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Braiding and nets of factors on the circle

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Abstract.

We review various properties of braiding in subfactor theory and their connection to nets of factors on S^1 particularly.

§1. Introduction

The notion of braiding has recently caught much attention in theory of quantum groups, 3-dimensional topological quantum field theory, and conformal field theory. Here we review the current status of results related to braiding in subfactor theory. We particularly focus on nets of factors on S^1, or chiral conformal field theories on S^1 here.

§2. Braiding in subfactor theory

Braiding plays an important role in subfactor theory. Rehren's early work [26] sets a fundamental base in the theory of braiding in the setting of subfactors and algebraic quantum field theory. He defined the notion of braiding and its non-degeneracy for a system of endomorphisms of a factor and showed that we have a unitary representation of $SL(2, \mathbf{Z})$ if and only if a braiding on a finite system of irreducible endomorphisms is non-degenerate.

In subfactor theory, we work on a certain algebraic system which is closed under algebraic operations such as "tensor product" and "conjugation". In an axiomatic approach, our "object" is just something satisfying certain set of axioms and one can study algebraic systems of such objects independently from operator algebras, but we are interested in operator algebraic viewpoints here. Then an object we study in such a theory is an M-N bimodule or a $*$-homomorphism from N into M where M and N are appropriate von Neumann algebras, usually factors of type II_1 or type III. Considering bimodules over factors of type II_1

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and $*$-homomorphisms form a factor of type III into another are essentially the same from a viewpoint of algebraic/combinatorial structures, but in this paper we deal with type III factors in connection to algebraic quantum field theory.

Let N be a factor of type III and $\Delta \subset \mathrm{End}(N)$ a finite system of endomorphisms of N in the following sense.

1. Each $\lambda \in \Delta$ is an irreducible endomorphism of N and has a finite statistical dimension.
2. Endomorphisms in Δ are mutually inequivalent.
3. The identity morphism is in Δ.
4. For any $\lambda \in \Delta$, we have a conjugate morphism $\bar\lambda$ in Δ.
5. For any $\lambda, \mu \in \Delta$, we have non-negative integers $N^{\nu}_{\lambda,\mu}$ satisfying $[\lambda][\mu] = \sum_{\nu\in\Delta} N^{\nu}_{\lambda,\mu}[\nu]$, where $[\lambda]$ denotes the unitary equivalence class of λ which is also called a sector.

A system of endomorphism naturally gives a fusion rule algebra with composition of endomorphisms as its multiplication, but there is no reason this multiplication is commutative (up to inner automorphisms) and it is very easy to construct a non-commutative example from an action of a finite non-commutative group, for example. But here we are interested in the commutative case.

When the composition of the endomorphisms in the system is commutative up to inner automorphism of N, a braiding, roughly speaking, means a "compatible choice" of such unitary intertwiners in each space $\mathrm{Hom}(\lambda\mu, \mu\lambda)$, $\lambda, \mu \in \Delta$. The following gives the precise definition of a braiding on a system of endomorphisms. (Even when such a commutative system is given, we do not have existence nor uniqueness of a braiding in general.)

Definition 2.1. We say that a system Δ of endomorphisms of N has a *braiding* if for any pair $\lambda, \mu \in \Delta$ there is a unitary operator $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$ satisfying the following properties.

1. We have $\varepsilon(\mathrm{id}_N, \mu) = \varepsilon(\lambda, \mathrm{id}_N) = 1$, for any $\lambda, \mu \in \Delta$.
2. Whenever $t \in \mathrm{Hom}(\lambda, \mu\nu)$ we have

$$\begin{aligned} \rho(t)\varepsilon(\lambda,\rho) &= \varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho))t, \\ t\varepsilon(\rho,\lambda) &= \mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu)\rho(t), \\ \rho(t)^*\varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho)) &= \varepsilon(\lambda,\rho)t^*, \\ t^*\mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu) &= \varepsilon(\rho,\lambda)\rho(t)^*, \end{aligned}$$

   for any $\lambda, \mu, \nu \in \Delta$.

The unitaries $\varepsilon(\lambda, \mu)$ are called *braiding operators*. We sometimes write ε^+ for ε with convention $\varepsilon^-(\lambda, \mu) = (\varepsilon(\mu, \lambda))^*$ for the opposite braiding. The following definition of non-degeneracy of a braiding means that ε^+ and ε^- are "really different". This notion is quite important for our study as well as study of topological invariants, since a braiding corresponds to a crossing of a planar picture of a link. (If overcrossing and undercrossing are not really distinguished, one can easily imagine that such a topological study is rather limited.)

Definition 2.2. We say that a braiding ε on a system Δ of endomorphisms of N is *non-degenerate*, if the equalities $\varepsilon^+(\lambda,\mu)=\varepsilon^-(\lambda,\mu)$ for all endomorphisms $\mu\in\Delta$ imply $\lambda=\mathrm{id}_N$.

If we have a braiding on a finite system Δ, we can define S- and T-matrices whose sizes are the number of endomorphisms in Δ, as in [26]. The above non-degeneracy is equivalent to unitarity of the S-matrix as proved in [26], and if it is non-degenerate, the S- and T-matrices give a unitary representation of $SL(2,\mathbf{Z})$.

The above setting is for endomorphisms of a single operator algebra N. We now discuss subfactors $N\subset M$. Suppose we start with an arbitrary subfactor $N\subset M$ of type III with finite index. Let $\iota: N\to M$ be the embedding map and $\bar\iota: M\to N$ be its conjugate morphism. We choose sets of morphisms ${}_N\mathcal{X}_N\subset \mathrm{Mor}(N,N)$, ${}_N\mathcal{X}_M\subset \mathrm{Mor}(M,N)$, ${}_M\mathcal{X}_N\subset \mathrm{Mor}(N,M)$ and ${}_M\mathcal{X}_M\subset \mathrm{Mor}(M,M)$ consisting of representative morphisms of irreducible subsectors of sectors of the form $[\bar\iota\iota\cdots\bar\iota\iota]$, $[\bar\iota\iota\cdots\bar\iota]$, $[\iota\cdots\bar\iota\iota]$ and $[\iota\bar\iota\cdots\iota\bar\iota]$ respectively. (We may and do choose $\mathrm{id}_M,\mathrm{id}_N$ in ${}_N\mathcal{X}_N, {}_M\mathcal{X}_M$ as the endomorphisms representing the trivial sectors.) Then ${}_N\mathcal{X}_N$ and ${}_M\mathcal{X}_M$ are systems of endomorphisms of N and M, respectively, in the above sense. We also assume that ${}_N\mathcal{X}_N$ is finite. This automatically implies that the subfactor $N\subset M$ is of finite depth. If ${}_N\mathcal{X}_N$ is braided in the above sense, we say that the subfactor $N\subset M$ is braided. (Note that this is not equivalent to the condition that ${}_M\mathcal{X}_M$ is braided.) More generally, we also consider a finite system of endomorphism containing ${}_N\mathcal{X}_N$ strictly as a subsystem, and such an extension is important in many aspects, but we do not care this matter very much in this article. Even when a subfactor $N\subset M$ is of type II, we can consider a subfactor $N\otimes R\subset M\otimes R$ for any type III factor R and this tensoring does not change any abstract structure of bimodules arising from the subfactor in which we are interested, so if the resulting subfactor $N\otimes R\subset M\otimes R$ is braided, we also say that $N\subset M$ is braided.

If we arbitrarily construct a subfactor, it is highly unlikely that it is braided. However, natural constructions of a braiding are well-known in theory of quantum groups and conformal field theory. We also have natural appearance of braided subfactors in theory of subfactors as follows.

1. Ocneanu's asymptotic inclusions in [22, 23].
2. Longo-Rehren subfactors in [18].
3. Goodman-de la Harpe-Jones subfactors in [11, Sect. 4.5].
4. Wassermann's loop group construction in [29].

The first and second construction give a new subfactor from a given one, and from a categorical viewpoint, they are identified as in [19]. They are very general constructions to produce a braiding from an arbitrary finite system of endomorphisms. In this sense, these constructions can be regarded as an analogue of the quantum double construction [7] in subfactor theory. (See [20, 21] for a more precise interpretation as a quantum double construction.) Both of these are special cases of Popa's construction of symmetric enveloping inclusion [25]. For the third, we need results from conformal field theory or quantum group theory in order to show that the system of N-N bimodules is indeed braided. For the fourth construction, we get more interesting examples in connection to conformal inclusions as in [30, 31, 2]. The non-degeneracy of the resulting braiding was claimed for (1) in [23] and proved for (2) in [12]. (Strictly speaking, we need connectedness of the fusion graph as in [9, Theorem 12.29]. Otherwise, we need to extend the system of endomorphisms in order to get the non-degeneracy. See [12] for more on this matter.) For (3), if we just consider the usual system of N-N bimodules, then the braiding on it is possibly degenerate, since the N-N bimodules correspond to the even vertices of the Dynkin diagram A_n. We need to extend the system of bimodules so that we have N-N bimodules corresponding to the odd vertices of A_n. Then the braiding there is non-degenerate.) For (4), non-degeneracy of the braiding is proved in [29].

If we have a non-degenerate braiding on a finite system of endomorphism, we can produce an invariant of colored links up to regular isotopy and a 3-dimensional topological quantum field theory of Reshetikhin-Turaev type [27]. See [28] for more details on topological quantum field theory. It has been extensively studied these years.

In subfactor theory, one of the most important applications of braiding is theory of α-induction. This construction was defined by [18] and used systematically in [30, 31]. For further development and unification with Ocneanu's graphical method in [24], see [1, 2, 3, 4, 5, 6]. With this method, one can pass from a braided system to a new system which is not braided in general. Other studies of non-degeneracy of braiding in subfactor theory can be found in [8, 12, 13]. Izumi [12] found that study of the Longo-Rehren subfactors can also be made from a viewpoint of extension (or restriction) of endomorphisms.

§3. Completely rational nets of factors on S^1

Longo [16, 17] has found a deep relation of algebraic quantum field theory to the Jones theory [14] of subfactors. Such a relation was also studied in [10]. Here we explain algebraic quantum field theory on S^1, which is regarded as a compactification of $\mathbf{R}$, and results in [15] in connection to theory of braiding as described above.

We denote by $\mathcal{I}$ the set of non-empty open connected proper subsets of S^1. Such a set is simply called an *interval* here. We study a local irreducible conformal net $\mathcal{A}$ of factors on S^1, which is axiomatized as follows.

For each interval I, we have a factor $\mathcal{A}(I)$ on a fixed Hilbert space H. We also have a strongly continuous unitary representation U on H of the Möbius group $PSU(1,1) = SU(1,1)/\{\pm 1\}$ which acts on S^1 as fractional linear transformations. For an arbitrary set $E \subset S^1$, we define $\mathcal{A}(E)$ to be the von Neumann algebra generated by all the $\mathcal{A}(I)$'s with I contained in E. For $E \subset S^1$, we denote the interior of the complement of E by E'. We then require that they satisfy the following properties. (Though there are slightly different versions of requirements, here we just list a simple set of axioms. Our results in [15] actually hold under a weaker set of assumptions.)

- Isotony: For intervals $I \subset J$, we have $\mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: For disjoint intervals I and J, we have $\mathcal{A}(I) \subset \mathcal{A}(J)'$.
- Irreducibility: The von Neumann algebra generated by all $\mathcal{A}(I)$'s is $B(H)$.
- Covariance: For $g \in PSU(1,1)$ and an interval I, we have

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

- Positive energy: The generator of the rotation subgroup of PSU $(1,1)$ is positive.
- Split property: If $\bar{I}$ and $\bar{J}$ do not intersect for intervals I and J, then $\mathcal{A}(I) \otimes \mathcal{A}(J)$ are naturally isomorphic to $\mathcal{A}(I) \vee \mathcal{A}(J)$.
- Strong additivity: For an interval I and its interior point p, we have $\mathcal{A}(I) = \mathcal{A}(I \setminus \{p\})$.
- Unique existence of vacuum: All the vectors in H fixed by the action of $PSU(1,1)$ are multiples of a fixed non-zero vector Ω.

We then acutally have a stronger form of locality, Haag duality, which says that for an interval I, we have $\mathcal{A}(I') = \mathcal{A}(I)'$. Factors $\mathcal{A}(I)$ are then automatically injective and of type III_1. Important examples of such nets of factors on S^1 have been constructed by A. Wassermann [29] using loop groups of $SU(n)$.

For an arbitrary set $E \subset S^1$, locality implies that $\mathcal{A}(E)$ and $\mathcal{A}(E')$ commute, thus we naturally have an inclusion $\mathcal{A}(E) \subset \mathcal{A}(E')'$. This inclusion can be non-trivial if E is not an interval. We are interested in this inclusion for the case E is a union of two intervals whose closures have no intersection.

A representation π of a net $\mathcal{A}$ on a Hilbert space K is a family $\pi = \{\pi_I\}_{I \subset S^1}$, where π_I is a representation of $\mathcal{A}(I)$ on K and we require that π_J is an extension of π_I for intervals $I \subset J$. A representation π is called *locally normal* if each π_I is normal. Since we deal with only representations on separable Hilbert spaces, the local normality automatically holds. There is also a notion of covariance for such a representation, which is defined as obvious compatibility with a unitary representation of the Möbius group on K, but we do not assume such a property on representations of a net. It turns out that this covariance property automatically holds for representations of a net which we are interested in.

Such a representation of a net is described as a localized transportable endomorphism λ of the quasi-local C^*-algebra as usual in the DHR-framework. See [10] for example. A unitary equivalence class of such representations (or localized endomorphisms) is called a *(superselection) sector* of the net $\mathcal{A}$. For an interval I, such λ gives a sector of $\mathcal{A}(I)$, which is a unitary equivalence class of endomorphisms of $\mathcal{A}(I)$. We are interested in structure of superselection sectors of a net $\mathcal{A}$.

Let E be any union of two intervals on S^1 whose closures have no intersection. Let $\hat{\mathcal{A}}(E) = \mathcal{A}(E')'$ for such E and consider the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. It turns out that if this subfactor has a finite index for some E, then we always have the same finite index for any E. When this finiteness holds, we say that the net $\mathcal{A}$ is *completely rational* and write $\mu_{\mathcal{A}}$ for the index value.

Let $\mathcal{A}$ be a completely rational net of factors on S^1 as above. Let E be a disjoint union of two intervals I, J whose closures have no intersection. Let λ and μ be irreducible endomorphisms of $\mathcal{A}$ localized in I and in J, respectively. Then $\lambda\mu$ restricts to an endomorphism of $\mathcal{A}(E)$. Let γ_E be the canonical endomorphism of $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$ and θ_E its restriction on $\mathcal{A}(E)$. We can prove as in [15] that $\lambda\mu$ restricted on $\mathcal{A}(E)$ is contained in θ_E if and only if λ and μ are mutually conjugate. Moreover, in this case, the multiplicity of $\lambda\mu|_{\mathcal{A}(E)}$ in θ_E is one. Using this, we can prove the following result as in [15], which gives a reason for the terminology "completely rational".

Theorem 3.1. *Let $\mathcal{A}$ be a completely rational net on S^1 as above. Then the net $\mathcal{A}$ is rational in the sense that we have only finitely many irreducible superselection sectors $[\lambda_0], [\lambda_1], \ldots, [\lambda_n]$ with finite dimension, and furthermore, we have $\sum_{i=0}^n d(\lambda_i)^2 = \mu_{\mathcal{A}}$.*

Fix an interval I and regard $[\lambda_0], [\lambda_1], \ldots, [\lambda_n]$ as sectors of $\mathcal{A}(I)$. Then $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ gives a system of endomorphisms of $\mathcal{A}(I)$ in the sense defined above. The Longo-Rehren construction [18] applies to such a system and we have a factor $\mathcal{A}(I) \otimes \mathcal{A}(I)^{\text{opp}} \subset B$. The index of this Longo-Rehren subfactor is equal to $\sum_{i=0}^n d(\lambda_i)^2 = \mu_{\mathcal{A}}$ and this equality suggests some relation between $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ and the Longo-Rehren subfactor. Actually, we have the following result [15], where the symbol "opp" means the opposite algebra.

Theorem 3.2. *The subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the Longo-Rehren subfactor $\mathcal{A}(I) \otimes \mathcal{A}(I)^{\text{opp}} \subset B$.*

Now we discuss a relation of this result to theory of braiding. It is well-known that we naturally have a braiding on the system of endomorphisms $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of $\mathcal{A}(I)$, and the construction of the braiding goes roughly as follows. (See [1, Section 2.2], for example.)

Take endomorphisms λ_j, λ_k localized in an interval I. Choose two intervals I_1, I_2 with empty intersection, and Then there are unitaries U_1 and U_2 such that $\lambda'_j = \text{Ad}(U_1) \circ \lambda_j$ and $\lambda'_k = \text{Ad}(U_2) \circ \lambda_k$ are localized in I_1 and I_2, respectively. Set $\varepsilon(\lambda_j, \lambda_k) = \lambda_k(U_1^*)U_2^*U_1\lambda_j(U_2)$. This unitary does not depend on choices of U_1, U_2, and it depends only on the "order" of I_1 and I_2 on S^1. In this way, we get two unitaries $\varepsilon^{\pm}(\lambda_j, \lambda_k)$ and these give a braiding on the system of endomorphisms $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of a type III factor $\mathcal{A}(I)$.

On one hand, the above theorem says that the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the Longo-Rehren subfactor arising from a braided system of endomorphisms. As mentioned above, the Longo-Rehren construction produces a non-degenerate braiding, but if we have a non-degenerate braiding from the beginning, the Longo-Rehren construction just produces a direct product system of the original braided system and its opposite system as in [23, 8, 12]. So if the original system $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ has a non-degenerate braiding, then the systems of endomorphisms of $\mathcal{A}(E)$ and $\hat{\mathcal{A}}(E)$ are isomorphic for the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. On the other hand, it is trivial from the construction that the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is self-dual. In comparison to the study of the Longo-Rehren subfactors (or asymptotic inclusions) arising from a non-degenerate system as mentioned above, this self-duality suggests that the braiding on the system $\{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is non-degenerate. We have proved in [15] that this is indeed the case.

Theorem 3.3. *The braiding on the system* $\{\lambda_0, \lambda_1, \dots, \lambda_n\}$ *is non-degenerate and thus we have a unitary representation of* $SL(2, \mathbf{Z})$.

As a final remark, we note that it is not very easy to verify the complete rationality since it involves the index computation, but Xu has verified this condition in several cases. In the case of Wassermann's net [29] arising from loop groups of $SU(n)$, Xu [32] computed the index of the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ using a brilliant idea and thus verified the complete rationality. He then also applied the above our results in various other contexts in [33, 34] by verifying the complete rationality.

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