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Topological conjugacy invariants of symbolic dynamics arising from C^*-algebra K-theory

Kengo Matsumoto

§1 Introduction

In [Wi], R. F. Williams introduced the notions of strong shift equivalence and shift equivalence between nonnegative square matrices and showed that two topological Markov shifts are topologically conjugate if and only if the associated matrices are strong shift equivalent. He also showed that strong shift equivalence implies shift equivalence (cf. [KimR]). There is a class of subshifts called sofic subshifts that are generalized class of Markov shifts and determined by square matrices with entries in formal sums of symbols (see [Kit],[Kr4],[LM],[We],etc.). Such a square matrix is called a symbolic matrix. It is an equivalent object to a labeled graph called a λ-graph. M. Nasu in [N],[N2] generalized the notion of strong shift equivalence to symbolic matrices. He showed that two sofic subshifts are topologically conjugate if and only if their canonical symbolic matrices are strong shift equivalent ([N],[N2],see also [HN]). M. Boyle and W. Krieger in [BK] introduced the notion of shift equivalence for symbolic matrices and studied topologically conjugacy for sofic subshifts.

In [Ma6], the notions of symbolic matrix system and λ-graph system have been introduced as presentations of subshifts. They are generalized notions of symbolic matrix and λ-graph for sofic subshifts. Let Σ be a finite set. A symbolic matrix system over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$. The matrices $\mathcal{M}_{l,l+1}$ have entries in formal sums of Σ and the matrices $I_{l,l+1}$ have entries in $\{0,1\}$. They satisfy the following commutation relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

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We assume that for i there exists j such that the (i,j)-component $I_{l,l+1}(i,j) = 1$ and for j there uniquely exists i such that $I_{l,l+1}(i,j) = 1$. We denote it by $(\mathcal{M}, I)$.

A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ is a labeled Bratteli diagrams with vertex set $V = V_1 \cup V_2 \cup \cdots$ and edge set $E = E_{1,2} \cup E_{2,3} \cup \cdots$ that naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The matrix $\mathcal{M}_{l,l+1}$ defines an edge e in $E_{l,l+1}$ from a vertex in V_l to a vertex in V_{l+1} whose label is denoted by $\lambda(e) \in \Sigma$. The matrix $I_{l,l+1}$ defines a surjection from V_{l+1} to V_l. The symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts. There is a canonical method to construct a symbolic matrix system from an arbitrary subshift. The obtained symbolic matrix system is said to be canonical for the subshift. If a subshift is sofic, the canonical symbolic matrix system corresponds to the symbolic matrix of its left Krieger cover graph. The notion of strong shift equivalence for nonnegative matrices and symbolic matrices has been generalized to symbolic matrix systems ([Ma6], cf. [Ma11]). We have proved (cf. [N],[Wi])

Theorem A ([Ma6]). *Two subshifts are topologically conjugate if and only if their canonical symbolic matrix systems are strong shift equivalent.*

Shift equivalence between two symbolic matrix systems has been defined in [Ma6] as a generalization of the corresponding notion for symbolic matrices defined by Boyle-Krieger in [BK].

For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the nonnegative rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{N}$. Then the resulting pair (M, I) still satisfies the following relations.

$$I_{l,l+1}M_{l+1,l+2} = M_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

We call (M, I) the nonnegative matrix system for $(\mathcal{M}, I)$. We say (M, I) to be canonical when $(\mathcal{M}, I)$ is canonical. More generally, for a sequence $M_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in nonnegative integers and a sequence $I_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in $\{0,1\}$, the pair (M, I) is called a nonnegative matrix system if they satisfy the relations above. For a single $n \times n$ nonnegative square matrix A, if we set $M_{l,l+1} = A$ and $I_{l,l+1} = I_n$: the $n \times n$ identity matrix for all $l \in \mathbb{N}$, the pair (M, I) is a nonnegative matrix system. We similarly formulate strong shift equivalence and shift equivalence between nonnegative matrix systems as generalizations of the corresponding equivalences for single nonnegative square matrices.

For nonnegative matrices, the dimension groups defined by Krieger in [Kr2], [Kr3] and the Bowen-Franks groups considered in [BF] are crucial shift equivalence invariants. They induce topological conjugacy invariants for the associated topological Markov shifts. We generalize them to nonnegative matrix systems. The following three kinds of objects for a nonnegative matrix system (M, I) are defined:

$$(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}), \qquad K_i(M, I), \qquad BF^i(M, I), \quad i = 0, 1.$$

The dimension triple $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)})$ consists of an abelian group $\Delta_{(M,I)}$ with positive cone $\Delta^+_{(M,I)}$ and an ordered automorphism $\delta_{(M,I)}$ on it. The K-groups $K_i(M, I), i = 0, 1$ and the Bowen-Franks groups $BF^i(M, I), i = 0, 1$ consist of a pair of abelian groups for each. The three kinds of objects above are invariant under shift equivalence in nonnegative matrix systems. Hence they are naturally induce topological conjugacy invariants for subshifts by taking their canonical nonnegative matrix systems. Relationships among the equivalences and the invariants for the matrix systems are as in the following way:

Theorem B ([Ma6]). *For two symbolic matrix systems* $(\mathcal{M}, I)$, $(\mathcal{M}', I')$ *and their respect nonnegative matrix systems* $(M, I), (M', I')$, *consider the following situations:*

(S1) $(\mathcal{M}, I) \approx (\mathcal{M}', I)$: *strong shift equivalence,*
(N1) $(M, I) \approx (M', I)$: *strong shift equivalence,*
(S2) $(\mathcal{M}, I) \sim (\mathcal{M}', I)$: *shift equivalence,*
(N2) $(M, I) \sim (M', I)$: *shift equivalence,*
(3) $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \Delta^+_{(M',I')}, \delta_{(M',I')})$: *isomorphic dimension triples,*
(4) $(\Delta_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \delta_{(M',I')})$: *isomorphic dimension pairs,*
(5) $K_*(M, I) \cong K_*(M', I)$: *isomorphic K-groups,*
(6) $BF^*(M, I) \cong BF^*(M', I)$: *isomorphic Bowen-Franks groups.*

Then we have the following implications:

$$\begin{array}{ccccccccccc} (S1) & \Longrightarrow & (S2) & & & & & & & & \\ \Downarrow & & \Downarrow & & & & & & & & \\ (N1) & \Longrightarrow & (N2) & \Longrightarrow & (3) & \Longrightarrow & (4) & \Longrightarrow & (5) & \Longrightarrow & (6). \end{array}$$

Two subshifts are said to be flow equivalent if their suspension flows act on homeomorphic spaces under some homeomorphism that preserves orbits in an orientation preserving way (cf.[PS]). The Bowen-Franks group $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ for nonnegative matrix A is known to be not only a topological conjugacy invariant but also a flow equivalence invariant for the associated topological Markov shift ([BF], cf. [Fr], [PS]). We generalize it to subshifts.

Theorem C ([Ma8], cf.[Ma3]). *The K-groups and the Bowen-Franks groups for canonical nonnegative matrix systems for subshifts are invariant under flow equivalence.*

In [Ma6], the eigenvalues and eigenvectors of a nonnegative matrix system (M, I) have been defined as a generalization of the nonzero spectrum of a single nonnegative matrix. We denote by $Sp^{\times}(M, I)$ the set of all nonzero eigenvalues of (M, I). Let $Sp_b^{\times}(M, I)$ be the set of all nonzero eigenvalues of (M, I) having a certain boundedness condition on the corresponding eigenvectors. We know that the both $Sp^{\times}(M, I)$ and $Sp_b^{\times}(M, I)$ are not empty and invariant under shift equivalence of (M, I). In particular, if (M, I) is the canonical nonnegative matrix system for a subshift, the set $Sp_b^{\times}(M, I)$ is bounded by the topological entropy of the subshift.

The author has constructed the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ ([Ma], cf. [CaM], [Ma10]) as a generalization of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with topological Markov shift Λ_A for matrix A with entries in $\{0, 1\}$. The C^*-algebra $\mathcal{O}_\Lambda$ has a canonical action of the one dimensional torus group, called gauge action and written as α_Λ. Let (M, I) be the canonical nonnegative matrix system for the subshift Λ. The invariants mentioned above are described in terms of the K-theoretic objects for the C^*-algebra as in the following way:

Theorem D ([Ma2], [Ma3], [Ma4], cf.[C3],[CK]).

$$
\begin{aligned}
(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) &\cong (K_0(\mathcal{F}_\Lambda), K_0(\mathcal{F}_\Lambda)_+, \hat{\alpha_\Lambda}_*),\\
K_i(M, I) &\cong K_i(\mathcal{O}_\Lambda), \qquad i = 0, 1,\\
BF^i(M, I) &\cong \mathrm{Ext}^{i+1}(\mathcal{O}_\Lambda), \qquad i = 0, 1
\end{aligned}
$$

where $\hat{\alpha_\Lambda}$ *denotes the dual action of* α_Λ *and* $\mathrm{Ext}^1(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda)$, $\mathrm{Ext}^0(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R}))$.

The normalized nonnegative eigenvectors of (M, I) exactly correspond to the KMS-states for α_Λ on the C^*-algebra $\mathcal{O}_\Lambda$. Hence the set of bounded spectrums with nonnegative eigenvectors are the set of inverse temperatures for the admitted KMS states ([MWY],cf.[EFW]).

§2 Symbolic matrix systems and λ-graph systems as presentations of subshifts

We fix a finite set Σ and call it the alphabet. Each element of Σ is called a symbol. We write the empty symbol $\emptyset$ in Σ as 0. We denote by $\mathfrak{S}_\Sigma$ the set of all finite formal sums of elements of Σ.

For two symbolic matrices $\mathcal{A}$ over alphabet Σ and $\mathcal{A}'$ over alphabet Σ' and bijection ϕ from a subset of Σ onto a subset of Σ', we say that $\mathcal{A}$ and $\mathcal{A}'$ are specified equivalence under specification ϕ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by replacing every symbol a appearing in $\mathcal{A}$ by $\phi(a)$. We write it as $\mathcal{A} \overset{\phi}{\simeq} \mathcal{A}'$. We call ϕ a specification from Σ to Σ'. These notions are due to M. Nasu in [N],[N2].

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$-square permutation matrix P_l for each $l \in \mathbb{N}$ such that

$$P_l \mathcal{M}_{l,l+1} \overset{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \qquad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \qquad \text{for} \quad l \in \mathbb{N}.$$

Two λ-graph systems (V, E, λ, ι) over alphabet Σ and $(V', E', \lambda', \iota')$ over alphabet Σ' are said to be isomorphic if there exist bijections $\Phi_V : V \to V'$, $\Phi_E : E \to E'$ and a specification $\phi : \Sigma \to \Sigma'$ such that

(1) $\Phi_V(V_l) = V'_l$ and $\Phi_E(E_{l,l+1}) = E'_{l,l+1}$ for $l \in \mathbb{N}$,
(2) $\Phi_V(s(e)) = s(\Phi_E(e))$ and $\Phi_V(r(e)) = r(\Phi_E(e))$ for $e \in E$,
(3) $\iota'(\Phi_V(v)) = \Phi_V(\iota(v))$ for $v \in V$,
(4) $\lambda'(\Phi_E(e)) = \phi(\lambda(e))$ for $e \in E$

where for an edge $e \in E_{l,l+1}$, $s(e) \in V_l$ and $r(e) \in V_{l+1}$ denote the source vertex of e and the range vertex of e respectively. Then we know that there exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ-graph systems.

We will see that any subshift comes from a symbolic matrix system and equivalently from a λ-graph system. We review on subshifts. Let Σ be an alphabet. Let $\Sigma^{\mathbb{Z}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $(\sigma(x_i)) = (x_{i+1}), i \in \mathbb{Z}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ and write the subshift as Λ for short. We denote by $X_\Lambda (\subset \prod_{i=1}^{\infty} \Sigma_i)$ the set of all right-infinite sequences that appear in Λ. A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word of length

k. A block $\mu = (\mu_1, ..., \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, ..., x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$.

We will construct subshifts from symbolic matrix systems.

Let $(\mathcal{M}, I)$ be a symbolic matrix system over Σ and $\mathfrak{L} = (V, E, \lambda, \iota)$ its corresponding λ-graph system. Let L_l for $l \in \mathbb{N}$ be the set of all label sequences of paths from V_1 to V_l, that is,

$$L_l = \{(\lambda(e_1), \lambda(e_2), \dots, \lambda(e_l)) \in \Sigma^l \mid e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \dots, l-1\}.$$

We set

$$X_{(\mathcal{M},I)} = \{(\lambda(e_1), \lambda(e_2), \dots) \in \Sigma^{\mathbb{N}} \mid e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i \in \mathbb{N}\}$$

the set of all right infinite sequences consisting of labels along infinite paths. The topology on $X_{(\mathcal{M},I)}$ is defined from open sets of the form

$$U_{(\mu_1,\dots,\mu_k)} = \{(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)} \mid \alpha_i = \mu_i \text{ for } i = 1, \dots, k\}$$

for $(\mu_1, \dots, \mu_k) \in L_k$ so that $X_{(\mathcal{M},I)}$ is a compact Hausdorff space. For $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, we have $(\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$. For $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, we may find a symbol $\alpha_0 \in \Sigma$ such that $(\alpha_0, \alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$. Hence the following map

$$S : (\alpha_1, \alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)} \to (\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$$

is well-defined, continuous and surjective. We set

$$\Lambda_{(\mathcal{M},I)} = \varprojlim \{S : X_{(\mathcal{M},I)} \to X_{(\mathcal{M},I)}\}$$

the projective limit in the category of compact Hausdorff spaces. Thus $\Lambda_{(\mathcal{M},I)}$ is identified with the set of all biinfinite sequences arising from the sequences in $X_{(\mathcal{M},I)}$. That is

$$\Lambda_{(\mathcal{M},I)} = \{(\dots, \alpha_2, \alpha_1, \alpha_0, \alpha_1, \alpha_2, \dots) \mid (\alpha_n, \alpha_{n+1}, \dots) \in X_{(\mathcal{M},I)} \text{ for } n \in \mathbb{Z}\}.$$

The map S induces a homeomorphism on it. We denote it by σ that satisfies $\sigma((\alpha_i)_{i\in\mathbb{Z}}) = (\alpha_{i+1})_{i\in\mathbb{Z}}$. Therefore we obtain a subshift $(\Lambda_{(\mathcal{M},I)}, \sigma)$.

We next construct symbolic matrix systems from subshifts.

For a subshift (Λ, σ) over Σ and a number $k \in \mathbb{N}$, let Λ^k be the set of all words of length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. For $l \in \mathbb{N}$, two points

$x, y \in X_\Lambda$ are said to be l-past equivalent if $\{\mu \in \Lambda^l | \mu x \in X_\Lambda\} = \{\nu \in \Lambda^l | \nu y \in X_\Lambda\}$. Let $F_i^l, i = 1, 2, \dots, m(l)$ be the set of all l-past equivalence classes of X_Λ. We define two rectangular $m(l) \times m(l+1)$ matrices $I_{l,l+1}^\Lambda, \mathcal{M}_{l,l+1}^\Lambda$ with entries in $\{0,1\}$ and entries in $\mathfrak{S}_\Sigma$ respectively as in the following way. For $i = 1, 2, \dots, m(l), j = 1, 2, \dots, m(l+1)$, the (i,j)-component $I_{l,l+1}^\Lambda(i,j)$ of $I_{l,l+1}^\Lambda$ is one if F_i^l contains F_j^{l+1} otherwise zero. Let $a_1, \dots, a_n$ be the set of all symbols in Σ for which $a_k x \in F_i^l$ for some $x \in F_j^{l+1}$. We then define the (i,j)-component of the matrix $\mathcal{M}_{l,l+1}^\Lambda(i,j)$ as $\mathcal{M}_{l,l+1}^\Lambda(i,j) = a_1 + \cdots + a_n$: the formal sum of $a_1, \dots, a_n$. We can show that the pair $(\mathcal{M}^\Lambda, I^\Lambda)$ becomes a symbolic matrix system. We call it the *canonical* symbolic matrix system for Λ. We denote its λ-graph system by $\mathfrak{L}^\Lambda = (V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$ and call it the *canonical* λ-graph system for Λ. The subshift $\Lambda_{(\mathcal{M}^\Lambda, I^\Lambda)}$ associated with $(\mathcal{M}^\Lambda, I^\Lambda)$ coincides with the original subshift Λ.

§3 Strong shift equivalence and shift equivalence

In this section, we define strong shift equivalence and shift equivalence between two symbolic matrix systems. For alphabets C, D, put $C \cdot D = \{cd | c \in C, d \in D\}$. For $x = \sum_j c_j \in \mathfrak{S}_C$ and $y = \sum_k d_k \in \mathfrak{S}_D$, define $xy = \sum_{j,k} c_j d_k \in \mathfrak{S}_{C \cdot D}$.

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ' respectively, where $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices.

Definition. Two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I)$ are said to be *strong shift equivalent in 1-step* and written as $(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}', I')$ if there exist alphabets C, D and specifications $\varphi : \Sigma \to C \cdot D$ and $\phi : \Sigma' \to D \cdot C$ such that for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix $\mathcal{H}_l$ over C and an $m'(l-1) \times m(l)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$I_{l-1,l}\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{H}_l \mathcal{K}_{l+1}, \qquad I'_{l-1,l}\mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{K}_l \mathcal{H}_{l+1}$$

and

$$\mathcal{H}_l I'_{l,l+1} = I_{l-1,l}\mathcal{H}_{l+1}, \qquad \mathcal{K}_l I_{l,l+1} = I'_{l-1,l}\mathcal{K}_{l+1}.$$

Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *strong shift equivalent in N-step* and written as $(\mathcal{M}, I) \underset{N-st}{\approx} (\mathcal{M}', I')$ if there exist symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-st}{\approx} (\mathcal{M}^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots$$
$$\cdots \underset{1-st}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathcal{M}', I').$$

We simply call it a *strong shift equivalence.*

We see the following theorem.

Theorem 3.1 ([Ma6]). *Two subshifts Λ and Λ' are topologically conjugate if and only if their respect canonical symbolic matrix systems $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are strong shift equivalent.*

In the proof given in [Ma6] of the only if part of Theorem 3.1, the bipartite λ-graph system has been introduced and M. Nasu's factorization theorem for topological conjugacy between subshifts into bipartite codes and symbolic conjugacies has been used. We can also prove Theorem 3.1 without using the Nasu's result, by considering the state splitting of λ-graph systems. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. Let $\mathcal{P}$ be a partition of Σ. We put $\Sigma^{[\mathcal{P}]} = \Sigma \times \Sigma/\mathcal{P}$ and $\Sigma_{[\mathcal{P}]} = \Sigma/\mathcal{P} \times \Sigma$ where $\Sigma/\mathcal{P}$ denotes the equivalence classes of Σ by the partition $\mathcal{P}$. Then we can construct the out-split λ-graph system $\mathfrak{L}^{[\mathcal{P}]} = (V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ over $\Sigma^{[\mathcal{P}]}$ and the in-split λ-graph system $\mathfrak{L}_{[\mathcal{P}]} = (V_{[\mathcal{P}]}, E_{[\mathcal{P}]}, \lambda_{[\mathcal{P}]}, \iota_{[\mathcal{P}]})$ over $\Sigma_{[\mathcal{P}]}$ such that

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]}) \quad \text{and} \quad (\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$$

where $(\mathcal{M}, I), (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$ and $(\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$ are the symbolic matrix systems for the λ-graph systems $\mathfrak{L}, \mathfrak{L}^{[\mathcal{P}]}$ and $\mathfrak{L}_{[\mathcal{P}]}$ respectively. Full detail of the construction will appear in [Ma11].

We will state the notion of shift equivalence between two symbolic matrix systems as a generalization of Williams's notion for nonnegative matrices and Boyle-Krieger's notion for symbolic matrices. Let $(\mathcal{M}, I), (\mathcal{M}', I')$ be two symbolic matrix systems over alphabets Σ, Σ' respectively. For $N \in \mathbb{N}$, we put $(\Sigma)^N = \Sigma \cdots \Sigma, (\Sigma')^N = \Sigma' \cdots \Sigma'$: the N-times products.

Definition. For $N \in \mathbb{N}$, two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I')$ are said to be *shift equivalent of lag N* if there exist alphabets C_N, D_N and specifications

$$\varphi_1 : \Sigma \cdot C_N \to C_N \cdot \Sigma', \qquad \varphi_2 : \Sigma' \cdot D_N \to D_N \cdot \Sigma$$

and

$$\psi_1 : (\Sigma)^N \to C_N \cdot D_N, \qquad \psi_2 : (\Sigma')^N \to D_N \cdot C_N$$

such that for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix $\mathcal{H}_l$ over C_N and an $m'(l) \times m(l+N)$ matrix $\mathcal{K}_l$ over D_N satisfying the following equations:

$$\mathcal{M}_{l,l+1}\mathcal{H}_{l+1} \overset{\varphi_1}{\simeq} \mathcal{H}_l\mathcal{M}'_{l+N,l+N+1}, \qquad \mathcal{M}'_{l,l+1}\mathcal{K}_{l+1} \overset{\varphi_2}{\simeq} \mathcal{K}_l\mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+N}\mathcal{M}_{l+N,l+2N} \overset{\psi_1}{\simeq} \mathcal{H}_l\mathcal{K}_{l+N}, \qquad I'_{l,l+N}\mathcal{M}'_{l+N,l+2N} \overset{\psi_2}{\simeq} \mathcal{K}_l\mathcal{H}_{l+N}$$

and

$$I_{l,l+1}\mathcal{H}_{l+1} = \mathcal{H}_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1}\mathcal{K}_{l+1} = \mathcal{K}_l I_{l+N,l+N+1}.$$

We denote this situation by

$$(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I') \quad \text{or} \quad (\mathcal{H}, \mathcal{K}) : (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$$

and simply call it a *shift equivalence.*

Proposition 3.2. *Strong shift equivalence in N-step implies shift equivalence of lag N.*

§4 Nonnegative matrix systems and dimension groups

A nonnegative matrix system consists of two sequences of rectangular matrices $(A_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$. The matrices $A_{l,l+1}$ have entries in nonnegative integers and the matrices $I_{l,l+1}$ have entries in $\{0,1\}$. They satisfy the following commutation relations

$$I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

We assume that for i there exists j such that the (i,j)-component $I_{l,l+1}(i,j) = 1$ and for j there uniquely exists i such that $I_{l,l+1}(i,j) = 1$. We denote it by (A, I).

Lemma 4.1. *For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the $m(l) \times m(l+1)$ rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1. Then the resulting pair (M, I) becomes a nonnegative matrix system.*

For nonnegative matrix systems we formulate strong shift equivalence and shift equivalence as follows.

Definition. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *strong shift equivalent in 1-step* and written as $(A, I) \underset{1-st}{\approx} (A', I')$ if for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix H_l with entries in nonnegative integers and an $m'(l-1) \times m(l)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$I_{l-1,l}A_{l,l+1} = H_l K_{l+1}, \qquad I'_{l-1,l}A'_{l,l+1} = K_l H_{l+1}$$

and

$$H_l I'_{l,l+1} = I_{l-1,l}H_{l+1}, \qquad K_l I_{l,l+1} = I'_{l-1,l}K_{l+1}.$$

Two nonnegative matrix systems (A, I) and (A', I') are said to be *strong shift equivalent in N-step* $(A, I) \underset{N-st}{\approx} (A', I')$ if there exist nonnegative matrix systems $(A^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$(A, I) \underset{1-st}{\approx} (A^{(1)}, I^{(1)}) \underset{1-st}{\approx} (A^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots$$
$$\cdots \underset{1-st}{\approx} (A^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (A', I').$$

We simply call it a *strong shift equivalence.*

For a nonnegative matrix system (A, I), we set the $m(l) \times m(l+k)$ matrices:

$$I_{l,l+k} = I_{l,l+1} \cdot I_{l+1,l+2} \cdots I_{l+k-1,l+k},$$
$$A_{l,l+k} = A_{l,l+1} \cdot A_{l+1,l+2} \cdots A_{l+k-1,l+k}.$$

Definition. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *shift equivalent of lag* N if for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix H_l with entries in nonnegative integers and an $m'(l) \times m(l+N)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$A_{l,l+1}H_{l+1} = H_l A'_{l+N,l+N+1}, \qquad A'_{l,l+1}K_{l+1} = K_l A_{l+N,l+N+1},$$

$$H_l K_{l+N} = I_{l,l+N}A_{l+N,l+2N}, \qquad K_l H_{l+N} = I'_{l,l+N}A'_{l+N,l+2N}$$

and

$$I_{l,l+1}H_{l+1} = H_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1}K_{l+1} = K_l I_{l+N,l+N+1}.$$

We write this situation as

$$(A, I) \underset{lagN}{\sim} (A', I') \quad \text{or} \quad (H, K) : (A, I) \underset{lagN}{\sim} (A', I')$$

and simply call it a *shift equivalence.*

Proposition 4.2. *If two symbolic matrix systems are strong shift equivalent in N-step (resp. shift equivalent of lag N), then the associated nonnegative matrix systems are strong shift equivalent in N-step (resp. shift equivalent of lag N).*

We describe the matrix relations appearing in the formulations of strong shift equivalence and shift equivalence between nonnegative matrix systems in terms of certain single homomorphisms between inductive limits of associated abelian groups. For a nonnegative matrix system (A, I), the transpose $I^t_{l,l+1}$ of the matrix $I_{l,l+1}$ naturally induces an ordered homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$, where the positive cone $\mathbb{Z}^{m(l)}_+$ of the group $\mathbb{Z}^{m(l)}$ is defined by

$$\mathbb{Z}^{m(l)}_+ = \{(n_1, n_2, \dots, n_{m(l)}) \in \mathbb{Z}^{m(l)} | n_i \geq 0, i = 1, 2 \dots m(l)\}.$$

We put the inductive limits:

$$\mathbb{Z}_{I^t} = \varinjlim_{l} \{I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\},$$

$$\mathbb{Z}^+_{I^t} = \varinjlim_{l} \{I^t_{l,l+1} : \mathbb{Z}^{m(l)}_+ \to \mathbb{Z}^{m(l+1)}_+\}.$$

The canonical homomorphism $\iota_l : \mathbb{Z}^{m(l)} \to \mathbb{Z}_{I^t}$ is injective. By the relation: $I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}$, the sequence of the transposed matrices $A^t_{l,l+1}, l \in \mathbb{N}$ of the matrices $A_{l,l+1}, l \in \mathbb{N}$ yields an endomorphism of the ordered group $\mathbb{Z}_{I^t}$. We write it as $\lambda_{(A,I)}$. For nonnegative matrix systems $(A, I), (A', I')$ and $L \in \mathbb{N}$, a homomorphism ξ from the group $\mathbb{Z}_{I^t}$ to the group $\mathbb{Z}_{I'^t}$ is said to be *finite homomorphism of lag L* if it satisfies the condition

$$\xi(\mathbb{Z}^{m(l)}) \subset \mathbb{Z}^{m'(l+L)} \qquad \text{for all } l \in \mathbb{N}$$

where $\mathbb{Z}^{m(l)}$ and $\mathbb{Z}^{m'(l)}$ are naturally imbedded into $\mathbb{Z}_{I^t}$ and $\mathbb{Z}_{I'^t}$ respectively.

Lemma 4.3. *Two nonnegative matrix systems (A, I) and (A', I') are shift equivalent of lag N if and only if there exist order preserving finite homomorphisms of lag N: $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ and $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ such that*

$$\lambda_{(A',I')} \circ \xi = \xi \circ \lambda_{(A,I)}, \qquad \lambda_{(A,I)} \circ \eta = \eta \circ \lambda_{(A',I')}$$

and

$$\eta \circ \xi = \lambda^N_{(A,I)}, \qquad \xi \circ \eta = \lambda^N_{(A',I')}.$$

In particular, (A, I) and (A', I') are strong shift equivalent in 1-step if and only if there exist order preserving finite homomorphisms of lag 1: $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ and $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ such that

$$\eta \circ \xi = \lambda_{(A,I)}, \qquad \xi \circ \eta = \lambda_{(A',I')}.$$

For nonnegative matrices, W. Krieger in [Kr2],[Kr3] showed that shift equivalence relation is the complete relation that defines the same dimension triples. We next formulate dimension groups for nonnegative matrix systems. Let (A, I) be a nonnegative matrix system. We set $\mathbb{Z}_{I^t}(k) = \mathbb{Z}_{I^t}$ and $\mathbb{Z}^+_{I^t}(k) = \mathbb{Z}^+_{I^t}$ for $k \in \mathbb{N}$. We define an abelian group and its positive cone by the following inductive limits:

$$\Delta_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)\},$$
$$\Delta^+_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}^+_{I^t}(k) \to \mathbb{Z}^+_{I^t}(k+1)\}.$$

The ordered group $(\Delta_{(A,I)}, \Delta^+_{(A,I)})$ is called the *dimension group for* (A, I). The map $\delta_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)$ defined by $\delta_{(A,I)}([X, k]) = ([X, k+1])$ for $X \in \mathbb{Z}_{I^t}$ yields an automorphism on $\Delta_{(A,I)}$ that preserves the positive cone $\Delta^+_{(A,I)}$. We also denote it by $\delta_{(A,I)}$ and call it the *dimension automorphism.* We call the triple $(\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)})$ the *dimension triple for* (A, I) and the pair $(\Delta_{(A,I)}, \delta_{(A,I)})$ the *dimension pair for* (A, I).

Proposition 4.4. *If two nonnegative matrix systems are shift equivalent, their dimension triples are isomorphic.*

§5 K-groups, Bowen-Franks groups and flow equivalence

Let (A, I) be a nonnegative matrix system. For $l \in \mathbb{N}$, we set the abelian groups

$$K^l_0(A, I) = \mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)},$$
$$K^l_1(A, I) = \operatorname{Ker}(I^t_{l,l+1} - A^t_{l,l+1}) \text{ in } \mathbb{Z}^{m(l)}.$$

Then the map $I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}$ naturally induces homomorphisms between the groups:

$$i^l_* : K^l_*(A, I) \to K^{l+1}_*(A, I) \qquad \text{for} \quad * = 0, 1.$$

Definition. The K-groups for (A, I) are defined as the following inductive limits of the abelian groups:

$$K_0(A,I) = \lim_{\substack{\longrightarrow \\ l}}\{i_0^l : K_0^l(A,I) \to K_0^{l+1}(A,I)\},$$

$$K_1(A,I) = \lim_{\substack{\longrightarrow \\ l}}\{i_1^l : K_1^l(A,I) \to K_1^{l+1}(A,I)\}.$$

The groups $K_*(A, I)$ are also represented as in the following way

Proposition 5.1.

(i) $K_0(A,I) = \mathbb{Z}_{I^t}/(\mathrm{id} - \lambda_{(A,I)})\mathbb{Z}_{I^t}$,
(ii) $K_1(A,I) = \mathrm{Ker}(\mathrm{id} - \lambda_{(A,I)})$ *in* $\mathbb{Z}_{I^t}$.

As the automorphism $\delta_{(A,I)}$ is given by $\lambda_{(A,I)} = \{A^t_{l,l+1}\}$ on $\Delta_{(A,I)}$, we have

Proposition 5.2.

(i) $K_0(A,I) = \Delta_{(A,I)}/(\mathrm{id} - \delta_{(A,I)})\Delta_{(A,I)}$,
(ii) $K_1(A,I) = \mathrm{Ker}(\mathrm{id} - \delta_{(A,I)})$ *in* $\Delta_{(A,I)}$.

Set the abelian group

$$\mathbb{Z}_I = \lim_{\substack{\longleftarrow \\ l}}\{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$$

the projective limit of the system: $I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}, l \in \mathbb{N}$. The sequence $A_{l,l+1}, l \in \mathbb{N}$ naturally acts on $\mathbb{Z}_I$ as an endomorphism that we denote by A. The identity on $\mathbb{Z}_I$ is denoted by I.
Definition. For a nonnegative matrix system (A, I),

$$BF^0(A,I) = \mathbb{Z}_I/(I - A)\mathbb{Z}_I, \qquad BF^1(A,I) = \mathrm{Ker}(I - A) \text{ in } \mathbb{Z}_I.$$

We call $BF^i(A,I), i = 0,1$ the Bowen-Franks groups for (A, I).

Theorem 5.3. *The K-groups and the Bowen-Franks groups are invariant under shift equivalence of nonnegative matrix systems. Hence the K-groups and the Bowen-Franks groups of the canonical symbolic matrix systems for subshifts are invariant under topological conjugacy of the subshifts.*

The following formulation of the universal coefficient type theorem comes from the Universal Coefficient Theorem for K-theory of C^*-algebras (cf.[Bro],[Rs]). It says that the Bowen-Franks groups are determined by the K-groups.

Theorem 5.4 ([Ma6]).

(i) *There exists a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}) \xrightarrow{\delta} BF^0(A,I) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(A,I),\mathbb{Z}) \longrightarrow 0$$

that splits unnaturally.

(ii)
$$BF^1(A,I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}).$$

Parry-Sullivan showed that the flow equivalence relation on homeomorphisms of Cantor sets is generated by topological conjugacies and operations called expansions ([PS]). For the case of topological Markov shifts, they also gave a description of the expansions in terms of a matrix operation. By using their result, Bowen-Franks in [BF] proved that for an $n \times n$ nonnegative matrix A the groups $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ and $\mathrm{Ker}(1-A)$ are invariant under flow equivalence of the topological Markov shifts Λ_A for the matrix A. We can generalize the Parry-Sullivan's argument and the Bowen-Franks's proof to the canonical nonnegative matrix systems for subshifts.

Theorem 5.5 ([Ma8]). *The K-groups and Bowen-Franks groups of the canonical nonnegative matrix systems for subshifts are invariant under flow equivalence of the subshifts.*

This result first has been shown by using a C^*-algebra technique under some conditions on subshifts in [Ma3] (cf.[Ma7]). For the case of topological Markov shifts, Cuntz-Krieger [CK] and Cuntz [C3] had discussed their flow equivalence by C^*-algebra approach and obtained the corresponding result to the above theorem (cf. [H], [H2]).

§6 Spectrum

We fix a nonnegative matrix system (A,I). A sequence $\{v^l\}_{l\in\mathbb{N}}$ of vectors $v^l = (v^l_1,\dots,v^l_{m(l)}) \in \mathbb{C}^{m(l)}, l \in \mathbb{N}$ is called an I-compatible vector if it satisfies the conditions:

$$v^l = I_{l,l+1}v^{l+1} \qquad \text{for all} \quad l \in \mathbb{N}.$$

An I-compatible vector $\{v^l\}_{l\in\mathbb{N}}$ is said to be nonzero if v^l is a nonzero vector for some l. If $v^l_i \geq 0$ for all $i = 1,\dots,m(l)$ and $l \in \mathbb{N}$, the sequence $\{v^l\}_{l\in\mathbb{N}}$ of vectors is said to be nonnegative. If there exists a number M such that $\sum_{i=1}^{m(l)} |v^l_i| \leq M$ for all $l \in \mathbb{N}$, $\{v^l\}_{l\in\mathbb{N}}$ is said to be bounded.

Definition. For a complex number β, a nonzero I-compatible vector $\{v^l\}$ is called an *eigenvector* of (A, I) for *eigenvalue* β if it satisfies the conditions:

$$A_{l,l+1} v^{l+1} = \beta v^l \qquad \text{for all} \quad l \in \mathbb{N}.$$

An eigenvalue β is said to be bounded if it is an eigenvalue for a bounded eigenvector. Let $Sp^{\times}(A, I)$ be the set of all nonzero eigenvalues of (A, I) and $Sp_b^{\times}(A, I)$ the set of all nonzero bounded eigenvalues of (A, I). We call them the *nonzero spectrum* of (A, I) and the *nonzero bounded spectrum* of (A, I) respectively.

Proposition 6.1. *The nonzero spectrum and the nonzero bounded spectrum are invariant under shift equivalence of nonnegative matrix systems.*

We denote by $\mathfrak{B}_I$ the set of all bounded I-compatible vectors. It is a complex Banach space with norm $\|\cdot\|_1$ where $\|v\|_1 = \sup_l \Sigma_{i=1}^{m(l)} |v_i^l|$ for $v = \{v^l\}_{l \in \mathbb{N}}, v^l = (v_i^l)_{i=1,\cdots,m(l)}$. The sequence $A_{l,l+1}, l \in \mathbb{N}$ of matrices gives rise to a bounded linear operator on the Banach space $\mathfrak{B}_I$. We denote it by L_A.

Proposition 6.2. *A complex number β belongs to $Sp_b(A, I)$ if and only if it satisfies $L_A v = \beta v$ for some nonzero $v \in \mathfrak{B}_I$. In particular, the spectral radius of the operator L_A on $\mathfrak{B}_I$ belongs to $Sp_b^{\times}(A, I)$.*

We say a symbolic matrix system $(\mathcal{M}, I)$ to be left resolving if a symbol appearing in $\mathcal{M}_{l,l+1}(i, j)$ can not appear in $\mathcal{M}_{l,l+1}(i', j)$ for other $i' \neq i$. A canonical symbolic matrix system is left resolving. The following proposition states a relation between spectrum and the topological entropy of subshift ([MWY], cf.[EFW]).

Proposition 6.3. *Let $(\mathcal{M}, I)$ be a left resolving symbolic matrix system and (M, I) its associated nonnegative matrix system. For any $\beta \in Sp_b(M, I)$, we have the inequalities:*

$$\log |\beta| \leq \log r_M \leq h_{\text{top}}(\Lambda_{(\mathcal{M},I)})$$

where r_M is the spectral radius of the operator L_M on $\mathfrak{B}_I$ and $\Lambda_{(\mathcal{M},I)}$ is the associated subshift with $(\mathcal{M}, I)$.

§7 Examples

Let M be an $n \times n$ nonnegative matrix . Put for each $l \in \mathbb{N}$

$$A_{l,l+1} = M, \qquad I_{l,l+1} = \text{ the } n \times n \text{ identity matrix.}$$

Then (A, I) is a nonnegative matrix system. We know

$$K_0(A,I) = \mathbb{Z}^n/(1-M^t)\mathbb{Z}^n, \qquad K_1(A,I) = \mathrm{Ker}(1-M^t) \text{ in } \mathbb{Z}^n,$$
$$BF^0(A,I) = \mathbb{Z}^n/(1-M)\mathbb{Z}^n, \qquad BF^1(A,I) = \mathrm{Ker}(1-M) \text{ in } \mathbb{Z}^n.$$

Hence we have

$$\begin{aligned} K_0(A,I) \cong & BF^0(A,I) \\ = & BF(M) : \text{the original Bowen-Franks group for } M, \\ K_1(A,I) \cong & BF^1(A,I) = \text{ the torsion-free part of } BF(M). \end{aligned}$$

Note that for a general nonnegative matrix systems (A, I), $BF^1(A, I)$ is not necessarily the torsion-free part of $BF^0(A, I)$ as in the following examples.

We will next present examples of the groups K_*, BF^* for canonical nonnegative matrix system of nonsofic subshifts (cf. [KMW]). Let Z be the subshift over $\{1, 2, 3\}$ whose forbidden words are $\{32^m 1^k 3 | m \neq k\}$ where the word 32^m1^k3 means $3\underbrace{2\cdots 2}_{m \text{ times}}\underbrace{1\cdots 1}_{k \text{ times}}3$. Let D be the Dyck shift over brackets $(,), [,]$ whose forbidden words consist of words that do not obey the standard bracket rules (cf. [AU], [Kr]). We denote by $(A^Z, I^Z), (A^D, I^D)$ and $(A^{D\times[n]}, I^{D\times[n]})$ the canonical nonnegative matrix systems of the subshifts Z, D and the product subshift between D and the full n-shift $[n]$ respectively.

Proposition 7.1 ([Ma5], [Ma9]).

(i)

$$\begin{aligned} K_0(A^Z, I^Z) = & BF^1(A^Z, I^Z) = \mathbb{Z}, \\ K_1(A^Z, I^Z) = & BF^0(A^Z, I^Z) = 0. \end{aligned}$$

(ii)

$$K_0(A^D, I^D) = \sum^{\infty} \mathbb{Z}, \qquad BF^1(A^D, I^D) = \prod^{\infty} \mathbb{Z},$$
$$K_1(A^D, I^D) = BF^0(A^D, I^D) = 0.$$

(iii)

$$
\begin{aligned}
K_0(A^{D\times[n]}, I^{D\times[n]}) &\cong \mathbb{Z}[\frac{1}{n}]^{\infty}, \quad K_1(A^{D\times[n]}, I^{D\times[n]}) \cong 0,\\
BF^0(A^{D\times[n]}, I^{D\times[n]}) &\cong \prod_{\mathbb{N}} (\underset{\substack{\longleftarrow \\ i}}{\lim} \mathbb{Z}/n^i\mathbb{Z})/\mathbb{Z}\\
&\cong \prod_{\mathbb{N}} \frac{n-\textit{adic infinite polynomials}}{n-\textit{adic finite polynomials}},\\
BF^1(A^{D\times[n]}, I^{D\times[n]}) &\cong 0
\end{aligned}
$$

where $\prod_{\mathbb{N}}(\underset{\substack{\longleftarrow \\ i}}{\lim}\mathbb{Z}/n^i\mathbb{Z})/\mathbb{Z}$ *is the countable infinite product of the quotient group by* $\mathbb{Z}$ *of the natural projective limit:* $\mathbb{Z}/n\mathbb{Z} \leftarrow \mathbb{Z}/n^2\mathbb{Z} \leftarrow \cdots$.

Corollary 7.2.

(i) *The subshift* Z *is not flow equivalent to any of the product subshifts* $D \times [n], n = 1, 2, \cdots$.
(ii) $D \times [n]$ *is not flow equivalent to* $D \times [m]$ *for* $n \neq m$.

§8 Connection to C^*-algebra K-theory

The author in [Ma] (cf.[CaM],[Ma10]) has constructed the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ as a generalization of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with topological Markov shift Λ_A for matrix A with entries in $\{0,1\}$ ([CK]). The C^*-algebra $\mathcal{O}_\Lambda$ has a canonical action of the one dimensional torus group, called gauge action and written as α_Λ. The fixed point algebra $\mathcal{F}_\Lambda$ of $\mathcal{O}_\Lambda$ under α_Λ is an AF-algebra which is stably isomorphic to the crossed product $\mathcal{O}_\Lambda \times_{\alpha_\Lambda} \mathbb{T}$ ([Ma2]).

Proposition 8.1 ([Ma7], [Ma12], [CK]). *If two subshifts* Λ *and* Λ' *are topologically conjugate, we have*

$$(\mathcal{O}_\Lambda \otimes \mathcal{K}, \alpha_\Lambda \otimes \mathrm{id}) \cong (\mathcal{O}_{\Lambda'} \otimes \mathcal{K}, \alpha_{\Lambda'} \otimes \mathrm{id})$$

where $\mathcal{K}$ *is the* C^**-algebra of all compact operators on separable infinite dimensional Hilbert space.*

Let (M, I) be the canonical nonnegative matrix system for the subshift Λ. The invariants mentioned above are described in terms of the K-theoretic objects for the C^*-algebras as in the following way, where if Λ is a topological Markov shift Λ_A the corresponding results have been seen in [C3], [CK].

Theorem 8.2.

$$(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) = (K_0(\mathcal{F}_\Lambda), K_0(\mathcal{F}_\Lambda)_+, \hat{\alpha_\Lambda}_*),$$
$$K_i(M,I) = K_i(\mathcal{O}_\Lambda), \qquad i = 0,1,$$
$$BF^i(M,I) = \mathrm{Ext}^{i+1}(\mathcal{O}_\Lambda), \qquad i = 0,1$$

where $\hat{\alpha_\Lambda}$ *denotes the dual action of* α_Λ *and* $\mathrm{Ext}^1(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda)$, $\mathrm{Ext}^0(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R}))$.

The normalized nonnegative eigenvectors of (M,I) exactly correspond to the KMS-states for α_Λ on the C^*-algebra $\mathcal{O}_\Lambda$. Hence the set of all bounded spectrums with nonnegative eigenvectors are the set of all inverse temperatures for the admitted KMS states ([MWY],cf.[EFW]).

As the K-groups and the Ext-groups for C^*-algebras are stably isomorphic invariant, it is possible to know that the dimension triples, the K-groups and the Bowen-Franks groups for the canonical nonnegative matrix systems for subshifts are topological conjugacy invariants of the subshifts by using Proposition 8.1 and Theorem 8.2 under some conditions on subshifts.

In [Ma10], as a generalization of the C^*-algebras associated with subshifts, construction of C^*-algebras from symbolic matrix systems are introduced.

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Department of Mathematical Sciences
Yokohama City University
Seto, Kanazawa-ku, Yokohama 236-0027
JAPAN
`kengo@yokohama-cu.ac.jp`