi}}[\|h^N - h_f\|^2_{\mathbb{L}^2(D)}] = 0$ (cf. (1.39) in [13]), and this implies $\lim_{N\to\infty} \tilde{\mu}_{N,f}^{\psi}(h^N \in \mathcal{O}) = 1$ for every open set $\mathcal{O} \subset \mathbb{L}^2(D)$ satisfying $h_f \in \mathcal{O}$. Combining these two facts with the entropy inequality (cf. [14, Lemma 5.4.21]), we obtain

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \tilde{\mu}_N^{\psi}(h^N \in \mathcal{O}) \geq - \inf_{\substack{f\in C_0^\infty(D) \\ \text{s.t. } h_f\in\mathcal{O}}} \widetilde{I}(h_f).$$

However, we can prove by (3.4) that if $h_{f_n} \to h$ in $H_0^1(D)$ as $n \to \infty$ for $\{f_n\} \subset C_0^\infty(D)$ then $\widetilde{I}(h_{f_n}) \to \widetilde{I}(h)$ as $n \to \infty$. This fact and the continuity of h_f in $H_0^1(D)$ with respect to $f \in \mathbb{L}^2(D)$ yield that $\inf_{\substack{f\in C_0^\infty(D) \\ \text{s.t. } h_f\in\mathcal{O}}} \widetilde{I}(h_f) = \inf_{h\in\mathcal{O}} \widetilde{I}(h)$ for every open set $\mathcal{O} \subset \mathbb{L}^2(D)$, which completes the proof of the LDP lower bound. Q.E.D.

§5. Proof of Lemma 4.1: Convergence of Average Profiles

Reduction to two lemmas (Lemmas 5.2 and 5.3).

In this section we shall prove Lemma 4.1. The following lemma follows from (3.4) (cf. [13, Lemma 3.7]).

Lemma 5.1. *Let $\{h_n\}_{n\geq 1}$ be a sequence of $H_0^1(D)$ and define $\widetilde{\Sigma}_f(h)$ $= \widetilde{\Sigma}(h) - \langle h, f\rangle$. If $\lim_{n\to\infty} \widetilde{\Sigma}_f(h_n) = \inf_{H_0^1(D)} \widetilde{\Sigma}_f$, then $h_n \to h_f$ in $H_0^1(D)$ as $n \to \infty$.*

Also by (3.4), we have

$$\begin{aligned}\widetilde{\Sigma}_f(q) - \widetilde{\Sigma}_f(\tilde{h}^{\psi}_{N,f}) \\ &\geq \int_D (\nabla q(\theta) - \nabla \tilde{h}^{\psi}_{N,f}(\theta)) \cdot (\nabla\sigma)(\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta))d\theta \\ &\quad - \int_D (q(\theta) - \tilde{h}^{\psi}_{N,f}(\theta))f(\theta)d\theta,\end{aligned}$$

for every $q \in C_0^\infty(D)$. Once we can prove that the right hand side goes to 0 as $N \to \infty$ for every $q \in C_0^\infty(D)$, we have $\lim_{N\to\infty} \widetilde{\Sigma}_f(\tilde{h}^{\psi}_{N,f}) = \inf_{H_0^1(D)} \widetilde{\Sigma}_f$. This combined with Lemma 5.1 completes the proof of Lemma 4.1. Hence, all we have to prove are the following two lemmas.

Lemma 5.2. *For every* $q \in C_0^\infty(D)$,

$$\lim_{N\to\infty} \int_D \nabla q(\theta) \cdot (\nabla\sigma)(\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta))d\theta = \int_D q(\theta)f(\theta)d\theta.$$

Lemma 5.3.

$$\lim_{N\to\infty} \Big\{ \int_D \nabla \tilde{h}^{\psi}_{N,f}(\theta) \cdot (\nabla\sigma)(\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta))d\theta - \int_D \tilde{h}^{\psi}_{N,f}(\theta)f(\theta)d\theta \Big\} = 0.$$

For the proof of Lemmas 5.2 and 5.3, we prepare several lemmas.

A priori bounds.

Lemma 5.4. *There exists some* $p \in (2, p_0)$ *such that*

$$\sup_{N\geq 1} \|\nabla \tilde{h}^{\psi}_{N,f}\|_{\mathbb{L}^p(D)} < \infty \qquad \textit{and} \qquad \sup_{N\geq 1} \|\nabla \bar{h}^{\psi}_{N,f}\|_{\mathbb{L}^p(D)} < \infty,$$

where $p_0 > 2$ *is the constant appearing in the condition* $(\psi 2)$.

Proof. We first prove the uniform $\mathbb{L}^p$ estimate for $\nabla \tilde{h}^{\psi}_{N,f}$. It is easy to see that

(5.1)

$$\begin{aligned}&V'\big(\nabla_j\phi(x) + \nabla_j(\xi \vee \psi)(x)\big) \\ &\quad - V'\big(\nabla_j\phi(x) - E^{\tilde{\mu}^{\psi}_{N,f}}[\nabla_j\phi(x)] + \nabla_j(\xi \vee \psi)(x)\big) \\ &= E^{\tilde{\mu}^{\psi}_{N,f}}[\nabla_j\phi(x)] \\ &\quad \times \int_0^1 V''\big(\nabla_j\phi(x) - (1-t)E^{\tilde{\mu}^{\psi}_{N,f}}[\nabla_j\phi(x)] + \nabla_j(\xi \vee \psi)(x)\big)dt,\end{aligned}$$

for every $1 \le j \le d$ and $x \in D_N$. For $x \in D_N$, define $A_N(x) = \{A_{N,i,j}(x)\}_{1\le i,j\le d}$ and $a_N(x) = \{a_{N,j}(x)\}_{1\le j\le d}$ by

$$\begin{aligned} A_{N,j,j}(x) &= E^{\tilde\mu^\psi_{N,f}}\Big[\int_0^1 V''\big(\nabla_j\phi(x) - (1-t)E^{\tilde\mu^\psi_{N,f}}[\nabla_j\phi(x)] \\ &\qquad\qquad + \nabla_j(\xi\vee\psi)(x)\big)dt\Big], \\ A_{N,i,j}(x) &= 0 \quad \text{if } i \ne j, \\ a_{N,j}(x) &= E^{\tilde\mu^\psi_{N,f}}\Big[V'\big(\nabla_j\phi(x) - E^{\tilde\mu^\psi_{N,f}}[\nabla_j\phi(x)] + \nabla_j(\xi\vee\psi)(x)\big)\Big], \end{aligned}$$

respectively. Then, taking $\mathrm{div}_N\{E^{\tilde\mu^\psi_{N,f}}[\,\cdot\,]\}$ of the both sides of (5.1), we have

$$\begin{aligned} &\mathrm{div}_N\{A_N(x)\nabla^N\tilde h^\psi_{N,f}(\tfrac{x}{N})\} \\ &= -\mathrm{div}_N\{a_N(x)\} + \mathrm{div}_N\{E^{\tilde\mu^\psi_{N,f}}[V'(\nabla\phi(x) + \nabla(\xi\vee\psi)(x))]\}, \end{aligned}$$

where $\mathrm{div}_N\alpha$ is defined by $\mathrm{div}_N\alpha(x) = N\sum_{j=1}^d(\alpha_j(x)-\alpha_j(x-e_j))$ for a vector lattice field $\alpha(x) = \{\alpha_j(x)\}_{1\le j\le d}$, $x\in\mathbb{Z}^d$. By calculating $\frac{\partial H^\psi_{N,f}}{\partial\phi(x)}$ and taking its expectation under $\mu^\psi_{N,f}$ as in the proof of (1.55) of [13], we obtain

$$\mathrm{div}_N\{E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))]\} = -f(\frac{x}{N}), \tag{5.2}$$

for every $x \in D_N$. By (3.1), the change of variable yields

$$\mathrm{div}_N\{E^{\tilde\mu^\psi_{N,f}}[V'(\nabla\phi(x) + \nabla(\xi\vee\psi)(x))]\} = -f(\frac{x}{N}).$$

Therefore, $\{\tilde h^\psi_{N,f}(\frac{x}{N})\}$ satisfies the following discrete elliptic equation:

$$\mathrm{div}_N\{A_N(x)\nabla^N\tilde h^\psi_{N,f}(\frac{x}{N})\} = -\mathrm{div}_N\{a_N(x)\} - f(\frac{x}{N}),$$

for every $x\in D_N$. However, by the assumption on V, $A_N(x)$ satisfies the uniform ellipticity condition $c_-I \le A_N(x) \le c_+I$ for every $x\in D_N$. Hence, by the proof of Lemma 3.4 of [13], we know that there exist some $p>2$ and $C<\infty$ such that

$$\|\nabla\tilde h^\psi_{N,f}\|_{\mathbb{L}^p(D)} \le C\big(\|a_N\|_{\mathbb{L}^p(D)} + \|f\|_{\mathbb{L}^p(D)}\big),$$

uniformly in N. Note that $\tilde\mu^\psi_{N,f}$ is endowed with 0-boundary condition.

Now, since V' is linearly growing, using the change of variable again, we have that

$$|a_{N,j}(x)| \le C\big(E^{\mu^{\psi}_{N,f}}\big[|\nabla_j\phi(x) - E^{\mu^{\psi}_{N,f}}[\nabla_j\phi(x)]|\big] + |\nabla_j(\xi \vee \psi)(x)|\big),$$

for some $C > 0$. Then, $\sum_{x\in D_N} |a_N(x)|^{p_0} = O(N^d)$ as $N \to \infty$ follows from the Brascamp-Lieb inequality and the assumptions on ψ as in the proof of Lemma 4.3. This proves the uniform $\mathbb{L}^p$ estimate for $\nabla\tilde{h}^{\psi}_{N,f}$.

The uniform $\mathbb{L}^p$ estimate for $\nabla\bar{h}^{\psi}_{N,f}$ follows from that for $\nabla\tilde{h}^{\psi}_{N,f}$, the change of variable and the assumptions on ψ. Q.E.D.

Lemma 5.5. *For every $e \in \mathbb{Z}^d$ with $|e| = 1$, we have*

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in D_N} \Big|\nabla^N\tilde{h}^{\psi}_{N,f}\Big(\frac{x+e}{N}\Big) - \nabla^N\tilde{h}^{\psi}_{N,f}\Big(\frac{x}{N}\Big)\Big|^2 = 0, \tag{5.3}$$

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in D_N} \Big|\nabla^N\bar{h}^{\psi}_{N,f}\Big(\frac{x+e}{N}\Big) - \nabla^N\bar{h}^{\psi}_{N,f}\Big(\frac{x}{N}\Big)\Big|^2 = 0. \tag{5.4}$$

Proof. We first prove (5.4) by following the argument for the proof of Lemma 3.1 of [13]. Define $I_N = \{x \in D_N; \mathrm{dist}(x, \mathbb{Z}^d \setminus D_N) \ge 2\}$, then the sum $\sum_{x\in D_N}$ in (5.4) can be divided into $\sum_{x\in I_N}$ and $\sum_{x\in D_N\setminus I_N}$. The boundary term $\sum_{x\in D_N\setminus I_N}$ is $o(N^d)$ as $N \to \infty$ by Lemma 5.4 and Hölder's inequality. For the interior term $\sum_{x\in I_N}$, the entropy argument (cf. [13, Proposition 2.10 and Lemma 3.2]) yields the desired result. Note that the variance of the field $\phi(x)$ does not depend on the boundary condition ψ under the Gaussian measure $\mu^{\psi,*}_N$.

Next, we shall prove (5.3). The boundary term $\sum_{x\in D_N\setminus I_N}$ is $o(N^d)$ as before. For the interior term, by (3.1), the change of variable yields

$$\nabla^N_j\bar{h}^{\psi}_{N,f}\Big(\frac{x}{N}\Big) = \nabla^N_j\tilde{h}^{\psi}_{N,f}\Big(\frac{x}{N}\Big) + \nabla_j(\xi \vee \psi)(x), \tag{5.5}$$

for every $1 \le j \le d$ and $x \in D_N$. The contribution from the first term is $o(N^d)$ by (5.4), while that coming from the second term: $\sum_{x\in I_N} \big|\nabla\xi(x+e) - \nabla\xi(x)\big|^2$ is also $o(N^d)$. This is because $\xi(x) = Ng(\frac{x}{N})$ and we have $\nabla_j\xi(x+e) - \nabla_j\xi(x) = \frac{1}{N}\nabla^N_j\nabla^N_e g(\frac{x}{N})$ for every $1 \le j \le d$ and $x \in D_N$. Q.E.D.

Local equilibria.

Next, let

$$\begin{aligned}\mathcal{X} &= \{\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}; \eta = \nabla\phi \text{ for some } \phi \in \mathbb{R}^{\mathbb{Z}^d}\},\\ \mathcal{X}_r &= \{\eta \in \mathcal{X}; \sum_{b\in(\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r|x_b|} < \infty\}, \quad r>0,\end{aligned}$$

and define $Q_N \in \mathcal{M}_+(D \times \mathcal{X})$ and $V_N \in \mathcal{M}_+(\mathbb{R}^d \times \mathcal{X})$ by

$$\begin{aligned}Q_N(d\theta d\eta) &= \frac{1}{N^d}\sum_{x\in D_N} \delta_{\frac{x}{N}}(d\theta)\mu_{N,f}^{\psi,\nabla}\circ\tau_x^{-1}(d\eta),\\ V_N(dvd\eta) &= \frac{1}{N^d}\sum_{x\in D_N} \delta_{\nabla^N \bar{h}_{N,f}^{\psi}(\frac{x}{N})}(dv)\mu_{N,f}^{\psi,\nabla}\circ\tau_x^{-1}(d\eta),\end{aligned}$$

where $\mathcal{M}_+(\mathcal{E})$ stands for the class of all non-negative measures on $\mathcal{E}$, $\mu_{N,f}^{\psi,\nabla}(d\eta)$ is the distribution of $\eta = \nabla\phi$ on $\mathcal{X}$ under $\mu_{N,f}^{\psi}$ and $\tau_x : \mathcal{X} \to \mathcal{X}$ denotes the shift on $\mathbb{Z}^d$ defined by $(\tau_x\eta)(b) = \eta(b-x)$ for $b \in (\mathbb{Z}^d)^*$. We regard $\mu_{N,f}^{\psi} \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$ by considering $\phi(x) = \psi(x)(= g(\frac{x}{N}))$ for $x \in \mathbb{Z}^d \setminus D_N$. We denote by $\mu_v^{\nabla}(d\eta)$, $v = (v_i)_{1\le i\le d} \in \mathbb{R}^d$ the unique $\nabla\phi$-Gibbs measure on $\mathcal{X}$ which is translation invariant, ergodic and satisfies $E^{\mu_v^{\nabla}}[\eta(b)^2] < \infty$ for every $b \in (\mathbb{Z}^d)^*$ and $E^{\mu_v^{\nabla}}[\eta(e_i)] = v_i$ for every $1 \le i \le d$ (cf. [18, Section 3]).

In a similar way to the proof of Lemma 4.3 of [13], we can prove the following lemma. Note again that the variance does not depend on the boundary condition ψ under the Gaussian measure $\mu_N^{\psi,*}$. The proof is omitted.

Lemma 5.6. *For each $r > 0$ both the families of measures $\{Q_N\}$ on $D\times\mathcal{X}_r$ and $\{V_N\}$ on $\mathbb{R}^d\times\mathcal{X}_r$ are tight. Moreover, for every limit point Q of $\{Q_N\}$, there exists $\nu_Q \in \mathcal{M}_+(D \times \mathbb{R}^d)$ such that Q is represented as*

$$Q(d\theta d\eta) = \int_{\mathbb{R}^d} \nu_Q(d\theta dv)\mu_v^{\nabla}(d\eta).$$

Similarly, for each limit point V of $\{V_N\}$, there exists $\nu_V \in \mathcal{M}_+(\mathbb{R}^d\times\mathbb{R}^d)$ such that V is represented as

$$V(dvd\eta) = \int_{\mathbb{R}^d} \nu_V(dvdu)\mu_u^{\nabla}(d\eta).$$

Now by Lemma 5.4, along some subsequence, $\{\nabla\tilde{h}^{\psi}_{N,f}(\theta)\}_N$ generates the family of Young measures $\tilde{\nu}(\theta, dv) \in \mathcal{P}(\mathbb{R}^d)$ i.e. it holds that

$$\lim_{N\to\infty}\int_D q(\theta)G(\nabla\tilde{h}^{\psi}_{N,f}(\theta))d\theta = \int_{D\times\mathbb{R}^d} q(\theta)G(v)\tilde{\nu}(\theta,dv)d\theta. \tag{5.6}$$

for every $q \in \mathbb{L}^\infty(D)$ and $G \in C_0(\mathbb{R}^d)$ (cf. [13, Section 4.3], [3]). Then, the following lemma holds.

Lemma 5.7. *If the subsequence $\{N\}$ is commonly taken, the limits ν_Q and ν_V which appear in Lemma 5.6 can be represented as*

$$\nu_Q(d\theta dv) = \tilde{\nu}(\theta, dv - \nabla g(\theta))d\theta, \tag{5.7}$$

and

$$\nu_V(dvdu) = \delta_v(du)\int_D \tilde{\nu}(\theta, dv - \nabla g(\theta))d\theta. \tag{5.8}$$

Proof. By following the argument in the proof of Lemma 4.4 of [13], we shall only prove (5.7). The second equality (5.8) can be proved in a similar manner. For (5.7), it is enough to show that

$$\int_{D\times\mathbb{R}^d} q(\theta)G(v)\nu_Q(d\theta dv) = \int_{D\times\mathbb{R}^d} q(\theta)G(v+\nabla g(\theta))\tilde{\nu}(\theta,dv)d\theta \tag{5.9}$$

for every $q \in C_0^\infty(D)$ and $G \in C_b^1(\mathbb{R}^d)$. In fact, since the ergodicity of μ_v^∇ implies

$$G(v) = \lim_{l\to\infty} E^{\mu_v^\nabla}[G(\mathrm{Av}_l\eta)],$$

where $\mathrm{Av}_l\eta = \frac{1}{(2l+1)^d}\sum_{x\in B_l}\eta(x) \in \mathbb{R}^d$, $B_l = [-l,\ l]^d \cap \mathbb{Z}^d$, we have by Lemma 5.6,

$$\int_{D\times\mathbb{R}^d} q(\theta)G(v)\nu_Q(d\theta dv)$$
$$= \lim_{l\to\infty}\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N} q(\frac{x}{N})E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[G(\mathrm{Av}_l\eta)].$$

If one can replace $E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[G(\mathrm{Av}_l\eta)]$ with $G\big(\nabla^N\tilde{h}^{\psi}_{N,f}(\frac{x}{N})+\nabla^N g(\frac{x}{N})\big)$, then the right hand side is equal to

$$\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N} q(\frac{x}{N})G\big(\nabla^N\tilde{h}^{\psi}_{N,f}(\frac{x}{N}) + \nabla^N g(\frac{x}{N})\big)$$
$$= \int_{D\times\mathbb{R}^d} q(\theta)G(v+\nabla g(\theta))\tilde{\nu}(\theta,dv)d\theta,$$

which implies (5.9). The last equality follows from Proposition 4.2 of [13], Lemma 5.5 and the fact that the equation (5.6) holds for $G = G(v+\nabla g(\theta))$ instead of $G = G(v)$ by p.213 remark 3 of [3].

For the replacement above, we have

$$\begin{aligned} &\Big| \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[G(\mathrm{Av}_l\eta)] \\ &\qquad - \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) G\left(\nabla^N \tilde{h}^{\psi}_{N,f}\left(\frac{x}{N}\right) + \nabla^N g\left(\frac{x}{N}\right)\right) \Big| \\ &\le S_1 + S_2 + S_3, \end{aligned}$$

where

$$\begin{aligned} S_1 &= \Big| \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) \left\{ E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[G(\mathrm{Av}_l\eta)] - G\left(E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[\mathrm{Av}_l\eta]\right)\right\} \Big|, \\ S_2 &= \Big| \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) \left\{ G\left(E^{\mu^{\psi,\nabla}_{N,f}\circ\tau_x^{-1}}[\mathrm{Av}_l\eta]\right) - G\left(\nabla^N \bar{h}^{\psi}_{N,f}\left(\frac{x}{N}\right)\right)\right\} \Big|, \\ S_3 &= \Big| \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) \left\{ G\left(\nabla^N \bar{h}^{\psi}_{N,f}\left(\frac{x}{N}\right)\right) \right. \\ &\qquad \left. - G\left(\nabla^N \tilde{h}^{\psi}_{N,f}\left(\frac{x}{N}\right) + \nabla^N g\left(\frac{x}{N}\right)\right)\right\} \Big| . \end{aligned}$$

In a similar way to the proof of Lemma 4.4 of [13], we can prove that $S_1, S_2 \to 0$ as $N \to \infty$, $l \to \infty$. Also by (5.5),

$$\begin{aligned} S_3 &= \Big| \frac{1}{N^d} \sum_{x\in D_N} q\left(\frac{x}{N}\right) \left\{ G\left(\nabla^N \tilde{h}^{\psi}_{N,f}\left(\frac{x}{N}\right) + \nabla(\xi\vee\psi)(x)\right) \right. \\ &\qquad \left. - G\left(\nabla^N \tilde{h}^{\psi}_{N,f}\left(\frac{x}{N}\right) + \nabla\xi(x)\right)\right\} \Big| \\ &\le \frac{1}{N^d} \sum_{x\in\partial^- D_N} \|q\|_\infty \|\nabla G\|_\infty |\nabla(\xi\vee\psi)(x) - \nabla\xi(x)|, \end{aligned}$$

where $\partial^- D_N = \{x \in D_N; \mathrm{dist}(x, \mathbb{Z}^d \setminus D_N) = 1\}$. This goes to 0 as $N \to \infty$ by the assumptions on ψ. Q.E.D.

Proof of Lemmas 5.2 and 5.3.

We are now in the position to prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. For every $q \in C_0^\infty(D)$, by (5.2) and summation by parts, we have

$$\int_D q(\theta)f(\theta)d\theta = \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N} q(\frac{x}{N})f(\frac{x}{N})$$
$$= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N} \nabla^N q(\frac{x}{N})\cdot E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))].$$

Now by the definition of Q_N, Lemmas 5.6, 5.7 and the property of the surface tension $\frac{\partial\sigma}{\partial v_i}(v) = E^{\mu_v^\nabla}[V'(\nabla_i\phi(0))]$ for every $1 \le i \le d$ (cf. [18, Theorem 3.4 (iii)]), we obtain

$$\int_D q(\theta)f(\theta)d\theta = \int_{D\times\mathcal{X}} \nabla q(\theta)\cdot E^{\mu_v^\nabla}[V'(\nabla\phi(0))]\nu_Q(d\theta dv)$$
$$= \int_{D\times\mathbb{R}^d} \nabla q(\theta)\cdot(\nabla\sigma)(v+\nabla g(\theta))\tilde{\nu}(\theta,dv)d\theta$$
$$= \lim_{N\to\infty}\int_D \nabla q(\theta)\cdot(\nabla\sigma)(\nabla\tilde{h}^\psi_{N,f}(\theta)+\nabla g(\theta))d\theta,$$

Note that we can apply (5.6) for $G = G(v,\theta) = (\nabla\sigma)(v+\nabla g(\theta))$ instead of $G = G(v)$ by p.213 remark 3 of [3] and the property of the surface tension $|(\nabla\sigma)(u)| \le c(1+|u|)$ (cf. [18, Theorem 3.4 (v)]).

Q.E.D.

Proof of Lemma 5.3. By (5.2), summation by parts and (5.5), we have

$$\lim_{N\to\infty}\int_D \tilde{h}^\psi_{N,f}(\theta)f(\theta)d\theta$$
$$= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N}\nabla^N\tilde{h}^\psi_{N,f}(\frac{x}{N})\cdot E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))]$$
$$= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N}\nabla^N\bar{h}^\psi_{N,f}(\frac{x}{N})\cdot E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))]$$
$$- \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N}\nabla(\xi\vee\psi)(x)\cdot E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))]$$
$$\equiv S_1 - S_2.$$

Now, by the assumptions on V and ψ, it is easy to see that

$$S_2 = \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in D_N}\nabla^N g(\frac{x}{N})\cdot E^{\mu^\psi_{N,f}}[V'(\nabla\phi(x))],$$

since $\xi(x) = Ng(\frac{x}{N})$. Hence, by the proof of Lemma 5.2, we obtain

$$S_2 = \lim_{N\to\infty} \int_D \nabla g(\theta) \cdot (\nabla\sigma)(\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta))d\theta.$$

Also, by Lemmas 5.6, 5.7 and the property of the surface tension σ, in a similar way to the proof of Lemma 5.2 we can prove that

$$S_1 = \lim_{N\to\infty} \int_D (\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta)) \cdot (\nabla\sigma)(\nabla \tilde{h}^{\psi}_{N,f}(\theta) + \nabla g(\theta))d\theta.$$

Therefore, the proof is completed. Q.E.D.

§6. Proof of Theorem 2.2: LDP for δ-Pinning

Schilder's theorem.

Throughout this section, we assume that $d = 1$ and $V(\eta) = \frac{1}{2}\eta^2$. We first notice that the large deviation principle holds for $\{h^N(\theta); \theta \in D\}$ under $\mu_N^{a,b}$ on $W_{a,b}(D)$. Recall that the space $W_{a,b}(D)$ is endowed with the topology determined by the sup-norm.

Lemma 6.1. *For the family of distributions on the space $W_{a,b}(D)$ under $\mu_N^{a,b}$ of $\{h^N(\theta); \theta \in D\}$, the large deviation principle holds with a rate functional $I^{a,b}(h) := \Sigma(h) - \frac{1}{2}(b-a)^2$ where $\Sigma(h) = \frac{1}{2}\int_0^1 (h')^2(\theta)d\theta$.*

Proof. Let $w = \{w(x); x \in [0, N]\}$ be the one-dimensional standard Brownian motion starting at 0 and set $\bar{h}^N(\theta) := w(N\theta)/N, \theta \in [0, 1]$. Then, by Schilder's theorem (see, e.g., Theorem 5.1 of [25]), the large deviation principle holds for $\{\bar{h}^N\}_N$ on $W_0 = \{h \in C([0, 1]; \mathbb{R}); h(0) = 0\}$ with the rate function $\Sigma(h)$. Define $\phi = \{\phi(x); x \in [0, N]\}$ from w as $\phi(x) = w(x) - xw(N)/N + (N-x)a + xb$. Then, $\{\phi(x); x \in D_N\}$ is $\mu_N^{a,b}$-distributed. Set $\tilde{h}^N(\theta) = \phi(N\theta)/N, \theta \in [0, 1]$, and consider a mapping $\Phi : \bar{h} \in W_0 \mapsto \tilde{h} \in W_{a,b}(D)$ defined by

$$\Phi(\bar{h})(\theta) = \bar{h}(\theta) - \theta\bar{h}(1) + (1-\theta)a + \theta b.$$

Then, Φ is continuous and $\tilde{h}^N = \Phi(\bar{h}^N)$ holds. Therefore, by the contraction principle, the large deviation principle holds for $\{\tilde{h}^N\}_N$ with the rate functional $\tilde{\Sigma}(h) = \inf_{\bar{h}\in W_0:\Phi(\bar{h})=h} \Sigma(\bar{h})$, which coincides with $I^{a,b}(h)$. The proof of lemma is completed by showing a super exponential estimate for the difference between h^N and $\tilde{h}^N$ as in p.17 of [25]: For every

$\delta > 0$,

$$P\left(\|h^N - \tilde{h}^N\|_\infty \ge \delta\right) = \exp\left[-\frac{N^2\delta^2}{8} + o(N^2)\right],$$

as $N \to \infty$. Q.E.D.

Proof of Theorem 2.2.

Proof of Theorem 2.2. Step1 (lower bound). Let $\delta > 0$ and $g \in W_{a,b}(D)$ which satisfies the condition:

$$\begin{aligned} &|\{\theta \in D; g(\theta) = 0\}| > 0 \text{ and there exist disjoint intervals} \\ (6.1)\quad &\{I^j\}_{1\le j\le K}, K < \infty \text{ such that } |\{\theta \in D; g(\theta) = 0\}| = \sum_{j=1}^K |I^j| \\ &\text{and } g(\theta) = 0 \text{ if } \theta \in \bigcup_{j=1}^K I^j, \end{aligned}$$

be fixed. Then, one can decompose $D \setminus \bigcup_{j=1}^K I^j = \bigcup_{j=1}^{K+1} L^j$ with disjoint intervals $\{L^j\}_{1\le j\le K+1}$. We define $I_N^j = NI^j \cap \mathbb{Z}, L_N^j = NL^j \cap \mathbb{Z}, I_N = \bigcup_{j=1}^K I_N^j$ and $L_N = \bigcup_{j=1}^{K+1} L_N^j$. By expanding the product $\prod_{x\in D_N}(e^J\delta_0(d\phi(x)) + d\phi(x))$, we have

$$\begin{aligned} &\frac{Z_N^{a,b,J}}{Z_N^{a,b}}\mu_N^{a,b,J}(h^N \in B_\infty(g,\delta)) \\ &= \sum_{\Lambda\subset D_N} e^{J|\Lambda^c|}\frac{Z_\Lambda^{a,b}}{Z_N^{a,b}}\mu_\Lambda^{a,b}(h^N \in B_\infty(g,\delta)) \\ &\ge \sum_{L_N\subset\Lambda\subset D_N} e^{J|\Lambda^c|}\frac{Z_\Lambda^{a,b}}{Z_N^{a,b}}\mu_\Lambda^{a,b}(h^N \in B_\infty(g,\delta)) \\ &= \sum_{A\subset I_N} e^{J|I_N\setminus A|}\frac{Z_{L_N\cup A}^{a,b}}{Z_N^{a,b}}\mu_{L_N\cup A}^{a,b}(h^N \in B_\infty(g,\delta)), \end{aligned}$$

where $B_\infty(g,\delta) = \{h \in W_{a,b}(D); \|h - g\|_\infty < \delta\}$ and $\mu_\Lambda^{a,b}$ is defined by

$$\begin{aligned} \mu_\Lambda^{a,b}(d\phi) =& \frac{1}{Z_\Lambda^{a,b}}\exp\{-\frac{1}{2}\sum_{b\in\bar{\Lambda}^*} V(\nabla(\phi\vee\tilde{\psi})(b))\} \\ &\times\prod_{x\in\Lambda} d\phi(x) \prod_{x\in\overline{D_N}\setminus\Lambda}\delta_{\tilde{\psi}(x)}(d\phi(x)), \end{aligned}$$

and $\tilde{\psi}(x) = \psi(x)$ if $x \in \partial^+ D_N = \{0, N\}$ (i.e. $\tilde{\psi}(0) = aN, \tilde{\psi}(N) = bN$), $\tilde{\psi}(x) = 0$ if $x \in D_N \setminus \Lambda$. The constant $Z_\Lambda^{a,b}$ is for normalization.

Now, write $I_N \setminus A = \{x_1, x_2, \cdots, x_k\}$, $1 \le x_1 < x_2 < \cdots < x_k \le N-1$ and define $l_1 = [1, x_1 - 1] \cap \mathbb{Z}$, $l_2 = [x_1 + 1, x_2 - 1] \cap \mathbb{Z}$, $\cdots$, $l_k = [x_{k-1}+1, x_k - 1] \cap \mathbb{Z}$, $l_{k+1} = [x_k + 1, N-1] \cap \mathbb{Z}$. Then, $\bigcup_{j=1}^{k+1} l_j = L_N \cup A$ and by the Markov property of the ϕ-field, we have

$$\begin{aligned}\mu_{L_N \cup A}^{a,b}(h^N \in B_\infty(g, \delta)) &\ge \mu_{L_N \cup A}^{a,b}\Big(\max_{x \in \bigcup_{j=1}^{k+1} l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \frac{1}{2}\delta\Big) \\ &= \prod_{j=1}^{k+1} \mu_{l_j}^{a_j, b_j}\big(\max_{x \in l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \frac{1}{2}\delta\big),\end{aligned}$$

for N large enough, where $a_j = a$ if $j = 1$, $a_j = 0$ otherwise, $b_j = b$ if $j = k+1$, $b_j = 0$ otherwise. We define $\Gamma = \{1 \le j \le k+1; l_j \supset L_N^i$ for some $1 \le i \le K+1\}$ and $\Gamma^c = \{1 \le j \le k+1\} \setminus \Gamma$. If $j \in \Gamma^c$, since $g(\frac{x}{N}) = 0$ for each $x \in l_j$, we have

$$\begin{aligned}\mu_{l_j}^{a_j, b_j}\big(\max_{x \in l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \frac{1}{2}\delta\big) &= \mu_{l_j}^{a_j, b_j}\big(\max_{x \in l_j} |\frac{1}{N}\phi(x)| < \frac{1}{2}\delta\big) \\ &\ge 1 - \sum_{x \in l_j} \mu_{l_j}^{0,0}\big(|\phi(x)| \ge \frac{1}{2}\delta N\big).\end{aligned}$$

However, it is easy to see that

$$\mu_{l_j}^{0,0}\big(|\phi(x)| \ge \frac{1}{2}\delta N\big) \le \exp\Big\{-\frac{(\frac{1}{2}\delta N)^2}{\mathrm{Var}_{\mu_{l_j}^{0,0}}(\phi(x))}\Big\} \le \exp\{-C\delta^2 N\},$$

for some $C > 0$ and we obtain

$$\prod_{j \in \Gamma^c} \mu_{l_j}^{a_j, b_j}\big(\max_{x \in l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \frac{1}{2}\delta\big) \ge 1 - N\exp\{-C\delta^2 N\}.$$

Next, for every closed interval $F \equiv [x_F, y_F] \subset [0, 1]$, define

$$\begin{aligned}&B_\infty(g, \delta; F) = \{h \in C(F; \mathbb{R}); \sup_{\theta \in F} |h(\theta) - g(\theta)| < \delta\}, \\ &W_{a,b}(F) = \{h \in C(F; \mathbb{R}); h(x_F) = a, h(y_F) = b\}, \\ &H_{a,b}^1(F) = \{h \in W_{a,b}(F); h \text{ is absolutely continuous}, h' \in \mathbb{L}^2(F)\}.\end{aligned}$$

We also define $\tilde{l}_j = \frac{l_j}{N} \subset [0,1]$ for $j \in \Gamma$. Then, by the LDP lower bound for $\mu_N^{a,b}$ (Lemma 6.1), we know that

$$\begin{aligned}\prod_{j\in\Gamma} \mu_{l_j}^{a_j,b_j}\big(\max_{x\in l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \frac{1}{2}\delta\big) \\ \geq \exp\Big\{-N\big(\sum_{j\in\Gamma} \inf_{h\in B_\infty(g,\frac{1}{2}\delta;\tilde{l}_j)} I_{\tilde{l}_j}^{a_j,b_j}(h) + \varepsilon\big)\Big\} \\ \geq \exp\Big\{-N\big(\Sigma(g) - \frac{1}{2}(\frac{a^2}{|\tilde{l}_1|} + \frac{b^2}{|\tilde{l}_{k+1}|}) + \varepsilon\big)\Big\},\end{aligned}$$

for every $\varepsilon > 0$ and N large enough, where

$$I_F^{a,b}(h) = \begin{cases} \Sigma_F(h) - \frac{(b-a)^2}{2|F|} & \text{if } h \in H_{a,b}^1(F), \\ +\infty & \text{otherwise,} \end{cases}$$

and $\Sigma_F(h) = \frac{1}{2}\int_F (h')^2(\theta)d\theta$ for closed interval $F \subset [0,1]$. Recall that $\Sigma_{[0,1]}(h)$ coincides with $\Sigma(h)$. Therefore, we obtain

$$\begin{aligned}&\mu_{L_N\cup A}^{a,b}(h^N \in B_\infty(g,\delta)) \\ &\quad\geq \exp\Big\{-N\big(\Sigma(g) - \frac{1}{2}(\frac{a^2}{|\tilde{l}_1|} + \frac{b^2}{|\tilde{l}_{k+1}|}) + \varepsilon\big)\Big\}(1 - Ne^{-CN\delta^2}).\end{aligned}$$

Note that this estimate holds for every choice of $A \subset I_N$ and for every N large enough, since $|\Gamma| \leq K+1$ is independent of N. Also, simple calculation yields that

$$\begin{aligned}Z_{L_N\cup A}^{a,b} &= Z_{L_N\cup A}^{0,0} \exp\{-\frac{N}{2}(\frac{a^2}{|\tilde{l}_1|} + \frac{b^2}{|\tilde{l}_{k+1}|})\}, \\ Z_N^{a,b} &= Z_N^{0,0} \exp\{-\frac{N}{2}(b-a)^2\}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}(6.2)\qquad &\frac{Z_N^{a,b,J}}{Z_N^{a,b}} \mu_N^{a,b,J}(h^N \in B_\infty(g,\delta)) \\ &\quad\geq \sum_{A\subset I_N} e^{J|I_N\setminus A|} \frac{Z_{L_N\cup A}^{0,0}}{Z_N^{0,0}} \exp\{-N(I^{a,b}(g) + 2\varepsilon)\},\end{aligned}$$

for every $\varepsilon > 0$ and N large enough.

Now, we can exactly calculate that $Z_N^{0,0} = \frac{(\sqrt{2\pi})^{N-1}}{\sqrt{N}}$ and this shows

$$1 \le \frac{Z_{L_N \cup A}^{0,0}}{Z_{L_N}^{0,0} Z_A^{0,0}} \le e^{a_N}, \tag{6.3}$$

for every $A \subset I_N$, where $a_N = o(N)$. Note that L_N consists of finite number of disjoint intervals of size $O(N)$. By using (6.3), it is easy to see that

$$\frac{Z_{I_N}^{0,0,J}}{Z_{I_N}^{0,0}} e^{-a_N} \le \sum_{A \subset I_N} e^{J|I_N \setminus A|} \frac{Z_{L_N \cup A}^{0,0}}{Z_N^{0,0}} \le \frac{Z_{I_N}^{0,0,J}}{Z_{I_N}^{0,0}} e^{a_N}. \tag{6.4}$$

The sub-additivity argument (cf. [8, Section 4.3], [18, Appendix II]) and the fact that $\frac{|I_N|}{N} \to |\{\theta \in D; g(\theta) = 0\}|$ as $N \to \infty$ yield that the limit $\tau(J)$ in (2.7) exists and it holds that

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_{I_N}^{0,0,J}}{Z_{I_N}^{0,0}} = -\tau(J)|\{\theta \in D; g(\theta) = 0\}|. \tag{6.5}$$

Combining (6.4), (6.5) with (6.2), we obtain

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{a,b,J}}{Z_N^{a,b}} \mu_N^{a,b,J}(h^N \in B_\infty(g,\delta)) \\ \ge -I^{a,b}(g) - \tau(J)|\{\theta \in D; g(\theta) = 0\}| \\ \equiv -I^{a,b;J}(g), \end{aligned} \tag{6.6}$$

for every $g \in W_{a,b}(D)$ satisfying the condition (6.1) and $\delta > 0$. In the case that $|\{\theta \in D; g(\theta) = 0\}| = 0$, we have only to take the sum $A = I_N$ in (6.2) and the same inequality as above is obtained.

However, for every open set $\mathcal{O}$ of $W_{a,b}(D)$, we have that

$$\inf_{g \in \mathcal{O}:(6.1)'} I^{a,b;J}(g) = \inf_{h \in \mathcal{O}} I^{a,b;J}(h), \tag{6.7}$$

where (6.1)′ means the condition (6.1) or $|\{\theta \in D; g(\theta) = 0\}| = 0$. Indeed, since the left hand side of (6.7) is larger than or equal to the right hand side, we may prove the reverse inequality only. To this end, for every $\varepsilon > 0$, take $h \in \mathcal{O}$ such that $I^{a,b;J}(h) \le \inf_{\mathcal{O}} I^{a,b;J} + \varepsilon$; note that $h \in H^1_{a,b}(D)$. Since $\mathcal{O}$ is open, one can find $\delta > 0$ such that $B_\infty(h,\delta) \subset \mathcal{O}$. Taking $n \ge 1$ such that $|\theta_1 - \theta_2| \le 1/n$ implies $|h(\theta_1) - h(\theta_2)| < \delta$, divide the interval $[0,1] = \cup_{k=1}^n \mathcal{J}_k$, $\mathcal{J}_k = [(k-1)/n, k/n]$ and set $\mathcal{J} = \cup_k \mathcal{J}_k$, the union of $\mathcal{J}_k$'s on which $h(\theta) \ne 0$. We now define a function $g = g(\theta)$, first on $\mathcal{J}$, by $g(\theta) = h(\theta)$. On $\mathcal{J}^c$, starting at points in $\partial\mathcal{J}$, $g(\theta) = h(\theta)$ up to $\bar{\theta}$'s such that $h(\bar{\theta}) = 0$, and set $g \equiv 0$ otherwise. Then, $g \in B_\infty(h,\delta) \subset \mathcal{O}$, $I^{a,b;J}(g) \le I^{a,b;J}(h)$ and g satisfies the condition (6.1)$'$. This proves (6.7). Therefore, from (6.6) and (6.7), we have

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{a,b,J}}{Z_N^{a,b}} \mu_N^{a,b,J}(h^N \in \mathcal{O}) \ge -\inf_{h\in\mathcal{O}} I^{a,b;J}(h), \tag{6.8}$$

for every open set $\mathcal{O}$ of $W_{a,b}(D)$.

Step2 (*upper bound*). Let $\delta > 0$ and $g \in W_{a,b}(D)$ which satisfies the condition:

$$\begin{aligned} &\text{for every } \gamma > 0 \text{ small enough, there exist disjoint} \\ &\text{intervals } \{I^j(\gamma)\}_{1\le j\le K}, K < \infty \text{ such that} \\ &\{\theta \in D; |g(\theta)| \le \gamma\} = \bigcup_{j=1}^K I^j(\gamma), \end{aligned} \tag{6.9}$$

be fixed. Then, one can write $\{\theta \in D; |g(\theta)| > \gamma\} = \bigcup_{j=1}^{K+1} L^j(\gamma)$ for disjoint intervals $\{L^j(\gamma)\}_{1\le j\le K+1}$. We define $I_N^j = NI^j(\delta) \cap \mathbb{Z}, L_N^j = NL^j(\delta) \cap \mathbb{Z}, I_N = \bigcup_{j=1}^K I_N^j$ and $L_N = \bigcup_{j=1}^{K+1} L_N^j$. Since $\mu_\Lambda^{a,b}(h^N \in B_\infty(g,\delta)) = 0$ for $\Lambda \subset D_N$ such that $\Lambda \not\supset L_N$, we have

$$\begin{aligned} &\frac{Z_N^{a,b,J}}{Z_N^{a,b}} \mu_N^{a,b,J}(h^N \in B_\infty(g,\delta)) \\ &\quad = \sum_{L_N\subset\Lambda\subset D_N} e^{J|\Lambda^c|} \frac{Z_\Lambda^{a,b}}{Z_N^{a,b}} \mu_\Lambda^{a,b}(h^N \in B_\infty(g,\delta)) \\ &\quad = \sum_{A\subset I_N} e^{J|I_N\setminus A|} \frac{Z_{L_N\cup A}^{a,b}}{Z_N^{a,b}} \mu_{L_N\cup A}^{a,b}(h^N \in B_\infty(g,\delta)). \end{aligned}$$

Now, let $I_N \setminus A = \{x_1, x_2, \cdots, x_k\}$, $1 \le x_1 < x_2 < \cdots < x_k \le N-1$ and define $l_1, l_2, \cdots, l_k, l_{k+1}$ and Γ in the same way as in the proof of lower bound. Then, by the Markov property of the ϕ-field and the LDP upper bound for $\mu_N^{a,b}$ (Lemma 6.1), we have

$$\begin{aligned} \mu_{L_N\cup A}^{a,b}(h^N \in B_\infty(g,\delta)) &\le \mu_{L_N\cup A}^{a,b}\Big(\max_{x\in\bigcup_{j=1}^{k+1} l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \delta\Big) \\ &\le \prod_{j\in\Gamma} \mu_{l_j}^{a_j,b_j}\big(\max_{x\in l_j} |\frac{1}{N}\phi(x) - g(\frac{x}{N})| < \delta\big) \end{aligned}$$

$$\le \exp\Big\{-N\Big(\sum_{j\in\Gamma} \inf_{h\in\bar{B}_\infty(g,2\delta;\tilde{l}_j)} I_{\tilde{l}_j}^{a_j,b_j}(h)-\varepsilon\Big)\Big\}$$

$$\le \exp\Big\{-N\Big(\inf_{\substack{h\in\bar{B}_\infty(g,2\delta)\\ h(0)=a,h(1)=b}} \Sigma(h)-\frac{1}{2}\Big(\frac{a^2}{|\tilde{l}_1|}+\frac{b^2}{|\tilde{l}_{k+1}|}\Big)-\varepsilon\Big)\Big\},$$

for every $\varepsilon>0$ and N large enough. Then, in a similar way to the proof of lower bound, we can prove that

$$\text{(6.10)}\qquad \limsup_{N\to\infty}\frac{1}{N}\log\frac{Z_N^{a,b,J}}{Z_N^{a,b}}\mu_N^{a,b,J}(h^N\in B_\infty(g,\delta))$$
$$\le -\inf_{h\in\bar{B}_\infty(g,2\delta)} I^{a,b}(g)-\tau(J)|\{\theta\in D;|g(\theta)|\le\delta\}|,$$

for every $g\in W_{a,b}(D)$ satisfying the condition (6.9) and $\delta>0$. Note that I_N is defined by $N\{\theta\in D;|g(\theta)|\le\delta\}\cap\mathbb{Z}$ in this case.

By using (6.10), the right-continuity of $|\{\theta\in D;|g(\theta)|\le\delta\}|$ in δ and the fact that the set of $g\in W_{a,b}(D)$ satisfying the condition (6.9) is dense in $W_{a,b}(D)$, the similar argument to the proof of the upper bound of Theorem 2.1 yields that for every $g\in W_{a,b}(D)$ and $\varepsilon>0$, there exists some $\delta>0$ such that

$$\limsup_{N\to\infty}\frac{1}{N}\log\frac{Z_N^{a,b,J}}{Z_N^{a,b}}\mu_N^{a,b,J}(h^N\in B_\infty(g,\delta))\le -I^{a,b;J}(g)+\varepsilon.$$

Since $\mu_N^{a,b,J}$ can be written as the superposition of $\mu_\Lambda^{a,b},\Lambda\subset D_N$, exponential tightness for $\mu_N^{a,b,J}$ follows from the similar argument as before and the standard argument yields

$$\text{(6.11)}\qquad \limsup_{N\to\infty}\frac{1}{N}\log\frac{Z_N^{a,b,J}}{Z_N^{a,b}}\mu_N^{a,b,J}(h^N\in\mathcal{C})\le -\inf_{h\in\mathcal{C}} I^{a,b;J}(h),$$

for every closed set $\mathcal{C}$ of $W_{a,b}(D)$. The lower and upper bounds (6.8) and (6.11) conclude the proof. Q.E.D.

Remark 6.1. *By the proof above and* [8, Lemma 2.3.1 (a)] (*note that the argument given there can be extended to all* $d\ge 1$), *we know that*

$$\frac{Z_N^{0,0,J}}{Z_N^{0,0}}=\sum_{\Lambda\subset D_N}e^{J|\Lambda^c|}\frac{Z_\Lambda^{0,0}}{Z_N^{0,0}}\ge\sum_{\Lambda\subset D_N}e^{J|\Lambda^c|}e^{-C|\Lambda^c|}=(1+e^{J-C})^{|D_N|}$$

for some constant $C>0$. *Therefore,* $\tau(J)<0$ *for every* $J\in\mathbb{R}$.

Acknowledgement.

The authors thank G.S. Weiss for pointing out the references [1], [2] and [26] to them. They also thank H. Spohn for helpful discussions.

References

[1] H.W. Alt and L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., **325** (1981), pp. 105–144.

[2] H.W. Alt, L.A. Caffarelli and A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc., **282** (1984), pp. 431–461.

[3] J.M. Ball, *A version of the fundamental theorem for Young measures*, in *PDEs and continuum models of phase transitions (Nice, 1988)*, Springer–Verlag, Lecture Notes in Physics, **344** (1989), pp. 207–215.

[4] G. Ben Arous and J.-D. Deuschel, *The construction of the $d+1$-dimensional Gaussian droplet*, Commun. Math. Phys., **179** (1996), pp. 467–488.

[5] T. Bodineau, D. Ioffe and Y. Velenik, *Rigorous probabilistic analysis of equilibrium crystal shapes*, J. Math. Phys., **41** (2000), pp. 1033–1098.

[6] E. Bolthausen, J.-D. Deuschel and O. Zeitouni, *Entropic repulsion of the lattice free field*, Commun. Math. Phys., **170** (1995), pp. 417–443.

[7] E. Bolthausen, J.-D. Deuschel and O. Zeitouni, *Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger*, J. Math. Phys., **41** (2000), pp. 1211–1223.

[8] E. Bolthausen and D. Ioffe, *Harmonic crystal on the wall: a microscopic approach*, Commun. Math. Phys., **187** (1997), pp. 523–566.

[9] E. Bolthausen and Y. Velenik, *Critical behavior of the massless free field at the depinning transition*, Commun. Math. Phys., **223** (2001), pp. 161–203.

[10] A. Caputo and Y. Velenik, *A note on wetting transition for gradient fields*, Stoch. Proc. Appl., **87** (2000), pp. 107–113.

[11] J. De Coninck, F. Dunlop and V. Rivasseau, *On the microscopic validity of the Wulff construction and of the generalized Young equation*, Commun. Math. Phys., **121** (1989), pp. 401–419.

[12] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, 2nd edition, Springer-Verlag, Applications of Mathematics **38**, 1998.

[13] J.-D. Deuschel, G. Giacomin and D. Ioffe, *Large deviations and concentration properties for $\nabla\varphi$ interface models*, Probab. Theory Relat. Fields, **117** (2000), pp. 49–111.

[14] J.-D. Deuschel and D.W. Stroock, *Large Deviations*, Academic Press, Vol. 137 in Pure and Applied Mathematics, 1989.

[15] J.-D. DEUSCHEL AND Y. VELENIK, *Non-Gaussian surface pinned by a weak potential*, Probab. Theory Relat. Fields, **116** (2000), pp. 359–377.

[16] F. DUNLOP, J. MAGNEN, V. RIVASSEAU AND P. ROCHE, *Pinning of an interface by a weak potential*, J. Statis. Phys., **66** (1992), pp. 71–98.

[17] M.E. FISHER, *Walks, walls, wetting, and melting*, J. Statis. Phys., **34** (1984), pp. 667–729.

[18] T. FUNAKI AND H. SPOHN, *Motion by mean curvature from the Ginzburg-Landau* $\nabla\phi$ *interface models*, Commun. Math. Phys., **185** (1997), pp. 1–36.

[19] G. GIACOMIN, *Anharmonic lattices, random walks and random interfaces*, Recent Research Developments in Statistical Physics, Transworld Research Network, **1** (2000), pp. 97–118.

[20] D. IOFFE AND Y. VELENIK, *A note on the decay of correlations under* δ*-pinning*, Probab. Theory Relat. Fields, **116** (2000), pp. 379–389.

[21] Y. ISOZAKI AND N. YOSHIDA, *Weakly pinned random walk on the wall: pathwise descriptions of the phase transition*, Stoch. Proc. Appl., **96** (2001), pp. 261–284.

[22] C. KIPNIS AND C. LANDIM, *Scaling Limits of Interacting Particle Systems*, Springer-Verlag, Grundlehern der math. Wiss. **320**, 1999.

[23] T. NISHIKAWA, *Hydrodynamic limit for the Ginzburg-Landau* $\nabla\phi$ *interface model with a boundary condition*, to appear in Probab. Theory Rel. Fields, 2003.

[24] C.-E. PFISTER AND Y. VELENIK, *Interface, surface tension and reentrant pinning transition in the 2D Ising model*, Commun. Math. Phys., **204** (1999), pp. 269–312.

[25] S.R.S. VARADHAN, *Large Deviations and Applications*, SIAM, 1984.

[26] G.S. WEISS, *A free boundary problem for non-radial-symmetric quasi-linear elliptic equations*, Adv. Math. Sci. Appl., **5** (1995), pp. 497–555.

[27] J. WLOKA, *Partial differential equations*, Cambridge Univ. Press, 1987.

Tadahisa Funaki
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN
E-mail address: funaki@ms.u-tokyo.ac.jp

Hironobu Sakagawa
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kouhoku-ku
Yokohama 223-8522
JAPAN
E-mail address: sakagawa@math.keio.ac.jp