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Non-commutative Markov operators arising from subfactors

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§1. Introduction

It is well-known that there exists a close relationship between subfactor theory and (ordinary or non-commutative) probability theory. Indeed, one may observe it already in V. F. R. Jones' original paper [12], where L^1-estimate of conditional expectations plays an important role in his proof of reducibility of Jones subfactors of index larger than 4. Since then, several authors discussed the relationship between these two fields [1] [2] [8] [9] [10] [15] [16] [17] [18]. Among other notions in probability theory, the most suitable one for subfactors so far is the theory of Poisson boundaries of random walks. It is well-known that the center of the core of a subfactor can be identified with the L^∞-space of the Poisson boundary of some random walk on the principal graph.

In [11], the author obtained a precise description of the relative commutant of the fixed point subalgebra under the infinite tensor product action of the quantum group $SU_q(2)$ on the Powers factor. Indeed, it may be regarded as "the function algebra" over "the Poisson boundary" of a non-commutative Markov operator (synonymously, a unital completely positive operator) on "the group algebra" of $SU_q(2)$.

Following the same philosophy, in this note we provide a general machinery to determine the structure of the (higher) relative commutants of the core inclusions of (not necessarily strongly amenable) subfactors. These relative commutants also may be regarded as "the function algebras" of "the Poisson boundaries" of some non-commutative Markov operators of finite type I von Neumann algebras. As an easy application, we give a new proof, based on a random walk on some ladder-like graph, to the above mentioned fact about Jones inclusions.

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§2. Preliminaries

In this section, we give a quick introduction to two main ingredients of this note: (1) Poisson boundaries for Markov operators (2) a bimodule description of higher relative commutants of subfactors. Our basic reference for the boundary theory of (ordinary) random walks is V.A. Kaimanovich's review article [13]. Here, we give an algebraic description of the Poisson boundaries, and also give their extension to the non-commutative setting. For subfactors, we freely use definitions and notations in D. E. Evans and Y. Kawahigashi's book [5].

2.1. Poisson Boundaries

We start with a simple and classical case. Let $\mathcal{X}$ be a countable set. A Markov operator P on the state space $\mathcal{X}$ is a unital normal positive map from $\ell^\infty(\mathcal{X})$ to itself. For a given Markov operator, the corresponding transition probability $p(s,t)$ from $s \in \mathcal{X}$ to $t \in \mathcal{X}$ is given by

$$P(\delta_t) = \sum_{s\in\mathcal{X}} p(s,t)\delta_s,$$

where δ_s is the characteristic function of the one point set $\{s\}$. A function f is called *harmonic* if the right-hand side of the following makes sense and it is satisfied:

$$f(s) = \sum_{t\in\mathcal{X}} p(s,t)f(t),$$

which is equivalent to $P(f) = f$ for bounded f. We denote by $H^\infty(\mathcal{X}, P)$ the set all bounded harmonic functions.

The Poisson boundary of $(\mathcal{X}, P)$ is, roughly speaking, a measure space (Ω, μ) describing $H^\infty(\mathcal{X}, P)$, as in an analogous manner that the boundary values on the unit circle determines harmonic functions on the unit disc through the classical Poisson integral formula. Though one can find in [13] a decent measure theoretic construction of the Poisson boundary of $(\mathcal{X}, P)$, in this note we adopt the following characterization as a local definition [14, pp. 462], which is more suitable for the non-commutative situation: For every pair $f, g \in H^\infty(\mathcal{X}, P)$, strong limit

$$s - \lim_{n\to\infty} P^n(fg)$$

always exists and harmonic. This introduces a new associative product into $H^\infty(\mathcal{X},\, P)$, and equips it with abelian von Neumann algebra structure. The Poisson boundary is characterized as a measure space (Ω, μ)

such that $L^\infty(\Omega, \mu)$ is isomorphic to the abelian von Neumann algebra $H^\infty(\mathcal{X}, P)$.

Now we consider the notion of "Poisson boundaries" in a more general situation. Let A be a von Neumann algebra and P be a normal unital completely positive map from A to itself. Sometimes, we call P a *non-commutative Markov operator* for an obvious reason. We say that $x \in A$ is *P-harmonic* or *harmonic with respect to P* if x is fixed by P. $H^\infty(A, P)$ denotes the set of P-harmonic elements. Note that $H^\infty(A, P)$ is a weakly closed operator system [4]: namely it is a unital self-adjoint subspace of A.

We show that $H^\infty(A, P)$ has a von Neumann algebra structure as in the classical case, though it is in general non-commutative and no underlying measure theoretic object exists. We fix a free ultrafilter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ and define a norm one projection E_ω from A to $H^\infty(A, P)$ by the weak limit

$$E_\omega(x) = w - \lim_{n \to \omega} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x).$$

Then, we can introduce von Neumann algebra structure into $H^\infty(A, P)$ by using the Choi-Effros product $E_\omega(xy)$ for $x, y \in H^\infty(A, P)$ [4]. The resulting von Neumann algebra may be considered as a non-commutative analogue of the function algebra over "the Poisson boundary" associated with (A, P). Note that the Choi-Effros product $E_\omega(xy)$ for $x, y \in H^\infty(A, P)$ does not depend on ω because every completely positive surjective isometry between two von Neumann algebras is actually an isomorphism.

As in the classical case, a natural and tempting question would be to identify this von Neumann algebra with known one for a given concrete example of P. The goal of this note is to show that some von Neumann algebra naturally appearing in a subfactor problem happens to be "the function algebra" of "the Poisson boundary" of some non-commutative Markov operator, and $H^\infty(A, P)$ with the Choi-Effros product gives a better description of the algebra.

2.2. Core Inclusions

Throughout this note, $N \subset M$ denotes an extremal inclusion of type II_1 factors with a finite Jones index $[M : N]$. Let

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots,$$

be the Jones tower for $N \subset M$. We set $A_n := M' \cap M_n$, $n = 0, 1, 2, \cdots$, and $B_n := N' \cap M_n$, $n = -1, 0, 1, \cdots$. Then, the standard invariant of the inclusion introduced by S. Popa [17] is the following nested commuting squares:

$$\begin{array}{ccccccccc} & & A_0 & \subset & A_1 & \subset & A_2 & \subset & \cdots \\ & & \cap & & \cap & & \cap & & \\ B_{-1} & \subset & B_0 & \subset & B_1 & \subset & B_2 & \subset & \cdots \end{array}$$

We denote by A_∞ and B_∞ the weak closures of $\bigcup_n A_n$ and $\bigcup_n B_n$ respectively in the GNS representations with respect to the natural traces. The inclusion $A_\infty \subset B_\infty$ is called the core of $M \subset M_1$, which is known to be anti-isomorphic to the original one if M is hyperfinite and $N \subset M$ is strongly amenable (See [17] for these terms). However, we focus on the non-strongly amenable case in this note.

As in [5], we identify A_n and B_n with appropriate endomorphism spaces of bimodules M_j; more precisely, we have the following identification:

$$A_{2n} = \mathrm{End}_M(M_n)_M,$$

$$A_{2n+1} = \mathrm{End}_M(M_n)_N,$$

$$B_{2n} = \mathrm{End}_N(M_n)_M,$$

$$B_{2n+1} = \mathrm{End}_N(M_n)_N.$$

These spaces have natural inclusion relations coming from taking tensor products with the basic bimodules ${}_N M_M$ and ${}_M M_N$ from either left or right, which are of course compatible with the inclusion relations of $\{A_n\}_n$ and $\{B_n\}_n$.

Let $\mathcal{G}$ and $\mathcal{H}$ be the principal graphs of $N \subset M$. We denote by $\mathcal{G}^0$ and $\mathcal{H}^0$ the set of vertices of $\mathcal{G}$ and $\mathcal{H}$ respectively. $\mathcal{G}$ and $\mathcal{H}$ are bipartite graphs and we denote by $\mathcal{G}^{\mathrm{even}}$, $\mathcal{G}^{\mathrm{odd}}$, $\mathcal{H}^{\mathrm{even}}$, $\mathcal{H}^{\mathrm{odd}}$ their even and odd vertices respectively. We identify $\mathcal{G}^{\mathrm{even}}$ (respectively $\mathcal{G}^{\mathrm{odd}}$, $\mathcal{H}^{\mathrm{even}}$, $\mathcal{H}^{\mathrm{odd}}$) with the set of irreducible $M-M$ (respectively $M-N$, $N-N$, $N-M$) bimodules contained in ${}_M {M_n}_M$ (respectively ${}_M {M_n}_N$, ${}_N {M_n}_N$, ${}_N {M_n}_M$) for some n. For even (respectively odd) n, we denote by $\mathcal{G}_n^0 \subset \mathcal{G}^0$ the set of even (respectively odd) vertices with distance from the distinguished vertex $*_M = {}_M M_M$ less than or equal to n. Note that each element of $\mathcal{G}_n^0$ is identified with a simple component of A_n.

It is well-known that the centers $Z(A_\infty)$ (respectively $Z(B_\infty)$) of A_∞ (respectively B_∞) can be identified with the L^∞-space of the Poisson boundaries of some random walk on $\mathcal{G}$ (respectively $\mathcal{H}$) [10] [17]. In this note, we give a similar description of the relative commutant $A'_\infty \cap B_\infty$ using a non-commutative Markov operator. While the random walk on $\mathcal{G}$ is determined only by the trace vector [10] [17], the Markov operator describing $A'_\infty \cap B_\infty$ is much more involved. Indeed, it is described in terms of intertwiners of bimodules.

In the rest of this section, we collect notations for bimodules and string algebras that will be used in this note.

Let A, B, and C be II_1 factors. For an $A-B$ bimodules ${}_AX_B$, we defined the statistical dimension of X by

$$d({}_AX_B) = \sqrt{\dim_A X \dim X_B}.$$

For irreducible bimodules ${}_AX_B$, ${}_BY_C$, and ${}_AZ_C$ with finite statistical dimensions, we denote by $\mathcal{H}^Z_{X,Y}$ the space of bimodule maps

$$\mathcal{H}^Z_{X,Y} = \mathrm{Hom}({}_AX \otimes_B Y_C, {}_AZ_C),$$

and by $N^Z_{X,Y}$ the multiplicity of ${}_AZ_C$ in ${}_AX \otimes_B Y_C$. In particular, we set $\beta := d({}_NM_M) = d({}_MM_N)$, and

$$\Gamma_{Z,X} := N^Z_{X,{}_MM_N}$$

for $A = B = M$, $C = N$ and bimodules X and Z associated with the inclusion $N \subset M$. Let $r_X \in \mathcal{H}^A_{X,\overline{X}}$ be the element defined by

$$r_X(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle_A,$$

where $\xi, \eta \in_A X$ are A-bounded elements and $\langle \cdot, \cdot \rangle_A$ is the A-valued inner product. Note that this is, up to constant, the Frobenius dual of the identity map $1_X \in \mathcal{H}^X_{A,X}$. Then, the right hand side Frobenius dual of $\sigma \in \mathcal{H}^Z_{X,Y}$ is expressed as

$$\sqrt{\frac{\dim X_B}{\dim Z_C}}(1_X \otimes r_Y) \cdot (\sigma^* \otimes 1_{\overline{Y}}).$$

For a graph $\mathcal{G}$ and a path ξ on $\mathcal{G}$, $s(\xi)$, $r(\xi)$, and $|\xi|$ denote the source, the range, and the length of ξ respectively. For a vertex $v \in \mathcal{G}^0$ and $n \in \mathbf{N}$, we denote by $\mathrm{Path}^n_v(\mathcal{G})$ the set of paths on $\mathcal{G}$ with source v and length n. We denote by $A^n_v(\mathcal{G})$ the string algebra spanned by the strings (ξ, η) with $\xi, \eta \in \mathrm{Path}^n_v(\mathcal{G})$, $r(\xi) = r(\eta)$.

§3. Main Result

Let $C_\infty := A'_\infty \cap B_\infty$, $C_n := A'_n \cap B_n$, $n = 0, 1, 2, \cdots$. We denote by E_n the trace preserving conditional expectation from B_∞ onto B_n.

Thanks to the commuting square condition, for a given $x \in C_\infty$, $x_n := E_n(x)$ belongs to C_n. The sequence $\{x_n\}_{n=0}^\infty$ converges to x in strong $*$-topology. On the other hand, if $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence satisfying $x_n \in C_n$ and $E_n(x_{n+1}) = x_n$, the sequence converges to some element $x \in C_\infty$ such that $x_n = E_n(x)$. Therefore, all information of $x \in C_\infty$ is encoded in the sequence $\{x_n\}_{n=1}^{\infty}$. Here, a possible difficulty in analyzing this sequence would be that all members of $\{x_n\}_{n=1}^{\infty}$ belong to different algebras C_n, $n = 0, 1, 2, \cdots$. We start with a description of C_n in terms of bimodules. The following lemma is just a translation from an algebra language to a bimodule language:

Lemma 3.1. *With the above notation, we have*

$$C_{2n} \cong \bigoplus_{X \in \mathcal{G}^0_{2n}} \mathrm{End}_N({}_N M \otimes_M X_M)_M,$$

$$C_{2n+1} \cong \bigoplus_{X \in \mathcal{G}^0_{2n+1}} \mathrm{End}_N({}_N M \otimes_M X_N)_N.$$

Proof. Using the string algebra expression of C_n with respect to the inclusions

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset B_n,$$

we can see that every element in C_n has the following form:

$$\sum_{|\sigma_+|=|\sigma_-|=1} \sum_{|\xi|=n} c_{\sigma_+,\sigma_-}(\xi \cdot \sigma_+, \xi \cdot \sigma_-), \quad c_{\sigma_+,\sigma_-} \in \mathbf{C}.$$

This means that we have isomorphisms

$$C_{2n} \cong \bigoplus_{X \in \mathcal{G}^0_{2n}} A^1_X(\mathcal{G}),$$

$$C_{2n+1} \cong \bigoplus_{X \in \mathcal{G}^0_{2n+1}} A^1_X(\mathcal{H}),$$

where $\mathcal{G}^{\mathrm{odd}}$ is identified with $\mathcal{H}^{\mathrm{odd}}$ though the contragredient map in the second equation. Thus, we get the result. Q.E.D.

In view of the above lemma, we set

$$D_X := \mathrm{End}_N({}_N M \otimes_M X_M)_M, \quad X \in \mathcal{G}^{\mathrm{even}},$$

$$D_X := \mathrm{End}_N({}_N M \otimes_M X_N)_N, \quad X \in \mathcal{G}^{\mathrm{odd}},$$

$$D_n := \bigoplus_{X \in \mathcal{G}^0_n} D_X,$$

$$D^{\text{even}} := \bigoplus_{X \in \mathcal{G}^{\text{even}}} D_X,$$

$$D^{\text{odd}} := \bigoplus_{X \in \mathcal{G}^{\text{odd}}} D_X,$$

$$D := D^{\text{even}} \oplus D^{\text{odd}},$$

where the direct sums are understood as von Neumann algebra direct sums. We regard D_n as a subalgebra of D in a natural way, and denote by $\pi_n : D \longrightarrow D_n$ the natural projection. Let $\theta_n : D_n \longrightarrow C_n$ be the isomorphism established in the above lemma. Note that θ_n is *not* compatible with the inclusion relations of $\{D_n\}$ and $\{C_n\}$ (in fact, there exists no inclusion relation between D_n and D_{n+1}).

We introduce a Markov operator P of D. For simplicity, the bimodule ${}_N M_M$ and ${}_M M_N$ will be denote by ρ and $\overline{\rho}$. For $x \in D_X$, $X \in \mathcal{G}^{\text{even}}$, we set

$$P(x) = \frac{d(X)}{\beta} \bigoplus_{Y \in \mathcal{G}^{\text{odd}}} \frac{1}{d(Y)} \sum_{i=1}^{\Gamma_{Y,X}} (1_\rho \otimes \nu_{Y,i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \nu_{Y,i}^*),$$

where $\{\nu_{Y,i}\}_{i=1}^{\Gamma_{Y,X}}$ is an orthonormal basis of $\mathcal{H}_{X,\overline{\rho}}^{Y}$. In a similar way, for $x \in D_X$ and $X \in \mathcal{G}^{\text{odd}}$, we set

$$P(x) = \frac{d(X)}{\beta} \bigoplus_{Y \in \mathcal{G}^{\text{even}}} \frac{1}{d(Y)} \sum_{i=1}^{\Gamma_{X,Y}} (1_\rho \otimes \nu_{Y,i}) \cdot (x \otimes 1_\rho) \cdot (1_\rho \otimes \nu_{Y,i}^*),$$

where $\{\nu_{Y,i}\}_{i=1}^{\Gamma_{X,Y}}$ is an orthonormal basis of $\mathcal{H}_{X,\rho}^{Y}$. It is easy to show that P restricted to D^{even} and D^{odd} are unital normal completely positive maps from one to the other. In fact, this is the right Markov operator that gives C_∞ as "the function algebra" of "the Poisson boundary".

Lemma 3.2. *Let E_n, θ_n, π_n, and P be as above. Then, they satisfy*

$$\theta_{n-1} \cdot \pi_{n-1} \cdot P = E_{n-1} \cdot \theta_n \cdot \pi_n, \quad n = 1, 2, \cdots .$$

Proof. It suffices to show the equality for $x \in D_X$, $X \in \mathcal{G}_n^0$. We may and do further assume that n is even, (the odd case can be treated in a similar way), and x has the form $x = \sigma_+^* \cdot \sigma_-$, where $\sigma_+, \sigma_- \in \mathcal{H}_{\rho,X}^{W}$, for some irreducible ${}_N W_M$. Let $\{\xi\}_i$ be an orthonormal basis of $\mathcal{H}_{Y,\rho}^{X}$. Since $N \subset M$ is extremal, we have

$$\dim {}_M X = \dim X_M = d(X),$$

$$\dim {}_M Y = \frac{d(Y)}{\beta}, \quad \dim Y_N = \beta d(Y).$$

Thus, the right hand side Frobenius dual of ξ_i is given by

$$\widetilde{\xi_i} = \sqrt{\frac{\beta d(Y)}{d(X)}} (1_Y \otimes r_\rho) \cdot (\xi_i^* \otimes 1_{\overline{\rho}}).$$

Therefore, the D_Y-component of $P(x)$ is given by

$$\begin{aligned}
&\frac{d(X)}{\beta d(Y)} \sum_{i=1}^{\Gamma_{Y,X}} (1_\rho \otimes \widetilde{\xi_i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \widetilde{\xi_i}^*) \\
&= \sum_{i=1}^{\Gamma_{Y,X}} (1_{\rho \otimes Y} \otimes r_\rho) \cdot (((1_\rho \otimes \xi_i^*) \cdot \sigma_+^* \cdot \sigma_- \cdot (1_\rho \otimes \xi_i)) \otimes 1_{\overline{\rho}}) \\
&\qquad \cdot (1_{\rho \otimes Y} \otimes r_\rho^*),
\end{aligned}$$

Let ${}_N V_N$ be an irreducible $N-N$ bimodule contained in ${}_N M \otimes_M Y_N$. We choose orthonormal bases $\{\eta_j\}_j$ and $\{\zeta_k\}_k$ of $\mathcal{H}_{\rho,Y}^V$ and $\mathcal{H}_{V,\rho}^W$ respectively. Using the connection and the basis of $\mathrm{Hom}({}_N M \otimes_M Y \otimes_N M_M, {}_N W_N)$ coming from these, we get

$$\begin{aligned}
&(1_\rho \otimes \xi_i^*) \cdot \sigma_+^* \cdot \sigma_- \cdot (1_\rho \otimes \xi_i) \\
&= \sum_{V,j,j',k,k'} \begin{array}{ccc} Y & \stackrel{\xi_i}{\rightarrow} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \underset{\zeta_k}{\rightarrow} & W \end{array} \overline{\begin{array}{ccc} Y & \stackrel{\xi_i}{\rightarrow} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \underset{\zeta_{k'}}{\rightarrow} & W \end{array}} (\eta_j^* \otimes 1_\rho) \cdot \zeta_k^* \cdot \zeta_{k'} \\
&\qquad \cdot (\eta_{j'} \otimes 1_\rho).
\end{aligned}$$

Using the Frobenius reciprocity again, we get

$$\begin{aligned}
&(1_{\rho \otimes Y} \otimes r_\rho) \cdot (((\eta_j^* \otimes 1_\rho) \cdot \zeta_k^* \cdot \zeta_{k'} \cdot (\eta_{j'} \otimes 1_\rho)) \otimes 1_{\overline{\rho}}) \cdot (1_{\rho \otimes Y} \otimes r_\rho^*) \\
&= \eta_j^* \cdot (1_V \otimes r_\rho) \cdot (\zeta_k^* \cdot \zeta_{k'} \otimes 1_{\overline{\rho}}) \cdot (1_V \otimes r_\rho^*) \cdot \eta_{j'} \\
&= \frac{d(W)}{\beta d(V)} \eta_j^* \cdot \widetilde{\zeta_k} \cdot \widetilde{\zeta_{k'}}^* \cdot \eta_{j'} \\
&= \frac{\delta_{k,k'} d(W)}{\beta d(V)} \eta_j^* \cdot \eta_{j'},
\end{aligned}$$

where $\widetilde{\zeta_k}$ is the right hand side Frobenius dual of ζ_k. Thus, the D_Y-component of $P(x)$ is

$$\sum_{V,i,j,j',k} \frac{d(W)}{\beta d(V)} \begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \xrightarrow[\zeta_k]{} & W \end{array} \quad \overline{\begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \xrightarrow[\zeta_k]{} & W \end{array}} \quad \eta_j^* \cdot \eta_{j'}.$$

Now, we compute $E_{n-1} \cdot \theta_n \cdot \pi_n(x)$. The string algebra expression of $\theta_n \cdot \pi_n(x)$ in terms of the inclusions $A_{n-1} \subset A_n \subset B_n$ is

$$\sum_{|\xi|=n} (\xi \cdot \sigma_+, \xi \cdot \sigma_-).$$

The same element can be expressed in terms of $A_{n-1} \subset B_{n-1} \subset B_n$ as

$$\sum_{|\nu|=n-1} \sum_{V,i,j,j',k,k'} \begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \xrightarrow[\zeta_k]{} & W \end{array} \quad \overline{\begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \xrightarrow[\zeta_{k'}]{} & W \end{array}} \quad (\nu \cdot \eta_j \cdot \zeta_k, \nu \cdot \eta_{j'} \cdot \zeta_{k'}).$$

Therefore, we can get the statement from the explicit formula of the conditional expectation from B_n to B_{n-1} in terms of the string algebra [5, Lemma 11.7]. Q.E.D.

Theorem 3.3. *There exists a unital completely positive surjective isometry $\theta_\infty : H^\infty(D, P) \longrightarrow C_\infty$ satisfying*

(1) *For every $x \in H^\infty(D, P)$, $\theta_\infty(x)$ is given by*

$$\theta_\infty(x) = s - \lim_{n \to \infty} \theta_n \cdot \pi_n(x).$$

(2) *For every pair $x, y \in H^\infty(D, P)$, $\{P^n(xy)\}_{n=1}^\infty$ converges to an element in $H^\infty(D, P)$ in strong operator topology, and*

$$\theta_\infty(x)\theta_\infty(y) = \theta_\infty(s - \lim_n P^n(xy)).$$

Except for surjectivity of θ_∞, Theorem 3.3 is a direct consequence of Lemma 3.2 and the non-commutative martingale convergence theorem mentioned at the beginning of this section. To show that θ_∞ is surjective, we need the following Fougel'type estimate as usual:

Lemma 3.4. *For the Markov operator P as above, we have*

$$\lim_{n \to \infty} ||P^{n+2} - P^n|| = 0.$$

In consequence, for every bounded sequence $\{x_n\}_{n=1}^\infty$ in D satisfying $x_n \in D_n$, $\pi_n \cdot P(x_{n+1}) = x_n$, $n = 0, 1, 2, \cdots$, there exists $x \in H^\infty(D, P)$ such that $\pi_n(x) = x_n$ for all n.

Proof. For $V \in \mathcal{G}^{\text{even}}$, we define a normal completely positive map Φ_V from D^{even} to itself in a similar way as P; for $x \in D_X$, we set

$$\Phi_V(x) = \bigoplus_{Y \in \mathcal{G}^{\text{even}}} \frac{d(X)}{d(Y)} \sum_{i=1}^{N^Y_{X,\overline{V}}} (1_\rho \otimes \xi_{Y,i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \xi^*_{Y,i}),$$

where $\{\xi_{Y,i}\}_{i=1}^{N^Y_{X,\overline{V}}}$ is an orthonormal basis of $\mathcal{H}^Y_{X,\overline{V}}$. Then, it is a routine work to show $\Phi_V(1) = d(V)$ and

$$\Phi_V \cdot \Phi_W = \sum_Z N^Z_{V,W} \Phi_Z.$$

In the same way, for $V \in \mathcal{H}^{\text{even}}$ we define a normal completely positive map Φ_V from D^{odd} to itself.

For a probability measure μ on $\mathcal{G}^{\text{even}}$ or on $\mathcal{H}^{\text{even}}$, we set

$$\phi_\mu = \sum_V \frac{\mu(V)}{d(V)} \Phi_V,$$

which is a non-commutative Markov operator. Then, we get $\phi_\mu \cdot \phi_\nu = \phi_{\mu*\nu}$, where $\mu * \nu$ is the convolution product of two probability measure μ and ν introduced in [10]. Moreover, the following holds:

$$P^2 = \frac{1}{\beta^2}(\Phi_{\overline{\rho}\rho} \oplus \Phi_{\rho\overline{\rho}}).$$

If we define two probability measures μ on $\mathcal{G}^{\text{even}}$ and ν on $\mathcal{H}^{\text{even}}$ by

$$\mu = \sum_V \frac{d(V) N^V_{\overline{\rho},\rho}}{\beta^2} \delta_V, \qquad \nu = \sum_V \frac{d(V) N^V_{\rho,\overline{\rho}}}{\beta^2} \delta_V,$$

we get $P^{2n} = \phi_{\mu^n} \oplus \phi_{\nu^n}$, where μ^n and ν^n are the n-fold convolution product of μ and ν. Thanks to Fougel's theorem [6] [10, Lemma 3.1], we have the following ℓ^1-norm estimate:

$$\lim_{n\to\infty} ||\mu^{n+1} - \mu^n||_1 = 0, \quad \lim_{n\to\infty} ||\nu^{n+1} - \nu^n||_1 = 0.$$

Therefore, we get $\lim_{n\to\infty} ||P^{n+2} - P^n|| = 0$. The rest of the statements is standard (see [13] for example). Q.E.D.

(2) of Theorem 3.3 implies the following:

Corollary 3.5. *If $\mathcal{G}$ or $\mathcal{H}$ has no multi-edges, C_∞ is abelian.*

Remark.

(1) Assume neither $\mathcal{G}$ or $\mathcal{H}$ has multi-edges. Then, since D is abelian, P induces a random walk on a graph. Let ${}_MX_M$, ${}_MY_N$, ${}_NV_N$, and ${}_NW_M$ be irreducible bimodules associated with the inclusion $N \subset M$ and $\xi \in \mathcal{H}^X_{Y,\rho}$, $\eta \in \mathcal{H}^V_{\bar\rho,Y}$, $\zeta \in \mathcal{H}^W_{V,\rho}$, $\sigma \in \mathcal{H}^W_{\rho,X}$ be normalized unique (up to phase) intertwiners. Then, we have

$$P(\sigma^*\sigma) = \sum_\eta \frac{d(W)}{\beta d(V)} \left| \begin{array}{ccc} Y & \xrightarrow{\xi} & X \\ \eta\downarrow & & \downarrow\sigma \\ V & \xrightarrow[\zeta]{} & W \end{array} \right|^2 \quad \eta^* \cdot \eta.$$

Therefore, if we regard D as the ℓ^∞-space over the set of vertical paths $\mathcal{X}$, the transition probability from η to σ is given by

$$p(\eta,\sigma) = \frac{d(W)}{\beta d(V)} \left| \begin{array}{ccc} Y & \xrightarrow{\xi} & X \\ \eta\downarrow & & \downarrow\sigma \\ V & \xrightarrow[\zeta]{} & W \end{array} \right|^2 .$$

This is a reversible random walk in the sense of [19] thanks to the renormalization rule of the connection. Let τ be the natural trace on B_∞. Then, for $f \in H^\infty(D,P)$ we get

$$\tau(\theta_\infty(f)) = \tau(E_0(\theta_\infty(f))) = \tau(\theta_0 \cdot \pi_0(f)).$$

Now we assume, for simplicity, that $N \subset M$ is irreducible and $\sigma_0 \in \mathcal{X}$ is the path corresponding to the intertwiner in $\mathrm{Hom}(\rho\otimes_M M_M, \rho)$. Since the dimension of D_0 is one, the above equation means that $\tau(\theta_\infty(f))$ is given by the evaluation of f at σ_0. Thus, the measure corresponding to the restriction of τ to C_∞ is nothing but the harmonic measure on the Poisson boundary of the Markov chain induce by P with the initial distribution δ_{σ_0}.

(2) Let

$$\cdots M_{-2} \subset M_{-1} = N \subset M_0 = M$$

be a tunnel. We set $A_{i,j} := M'_{-i} \cap M_j$ and define $A_{i,\infty}$ to be the weak closure of $\cup_j A_{i,j}$. Then, the same machinery works in order to obtain $C_{n,\infty} := A'_{0,\infty} \cap A_{n,\infty}$. Indeed, there are obvious elements in $C_{n,\infty}$ coming from $A_{n,0}$, and what we really need to obtain is $pC_{n,\infty}q$ where p and q are minimal projections in $A_{n,0}$. Let ${}_AY_M$ and ${}_AZ_M$ be bimodules corresponding p and q respectively, where A is either M or N depending on the parity of n. Instead of D, we need to work on

$$\begin{aligned} D^{Y,Z} : &= \bigoplus_{X \in \mathcal{G}^{\mathrm{even}}} \mathrm{Hom}_A(Y \otimes_M X, Z \otimes_M X)_M \\ &\oplus \bigoplus_{X \in \mathcal{G}^{\mathrm{odd}}} \mathrm{Hom}_A(Y \otimes_M X, Z \otimes_M X)_N, \end{aligned}$$

as an object that the Markov operator P acts on. Or to make it an algebra, we can put it into

$$\bigoplus_{Y,Z} D^{Y,Z},$$

where the product is given by the composition (the product of not composable two elements is understood as 0).

(3) Let A be a von Neumann algebra and P be a unital normal completely positive map form A to itself. As we stated in the last section, we can always discuss the "Poisson boundary" using the Choi-Effros product. However, those P coming from natural examples, such as the classical examples or the ones discussed here, seem to have an additional property: namely, for every pair $x, y \in H^\infty(A, P)$, the sequence $\{P^n(xy)\}_n$ converges to an element in $H^\infty(A, P)$ in the strong operator topology. Does this hold for every unital normal completely positive map? If it is not the case, only those with the above property maybe deserve to be called "non-commutative Markov operators".

§4. Examples

In this section, we take the most fundamental example among non-trivial ones: a subfactor with the principal graph A_∞ and index larger than 4 (A_∞ should not be confused with the algebra A_∞ in the previous two sections). Another example, for which $H^\infty(D, P)$ may be explicitly obtained, would be the free composition of the A_3 and A_4 subfactors [3] [7] (see also [10]), though computation would be more complicated.

Let $N \subset M$ be a subfactor with the principal graph A_∞ and index larger than 4. Then, the core of this subfactor is the Jones inclusion, whose reducibility was first proven in Jones paper [12]. Another proof is available in Pimsner and Popa's paper [15]. We choose $0 < q < 1$ satisfying $\beta = [2]_q$, where

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 1, 2, \cdots .$$

For the edges of $\mathcal{G}$ and $\mathcal{H}$, we use the following labeling:

$$\mathcal{G}: 1 — 2 — 3 — 4 — \cdots,$$

$$\mathcal{H}: 1' — 2' — 3' — 4' — \cdots,$$

The statistical dimension of the bimodules corresponding to n and n' is $[n]_q$.

There exists only one connection, up to gauge freedom, for A_∞ graph, which is given by

$$\begin{array}{ccc} n & \to & n+1 \\ \downarrow & & \downarrow \\ n+1' & \to & n' \end{array} = \frac{(-1)^{n+1}}{[n]_q},$$

$$\begin{array}{ccc} n & \to & n-1 \\ \downarrow & & \downarrow \\ n-1' & \to & n' \end{array} = \frac{(-1)^{n}}{[n]_q},$$

$$\begin{array}{ccc} n & \to & n+1 \\ \downarrow & & \downarrow \\ n+1' & \to & n+2' \end{array} = \begin{array}{ccc} n+2 & \to & n+1 \\ \downarrow & & \downarrow \\ n+1' & \to & n' \end{array} = 1,$$

$$\begin{array}{ccc} n & \to & n+1 \\ \downarrow & & \downarrow \\ n-1' & \to & n' \end{array} = \begin{array}{ccc} n & \to & n-1 \\ \downarrow & & \downarrow \\ n+1' & \to & n' \end{array} = \frac{\sqrt{[n-1]_q[n+1]_q}}{[n]_q}.$$

We use the following labeling of the vertical paths of length 1:

$$a_n = \begin{array}{c} n \\ \downarrow \\ n+1' \end{array}, \qquad b_n = \begin{array}{c} n+1 \\ \downarrow \\ n' \end{array}.$$

Then, D is identified with the ℓ^∞-space over

$$\mathcal{X} = \{a_n\}_{n=1}^\infty \cup \{b_n\}_{n=1}^\infty.$$

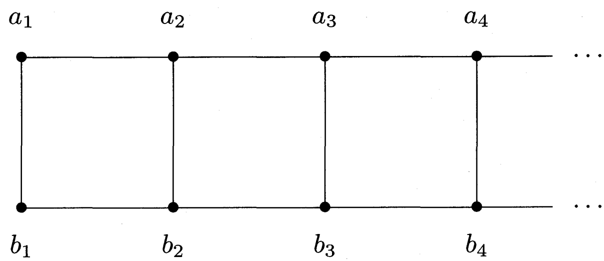

Fig. 1. Graph $\widetilde{\mathcal{X}}$

Thanks to the formula obtained in the remark of the last section, the transition probabilities corresponding to P are given as follows:

$$
\begin{aligned}
p(a_n, b_n) &= \frac{1}{[2]_q[n]_q[n+1]_q}, \quad n \geq 1, \\
p(a_n, a_{n+1}) &= \frac{[n+2]_q}{[2]_q[n+1]_q}, \quad n \geq 1, \\
p(a_n, a_{n-1}) &= \frac{[n-1]_q}{[2]_q[n]_q}, \quad n \geq 2, \\
p(b_n, a_n) &= \frac{1}{[2]_q[n]_q[n+1]_q}, \quad n \geq 1, \\
p(b_n, b_{n+1}) &= \frac{[n+2]_q}{[2]_q[n+1]_q}, \quad n \geq 1, \\
p(b_n, b_{n-1}) &= \frac{[n-1]_q}{[2]_q[n]_q}, \quad n \geq 2.
\end{aligned}
$$

All the other transition probabilities are 0. Therefore, we can regard $\mathcal{X}$ as the vertex set of the graph $\widetilde{\mathcal{X}}$ as in Figure 1, such that transitions occur only to the nearest neighbors.

An important feature of this random walk is that the vertical bonds decay exponentially fast as n tends to infinity, while we have asymptotics

$$p(a_n, a_{n+1}) = p(b_n, b_{n+1}) \sim \frac{1}{1+q^2} > \frac{1}{2}, \quad (n \to \infty).$$

More intuitively, when n is sufficiently large, the graph looks like splitting into two straight lines, while our random walk quickly goes to infinity. In consequence, we get exactly two points in the Poisson boundary, or in other words, $C_\infty \cong \mathbf{C} \oplus \mathbf{C}$.

To make the above intuitive argument rigorous, we explicitly calculate the harmonic functions. There are exactly two independent (not necessary bounded) harmonic functions (even when $q = 1$). We choose a basis of them consisting of the constant function 1 and h satisfying $h(a_n) = -h(b_n)$, $n = 1, 2, \cdots$, $h(a_1) = 1$. Let $x_n = h(a_n)$ and $x_0 = 0$. Then, the sequence $\{x_n\}_{n=0}^\infty$ is determined by the following three-term recurrence relation:

$$x_0 = 0, \qquad x_1 = 1.$$

$$\begin{aligned} (1) \qquad & (1 + [2]_q [n]_q [n+1]_q) x_n \\ & \quad = [n+2]_q [n]_q x_{n+1} + [n+1]_q [n-1]_q x_{n-1}, \quad n \geq 1, \end{aligned}$$

We show that this sequence is monotone increasing, and obtain the limit $\lim_n a_n$ for $0 < q < 1$. Equation (1) can be expressed as

$$\begin{aligned} (2) \qquad 2x_n &= [n+2]_q [n]_q (x_{n+1} - x_n) \\ & \quad - [n+1]_q [n-1]_q (x_n - x_{n-1}), \quad n \geq 1. \end{aligned}$$

Thus, by induction we can show that $\{x_n\}_{n=0}^\infty$ is positive and monotone increasing. We set $y_n := x_{n+1} - x_n$, $n = 0, 1, 2, \cdots$. Then, Equation (2) implies

$$y_0 = 1, \qquad y_1 = \frac{2}{[3]_q},$$

$$(3) \qquad 2[n+1]_q y_n = [n+3]_q y_{n+1} + [n-1]_q y_{n-1}, \quad n \geq 1.$$

When $q = 1$, it is easy to solve Equation (3), and we can see that $\{x_n\}_{n=1}^\infty$ is not bounded and C_∞ is trivial. When, $0 < q < 1$, we introduce an analytic function $g(z)$ defined on a neighborhood of 0 as follows (it is easy to show that the radius of convergence is positive):

$$g(z) = \sum_{n=0}^\infty y_n z^n.$$

Equation (3) implies that the following function equation holds:

$$(4) \qquad (z - q)^2 g(qz) - (z - q^{-1})^2 g(q^{-1} z) = (q^2 - q^{-2}).$$

This means that the radius of convergence of $g(z)$ is larger than or equal to q^{-2} (in fact it is q^{-2}). Setting $z = q$ in Equation (4), we get

$$g(1) = \frac{q + q^{-1}}{q^{-1} - q},$$

and so,

$$\lim_{n \to \infty} x_n = \sum_{n=0}^{\infty} y_n = g(1) = \frac{q + q^{-1}}{q^{-1} - q}.$$

Therefore, h is bounded and $\dim C_\infty = 2$.

We set

$$f_1 = \frac{g(1) + h}{2g(1)}, \quad f_2 = \frac{g(1) - h}{2g(1)}.$$

Then, f_1 and f_2 are two extremal positive harmonic functions of norm 1, and so $\theta_\infty(f_1)$ and $\theta_\infty(f_2)$ are two minimal projections in C_∞. The trace evaluations of these projections are given by

$$\tau(\theta_\infty(f_1)) = f_1(a_1) = \frac{q^{-1}}{q + q^{-1}},$$

$$\tau(\theta_\infty(f_2)) = f_2(a_1) = \frac{q}{q + q^{-1}}.$$

Of course, this agrees with the result in [15].

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