

## On quasidiagonal $C^*$ -algebras

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### Abstract.

We give a detailed survey of the theory of quasidiagonal  $C^*$ -algebras. The main structural results are presented and various functorial questions around quasidiagonality are discussed. In particular we look at what is currently known (and not known) about tensor products, quotients, extensions, free products, etc. of quasidiagonal  $C^*$ -algebras. We also point out how quasidiagonality is connected to some important open problems.

### §1. Introduction

Quasidiagonal  $C^*$ -algebras have now been studied for more than 20 years. They are a large class of algebras which arise naturally in many contexts and include many of the basic examples of finite  $C^*$ -algebras. Notions around quasidiagonality have also played an important role in BDF/KK-theory and are connected to some important open questions. For example, whether every nuclear  $C^*$ -algebra satisfies the Universal Coefficient Theorem, Elliott's Classification Program and whether or not  $\text{Ext}(C_r^*(\mathbb{F}_2))$  is a group.

In these notes we give a detailed survey of the basic theory of quasidiagonal  $C^*$ -algebras. At present there is only one survey article in the literature which deals with this subject (cf. [Vo4]). While there is certainly overlap between this article and [Vo4], the focus of the present paper is quite different. We will spend a fair amount of time giving detailed proofs of a number of basic facts about quasidiagonal  $C^*$ -algebras. Some of these results have appeared in print, some are well known to the experts but have not (explicitly) appeared in print and some of them are new. Moreover, there have been a number of important advances since the writing of [Vo4]. We will not give proofs of most of the more difficult recent results. However, we have tried to at least give precise statements of these results and have included an extensive bibliography so that the interested reader may track down the original papers.

In this paper we are primarily concerned with basic structural questions. In particular this means that many interesting topics have been left out or only briefly touched upon. For example, we do not explore the connections between (relative) quasidiagonality and BDF/KK-theory (found in the work of Salinas, Kirchberg, S. Wassermann, Dădărlat-Eilers, Schochet and others) or the generalized inductive limit approach (introduced by Blackadar and Kirchberg). But, for the interested reader, we have included a section containing references to a number of these topics.

Throughout the main body of these notes we will only be concerned with separable  $C^*$ -algebras and representations on separable Hilbert spaces. It turns out that one can usually reduce to this case so we don't view this as a major loss of generality. However, it causes one problem in that certain nonseparable  $C^*$ -algebras naturally arise in the (separable) theory. Hence we have included an appendix which deals with the nonseparable case.

A brief overview of this paper is as follows. In section 2 we collect a number of facts that will be needed in the rest of these notes. This is an attempt to keep the paper self contained, but these results include some of the deepest and most important tools in  $C^*$ -algebra theory and no proofs of well known results are given.

Section 3 contains the definitions and some basic properties of quasidiagonal operators, quasidiagonal sets of operators and quasidiagonal (QD)  $C^*$ -algebras. We also give some examples of QD and non-QD  $C^*$ -algebras. We end the section with the well known fact that quasidiagonality implies stable finiteness.

In section 4 we prove the abstract characterization of QD  $C^*$ -algebras which is due to Voiculescu.

Section 5 deals with the local approximation of QD  $C^*$ -algebras. We show that every such algebra can be locally approximated by a residually finite dimensional algebra. We also state a result of Dădărlat showing that every exact QD  $C^*$ -algebra can be locally approximated by finite dimensional  $C^*$ -algebras.

Section 6 contains the simple fact that every unital QD  $C^*$ -algebra has a tracial state.

Sections 7 - 11 deal with how quasidiagonality behaves under some of the standard operator algebra constructions. Section 7 discusses the easiest of these questions. Namely what happens when taking subalgebras, direct products and minimal tensor products of QD  $C^*$ -algebras. Quotients of QD algebras are treated in section 8, inductive limits in section 9, extensions in section 10 and crossed products in section 11.

Section 12 discusses the relationship between quasidiagonality and nuclearity. We state a result of Popa which led many experts to believe that simple QD  $C^*$ -algebras with ‘sufficiently many projections’ are always nuclear. We then state a result of Dădărlat which shows that this is not the case. We also discuss a certain converse to this question and its relationship to the classification program. Namely the question (due to Blackadar and Kirchberg) of whether every nuclear stably finite  $C^*$ -algebra must be QD.

Section 13 contains miscellaneous results which didn’t quite fit anywhere else. We state results of Boca and Voiculescu which concern full free products and homotopy invariance, respectively. We observe that all projective algebras and semiprojective MF algebras must be residually finite dimensional. Finally, we discuss how quasidiagonality relates to the question of when the classical BDF  $Ext(\cdot)$  semigroups are actually groups and the question of whether all nuclear  $C^*$ -algebras satisfy the Universal Coefficient Theorem.

In section 14 we point out where the interested reader can go to learn more about some of the things that are not covered thoroughly here.

Finally at the end we have an appendix which treats the case of nonseparable QD  $C^*$ -algebras. The main result being that a  $C^*$ -algebra is QD if and only if all of its separable  $C^*$ -subalgebras are QD.

## §2. Preliminaries

Central to much of what will follow is the theory of *completely positive* maps. We refer the reader to [Pa] for a comprehensive treatment of these important maps. Perhaps the single most important result about these maps is Stinespring’s Dilation Theorem (cf. [Pa, Thm. 4.1]). We will not state the most general version; for our purposes the following result will suffice.

**Theorem 2.1.** (*Stinespring*) *Let  $A$  be a unital separable  $C^*$ -algebra and  $\varphi : A \rightarrow B(H)$  be a unital completely positive map. Then there exists a separable Hilbert space  $K$ , an isometry  $V : H \rightarrow K$  and a unital representation  $\pi : A \rightarrow B(K)$  such that  $\varphi(a) = V^*\pi(a)V$  for all  $a \in A$ .*

Throughout most of [Pa], only the unital case is treated. The following result shows that this is not a serious problem.

**Proposition 2.2.** (*cf. [CE2, Lem. 3.9]*) *Let  $\varphi : A \rightarrow B$  be a contractive completely positive map. Then the unique unital extension  $\tilde{\varphi} : \tilde{A} \rightarrow \tilde{B}$  is also completely positive, where  $\tilde{A}$ ,  $\tilde{B}$  are the  $C^*$ -algebras obtained by adjoining new units.*

Another fundamental result concerning completely positive maps is Arveson's Extension Theorem. To state the theorem we need the following definition.

**Definition 2.3.** Let  $A$  be a unital  $C^*$ -algebra and  $X \subset A$  be a closed linear subspace. Then  $X$  is called an *operator system* if  $1_A \in X$  and  $X = X^*$ .

**Theorem 2.4.** (*Arveson's Extension Theorem*) If  $A$  is a unital  $C^*$ -algebra,  $X \subset A$  is an operator system and  $\varphi : X \rightarrow C$  is a contractive completely positive map with  $C = B(H)$  or  $\dim(C) < \infty$  then there exists a completely positive map  $\Phi : A \rightarrow C$  which extends  $\varphi$  (i.e.  $\Phi|_X = \varphi$ ). If  $X$  is a  $C^*$ -subalgebra of  $A$  then there always exists a unital completely positive extension of  $\varphi$  (whether or not  $X$  contains the unit of  $A$ ).

A proof of the unital statement above can be found in [Pa] while the nonunital statement is due to Lance [La1, Thm. 4.2].

Representations of quasidiagonal  $C^*$ -algebras will be important in what follows and hence we will need Voiculescu's Theorem (cf. [Vo1]). In fact, we will need a number of different versions of this result. It will be convenient to have Hadwin's formulation in terms of rank.

**Definition 2.5.** If  $T \in B(H)$  then let  $\text{rank}(T) = \dim(\overline{TH})$ .

**Theorem 2.6.** Let  $A$  be a unital  $C^*$ -algebra and  $\pi_i : A \rightarrow B(H_i)$  be unital  $*$ -representations for  $i = 1, 2$ . Then there exists a net of unitaries  $U_\lambda : H_1 \rightarrow H_2$  such that  $\|\pi_2(a) - U_\lambda \pi_1(a) U_\lambda^*\| \rightarrow 0$  for all  $a \in A$  if and only if  $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$  for all  $a \in A$ . If  $A$  is nonunital then there exists such a net of unitaries if and only if  $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$  for all  $a \in A$  and  $\dim(H_1) = \dim(H_2)$ .

When such unitaries exist we say that  $\pi_1$  and  $\pi_2$  are *approximately unitarily equivalent*. When both  $A$  and the underlying Hilbert spaces are separable one can even arrange the stronger condition that  $\pi_2(a) - U_n \pi_1(a) U_n^*$  is a compact operator for each  $a \in A$ ,  $n \in \mathbb{N}$  (of course, we can take a sequence of unitaries when  $A$  is separable). When this is the case we say that  $\pi_1$  and  $\pi_2$  are *approximately unitarily equivalent modulo the compacts*. A proof of this stronger (in the separable case) result can be found in [Dav, Thm. II.5.8] or a proof of the general result can be found in [Had1].

It turns out that one can usually reduce to the case of separable  $C^*$ -algebras and Hilbert spaces. In this case, the following version of Voiculescu's theorem will be convenient (cf. [Dav, Cor. II.5.5]).

**Theorem 2.7.** *Let  $H$  be a separable Hilbert space and  $C \subset B(H)$  be a unital separable  $C^*$ -algebra such that  $1_H \in C$ . Let  $\iota : C \hookrightarrow B(H)$  denote the canonical inclusion and let  $\rho : C \rightarrow B(K)$  be any unital representation such that  $\rho(C \cap \mathcal{K}(H)) = 0$ . Then  $\iota$  is approximately unitarily equivalent modulo the compacts to  $\iota \oplus \rho$ .*

We will be particularly interested in the case that  $C \cap \mathcal{K}(H) = 0$ .

**Definition 2.8.** Let  $\pi : A \rightarrow B(H)$  be a faithful representation of a  $C^*$ -algebra  $A$ . Then  $\pi$  is called *essential* if  $\pi(A)$  contains no nonzero finite rank operators.

**Corollary 2.9.** *Let  $A$  be a separable  $C^*$ -algebra and  $\pi_i : A \rightarrow B(H_i)$  be faithful essential representations with  $H_i$  separable for  $i = 1, 2$ . If  $A$  is unital and both  $\pi_1, \pi_2$  are unital then  $\pi_1$  and  $\pi_2$  are approximately unitarily equivalent modulo the compacts. If  $A$  is nonunital then  $\pi_1$  and  $\pi_2$  are always approximately unitarily equivalent modulo the compacts.*

We will need one more form of Voiculescu's Theorem. We have not been able to find the following version written explicitly in the literature. However, the main idea is essentially due to Salinas (see the proofs of [Sa1, Thm. 2.9] and [DHS, Thm. 4.2]).

If  $A$  is a separable, unital  $C^*$ -algebra and  $\varphi : A \rightarrow B(H)$  (with  $H$  separable and infinite dimensional) is a unital completely positive map then we say that  $\varphi$  is a *representation modulo the compacts* if  $\pi \circ \varphi : A \rightarrow Q(H)$  is a  $*$ -homomorphism, where  $\pi$  is the quotient map onto the Calkin algebra. If  $\pi \circ \varphi$  is injective then we say that  $\varphi$  is a *faithful representation modulo the compacts*. In this situation we define constants  $\eta_\varphi(a)$  by

$$\eta_\varphi(a) = 2 \max(\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2})$$

for every  $a \in A$ .

**Theorem 2.10.** *Let  $A$  be a separable, unital  $C^*$ -algebra and  $\varphi : A \rightarrow B(H)$  be a faithful representation modulo the compacts. If  $\sigma : A \rightarrow B(K)$  is any faithful, unital, essential representation then there exist unitaries  $U_n : H \rightarrow K$  such that*

$$\limsup_{n \rightarrow \infty} \|\sigma(a) - U_n \varphi(a) U_n^*\| \leq \eta_\varphi(a)$$

for every  $a \in A$ .

*Proof.* Note that by Corollary 2.9 it suffices to show that there exists a representation  $\sigma$  satisfying the conclusion of the theorem since all such representations are approximately unitarily equivalent.

Let  $\rho : A \rightarrow B(L)$  be the Stinespring dilation of  $\varphi$ ; i.e.  $\rho$  is a unital representation of  $A$  and there exists an isometry  $V : H \rightarrow L$  such that  $\varphi(a) = V^*\rho(a)V$ , for all  $a \in A$ . Let  $P = VV^* \in B(L)$  and  $P^\perp = 1_L - P$ . We claim that for every  $a \in A$ ,

$$\|P^\perp \rho(a)P\| \leq \|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}.$$

This follows from a simple calculation:

$$\begin{aligned} (P^\perp \rho(a)P)^*(P^\perp \rho(a)P) &= P\rho(a^*)P^\perp \rho(a)P \\ &= VV^*\rho(a^*a)VV^* - VV^*\rho(a^*)VV^*\rho(a)VV^* \\ &= V(\varphi(a^*a) - \varphi(a^*)\varphi(a))V^*. \end{aligned}$$

Now write  $L = PL \oplus P^\perp L$  and decompose the representation  $\rho$  accordingly. That is, consider the matrix decomposition

$$\rho(a) = \begin{pmatrix} \rho(a)_{11} & \rho(a)_{12} \\ \rho(a)_{21} & \rho(a)_{22} \end{pmatrix},$$

where  $\rho(a)_{21} = P^\perp \rho(a)P$  and  $\rho(a)_{12} = \rho(a^*)_{21}^*$ . Hence the norm of the matrix

$$\begin{pmatrix} 0 & \rho(a)_{12} \\ \rho(a)_{21} & 0 \end{pmatrix}$$

is bounded above by  $1/2\eta_\varphi(a)$  because of

$$\|P^\perp \rho(a)P\| \leq \|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}.$$

Now comes the trick. We consider the Hilbert space  $P^\perp L \oplus PL$  and the representation  $\rho' : A \rightarrow B(P^\perp L \oplus PL)$  given in matrix form as

$$\rho'(a) = \begin{pmatrix} \rho(a)_{22} & \rho(a)_{21} \\ \rho(a)_{12} & \rho(a)_{11} \end{pmatrix}.$$

Now using the obvious identification of the Hilbert spaces

$$PL \oplus \left( \bigoplus_{\mathbb{N}} P^\perp L \oplus PL \right) \quad \text{and} \quad \bigoplus_{\mathbb{N}} L = \bigoplus_{\mathbb{N}} (PL \oplus P^\perp L)$$

a standard calculation shows that

$$\|\rho(a)_{11} \oplus \rho'^\infty(a) - \rho^\infty(a)\| \leq \eta_\varphi(a)$$

for all  $a \in A$ , where  $\rho'^\infty = \bigoplus_{\mathbb{N}} \rho'$  and  $\rho^\infty = \bigoplus_{\mathbb{N}} \rho$ . Note also that  $\rho(a)_{11} = V\varphi(a)V^*$ .

Now, let  $C$  be the linear space  $\varphi(A) + \mathcal{K}(H)$ . Note that  $C$  is actually a separable, unital  $C^*$ -subalgebra of  $B(H)$  with  $\pi(C) = A$  where  $\pi : B(H) \rightarrow Q(H)$  is the quotient map onto the Calkin algebra. By Theorem 2.7 we have that  $\iota \oplus \rho'^\infty \circ \pi$  is approximately unitarily equivalent modulo the compacts to  $\iota$ , where  $\iota : C \hookrightarrow B(H)$  is the inclusion. Let  $W_n : H \rightarrow H \oplus (\bigoplus_{\mathbb{N}}(P^\perp L \oplus PL))$  be unitaries such that

$$\|\varphi(a) \oplus \rho'^\infty(a) - W_n \varphi(a) W_n^*\| \rightarrow 0$$

for all  $a \in A$ .

We now let

$$\tilde{V} : H \oplus \left(\bigoplus_{\mathbb{N}}(P^\perp L \oplus PL)\right) \rightarrow \bigoplus_{\mathbb{N}} L$$

be the unitary  $V \oplus 1$  (again using the obvious identification of  $PL \oplus (\bigoplus_{\mathbb{N}}(P^\perp L \oplus PL))$  and  $\bigoplus_{\mathbb{N}} L$ ). Note that  $\tilde{V}(\varphi(a) \oplus \rho'^\infty(a))\tilde{V}^* = V\varphi(a)V^* \oplus \rho'^\infty(a) = \rho(a)_{11} \oplus \rho'^\infty(a)$ . We now complete the proof by defining

$$K = \bigoplus_{\mathbb{N}} L, \quad \sigma = \rho^\infty = \bigoplus_{\mathbb{N}} \rho, \quad U_n = \tilde{V} W_n : H \rightarrow \bigoplus_{\mathbb{N}} L = K. \quad \text{Q.E.D.}$$

Finally, we will need a basic result concerning quotient maps of locally reflexive  $C^*$ -algebras. The notion of local reflexivity in the category of  $C^*$ -algebras was first introduced by Effros and Haagerup (cf. [EH]).

**Definition 2.11.** A unital  $C^*$ -algebra  $A$  is called *locally reflexive* if each unital completely positive map  $\varphi : X \rightarrow A^{**}$  is the limit (in the point-weak\* topology) of a net of unital completely positive maps  $\varphi_\lambda : X \rightarrow A$ , where  $X$  is an arbitrary finite dimensional operator system and  $A^{**}$  denotes the enveloping von Neumann algebra of  $A$ .

**Definition 2.12.** Let  $\pi : A \rightarrow B$  be a surjective  $*$ -homomorphism with  $A$  unital. Then  $\pi$  is called *locally liftable* if for each finite dimensional operator system  $X \subset B$  there exists a unital completely positive map  $\varphi : X \rightarrow A$  such that  $\pi \circ \varphi = id_X$ .

Of course, if either  $A$  or  $B$  is *nuclear* then the Choi-Effros Lifting Theorem (cf. [CE2, Thm. 3.10]) implies that  $\pi$  is more than just locally liftable; there then exists a completely positive splitting defined on all of  $B$ . However, local liftability is usually all we will need. The following result will be used several times and is a consequence of [EH; 3.2, 5.1, 5.3 and 5.5].

**Theorem 2.13.** Let  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  be an exact sequence with  $E$  unital. Then  $E$  is locally reflexive if and only if both  $I$  and  $B$  are locally reflexive and the morphism  $\pi$  is locally liftable.

Local reflexivity plays an important role in the theory of operator spaces. We will not need any more results about local reflexivity. However, we do wish to point out the following implications:

$$\text{Nuclear} \implies \text{Exact} \implies \text{Locally Reflexive}.$$

These results (together with the definitions of nuclear and exact  $C^*$ -algebras) can essentially be found in S. Wassermann's monograph [Wa3]. ([Wa3] Propositions 5.5 and 5.4 give the first implication while [Wa3, Remark 9.5.2] states that exactness is equivalent to property C of Archbold and Batty. However, property C implies property  $C''$ , as defined in [EH, pg. 120], which in turn is equivalent to local reflexivity by [EH, Thm. 5.1].) Since the pioneering work of E. Kirchberg, exactness has played a central role in  $C^*$ -algebras. However, since we will only need the local liftability statement of Theorem 2.13, we will also consider the class of locally reflexive  $C^*$ -algebras. (It is not known if this is really a larger class of algebras – i.e. it is not known if every locally reflexive  $C^*$ -algebra is exact [Kir2, pg. 75].)

### §3. Definitions, Basic Results and Examples

Recall that throughout the body of these notes all Hilbert spaces and  $C^*$ -algebras are assumed to be separable.

We begin this section by recalling the notions of *block diagonal* and *quasidiagonal* operators on a Hilbert space. In Proposition 3.4 we show that the notion of a quasidiagonal operator can be expressed in terms of a local finite dimensional approximation property. This local version then extends to a suitable definition of a quasidiagonal (QD)  $C^*$ -algebra (Definition 3.8). In Theorem 3.11 we prove a fundamental result about representations of QD  $C^*$ -algebras. At the end of this section we give some examples of QD (and non-QD)  $C^*$ -algebras and observe that QD  $C^*$ -algebras are always *stably finite* (cf. Proposition 3.19).

**Definition 3.1.** A bounded linear operator  $D$  on a Hilbert space  $H$  is called *block diagonal* if there exists an increasing sequence of finite rank projections,  $P_1 \leq P_2 \leq P_3 \cdots$ , such that  $\|[D, P_n]\| = \|DP_n - P_nD\| = 0$  for all  $n \in \mathbb{N}$  and  $P_n \rightarrow 1_H$  (in the strong operator topology) as  $n \rightarrow \infty$ .

Note that if  $\|[D, P_n]\| = 0$  then  $\|[D, (P_n - P_{n-1})]\| = 0$  as well. Thus the matrix for  $D$  with respect to the decomposition  $H = P_1H \oplus (P_2 - P_1)H \oplus (P_3 - P_2)H \oplus \cdots$  is block diagonal.

The notion of a quasidiagonal operator is due to Halmos and is a natural generalization of a block diagonal operator.



**Definition 3.2.** A bounded linear operator  $T$  on a Hilbert space  $H$  is called *quasidiagonal* if there exists an increasing sequence of finite rank projections,  $P_1 \leq P_2 \leq P_3 \cdots$ , such that  $\|[T, P_n]\| = \|TP_n - P_nT\| \rightarrow 0$  and  $P_n \rightarrow 1_H$  (in the strong operator topology) as  $n \rightarrow \infty$ .

Halmos observed the following relationship between quasidiagonal and block diagonal operators.

**Proposition 3.3.** *If  $T \in B(H)$  then  $T$  is quasidiagonal if and only if there exist a block diagonal operator  $D \in B(H)$  and a compact operator  $K \in \mathcal{K}(H)$  such that  $T = D + K$ .*

We will not give the proof of this proposition here as it is a special case of Theorem 5.2. Note, however, that one direction is easy. Namely, if  $T = D + K$  as above then  $T$  must be quasidiagonal since any increasing sequence of finite rank projections converging to  $1_H$  (s.o.t.) will form an approximate identity for  $\mathcal{K}(H)$  and hence will asymptotically commute with every compact operator.

It is an important fact that the seemingly global notion of quasidiagonality can be expressed in a local way.

**Proposition 3.4.** *Let  $T \in B(H)$ . Then  $T$  is quasidiagonal if and only if for each finite set  $\chi \subset H$  and  $\varepsilon > 0$  there exists a finite rank projection  $P \in B(H)$  such that  $\|[T, P]\| \leq \varepsilon$  and  $\|P(x) - x\| \leq \varepsilon$  for all  $x \in \chi$ .*

*Proof.* We may assume that  $\|T\| \leq 1$ . It is clear that the definition of a quasidiagonal operator implies the condition stated above. To prove the converse, it suffices to show that for each finite set  $\chi \subset H$  and  $\varepsilon > 0$  there exists a finite rank projection  $P$  such that  $\|[P, T]\| < \varepsilon$  and  $P(x) = x$  for all  $x \in \chi$ . Having established this it is not hard to construct finite rank projections  $P_1 \leq P_2 \leq P_3 \cdots$ , such that  $\|[T, P_n]\| \rightarrow 0$  and  $P_n \rightarrow 1_H$  in the strong operator topology.

So let  $\chi \subset H$  be a finite set,  $\varepsilon > 0$  and let  $R$  be the orthogonal projection onto  $K = \text{span}\{\chi\}$ . By compactness of the unit ball of  $K$  there is a finite set  $\tilde{\chi} \subset K$  which is  $\varepsilon$ -dense in the unit ball of  $K$ . Now let  $Q$  be a finite rank projection such that  $\|[Q, T]\| < \varepsilon$  and  $\|Q(x) - x\| < \varepsilon$  for all  $x \in \tilde{\chi}$ . Then for all  $y \in K$  we have  $\|Q(y) - y\| < 3\varepsilon\|y\|$  and hence  $\|(1 - R)QR\| < 3\varepsilon$ .

Now consider the positive contraction  $X = RQR + (1 - R)Q(1 - R)$ . Observe that  $X$  is actually very close to  $Q$ :

$$\begin{aligned} \|Q - X\| &= \|RQ(1 - R) + (1 - R)QR\| \\ &= \max\{\|RQ(1 - R)\|, \|(1 - R)QR\|\} \\ &= \|(1 - R)QR\| \\ &< 3\varepsilon. \end{aligned}$$

Hence  $X$  is almost a projection (i.e. its spectrum is contained in  $[0, 3\varepsilon] \cup (1 - 3\varepsilon, 1]$ ). Let  $P$  be the projection obtained from functional calculus on  $X$ . Then  $\|P - Q\| \leq 6\varepsilon$  and hence  $\|[P, T]\| \leq 13\varepsilon$ . Finally we claim that  $P(x) = x$  for all  $x \in \chi$ . To see this, first note that  $X$  commutes with  $R$  and hence so does  $P$ . This implies that  $PR = RPR$  is a projection with support contained in  $K$ . However, for each  $y \in K$  we also have  $\|PR(y) - y\| = \|R(P(y) - y)\| \leq \|P(y) - Q(y)\| + \|Q(y) - y\| \leq 9\varepsilon\|y\|$ . Hence the support of  $PR$  is all of  $K$ ; i.e.  $PR = R$ . Q.E.D.

With this local characterization in hand we now define the following generalization of a quasidiagonal operator.

**Definition 3.5.** A subset  $\Omega \subset B(H)$  is called a *quasidiagonal set of operators* if for each finite set  $\omega \subset \Omega$ , finite set  $\chi \subset H$  and  $\varepsilon > 0$  there exists a finite rank projection  $P \in B(H)$  such that  $\|[T, P]\| \leq \varepsilon$  and  $\|P(x) - x\| \leq \varepsilon$  for all  $T \in \omega$  and  $x \in \chi$ .

It is easy to see that a set  $\Omega \subset B(H)$  is a quasidiagonal set of operators if and only if the  $C^*$ -algebra generated by  $\Omega$ ,  $C^*(\Omega) \subset B(H)$ , is a quasidiagonal set of operators.

The proof of the next proposition is a straightforward adaptation of the proof of Proposition 3.4.

**Proposition 3.6.** *If  $A \subset B(H)$  is separable then  $A$  is a quasidiagonal set of operators if and only if there exists an increasing sequence of finite rank projections,  $P_1 \leq P_2 \leq P_3 \dots$ , such that for all  $a \in A$ ,  $\|[a, P_n]\| \rightarrow 0$  and  $P_n \rightarrow 1_H$  (s.o.t.) as  $n \rightarrow \infty$ .*

**Remark 3.7.** *The previous proposition is often used when defining quasidiagonal  $C^*$ -algebras. However, L. Brown has pointed out to us that the previous proposition is not true if  $A$  is not separable (even if  $H$  is separable). Definition 3.5 allows one to use Zorn's lemma to construct maximal quasidiagonal subsets of  $B(H)$  and we claim that they provide counterexamples. The proof goes by contradiction. So assume that  $\Omega \subset B(H)$  is a maximal quasidiagonal set of operators and there exist*

finite rank projections such that  $\|[x, P_n]\| \rightarrow 0$  for all  $x \in \Omega$ . Construct a block diagonal operator  $T$  such that  $[T, P_{2n}] = 0$  and  $\|[T, P_{2n+1}]\| = 1$  for all  $n \in \mathbb{N}$ . Since  $[T, P_{2n}] = 0$  for all  $n$  we see that  $\Omega \cup \{T\}$  is a quasidiagonal set of operators and hence (by maximality)  $T \in \Omega$ . This gives the contradiction since  $\|[T, P_{2n+1}]\| = 1$  for all  $n \in \mathbb{N}$ . Hence it is important to take Definition 3.5 for general quasidiagonal questions.

We finally come to the definition of a quasidiagonal  $C^*$ -algebra.

**Definition 3.8.** Let  $A$  be a  $C^*$ -algebra. Then  $A$  is called *quasidiagonal* (QD) if there exists a faithful representation  $\pi : A \rightarrow B(H)$  such that  $\pi(A)$  is a quasidiagonal set of operators.

Some remarks regarding this definition are in order. First we should point out that some authors (e.g. [Had2]) refer to  $C^*$ -algebras satisfying Definition 3.8 as ‘weakly quasidiagonal’  $C^*$ -algebras. There is good reason for this terminology as it emphasizes the distinction between abstract and concrete  $C^*$ -algebras. It is important to make this distinction since every  $C^*$ -algebra has a representation  $\pi$  such that  $\pi(A)$  is a quasidiagonal set of operators (namely the zero representation). On the other hand, it is possible to give examples of  $C^*$ -algebras  $A$  and faithful representations  $\pi : A \rightarrow B(H)$  such that  $A$  is QD but  $\pi(A)$  is *not* a quasidiagonal set of operators. Perhaps the most extreme case of this is an example of L. Brown. In [BrL2] it was shown that there exists an operator  $T$  on a separable Hilbert space such that  $T \oplus T$  is quasidiagonal while  $T$  is not! Hence  $C^*(T)$  is a QD  $C^*$ -algebra but is *not* a quasidiagonal set of operators in its natural representation. Thus it is indeed very important to distinguish between abstract QD  $C^*$ -algebras and concrete quasidiagonal sets of operators. (The reader is cautioned that this is not always done carefully in the literature.) Other authors prefer to say that a *representation* is quasidiagonal if its image is a quasidiagonal set of operators. Definition 3.8 then becomes equivalent to the statement that  $A$  admits a faithful quasidiagonal representation.

Definitions 3.5 and 3.8 are the correct definitions in the nonseparable case as well (see the Appendix). We will see that certain nonseparable  $C^*$ -algebras (namely  $\Pi M_n(\mathbb{C})$ ) naturally arise in the separable theory and hence it will be logically necessary to treat this case also.

Finally note that Definition 3.8 does not require the representation to be nondegenerate. Of course, this can always be arranged. Note, however, that this is actually a deep fact as the proof of Lemma 3.10 below depends on Voiculescu’s Theorem (at least in the nonunital case).

**Definition 3.9.** If  $\pi : A \rightarrow B(H)$  is a representation and  $L \subset H$  is a  $\pi(A)$ -invariant subspace then  $\pi_L : A \rightarrow B(L)$  denotes the restriction representation (i.e.  $\pi_L(a) = P_L\pi(a)|_L$ , where  $P_L$  is the orthogonal projection from  $H \rightarrow L$ ).

**Lemma 3.10.** *Let  $\pi : A \rightarrow B(H)$  be a faithful representation and  $L \subset H$  be the nondegeneracy subspace of  $\pi(A)$ . Then  $\pi(A)$  is a quasidiagonal set of operators if and only if  $\pi_L(A)$  is a quasidiagonal set of operators.*

*Proof.* Assume first that  $\pi_L(A)$  is a quasidiagonal set of operators. Then write  $H = L \oplus \tilde{L}$ . Since  $\pi(a) = \pi_L(a) \oplus 0$ , any finite rank projection  $P \in B(L)$  can be extended to a finite rank projection  $P \oplus \tilde{P}$  such that  $\|[\pi(a), P \oplus \tilde{P}]\| = \|[\pi_L(a), P]\|$ . From this one deduces that  $\pi(A)$  must also be a quasidiagonal set of operators.

Now assume that  $\pi(A)$  is a quasidiagonal set of operators. If  $A$  is unital then  $\pi(1_A) = P_L$ . If  $R \in B(H)$  is any finite rank projection that almost commutes with  $\pi(1_A) = P_L$  then  $P_L R P_L \in B(L)$  is very close to a projection which does commute with  $P_L$ . We leave the details to the reader, but some standard functional calculus then implies that  $\pi_L(A)$  is also a quasidiagonal set of operators.

In the case that  $\pi(A)$  is a quasidiagonal set of operators and  $A$  is nonunital, we have to call on Voiculescu's Theorem (version 2.6). Since it is clear that  $\text{rank}(\pi(a)) = \text{rank}(\pi_L(a))$  for all  $a \in A$  we have that  $\pi$  and  $\pi_L$  are approximately unitarily equivalent. However, it is an easy exercise to verify that if  $\rho$  and  $\tilde{\rho}$  are two approximately unitarily equivalent representations then  $\rho(A)$  is a quasidiagonal set of operators if and only if  $\tilde{\rho}(A)$  is a quasidiagonal set of operators. Q.E.D.

We now give the fundamental theorem on representations of QD  $C^*$ -algebras.

**Theorem 3.11.** (cf. [Vo4, 1.7]) *Let  $\pi : A \rightarrow B(H)$  be a faithful essential (cf. Definition 2.8) representation. Then  $A$  is QD if and only if  $\pi(A)$  is a quasidiagonal set of operators.*

*Proof.* If  $\pi(A)$  is a quasidiagonal set of operators then, of course,  $A$  is QD. Conversely, if  $A$  is QD then there exists a faithful representation  $\rho : A \rightarrow B(K)$  such that  $\rho(A)$  is a quasidiagonal set of operators. In light of Lemma 3.10, we may assume that both  $\pi$  and  $\rho$  are nondegenerate. Defining  $\rho_\infty = \bigoplus_{\mathbb{N}} \rho : A \rightarrow B(\bigoplus_{\mathbb{N}} K)$  it is easy to see that  $\rho_\infty(A)$  is also a quasidiagonal set of operators. But, since  $\rho_\infty$  is an essential representation, Voiculescu's Theorem (version 2.9) implies that  $\pi$  and  $\rho_\infty$  are approximately unitarily equivalent. Hence  $\pi(A)$  is also a quasidiagonal set of operators. Q.E.D.

We now give some examples of QD  $C^*$ -algebras and non-QD  $C^*$ -algebras.

**Example 3.12.** *Every commutative  $C^*$ -algebra is QD. Indeed, if  $A = C_0(X)$  and for each  $x \in X$  we let  $ev_x : A \rightarrow \mathbb{C}$  be evaluation at  $x$  then  $\pi = \bigoplus_{x \in F} ev_x$ , where  $F \subset X$  is a countable dense set, is a faithful representation and it is easy to see that  $\pi(A)$  is a quasidiagonal (in fact, diagonal) set of operators.*

**Example 3.13.** *Approximately finite dimensional (AF) algebras are QD. Let  $A = \overline{\bigcup_n A_n}$  be AF with each  $A_n \subset A_{n+1}$  finite dimensional. Let  $\pi : A \rightarrow B(H)$  be a faithful nondegenerate representation and write  $H = \overline{\bigcup_n H_n}$  where each  $H_n \subset H_{n+1}$  is a finite dimensional subspace. Then define  $P_n \in B(H)$  to be the (finite rank) projection onto the subspace  $\pi(A_n)H_n$ . Then we evidently have that  $\|[\pi(a), P_n]\| \rightarrow 0$  for all  $a \in A$  and  $P_n \rightarrow 1_H$  in the strong topology.*

**Example 3.14.** *Irrational rotation algebras are QD. That is, if  $A_\theta$  is the universal  $C^*$ -algebra generated by two unitaries  $U, V$  subject to the relation  $UV = (\exp(2\pi\theta i))VU$  for some irrational number  $\theta \in [0, 1]$  then  $A_\theta$  is QD. This was first proved by Pimsner and Voiculescu when they showed how to embed  $A_\theta$  into an AF algebra (cf. [PV2]). This was later generalized by Pimsner in [Pi] (see also section 11 of these notes).*

**Example 3.15.** *Perhaps the most important class of QD  $C^*$ -algebras are the so-called residually finite dimensional (RFD)  $C^*$ -algebras. A  $C^*$ -algebra  $R$  is called RFD if for each  $x \in R$  there exists a  $*$ -homomorphism  $\pi : R \rightarrow B$  such that  $\dim(B) < \infty$  and  $\pi(x) \neq 0$ . That such algebras have a faithful representation whose image is a quasidiagonal (in fact, block diagonal) set of operators is proved similar to the case of abelian algebras. Often times general questions about QD  $C^*$ -algebras can be reduced to the case of RFD algebras.*

**Example 3.16.** *Both the cone ( $CA = C_0((0, 1]) \otimes A$ ) and suspension ( $SA = C_0((0, 1)) \otimes A$ ) over any  $C^*$ -algebra  $A$  are QD. Since  $SA \subset CA$  and  $CA$  is homotopic to  $\{0\}$ , this can be deduced from the homotopy invariance of quasidiagonality (cf. [Vo3] or Theorem 13.1 of these notes). From this we see that every  $C^*$ -algebra is a quotient of a QD  $C^*$ -algebra (since  $A \cong CA/SA$ ).*

**Example 3.17.** *A  $C^*$ -algebra which contains a proper (i.e. non-unitary) isometry is not QD. Since it is clear that a subalgebra of a QD  $C^*$ -algebra is again QD, it suffices to show that the Toeplitz algebra is not QD. (Recall that Coburn's Theorem states that the  $C^*$ -algebras generated by any two proper isometries are isomorphic.) We*

let  $C^*(S)$  denote the Toeplitz algebra, where  $S$  is a proper isometry, and let  $\pi : C^*(S) \rightarrow B(H)$  be any faithful unital essential representation. Then  $\pi(S)$  is a semi-Fredholm operator with index  $-\infty$ . On the other hand, it follows from Proposition 3.3 that any semi-Fredholm quasidiagonal operator on  $H$  must have index zero (since any semi-Fredholm block diagonal operator must have index zero) and hence  $\pi(S)$  is not a quasidiagonal operator. Hence, by Theorem 3.11,  $C^*(S)$  is not QD. (See [Hal1] for generalizations of this result.)

The previous example implies a more general result.

**Definition 3.18.** Let  $A$  be a unital  $C^*$ -algebra. Then  $A$  is said to be *stably finite* if  $A \otimes M_n(\mathbb{C})$  contains no proper isometries for all  $n \in \mathbb{N}$ . If  $A$  is nonunital, then  $A$  is called *stably finite* if the unitization  $\tilde{A}$  is stably finite.

**Proposition 3.19.** *QD  $C^*$ -algebras are stably finite.*

*Proof.* It is easy to see that if  $A$  is nonunital and QD then the unitization  $\tilde{A}$  is also QD. Furthermore, it is a good exercise to verify that if  $A$  is QD then for all  $n \in \mathbb{N}$ ,  $M_n(\mathbb{C}) \otimes A$  is also QD. From these observations and Example 3.17 we see that if  $A$  is QD then  $M_n(\mathbb{C}) \otimes A$  (or  $M_n(\mathbb{C}) \otimes \tilde{A}$  in the non-unital case) has no proper isometries for all  $n \in \mathbb{N}$ . Hence  $A$  is stable finite. Q.E.D.

The converse is not true. S. Wassermann has given examples of non-QD MF algebras (cf. Definition 9.1 and Example 8.6 of these notes). But every MF algebras is stably finite (cf. [BK1, Prop. 3.3.8]). Hence, in general, QD is not equivalent to stably finite. However, Blackadar and Kirchberg have asked whether or not they are equivalent within the category of nuclear  $C^*$ -algebras (see Question 12.5).

#### §4. Voiculescu's Abstract Characterization

In this section we prove an abstract (i.e. representation free) characterization of QD  $C^*$ -algebras (cf. [Vo3, Thm. 1]). This fundamental result will be crucial in sections 8 - 10.

Consider the following property of an arbitrary  $C^*$ -algebra  $A$ .

(\*) For each finite set  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  there exists a contractive completely positive map  $\varphi : A \rightarrow B$  such that *i*)  $\dim(B) < \infty$ , *ii*)  $\|\varphi(x)\| \geq \|x\| - \varepsilon$  for all  $x \in \mathcal{F}$  and *iii*)  $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon$  for all  $x, y \in \mathcal{F}$ .

For a unital algebra  $A$  we have a related property.

(\*\*) For each finite set  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  there exists a unital completely positive map  $\varphi : A \rightarrow B$  such that *i*)  $B \cong M_n(\mathbb{C})$  for some

$n \in \mathbb{N}$ , *ii*)  $\|\varphi(x)\| \geq \|x\| - \varepsilon$  for all  $x \in \mathcal{F}$  and *iii*)  $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon$  for all  $x, y \in \mathcal{F}$ .

We will refer to such maps as  $\varepsilon$ -isometric and  $\varepsilon$ -multiplicative on  $\mathcal{F}$ .

**Lemma 4.1.** *If  $A$  is a unital  $C^*$ -algebra then  $A$  satisfies  $(*)$  if and only if  $A$  satisfies  $(**)$ .*

*Proof.* ( $\Leftarrow$ ) This is obvious.

( $\Rightarrow$ ) We only sketch the main idea. First, we identify  $B$  with a unital subalgebra of  $M_m(\mathbb{C}) = B(\mathbb{C}^m)$  for some  $m \in \mathbb{N}$ . Let  $1_A$  denote the unit of  $A$  and let  $P \in M_m(\mathbb{C})$  be the projection onto the range of  $\varphi(1_A)$ . Then one shows that  $\varphi(a) = P\varphi(a) = \varphi(a)P$  for all  $a \in A$ . Moreover, if  $\varphi$  is very multiplicative on  $1_A$  then  $\varphi(1_A)$  is close to  $P$ .

Now let  $\psi : A \rightarrow PM_m(\mathbb{C})P \cong M_n(\mathbb{C})$  (for some  $n \leq m$ ) be given by  $\psi(a) = P\varphi(a)P$  and clearly  $\psi$  has the same multiplicativity and isometric properties (up to  $\varepsilon$ ) that  $\varphi$  does. Moreover, since  $\varphi(1_A)$  is close to  $P$ ,  $\psi(1_A)$  is invertible in  $PM_m(\mathbb{C})P \cong M_n(\mathbb{C})$ . Thus, to get a unital complete positive map into a matrix algebra, we replace  $\psi$  with the map  $a \mapsto (\psi(1_A))^{-1/2}\psi(a)(\psi(1_A))^{-1/2}$ . The multiplicativity and isometric properties of this new map are not quite as good as those of  $\varphi$ , but they are good enough. Q.E.D.

We are now ready for Voiculescu's abstract characterization of QD  $C^*$ -algebras. Our proof is based on the proof of [DHS, Thm. 4.2] and, hopefully, is easier to follow than the original. However, the main ideas are the same. We have simply isolated the hard part in Theorem 2.10.

**Theorem 4.2** (Voiculescu). *Let  $A$  be a  $C^*$ -algebra. Then  $A$  is QD if and only if  $A$  satisfies  $(*)$ .*

*Proof.* From Proposition 2.2 it is easy to see that  $A$  satisfies  $(*)$  if and only if  $\tilde{A}$  satisfies  $(*)$ . Similarly it is clear that  $A$  is QD if and only if  $\tilde{A}$  is QD and hence we may assume that  $A$  is unital.

( $\Rightarrow$ ) Let  $\pi : A \rightarrow B(H)$  be a unital faithful essential representation on a separable Hilbert space. We can then find an increasing sequence of finite rank projections,  $P_1 \leq P_2 \leq P_3 \dots$ , such that for all  $a \in A$ ,  $\|[\pi(a), P_n]\| \rightarrow 0$  and  $P_n \rightarrow 1_H$  in the strong topology. Then for all  $n$ ,  $P_n B(H) P_n$  is isomorphic to a matrix algebra and the unital completely positive maps  $\varphi_n : A \rightarrow P_n B(H) P_n$ ,  $a \mapsto P_n \pi(a) P_n$  are easily seen to be asymptotically multiplicative and isometric.

( $\Leftarrow$ ) By Lemma 4.1 we can find a sequence of unital completely positive maps  $\varphi_i : A \rightarrow M_{n(i)}(\mathbb{C})$  which are asymptotically multiplicative and asymptotically isometric. Let

$$H_m = \bigoplus_{i=m}^{\infty} \mathbb{C}^{n(i)}, \quad \Phi_m = \bigoplus_{i=m}^{\infty} \varphi_i : A \rightarrow B(H_m).$$

Evidently each  $\Phi_m$  is a faithful representation modulo the compacts (as in Theorem 2.10). Let  $\sigma : A \rightarrow B(K)$  be any faithful, unital, essential representation and by Theorem 2.10 we can find unitaries  $U_m : H_m \rightarrow K$  such that  $\|\sigma(a) - U_m \Phi_m(a) U_m^*\| \rightarrow 0$  as  $m \rightarrow \infty$  for all  $a \in A$ . Since it is clear that  $\Phi_m(A)$  is a quasidiagonal (in fact, block diagonal) set of operators for every  $m$  it is easy to see that  $\sigma(A)$  is also a quasidiagonal set of operators and hence  $A$  is QD. Q.E.D.

In addition to being a very useful tool in establishing the quasidiagonality of a given  $C^*$ -algebra this result also shows that QD  $C^*$ -algebras are a very natural *abstract* class of algebras. Indeed, this result shows that QD  $C^*$ -algebras are precisely those which have ‘good matrix models’ in the sense that all of the relevant structure (order, adjoints, multiplication, norms) can approximately be seen in a matrix.

We wish to note a minor generalization which will be useful later on.

**Definition 4.3.** If  $A$  is a unital  $C^*$ -algebra and  $\mathcal{F} \subset A$  is a finite set then we will let  $X_{\mathcal{F}\mathcal{F}}$  denote the smallest operator system (cf. Definition 2.3) containing  $\mathcal{F}$  and  $\{ab : a, b \in \mathcal{F}\}$ .

**Definition 4.4.** If  $F, B \subset B(H)$  are sets of operators then we say  $F$  is  $\varepsilon$ -contained in  $B$  if for each  $x \in F$  there exists  $y \in B$  such that  $\|x - y\| < \varepsilon$ . When this is the case we write  $F \subset^\varepsilon B$ .

**Corollary 4.5.** Assume that  $A$  is unital and for every finite subset  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  there exists a contractive completely positive map  $\varphi : X_{\mathcal{F}\mathcal{F}} \rightarrow B(H)$  such that  $\varphi$  is  $\varepsilon$ -isometric and  $\varepsilon$ -multiplicative on  $\mathcal{F}$  and  $\varphi(\mathcal{F})$  is  $\varepsilon$ -contained in a QD  $C^*$ -algebra  $B \subset B(H)$ . Then  $A$  is QD.

*Proof.* That  $A$  satisfies (\*) follows from Arveson’s Extension Theorem applied to  $\varphi$  and to the almost isometric and multiplicative maps from  $B$  to finite dimensional  $C^*$ -algebras. Q.E.D.

**Remark 4.6.** Note that the hypotheses of the previous corollary can be relaxed further. Indeed, one only needs such  $\varepsilon$ -isometric and  $\varepsilon$ -multiplicative maps on a sequence of finite sets which are suitably dense in  $A$  (e.g. generate a dense  $*$ -subalgebra of  $A$ ).

## §5. Local Approximation

We observe that every QD  $C^*$ -algebra can be locally approximated by a residually finite dimensional (RFD)  $C^*$ -algebra (cf. Example 3.15). The proof is a simple adaptation of Halmos’ original proof that every quasidiagonal operator can be written as a block diagonal operator plus



a compact. We also recall a result of M. Dădărlat which gives a much stronger approximation in the case of exact QD  $C^*$ -algebras.

**Definition 5.1.** (cf. Definition 3.1) Let  $B \subset B(H)$  be a  $C^*$ -algebra. Then  $B$  is called a *block diagonal algebra* if there exists an increasing sequence of finite rank projections,  $P_1 \leq P_2 \leq P_3 \dots$ , such that  $\|[b, P_n]\| = 0$  for all  $b \in B$ ,  $n \in \mathbb{N}$  and  $P_n \rightarrow 1_H$  (s.o.t.).

It is relatively easy to see that a  $C^*$ -algebra  $R$  is RFD if and only if there exists a faithful representation  $\pi : R \rightarrow B(H)$  such that  $\pi(R)$  is a block diagonal algebra. The next result, which is well known to the experts, shows that every QD  $C^*$ -algebra can be locally approximated by an RFD algebra.

**Theorem 5.2.** Let  $A \subset B(H)$  be a  $C^*$ -algebra. Then  $A$  is a quasidiagonal set of operators if and only if for every finite set  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  there exists a block diagonal algebra  $B \subset B(H)$  such that  $\mathcal{F} \subset^\varepsilon B$  (cf. Definition 4.4) and  $A + \mathcal{K}(H) = B + \mathcal{K}(H)$ .

*Proof.* Clearly we only have to prove the necessity since  $B + \mathcal{K}(H)$  is a quasidiagonal set of operators. Our proof follows closely the proof of [Ar, Thm. 2] where a similar result is obtained for general quasicontral approximate units.

So let  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  be given. We may assume that  $\mathcal{F}$  is contained in the unit ball of  $A$ . Let  $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \dots$  be a sequence of finite sets such that  $\mathcal{F} \subset \mathcal{F}_1$  and whose union is dense in the unit ball of  $A$ . Since  $A$  is a quasidiagonal set of operators we can use Proposition 3.6 to find finite rank projections  $P_1 \leq P_2 \dots$  converging to  $1_H$  (strongly) and such that  $\|[P_n, a]\| \rightarrow 0$  for all  $a \in A$ . By passing to a subsequence we may assume that  $\|[P_n, a]\| < \varepsilon/(2^n)$  for all  $a \in \mathcal{F}_n$ . Now, let  $E_n = P_n - P_{n-1}$  for  $n = 1, 2, \dots$  where  $P_0 = 0$ . Note that  $\sum_{n=1}^{\infty} E_n = 1_H$ .

Then one defines completely positive maps  $\delta_k : A \rightarrow B(H)$  via the formula

$$\delta_k(a) = \sum_{n=1}^k E_n a E_n.$$

We leave it to the reader to verify that the  $\delta_k$ 's converge in the point strong operator topology (i.e.  $\delta_k(a)$  is strongly convergent for each  $a \in A$ ) and hence

$$\delta(a) = \sum_{n=1}^{\infty} E_n a E_n.$$

is a well defined completely positive map. Now let  $B = C^*(\delta(A))$  and clearly  $B$  is a block diagonal set of operators. Moreover, for each  $a \in A$

we have

$$\begin{aligned}
 a - \delta(a) &= \sum_{n=1}^{\infty} aE_n - \sum_{n=1}^{\infty} E_n a E_n \\
 &= \sum_{n=1}^{\infty} (aE_n - E_n a E_n) \\
 &= \sum_{n=1}^{\infty} (aE_n - E_n a) E_n
 \end{aligned}$$

where convergence of these sums is again taken in the strong operator topology. However, for each  $a \in \cup \mathcal{F}_n$  the last summation above is actually convergent in the norm topology and is compact since the  $E_n$ 's are finite rank. Note that by construction we have  $\|a - \delta(a)\| \leq \sum \varepsilon / (2^n) = \varepsilon$  for all  $a \in \mathcal{F}_1$ . Now since  $\delta$  is norm continuous (being completely positive) we then conclude that  $a - \delta(a)$  is a compact operator for all  $a \in A$ . It follows that  $A + \mathcal{K}(H) = B + \mathcal{K}(H)$ . Q.E.D.

Theorem 5.2 fails when  $A$  is not separable (cf. Remark 3.7).

**Corollary 5.3.** (cf. [GM]) *Every (separable)  $C^*$ -algebra  $A$  is a quotient of an RFD algebra. If  $A$  is nuclear (resp. exact) then the RFD algebra can be chosen nuclear (resp. exact).*

*Proof.* Let  $\pi : CA \rightarrow B(H)$  be a faithful essential representation of the cone over  $A$  (cf. Example 3.16). If  $A$  is nuclear (resp. exact) then so is  $CA$  and hence so is  $\pi(CA) + \mathcal{K}(H)$  (cf. [CE1, Cor. 3.3], [Kir2, Prop. 7.1]). Let  $R \subset B(H)$  be an RFD algebra such that  $\pi(CA) + \mathcal{K}(H) = R + \mathcal{K}(H)$ . Passing to the Calkin algebra we see that  $CA$ , and hence  $A$ , is a quotient of  $R$ . Since exactness passes to subalgebras ([Kir2, Prop. 7.1]), it is clear that  $R$  is exact whenever  $A$  is exact. When  $A$  is nuclear we deduce that  $R$  is also nuclear from [CE1, Cor. 3.3] and the exact sequence

$$0 \rightarrow R \cap \mathcal{K}(H) \rightarrow R \rightarrow CA \rightarrow 0,$$

since  $\mathcal{K}(H)$  is type I and hence all of its subalgebras are nuclear (cf. [B11]). Q.E.D.

The next result of Dădărlat is a vast improvement under the additional assumption of exactness. We will not prove this here; see [Dăd3, Thm. 6]. However we remark that the proof depends in an essential way on Theorem 5.2 as it allows one to reduce to the case of RFD algebras.

**Theorem 5.4** (Dădărlat). *Let  $A \subset B(H)$  be such that  $A \cap \mathcal{K}(H) = 0$ . Then  $A$  is exact and  $QD$  if and only if for every finite set  $\mathcal{F} \subset A$  and*

$\varepsilon > 0$  there exists a finite dimensional subalgebra  $B \subset B(H)$  such that  $\mathcal{F} \subset^\varepsilon B$ .

Note the similarity with the definition of an AF algebra. The difference, of course, is that we have had to go outside the algebra to get the finite dimensional approximation. We regard this as very strong evidence in favor of an affirmative answer to the following conjecture. (See also [BK1, Question 7.3.3])

**Conjecture 5.5.** *Every (separable) exact QD  $C^*$ -algebra is isomorphic to a subalgebra of an AF algebra.<sup>1</sup>*

## §6. Traces

**Proposition 6.1.** (cf. [Vo4, 2.4]) *If  $A$  is a unital QD  $C^*$ -algebra then  $A$  has a tracial state.*

*Proof.* By Theorem 4.2 and Lemma 4.1 we can find a sequence of unital completely positive maps  $\varphi_i : A \rightarrow M_{n(i)}(\mathbb{C})$  such that  $\|a\| = \lim_i \|\varphi_i(a)\|$  and  $\|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\| \rightarrow 0$  for all  $a, b \in A$ . Let  $\tau_{n(i)}$  denote the tracial state on  $M_{n(i)}(\mathbb{C})$  and let  $\tau \in S(A)$  be a weak limit point of the sequence  $\{\tau_{n(i)} \circ \varphi_i\} \subset S(A)$ . An easy calculation shows that  $\tau$  is a tracial state. Q.E.D.

One should not be tempted to think that the trace constructed above is faithful. Of course some very nice unital QD  $C^*$ -algebras, like the unitization of the compact operators, can't have a faithful tracial state. But we do have the following immediate corollary.

**Corollary 6.2.** *Every simple unital QD  $C^*$ -algebra has a faithful trace.*

## §7. Easy Functorial Properties

The following two facts are immediate from the definition.

**Proposition 7.1.** *A subalgebra of a QD  $C^*$ -algebra is also QD.*

**Proposition 7.2.** *The unitization of a QD  $C^*$ -algebra is also QD.*

We need some notation before going further.

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<sup>1</sup>Some exciting progress on this conjecture has been made by Ozawa who showed that the cone over any exact algebra is AF-embeddable (cf. [Oz], [Rør3]).

**Definition 7.3.** Let  $\{A_n\}$  be a sequence of  $C^*$ -algebras. Then  $\prod_{n \in \mathbb{N}} A_n = \{(a_n) : \sup_n \|a_n\| < \infty\}$ , where  $(a_n)$  is an element of the set theoretic product of the  $A_n$ 's. We let  $\bigoplus_{n \in \mathbb{N}} A_n$  denote the ideal of  $\prod_{n \in \mathbb{N}} A_n$  which consists of elements  $(a_n)$  with the property that  $\lim_{n \rightarrow \infty} \|a_n\| = 0$ .

If  $A$  and  $B$  are QD and  $\pi : A \rightarrow B(H)$ ,  $\rho : B \rightarrow B(K)$  are faithful representations whose ranges are quasidiagonal sets of operators then one easily checks that  $A \oplus B$  is QD by considering the representation  $\pi \oplus \rho$ . The following fact is an easy extension of this argument.

**Proposition 7.4.** *The direct product of QD  $C^*$ -algebras is QD. That is, if  $\{A_n\}$  is a sequence of  $C^*$ -algebras then  $\prod_{n \in \mathbb{N}} A_n$  is QD if and only if each  $A_n$  is QD.*

Recall that if  $A$  and  $B$  are  $C^*$ -algebras with faithful representations  $\pi : A \rightarrow B(H)$  and  $\rho : B \rightarrow B(K)$  then the minimal (or 'spatial') tensor product is defined to be the  $C^*$ -algebra generated by the image of the algebraic tensor product representation  $\pi \odot \rho : A \odot B \rightarrow B(H) \odot B(K) \subset B(H \otimes K)$ . The following result, which appeared first in [Had2], is left as an easy exercise. The proof only depends on the fact that the tensor product of two finite rank projections is again a finite rank projection.

**Proposition 7.5.** *The minimal tensor product of QD  $C^*$ -algebras is again QD.*

If both  $A$  and  $B$  contain projections and  $A \otimes_{\min} B$  is QD then both  $A$  and  $B$  must be QD as well. But in general the converse of Proposition 7.5 is not true (since cones and suspensions are always QD).

When one of the algebras happens to be nuclear then there is only one possible tensor product and hence quasidiagonality is always preserved in this case. In particular this fact implies that quasidiagonality is even invariant under the weaker notion of *stable isomorphism* (cf. [BrL1]). (Recall that  $A$  and  $B$  are stably isomorphic if  $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ , where  $\mathcal{K}$  denotes the compact operators on an infinite dimensional separable Hilbert space. Recall also that for separable algebras this is the same as strong Morita equivalence; cf. [BGR].)

It is not known whether or not Proposition 7.5 holds for other tensor products. In particular the following question is still open.

**Question 7.6.** *If  $A$  and  $B$  are QD then is  $A \otimes_{\max} B$  also QD?*

## §8. Quotients

We already pointed out in Example 3.16 that every  $C^*$ -algebra is a quotient of a QD  $C^*$ -algebra. Thus quasidiagonality does not pass to

quotients in general. In this section we give a sufficient condition for a quotient of a QD algebra to be QD. However, this condition is far from necessary and it is not clear what the real obstruction is. Also, at the end of this section we give another proof of Corollary 5.3 (i.e. one which does not depend on the fact that cones are always QD).

To state our result we first need a definition. The notion of relative quasidiagonality was introduced by Salinas in connection with KK-theory (cf. [Sa2]).

**Definition 8.1.** Let  $A$  be a  $C^*$ -algebra with (closed, 2-sided) ideal  $I$ . Then  $A$  is said to be *quasidiagonal relative to  $I$*  if  $I$  has an approximate unit consisting of projections which is quasicentral in  $A$ .

**Example 8.2.** In general, an algebra can be quasidiagonal relative to an ideal without itself (or the ideal) being QD. For example, let  $\{A_i\}_{i \in \mathbb{N}}$  be a sequence of unital (non-QD)  $C^*$ -algebras. Then  $A = \Pi_i A_i$  is quasidiagonal relative to the ideal  $I = \bigoplus_i A_i$ . But the terminology is inspired by a close connection in the case that the ideal is the compact operators. Indeed, if  $B \subset B(H)$  is a  $C^*$ -algebra then it is easy to see that  $B$  is a quasidiagonal set of operators if and only if  $B + \mathcal{K}(H)$  is quasidiagonal relative to  $\mathcal{K}(H)$  (cf. Proposition 3.6).

**Proposition 8.3.** Assume  $A$  is unital, QD, quasidiagonal relative to an ideal  $I$  and  $\pi : A \rightarrow A/I$  is locally liftable (cf. Definition 2.12). Then  $A/I$  is also QD.

*Proof.* Let  $\mathcal{F} \subset A/I$  be a finite set and  $\varepsilon > 0$ . In the notation of Corollary 4.5 we let  $\varphi : X_{\mathcal{F}\mathcal{F}} \rightarrow A$  be a unital completely positive splitting.

Now take a quasicentral approximate unit of projections, say  $\{p_n\}$  and consider the (isometric – though no longer unital) completely positive splittings  $\varphi_n(x) = (1-p_n)\varphi(x)(1-p_n)$ . We claim that for sufficiently large  $n$ , these maps are  $\varepsilon$ -multiplicative on  $\mathcal{F}$  and hence from Corollary 4.5 we will have that  $A/I$  is QD.

To see the  $\varepsilon$ -multiplicativity we first recall that if  $a \in A$  and  $\dot{a}$  denotes its image in  $A/I$  then  $\|\dot{a}\| = \lim \|(1-p_n)a\|$  since  $\{p_n\}$  is an approximate unit for  $I$ . However, since the  $p_n$ 's are projections and quasicentral for  $A$  we see that  $\|\dot{a}\| = \lim \|(1-p_n)a(1-p_n)\|$  as well. Now for  $a, b \in A$  consider the following estimates (see also the proof of Lem. 3.1 in [Ar] where these estimates are given in greater generality):

$$\begin{aligned}
& \|\varphi_n(\dot{a}\dot{b}) - \varphi_n(\dot{a})\varphi_n(\dot{b})\| \\
&= \|(1-p_n)\varphi(\dot{a}\dot{b})(1-p_n) - (1-p_n)\varphi(\dot{a})(1-p_n)\varphi(\dot{b})(1-p_n)\| \\
&\leq \|(1-p_n)\left(\varphi(\dot{a}\dot{b}) - \varphi(\dot{a})\varphi(\dot{b})\right)(1-p_n)\| \\
&\quad + \|(1-p_n)\left((1-p_n)\varphi(\dot{a}) - \varphi(\dot{a})(1-p_n)\right)\varphi(\dot{b})(1-p_n)\|.
\end{aligned}$$

Finally since  $\varphi$  is a splitting,  $\|\dot{x}\| = \lim \|(1-p_n)x(1-p_n)\|$  and  $\{p_n\}$  is quasicentral we see that  $\varphi_n$  is  $\varepsilon$ -multiplicative on  $\mathcal{F}$  for sufficiently large  $n$ . Q.E.D.

**Corollary 8.4.** *If  $A$  is unital, locally reflexive (e.g. exact or nuclear), QD and quasidiagonal relative to an ideal  $I$  then  $A/I$  is also QD.*

*Proof.* Use the previous proposition together with Theorem 2.13. Q.E.D.

**Remark 8.5.** *The proof of Proposition 8.3 given here is simply a formalization of a well known argument in the case the ideal is the compact operators. (cf. [Dăd1, Prop. 4.5].)*

**Example 8.6.** *Proposition 8.3 is no longer true without the ‘local liftability’ hypothesis. Indeed, S. Wassermann gave the first examples of quasidiagonal sets of operators whose image in the Calkin algebra was a non-QD  $C^*$ -algebra (cf. [Was1,2]). Hence, by the remarks in Example 8.2, Wassermann’s examples show that the ‘local liftability’ hypothesis can’t be dropped in Proposition 8.3.*

In section 10 we will see that the ‘right’ obstruction to look at for extensions is probably given in K-theoretic terms (for a large class of algebras). However, the following example shows that this is not the case for the quotient question. Indeed, it is not at all clear what type of obstruction one should be looking at in relation to the quotient question.

**Example 8.7.** *Let  $A = \mathcal{O}_2$  be the Cuntz algebra on two generators (cf. [Cu]). Then we have the short exact sequence  $0 \rightarrow SA \rightarrow CA \rightarrow A \rightarrow 0$ , where  $SA$  and  $CA$  denote the suspension and cone, respectively. The point we wish to make is that any potential K-theoretic obstruction would vanish for this example since the six term exact sequence is trivial. However,  $CA$  is QD while  $A$  is not.*

Finally we give another proof of the fact that every separable  $C^*$ -algebra is a quotient of an RFD algebra (cf. Corollary 5.3). The simplest proof of this fact is the original (cf. [GM]), however it does not yield the useful fact that this can be done within certain categories (e.g. nuclear, exact, etc.) Rather than use Voiculescu's result on the quasidiagonality of cones, we now exploit a basic fact from noncommutative topology. We will need the following generalization of the Tietze Extension Theorem.

**Theorem 8.8.** (cf. [We, Thm. 2.3.9]) *Let  $A$  be a separable  $C^*$ -algebra and  $\pi : A \rightarrow B$  be a surjective  $*$ -homomorphism. Then  $\pi$  extends to a surjective  $*$ -homomorphism  $\tilde{\pi} : M(A) \rightarrow M(B)$  of multiplier algebras.*

**Proposition 8.9.** *Let  $R$  be an RFD algebra. Then the multiplier algebra,  $M(R)$ , is also RFD.*

*Proof.* Let  $\pi_n : R \rightarrow A_n$  be a sequence of surjective  $*$ -homomorphisms such that each  $A_n$  is unital and QD (e.g. finite dimensional) and the map  $\bigoplus_{n \in \mathbb{N}} \pi_n$  is faithful. Construct extending morphisms  $\tilde{\pi}_n : M(R) \rightarrow A_n$ . (In this case the extensions are easy to construct. For each  $n$  let  $e_n \in R$  be a lift of the unit of  $A_n$  and simply define  $\tilde{\pi}_n(x) = \pi_n(xe_n)$  for all  $x \in M(R)$ .)

In general, if  $I \subset A$  is an essential ideal and  $\varphi : A \rightarrow B$  is a  $*$ -homomorphism such that  $\varphi|_I$  is injective then  $\varphi$  must be injective on all of  $A$  (since any nonzero ideal of  $A$  must have nonzero intersection with  $I$ ). Hence we see that the  $*$ -homomorphism  $\bigoplus_n \tilde{\pi}_n : M(R) \rightarrow \prod A_n$  is also injective. Q.E.D.

**Corollary 8.10.** *Let  $H$  be a (separable) Hilbert space. Then  $B(H)$  is a quotient of a RFD algebra.*

*Proof.* Since the compact operators on a separable Hilbert space are a quotient of an RFD algebra (note that for QD  $C^*$ -algebras we do not have to use cones in the proof of Corollary 5.3), it follows from the Tietze Extension Theorem and proposition 8.9 that  $B(H) = M(\mathcal{K})$  is a quotient of an RFD algebra. Q.E.D.

## §9. Inductive Limits

It follows easily from Theorem 3.11 that an inductive limit of QD  $C^*$ -algebras where the connecting maps are all *injective* will again be a QD  $C^*$ -algebra. We will see that, in general, inductive limits of QD algebras need not be QD. However, if the algebras in the sequence are also locally reflexive (e.g. exact or nuclear) then the limit algebra must be QD.

In [BK1] the notion of MF  $C^*$ -algebra was introduced. There are a number of characterizations of these algebras and hence we can choose the most convenient as our definition (though it is actually a theorem).

**Definition 9.1.** (cf. [BK1, Thm. 3.2.2]) A  $C^*$ -algebra  $A$  is MF if and only if  $A$  is isomorphic to a subalgebra of  $\prod M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C})$  for some sequence  $\{n(i)\}$ .

**Proposition 9.2.** (cf. [BK1, Prop. 3.1.3]) Let  $C \subset B(H)$  be a  $C^*$ -algebra which is also a quasicentral set of operators and  $\pi : B(H) \rightarrow Q(H)$  denote the quotient map onto the Calkin algebra. Then  $\pi(C)$  is MF.

*Proof.* By Theorem 5.2 we can find a block diagonal algebra  $B \subset B(H)$  such that  $C + \mathcal{K} = B + \mathcal{K}$ . If  $P_1 \leq P_2 \leq P_3 \leq \dots$  are finite rank projections which commute with  $B$  and converge to  $1_H$  then there is a canonical identification  $\prod_i M_{n(i)}(\mathbb{C}) \hookrightarrow B(H) = B(\oplus_{i \in \mathbb{N}} \mathbb{C}^{n(i)})$ , where  $n(i) = \text{rank}(P_i) - \text{rank}(P_{i-1})$  (and  $P_0 = 0$ ). Note that under this identification we have  $\prod M_{n(i)}(\mathbb{C}) \cap \mathcal{K} = \oplus M_{n(i)}(\mathbb{C})$  and  $B \subset \prod M_{n(i)}(\mathbb{C})$ . Hence

$$\pi(C) \cong B/(B \cap \mathcal{K}) \hookrightarrow \prod M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C}). \quad \text{Q.E.D.}$$

Evidently every MF algebra has such an extension by the compacts.

It follows that every QD  $C^*$ -algebra is MF (cf. Theorem 3.11). The converse is not true by Example 8.6.

The following simple result shows that MF algebras can also be described as the class of  $C^*$ -algebras arising as inductive limits of RFD  $C^*$ -algebras.

**Proposition 9.3.** A  $C^*$ -algebra  $A$  is MF if and only if  $A$  is isomorphic to an inductive limit of RFD algebras.

*Proof.* That an inductive limit of MF algebras (e.g. RFD algebras) is again MF is a bit out of the scope of this article. Please see [BK1, Cor. 3.4.4] for the proof. We will prove the converse however.

By pulling back the embedding  $A \hookrightarrow \prod M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C})$  we can find an RFD algebra  $R$  with ideal  $I = \oplus M_{n(i)}(\mathbb{C})$  such that  $A \cong R/I$ . Consider the finite dimensional ideals  $I_k = \oplus_{i=1}^k M_{n(i)}(\mathbb{C})$ . Evidently  $R/I_k$  is again an RFD algebra (being a direct summand of  $R$ ) and hence the natural inductive system

$$R \rightarrow R/I_1 \rightarrow R/I_2 \rightarrow R/I_3 \rightarrow \dots$$

consists of RFD algebras. Moreover, since  $I = \overline{\cup I_k}$  it is routine to verify that  $A \cong R/I$  is isomorphic to the inductive limit of the above sequence. Q.E.D.



**Remark 9.4.** *It follows that inductive limits of QD  $C^*$ -algebras need not be QD. To get such examples, we let  $A \subset B(H)$  be a  $C^*$ -algebra which is a quasidiagonal set of operators and such that the image,  $B \subset Q(H)$ , in the Calkin algebra is non-QD. (We mentioned in Example 8.6 that Wassermann has constructed such algebras.) By Propositions 9.2 and 9.3,  $B$  is an inductive limit of RFD algebras which is not QD.*

In contrast to the previous remark, the next result shows that mild assumptions will ensure the quasidiagonality of the limit.

**Theorem 9.5.** *Let  $\{A_m, \varphi_{n,m}\}_{m \in \mathbb{N}}$  be an inductive system of unital locally reflexive QD  $C^*$ -algebras with limit  $A = \varinjlim A_i$ . Then  $A$  is QD.*

*Proof.* To clarify our notation, we mean that for each  $n \geq m$  there is a  $*$ -homomorphism  $\varphi_{n,m} : A_m \rightarrow A_n$  and we have the usual compatibility condition that  $\varphi_{n,m} \circ \varphi_{m,l} = \varphi_{n,l}$  whenever  $l \leq m \leq n$ . We also let  $\Phi_n : A_n \rightarrow A$  denote the induced  $*$ -homomorphism.

Unitizing the inductive system, if necessary, we may assume that all the connecting maps are unit preserving. Now let  $\Psi_m : A_m \rightarrow \Pi A_i$  be the  $*$ -monomorphism defined by

$$\Psi_m(x) = 0 \oplus \cdots \oplus 0 \oplus x \oplus \varphi_{m+1,m}(x) \oplus \cdots$$

and  $B = C^*(\cup \Psi_m(A_m)) + \oplus A_i \subset \Pi A_i$ . Then it is easy to see that  $B$  is QD, quasidiagonal relative to the ideal  $\oplus A_i$  and  $A \cong B/(\oplus A_i)$ . Thus it suffices to see (by Proposition 8.3 and Remark 4.6) that the quotient map  $B \rightarrow B/(\oplus A_i)$  is locally liftable on a dense set. But this follows from the fact that each  $A_n$  is locally reflexive (cf. Theorem 2.13), the maps  $\Psi_n$  are injective, the exact sequences  $0 \rightarrow (\Psi_n(A_n) \cap \oplus A_i) \rightarrow \Psi_n(A_n) \rightarrow \Phi_n(A_n) \rightarrow 0$  and the fact that the union of the  $\Phi_n(A_n)$ 's is dense in  $A$ . Q.E.D.

**Remark 9.6.** *Blackadar and Kirchberg have shown that generalised inductive limits (where the connecting maps are completely positive contractions) of nuclear QD algebras are again QD (cf. [BK1, Cor. 5.3.5]).*

Inductive limit decompositions have played a crucial role in (the finite case of) Elliott's Classification Program. The next result of Blackadar and Kirchberg may turn out to have important consequences in this program. This theorem follows immediately from [BK1, Prop. 6.1.6] and [BK2, Cor. 5.1].

**Theorem 9.7.** *Let  $A$  be a unital simple nuclear QD  $C^*$ -algebra. Then  $A = \cup R_i$  where  $R_i \subset R_{i+1}$  are nuclear RFD algebras.*

The remarkable point of this theorem is that the connecting maps in the inductive system are all *injective*. Indeed, if one relaxes this condition then we can easily get every nuclear QD  $C^*$ -algebra.

**Proposition 9.8.** *Let  $A$  be a nuclear  $C^*$ -algebra. Then  $A$  is QD if and only if  $A$  is isomorphic to an inductive limit of nuclear RFD  $C^*$ -algebras.*

*Proof.* ( $\Leftarrow$ ) This follows from Theorem 9.5.

( $\Rightarrow$ ) This follows from the proof of Proposition 9.3 since extensions of nuclear  $C^*$ -algebras are nuclear. Q.E.D.

## §10. Extensions

Since the Toeplitz algebra is an extension of the compacts by  $C(\mathbb{T})$ , it follows that extensions of QD algebras need not be QD. Indeed, as with the quotient question, the general extension problem for QD algebras appears to be very hard. As we will see, it is not even clear whether or not a split extension of QD algebras should be QD.

We begin, however, with two simple positive results. The first states that if the *ideal* is sufficiently quasidiagonal then the middle algebra is always QD. The second states that if the *extension* is sufficiently quasidiagonal then the middle algebra is always QD.

**Proposition 10.1.** *Assume  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  is exact with  $I$  an RFD algebra and  $B$  a QD algebra. Then  $E$  is QD.*

*Proof.* Let  $\varphi : E \rightarrow M(I)$  be the natural extension of the inclusion  $I \hookrightarrow M(I)$  (cf. [We, 2.2.14]). Then the map  $\varphi \oplus \pi : E \rightarrow M(I) \oplus B$  is injective. But Proposition 8.9 states that  $M(I)$  is QD (even RFD) and hence  $E$  is a QD  $C^*$ -algebra. Q.E.D.

Note that the proof of Proposition 8.9 actually shows that  $M(I)$  is QD whenever  $I$  has a separating family of *unital* QD quotients. Hence the proposition above remains true for ideals of the form  $I = R \otimes_{\min} B$  where  $R$  is RFD and  $B$  is unital and QD. Hence a natural question is the following.

**Question 10.2.** *Which (nonunital)  $C^*$ -algebras have QD multiplier algebras?*

**Definition 10.3.** Let  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  be a short exact sequence of  $C^*$ -algebras. Such a sequence is called a *quasidiagonal extension* if  $E$  is quasidiagonal relative to  $\iota(I)$  (cf. Definition 8.1).

**Remark 10.4.** *It is important to note that in general an extension being quasidiagonal has nothing to do with whether or not the middle algebra  $E$  is QD (see Example 8.2).*

**Proposition 10.5.** *Let  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  be a quasidiagonal extension where both  $I$  and  $B$  are QD. Then  $E$  is QD.*

*Proof.* To ease notation somewhat, we identify  $I$  with  $\iota(I)$  and let  $\{P_n\} \subset I$  be an approximate unit of projections which is quasicontral in  $E$ . Now consider the contractive completely positive maps  $\varphi_n : E \rightarrow I \oplus B$ ,  $\varphi_n(x) = P_n x P_n \oplus \pi(x)$ . Evidently these maps are asymptotically multiplicative. So we may appeal to Corollary 4.5 and deduce that  $E$  is QD as soon as we verify the following assertion:

*Claim.* If  $x \in E$  then  $\|x\| = \max\{ \liminf_n \|P_n x P_n\|, \|\pi(x)\| \}$ .

To prove the claim we pass to the double dual  $E^{**}$ . Let  $P \in I^{**} \subset E^{**}$  be the (weak) limit of the  $P_n$ 's. Then  $P$  is central in  $E^{**}$  and we have a decomposition  $E^{**} = I^{**} \oplus B^{**}$ . Hence (regarding  $E \subset E^{**}$ ) for each  $x \in E$  we have  $\|x\| = \max\{ \|PxP\|, \|(1-P)x(1-P)\| \}$ . But  $\|\pi(x)\| = \|(1-P)x(1-P)\|$  and  $\|PxP\| \leq \liminf_n \|P_n x P_n\|$  since  $P_n x P_n \rightarrow PxP$  in the strong operator topology. But this proves the claim since the inequality  $\|x\| \geq \max\{ \liminf_n \|P_n x P_n\|, \|\pi(x)\| \}$  is obvious. Q.E.D.

**Remark 10.6.** *As mentioned previously, Propositions 10.1 and 10.5 can be regarded as saying that quasidiagonality is always preserved, provided that either the ideal or the extension is sufficiently quasidiagonal. This is not true if only the quotient is highly QD (e.g. the Toeplitz algebra). Instead a  $K$ -theoretic obstruction appears to govern in general.*

We would now like to discuss the general question of when quasidiagonality is preserved in extensions. However, to illustrate the difficulty of this problem we first pose two basic (open) questions.

**Question 10.7.** *Let  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  be a split exact sequence (i.e. there exists a  $*$ -homomorphism  $\rho : B \rightarrow E$  such that  $\pi \circ \rho = id_B$ ) with  $I$  and  $B$  QD. Is  $E$  necessarily QD?*

**Question 10.8.** *Let  $I$  and  $B$  be QD  $C^*$ -algebras and  $\pi : B \rightarrow M(I \otimes \mathcal{K})$  be a  $*$ -monomorphism such that  $\pi(B) \cap (I \otimes \mathcal{K}) = \{0\}$ . Is  $\pi(B) + I \otimes \mathcal{K}$  necessarily QD?*

Clearly an affirmative answer to Question 10.7 would imply an affirmative answer to Question 10.8. In fact the converse is true.

**Lemma 10.9.** *Questions 10.7 and 10.8 are equivalent.*

*Proof.* Assume Question 10.8 has an affirmative answer and let  $0 \rightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} B \rightarrow 0$  be a split exact sequence and  $\rho : B \rightarrow E$  be such that  $\pi \circ \rho = id_B$ . Identify  $I$  with  $\iota(I)$ . Let  $\eta : E \rightarrow B(H)$  be a faithful essential representation. Then from Theorem 3.11,  $\eta(I) + \mathcal{K}$  is QD.

Moreover,  $\eta(I) + \mathcal{K}$  is an essential ideal in  $\eta(E) + \mathcal{K}$ . So replacing  $I$  by  $\eta(I) + \mathcal{K}$  and  $E$  by  $\eta(E) + \mathcal{K}$  we may further assume that  $I$  is essential in  $E$ . But then  $0 \rightarrow I \otimes \mathcal{K} \xrightarrow{\iota} E \otimes \mathcal{K} \xrightarrow{\pi} B \otimes \mathcal{K} \rightarrow 0$  is still a split exact sequence with  $I \otimes \mathcal{K}$  essential in  $E \otimes \mathcal{K}$ . Hence  $E \otimes \mathcal{K}$  may be regarded as a subalgebra of  $M(I \otimes \mathcal{K})$  (cf. [We, 2.2.14]) and thus an affirmative answer to Question 10.8 would imply that  $E \otimes \mathcal{K}$  is QD. Q.E.D.

In [BND] it is shown that Question 10.8 has an affirmative answer under the additional hypothesis that either  $I$  or  $B$  is nuclear. Note, however, that even in the case that  $I = \mathbb{C}$ , Question 10.8 is not trivial (an affirmative answer still depends on the full power of Voiculescu's Theorem; cf. Theorem 3.11). Hence it is not clear whether or not we should expect an affirmative answer to these questions in general.

If we restrict to the class of nuclear  $C^*$ -algebras then some progress can be made on the general extension problem. Blackadar and Kirchberg have asked whether or not every nuclear stably finite  $C^*$ -algebra is QD (cf. [BK1, Question 7.3.1]). Hence one may ask whether the extension problem can be solved for stably finite  $C^*$ -algebras. J. Spielberg has given a complete answer to this question in his work on the AF embeddability of extensions of  $C^*$ -algebras.

**Proposition 10.10** (Sp, Lem. 1.5). *Let  $0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$  be an exact sequence with both  $I$  and  $B$  stably finite. If  $\partial : K_1(B) \rightarrow K_0(I)$  denotes the boundary map of this sequence then  $E$  is stably finite if and only if  $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$ , where  $K_0^+(I)$  is the canonical positive cone of  $K_0(I)$ .*

Though the proof is fairly straightforward, we will not prove this result here as we do not wish to introduce the K-theory which is needed.

In light of the previous proposition and the question of whether or not the notions of quasidiagonality and stable finiteness coincide in the class of nuclear  $C^*$ -algebras, the following question becomes quite natural.

**Question 10.11.** *Let  $0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$  be an exact sequence with both  $I$  and  $B$  nuclear QD  $C^*$ -algebras. Is it true that  $E$  is QD if and only if  $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$ ?*

If one approaches this problem via KK-theory then it is probably necessary to further assume that  $B$  satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet (cf. [RS]). In [BND] it is shown that this question is equivalent to some very natural questions concerning the K-theory of nuclear QD  $C^*$ -algebras. Moreover, it seems

likely that an affirmative answer to the question above could have important consequences in the classification program (specifically to the classification of Lin's TAF algebras; [Li1,2]).

In [BND] we also give a partial solution to the question above. The techniques used to prove the following result are similar to those from [Sp]. (See also [ELP] for the case that the quotient is AF.)

**Theorem 10.12** (BND). *Let  $0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$  be an exact sequence with  $I$  QD and  $B$  nuclear, QD and satisfying the UCT. If  $\partial : K_1(B) \rightarrow K_0(I)$  is the zero map then  $E$  is QD*

## §11. Crossed Products

In this section we discuss when crossed products of QD  $C^*$ -algebras are again QD. This is not always the case since the (purely infinite) Cuntz algebras are stably isomorphic to crossed products of AF algebras by  $\mathbb{Z}$ . The basic theory of crossed products by locally compact groups can be found in [Pe1, Chpt. 7]. (See also [Dav, Chpt. 8] for a nice treatment of the discrete case.)

We begin with a corollary of an imprimitivity theorem of P. Green. To state the result we will need to introduce some notation. So, let  $G$  be a separable locally compact group and  $H \subset G$  be a closed subgroup. Then  $G/H$  (the space of left cosets) is a separable locally compact space. There is a natural action  $\gamma$  of  $G$  on  $C_0(G/H)$  defined by  $\gamma_g(f)(xH) = f(g^{-1}xH)$  for all  $xH \in G/H$  and  $f \in C_0(G/H)$ . The crossed products below are the full crossed products and all groups actions  $\alpha : G \rightarrow \text{Aut}(A)$  are assumed to be suitably continuous (i.e. for each  $a \in A$  the map  $g \mapsto \alpha_g(a)$  is continuous).

**Theorem 11.1.** (*[Gr2, Cor. 2.8 ]*) *Let  $\alpha : G \rightarrow \text{Aut}(A)$  be a homomorphism from the separable locally compact group  $G$ . For each closed subgroup  $H \subset G$  there is an isomorphism*

$$A \otimes C_0(G/H) \rtimes_{\alpha \otimes \gamma} G \cong (A \rtimes_{\alpha|_H} H) \otimes \mathcal{K},$$

where  $\mathcal{K}$  denotes the compact operators on a separable (finite dimensional if and only if  $G/H$  is finite) Hilbert space.

For the rest of this section we will only be dealing with amenable groups (cf. [Pe1, 7.3]) and hence we do not need to distinguish between reduced and full crossed products (cf. [Pe1, Thm. 7.7.7]).

**Corollary 11.2.** *Let  $A$  be QD and  $\alpha : G \rightarrow \text{Aut}(A)$  be a homomorphism with  $G$  a separable compact group. Then  $A \rtimes_{\alpha} G$  is QD.*

*Proof.* Let  $H \subset G$  be the zero subgroup. The previous theorem then asserts that  $A \otimes C(G) \rtimes_{\alpha \otimes \gamma} G \cong A \otimes \mathcal{K}$ . But  $A \otimes \mathcal{K}$  is QD and there is a natural embedding  $A \rtimes_{\alpha} G \hookrightarrow A \otimes C(G) \rtimes_{\alpha \otimes \gamma} G$  since  $G$  amenable implies that the full and reduced crossed products are isomorphic (cf. [Pe1, 7.7.7 and 7.7.9]). Q.E.D.

For non-compact discrete groups the problem is considerably harder. However, Rosenberg has shown that we must restrict to the class of amenable groups.

**Theorem 11.3** (Ros, Thm. A1). *If  $G$  is discrete and  $C_r^*(G)$  is QD then  $G$  is amenable.*

It is not known whether the converse of this theorem holds (cf. [Vo4, 3.1]), but Følner's characterization of amenable groups in terms of almost shift invariant finite subsets leads one to believe that the converse should be true.

For actions of  $\mathbb{Z}$  there are only two classes of  $C^*$ -algebras where we currently have complete information on the quasidiagonality of  $A \rtimes_{\alpha} \mathbb{Z}$ ; when  $A$  is abelian or AF. Before stating the theorems we first give a definition.

**Definition 11.4.** Let  $A$  be a  $C^*$ -algebra. Then  $A$  is called *AF embeddable* if there exists a  $*$ -monomorphism  $\rho : A \rightarrow B$  where  $B$  is AF.

Of course AF embeddable  $C^*$ -algebras are QD. However, it is a non-trivial fact that the converse is not true. In fact, even RFD algebras need not be AF embeddable. The best known example is the full group  $C^*$ -algebra  $C^*(\mathbb{F}_2)$ . This is RFD but is not exact and hence cannot be embedded into *any* nuclear (in particular, AF) algebra (cf. [Was3]). However, for crossed products of abelian or AF algebras by  $\mathbb{Z}$ , quasidiagonality does imply AF embeddability.

**Theorem 11.5.** ([Pi, Thm. 9]) *Let  $\varphi : X \rightarrow X$  be a homeomorphism of the compact metric space  $X$  and  $\Phi \in \text{Aut}(C(X))$  denote the induced automorphism. Then the following are equivalent:*

1.  $C(X) \rtimes_{\Phi} \mathbb{Z}$  is AF embeddable,
2.  $C(X) \rtimes_{\Phi} \mathbb{Z}$  is QD,
3.  $C(X) \rtimes_{\Phi} \mathbb{Z}$  is stably finite,
4. ' $\varphi$  compresses no open sets.' (That is, if  $U \subset X$  is open and  $\varphi(\bar{U}) \subset U$  then  $\varphi(U) = U$ .)

**Theorem 11.6.** ([BrN1, Thm. 0.2]) *Let  $A$  be AF and  $\alpha \in \text{Aut}(A)$  be given. Then the following are equivalent:*

1.  $A \rtimes_{\alpha} \mathbb{Z}$  is AF embeddable,
2.  $A \rtimes_{\alpha} \mathbb{Z}$  is QD,

3.  $A \rtimes_{\alpha} \mathbb{Z}$  is stably finite,
4. ' $\alpha_* : K_0(A) \rightarrow K_0(\tilde{A})$  compresses no elements.' (That is, if  $x \in K_0(A)$  and  $\alpha_*(x) \leq x$  in the natural order then  $\alpha_*(x) = x$ .)

We have chosen to formulate the above results in a way that illustrates their similarities. In both cases the hard implications are  $4 \Rightarrow 1$ . Also in both cases it is not at all clear that the techniques in the proof will be of much use in general. Before going beyond actions of  $\mathbb{Z}$  we wish to point out that there is no harm in assuming unital algebras.

**Proposition 11.7.** *Let  $A$  be nonunital,  $\alpha \in \text{Aut}(A)$ ,  $\tilde{A}$  be the unitization of  $A$  and  $\tilde{\alpha} \in \text{Aut}(\tilde{A})$  the unique unital extension of  $\alpha$ . Then  $A \rtimes_{\alpha} \mathbb{Z}$  is QD if and only if  $\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}$  is QD.*

*Proof.* Recall that we always have a split exact sequence

$$0 \rightarrow A \rtimes_{\alpha} \mathbb{Z} \rightarrow \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z} \rightarrow C(\mathbb{T}) \rightarrow 0.$$

Thus the implication  $(\Leftarrow)$  is immediate and  $(\Rightarrow)$  follows from Theorem 10.12 since abelian algebras are nuclear, QD and satisfy the Universal Coefficient Theorem (cf. [RS]). Q.E.D.

Another natural direction to consider would be to try crossed products of well behaved  $C^*$ -algebras by more general groups. (We must stay within the class of amenable groups, though, because of Rosenberg's result; cf. Theorem 11.3) However, even for actions of  $\mathbb{Z}^2$  this is a problem. Indeed the following question of Voiculescu remains open even now – more than 15 years after Pimsner's result for  $C(X) \rtimes_{\Phi} \mathbb{Z}$ .

**Question 11.8.** (cf. [Vo4, 4.6]) *When is  $C(X) \rtimes_{\Phi} \mathbb{Z}^2$  AF embeddable?*<sup>2</sup>

For crossed products of certain simple AF algebras the question is more manageable.

**Theorem 11.9** (BrN2, Thm. 1). *If  $A$  is UHF and  $\alpha : \mathbb{Z}^n \rightarrow \text{Aut}(A)$  is a homomorphism then there always exists a  $*$ -monomorphism  $\rho : A \rtimes_{\alpha} \mathbb{Z}^n \rightarrow B$  where  $B$  is AF.*

The proof of this result (and Theorem 11.6 above) depends in an essential way on a technical notion known as the *Rohlin property* for automorphisms. This notion has been used by Connes, Kishimoto, Evans, Nakamura and others (with great success!) in classifying automorphisms of operator algebras. Moreover, Kishimoto has used these

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<sup>2</sup>Some nice progress, in the case where  $X$  is a Cantor set, has recently been made by Matui [Ma].

ideas to prove that many crossed products of certain simple AT algebras by automorphisms with the Rohlin property will again be AT (which is much stronger than just saying they are QD). See, for example, [Kis1-4].

**Remark 11.10.** *One nice consequence of Green's theorem (Theorem 11.1) is that understanding crossed products by  $\mathbb{Z}^n$  gives results about much more general groups. For example, if  $G$  is a finitely generated discrete abelian group then  $G \cong \mathbb{Z}^n \oplus F$  where  $F$  is a finite (hence compact) abelian group then by Green's result we have an embedding  $A \rtimes_{\alpha} G \hookrightarrow (A \rtimes_{\alpha|_{\mathbb{Z}^n}} \mathbb{Z}^n) \otimes \mathcal{K}$ . Writing a general discrete abelian group as an inductive limit of finitely generated such groups one can then handle crossed products by arbitrary discrete abelian groups. One can then proceed to take extensions by arbitrary separable compact groups and build a very large class of groups for which it suffices to consider crossed products by  $\mathbb{Z}^n$ . (See Def. 3.4 and the proof of Thm. 2 in [BrN2] for more details).*

## §12. Relationship with Nuclearity

It was an open question for quite some time whether or not quasidiagonality implied nuclearity. In [Had2], Hadwin asked whether or not every 'strongly' quasidiagonal (e.g. simple QD)  $C^*$ -algebra was nuclear. Then in [Po], Popa asked whether every simple unital QD  $C^*$ -algebra with 'sufficiently many projections' (e.g. real rank zero) was nuclear. There was some evidence supporting a positive answer to these questions. The strongest was the following theorem of Popa.

**Theorem 12.1** (Po, Thm. 1.2). *Let  $A$  be a simple unital  $C^*$ -algebra with 'sufficiently many projections' (e.g. real rank zero). Then  $A$  is QD if and only if for each finite set  $\mathcal{F} \subset A$  and  $\varepsilon > 0$  there exists a (non-zero) finite dimensional subalgebra  $B \subset A$  with unit  $P = 1_B$  such that  $\|[a, P]\| \leq \varepsilon$  for all  $a \in \mathcal{F}$  and  $PFP \subset^{\varepsilon} B$  (cf. Definition 4.4).*

The necessity of the technical condition above is quite hard, however the sufficiency is easily seen. Indeed, if one assumes the technical condition then we can find a sequence of finite dimensional subalgebras  $B_n \subset A$  with units  $P_n$  such that  $\|[a, P_n]\| \rightarrow 0$  and  $d(P_n a P_n, B) \rightarrow 0$  for all  $a \in A$ . Now let  $\Phi_n : A \rightarrow B_n$  be a conditional expectation and consider the maps  $\varphi_n : A \rightarrow B_n$  defined by  $\varphi_n(a) = \Phi_n(P_n a P_n)$ . This sequence of maps is evidently asymptotically multiplicative and hence defines a  $*$ -homomorphism

$$A \rightarrow \Pi B_n / \oplus B_n.$$



Since  $A$  is unital this morphism is nonzero and since  $A$  is simple, this morphism is injective. Hence the maps  $\varphi_n$  are also asymptotically isometric which implies (by Theorem 4.2) that  $A$  is QD. (Note that the hypothesis of a unit can't be dropped here. Indeed, the stabilization  $\mathcal{K} \otimes A$  of any unital  $C^*$ -algebra  $A$  satisfies the technical condition stated above. Simply take  $B_n$  of the form  $C_n \otimes 1_A$  where  $C_n$  is almost orthogonal to a large part of  $\mathcal{K}$ .)

The above result gave one hope of deducing nuclearity via the characterization in terms of injective enveloping von Neumann algebras (cf. [CE1]). However, it turns out that this is not possible as the following result of Dădărlat shows.

**Theorem 12.2** (Dăd2, Prop. 9). *There exists a unital, separable, simple, QD  $C^*$ -algebra with real rank zero, stable rank one and unique tracial state which is not exact (and hence not nuclear).*

The converse of the question we have been considering above is also interesting and worth discussion. Namely, what sort of general conditions on a  $C^*$ -algebra imply quasidiagonality?

**Example 12.3.** *A Cuntz algebra  $\mathcal{O}_n$  (cf. [Cu]) is simple, separable, unital, nuclear, has real rank zero and is not QD (since it is purely infinite; cf. Proposition 3.19).*

To get a finite non-QD example is a bit more delicate. Recall that  $C_r^*(G)$ , where  $G$  is a discrete group, is always stably finite since it has a faithful tracial state. Also recall that Rosenberg has shown that if the reduced group  $C^*$ -algebra of a discrete group is QD then the group must be amenable (cf. Theorem 11.3).

**Example 12.4.** *Let  $\mathbb{F}_2$  denote the free group on two generators. Then  $C_r^*(\mathbb{F}_2)$  is simple, unital, separable, exact, has stable rank one (cf. [DHR]) and a unique tracial state but is not QD since  $\mathbb{F}_2$  is not amenable.*

To get an example with the added property of real rank zero one can simply consider  $C_r^*(\mathbb{F}_2) \otimes U$ , where  $U$  is some UHF algebra (cf. [Rør, Thm. 7.2]).

It is also interesting to note that there are no known examples of finite nuclear non-QD  $C^*$ -algebras. (Recall that  $C_r^*(\mathbb{F}_2)$  is only exact.) In fact, as noted in Section 10, Blackadar and Kirchberg have formulated the following question.

**Question 12.5** (BK1, Question 7.3.1). *If  $A$  is nuclear and stably finite then must  $A$  necessarily be QD?*

This question is of particular interest in Elliott's classification program (cf. [E11]). Indeed, if this question turns out to have an affirmative answer then classifying simple unital nuclear *finite*  $C^*$ -algebras may be equivalent to classifying simple unital nuclear QD  $C^*$ -algebras. (One would still have to resolve the important open question of whether every simple finite algebra is stably finite - which is equivalent to the open question of whether every simple infinite  $C^*$ -algebra is purely infinite.<sup>3</sup>) The point is that for simple QD algebras (with enough projections) one has the structure theorem of Popa to work with. In fact, Lin has introduced a class of  $C^*$ -algebras (the so-called TAF algebras; [Li1]) whose definition is inspired by - and looks very similar to - Popa's structure theorem. Moreover, there are classification results for some of these TAF algebras (cf. [Li2,4], [DE1]) and it is not unreasonable to think that someday the general QD case can be handled in ways similar to the current strategies being applied to the TAF case.

### §13. More Advanced Topics

In our final section we will present some miscellaneous results which don't quite fit into any of the previous sections. The first is a very important result of Voiculescu which shows that quasidiagonality is a homotopy invariant. Recall that two  $C^*$ -algebras  $A$  and  $B$  are called *homotopic* if there exist  $*$ -homomorphisms  $\varphi : A \rightarrow B$  and  $\psi : B \rightarrow A$  such that  $\varphi \circ \psi$  is homotopic to  $id_B$  and  $\psi \circ \varphi$  is homotopic to  $id_A$  (cf. [B12], [We]).

**Theorem 13.1.** *Let  $A$  and  $B$  be homotopic  $C^*$ -algebras. Then  $A$  is QD if and only if  $B$  is QD.*

Voiculescu actually proved a more general result (cf. [Vo3, Thm. 5]). In [Dăd1, Thm. 1.1] Dădărlat generalized this to show that quasidiagonality is even an invariant of the weaker notion of 'asymptotic completely positive homotopy equivalence'. As mentioned previously, this result implies that the cone over *any*  $C^*$ -algebra is QD since cones are homotopic to  $\{0\}$ .

Free products of  $C^*$ -algebras were introduced in [Av] and independently in [Vo5]. (See also [VDN].) Reduced free products are rarely QD. The standard example of a reduced free product is  $C_r^*(\mathbb{F}_2) = C^*(\mathbb{Z}) * C^*(\mathbb{Z})$ , where the reduced free product is taken with respect

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<sup>3</sup>Rørørdam has recently answered this question - there exist simple, nuclear algebras which are infinite (resp. finite) but not purely infinite (resp. stably finite - hence not QD) [Rør2].

to Haar measure on the circle. The next result of F. Boca is in stark contrast. (See also [ExLo] where the class of RFD algebras is shown to be closed under full free products.)

**Theorem 13.2** (Bo, Prop. 13). *If  $A$  and  $B$  are unital QD  $C^*$ -algebras, then the full free product (amalgamating over the units)  $A * B$  is also QD.*

We next point out the connection between quasidiagonality and the notions of projectivity and semiprojectivity. These notions are studied at length in [Lo].

**Definition 13.3.** Let  $A$  be a  $C^*$ -algebra. Then  $A$  is called *projective* if for every  $C^*$ -algebra  $B$ , closed 2-sided ideal  $I \subset B$  and  $*$ -homomorphism  $\varphi : A \rightarrow B/I$  there exists a lifting  $*$ -homomorphism  $\psi : A \rightarrow B$ .  $A$  is called *semiprojective* if for every  $C^*$ -algebra  $B$ , closed 2-sided ideal  $I \subset B$  such that  $I = \overline{\cup_n I_n}$  for ideals  $I_1 \subset I_2 \subset \dots$  and  $*$ -homomorphism  $\varphi : A \rightarrow B/I$  there exists an  $n$  and a lifting  $*$ -homomorphism  $\psi : A \rightarrow B/I_n$  (that is, a lifting for the canonical quotient map  $B/I_n \rightarrow B/I$ ).

The projective case in our next result is well known. The semiprojective case was pointed out by B. Blackadar, though his proof was different.

**Proposition 13.4.** *If  $A$  is projective then  $A$  is RFD. If  $A$  is MF and semiprojective then  $A$  is RFD.*

*Proof.* First assume that  $A$  is projective. By Corollary 5.3  $A$  is a quotient of an RFD algebra. But then the definition of projectivity implies that  $A$  embeds into an RFD algebra and hence is itself RFD.

Now assume that  $A$  is semiprojective and MF. By the proof of Proposition 9.3 we can find an RFD algebra  $R$  with finite dimensional ideals  $I_n \subset I_{n+1}$  such that  $A \cong R/I$  where  $I = \overline{\cup_n I_n}$ . The definition of semiprojectivity then provides an embedding  $A \hookrightarrow R/I_n \subset R$  for some  $n$ . Q.E.D.

We now discuss a beautiful connection between quasidiagonality and the question of whether or not 'Ext is a group'. (See also the discussion in [Vo4].) Here we mean the classical BDF Ext semigroups. Recall that if  $A$  is nuclear then the Choi-Effros lifting theorem implies that  $Ext(A)$  is a group. (See [Ar] for a very nice treatment of this theory.) But it is known that there exist  $C^*$ -algebras  $A$  for which  $Ext(A)$  is not a group (cf. [An], [Was1,2], [Kir1]). For example, Kirchberg has shown that if  $A$  is the unitization of the cone over  $C_r^*(\mathbb{F}_2)$  then  $Ext(A)$  is not a group. However, it has been a long standing open problem to determine

whether or not  $\text{Ext}(C_r^*(\mathbb{F}_2))$  is a group. It is believed that  $\text{Ext}(C_r^*(\mathbb{F}_2))$  is not a group and we now outline one approach to proving this.

We described the class of MF algebras in Section 9. Recall that these algebras can be characterized as those which appear as the image in the Calkin algebra of a quasidiagonal set of operators in  $B(H)$  (cf. Proposition 9.2).

**Corollary 13.5.** *Let  $A$  be MF and assume  $\text{Ext}(A)$  is a group. Then  $A$  is QD.*

*Proof.* If  $\text{Ext}(A)$  is a group then every \*-monomorphism  $\varphi : A \rightarrow B(H)/\mathcal{K}$  has a completely positive lifting (cf. [Ar, pg. 353]). But then from Propositions 8.3 we see that  $A$  must be QD. Q.E.D.

It follows then that every nuclear MF algebra is QD. Recall, though, that there exist non-QD MF algebras. But it is not known whether Wassermann's examples are exact. The following question remains open.

**Question 13.6.** *Do there exist exact non-QD MF algebras? In particular, is  $C_r^*(\mathbb{F}_2)$  MF?*<sup>4</sup>

Kirchberg has also proved some remarkable results connecting quasidiagonality,  $\text{Ext}$  and various lifting properties of  $C^*$ -algebras (see [Kir1]).

Finally, we wish to point out a connection with one of the most important questions in  $C^*$ -algebras. Namely, whether or not the Universal Coefficient Theorem (UCT) holds for all nuclear separable  $C^*$ -algebras (cf. [RS]). We will not formulate this question precisely as it is well out of the scope of these notes. However, the experts will have no problem following our argument. The main ingredient is the following 'two out of three principle' for the UCT.

**Theorem 13.7.** *(cf. [RS, Prop. 2.3 and Thm. 4.1]) Let  $0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0$  be a short exact sequence with  $E$  nuclear and separable. If any two of  $\{I, E, B\}$  satisfy the UCT then so does the third. In particular, if  $I$  and  $E$  satisfy the UCT then so does  $B$ .*

Our final result has been noticed by several experts.

**Corollary 13.8.** *If the UCT holds for all separable nuclear RFD algebras then the UCT holds for all separable nuclear  $C^*$ -algebras.*

*Proof.* By the two out of three principle, it suffices to show that every separable nuclear  $C^*$ -algebra is a quotient of a separable nuclear RFD algebra. But this is contained in Corollary 5.3 Q.E.D.

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<sup>4</sup>We have been informed that Haagerup and Thorbjørnsen have now resolved this question affirmatively and hence  $\text{Ext}(C_r^*(\mathbb{F}_2))$  is not a group.

### §14. Further Reading

Below are references to some of the topics around quasidiagonality which are only briefly discussed (or not discussed at all) in these notes.

*AF embeddability.* [BrN1,2], [Dăd4,5], [Ka], [Li3], [Ma], [Oz], [Pi], [PV2], [Rør3], [Sp], [Vo2].

*Ext and KK-theory.* [BrL2], [DE2], [DHS], [Kir1], [PV1], [Sa1,2], [Sc1,2], [Wa1,2].

*Classification.* [DE1], [Ell], [Li2,4] and their bibliographies.

*MF, (strong) NF, and inner quasidiagonal algebras.* [BK1,2], [KW].

*General.* [Had2], [Th], [Vo4].

### §15. Appendix: Nonseparable QD $C^*$ -algebras

In this appendix we treat the case of nonseparable  $C^*$ -algebras. Hence we no longer require the Hilbert spaces in this section to be separable either. The results of this section (in particular Corollary 15.7) are necessary for the general case of Voiculescu's characterization of QD  $C^*$ -algebras. Though we have seen some of these results stated in the literature, we have been unable to find any proofs and hence complete proofs will be given.

**Definition 15.1.** A subset  $\Omega \subset B(H)$  is called a *quasidiagonal set of operators* if for each finite set  $\omega \subset \Omega$ , finite set  $\chi \subset H$  and  $\varepsilon > 0$  there exists a finite rank projection  $P \in B(H)$  such that  $\|[T, P]\| \leq \varepsilon$  and  $\|P(x) - x\| \leq \varepsilon$  for all  $T \in \omega$  and  $x \in \chi$ .

It is still easy to see that a set  $\Omega \subset B(H)$  is a quasidiagonal set of operators if and only if the  $C^*$ -algebra generated by  $\Omega$ ,  $C^*(\Omega) \subset B(H)$ , is a quasidiagonal set of operators.

We may finally give the general definition of a quasidiagonal  $C^*$ -algebra.

**Definition 15.2.** Let  $A$  be a  $C^*$ -algebra. Then  $A$  is called *quasidiagonal (QD)* if there exists a faithful representation  $\pi : A \rightarrow B(H)$  such that  $\pi(A)$  is a quasidiagonal set of operators.

There is one subtle point that needs resolved here. Namely we must show that the previous definition is equivalent to Definition 3.8 in the case that  $A$  is a separable  $C^*$ -algebra.

**Lemma 15.3.** *Let  $A$  be a separable  $C^*$ -algebra and assume that there exists a faithful representation  $\pi : A \rightarrow B(H)$  such that  $\pi(A)$  is a*

quasidiagonal set of operators. Then there exists a faithful representation  $\rho : A \rightarrow B(K)$  such that  $K$  is a separable Hilbert space and  $\rho(A)$  is a quasidiagonal set of operators.

*Proof.* Let  $\pi : A \rightarrow B(H)$  be a faithful representation such that  $\pi(A)$  is a quasidiagonal set of operators. We will show that there exists a separable subspace  $K \subset H$  which is  $\pi(A)$ -invariant and such that the restriction representation  $\rho = \pi_K = P_K \pi(\cdot) P_K : A \rightarrow B(K)$  (cf. Definition 3.9) is faithful and has the property that  $\rho(A)$  is a quasidiagonal set of operators.

The idea is to construct an increasing sequence of separable  $\pi(A)$ -invariant subspaces  $\tilde{K}_1 \subset \tilde{K}_2 \subset \tilde{K}_3 \dots$  and finite rank projections  $Q_n$  such that  $Q_n(H) \subset \tilde{K}_{n+1}$ ,  $\|[Q_n, \pi(a)]\| \rightarrow 0$  for all  $a \in A$  and  $\|Q_n(\xi) - \xi\| \rightarrow 0$  for all  $\xi \in \cup \tilde{K}_i$ . If we further arrange that the restriction of  $\pi(A)$  to  $\tilde{K}_1$  is faithful then it is clear that  $K = \cup \tilde{K}_i$  is the desired subspace.

We begin by choosing a sequence  $\{a_i\} \subset A$  which is dense in the unit ball of  $A$ . For each  $n \in \mathbb{N}$  we then choose a sequence of unit vectors  $\{\xi_i^{(n)}\}_{i \in \mathbb{N}} \subset H$  such that  $\|\pi(a_i)\xi_i^{(n)}\| > \|a_i\| - 1/2^n$ . Let  $K_1 \subset H$  be the closure of the span of  $\{\xi_i^{(n)}\}_{i, n \in \mathbb{N}}$  and let  $\tilde{K}_1$  be the closure of  $\pi(A)K_1$ . Then it is clear that  $\tilde{K}_1$  is separable,  $\pi(A)$ -invariant and the restriction of  $\pi(A)$  to  $\tilde{K}_1$  is faithful (since it is isometric on  $\{a_i\}$ ).

One then constructs the desired  $\tilde{K}_i$  and  $Q_i$  recursively as follows. Let  $\{h_i^{(1)}\}$  be an orthonormal basis for  $\tilde{K}_1$ . Choose a finite rank projection  $Q_1 \in B(H)$  such that  $\|[Q_1, \pi(a_1)]\| < 1/2$  and  $Q_1(h_1^{(1)}) = h_1^{(1)}$ . Recall from the proof of Proposition 3.4 that we can always arrange the stronger condition  $Q_1(h_1^{(1)}) = h_1^{(1)}$ .

Next let  $X_2 = \text{span}\{Q_1(H), \tilde{K}_1\}$ ,  $\tilde{K}_2$  be the closure of  $\pi(A)X_2$  and let  $\{h_i^{(2)}\}$  be an orthonormal basis for  $\tilde{K}_2$ . Now choose a finite rank projection  $Q_2 \in B(H)$  such that  $\|[Q_2, \pi(a_i)]\| < 1/(2^2)$  for  $i = 1, 2$ ,  $Q_2(h_i^{(j)}) = h_i^{(j)}$  for  $i, j = 1, 2$  and  $Q_1 \leq Q_2$  (this is arranged by requiring that  $Q_2(h) = h$  for a (finite) basis of  $Q_1(H)$ ).

Next let  $X_3 = \text{span}\{Q_2(H), \tilde{K}_2\}$ ,  $\tilde{K}_3$  be the closure of  $\pi(A)X_3$  and let  $\{h_i^{(3)}\}$  be an orthonormal basis for  $\tilde{K}_3$ , etc. Proceeding in this way we get an increasing sequence of separable  $\pi(A)$ -invariant subspaces  $\tilde{K}_1 \subset \tilde{K}_2 \subset \tilde{K}_3 \dots$  with orthonormal bases  $\{h_i^{(n)}\}_{i \in \mathbb{N}}$  and finite rank projections  $Q_n \leq Q_{n+1}$  such that  $Q_n(h_i^{(j)}) = h_i^{(j)}$  for  $i, j = 1, \dots, n$  and  $Q_n(H) \subset \tilde{K}_{n+1}$ ,  $\|[Q_n, \pi(a)]\| \rightarrow 0$  for all  $a \in A$ . Evidently this proves the lemma. Q.E.D.

Hence we see that Definitions 3.8 and 14.2 are equivalent for separable  $C^*$ -algebras. Indeed, it is clear that if  $A$  is separable and satisfies

Definition 3.8 then  $A$  also satisfies Definition 14.2. On the other hand, if  $A$  is separable and satisfies Definition 14.2 then by the previous lemma we can find a representation of  $A$  on a *separable* Hilbert space which gives a quasidiagonal set of operators and hence  $A$  satisfies Definition 3.8 as well.

We will need the following elementary, but technical, lemma.

**Lemma 15.4.** *Let  $\pi : A \rightarrow B(H)$  be a faithful representation where  $A$  is separable (but  $H$  is not). Then there exists a separable  $\pi(A)$ -invariant subspace  $K \subset H$  with the property that  $\pi_K : A \rightarrow B(K)$  is faithful,  $\pi_K(a)$  is a finite rank operator if and only if  $\pi(a)$  is a finite rank operator and in this case  $\dim(\pi(a)H) = \dim(\pi_K(a)K)$ .*

*Proof.* The idea is to find a sequence of  $\pi(A)$ -invariant separable subspaces,  $\tilde{H}_i$ , with the following properties:

1. The restriction of  $\pi(A)$  to  $\tilde{H}_1$  is faithful.
2. If  $a \in A$  is such that  $\pi(a)$  is a finite rank operator then  $\pi(a)H \subset \tilde{H}_1$ .
3.  $\tilde{H}_m \perp (\tilde{H}_1 \oplus \tilde{H}_2 \oplus \dots \oplus \tilde{H}_{m-1})$
4. If  $P_{\tilde{H}_m} \pi(a) P_{\tilde{H}_m} = 0$  then

$$(1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}}) \pi(a) (1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}}) = 0$$

for all  $a \in A$ , where for any subspace  $L \subset H$ ,  $P_L$  denotes the orthogonal projection onto  $L$ .

Having the subspaces  $\{\tilde{H}_i\}$  we define

$$K = \bigoplus_{i=1}^{\infty} \tilde{H}_i \subset H$$

and note that  $\pi_K : A \rightarrow B(K)$  is faithful (since this was already arranged on  $\tilde{H}_1$ ). Moreover, condition 2 ensures that if  $\pi(a)$  is a finite rank operator then  $\dim(\pi(a)H) = \dim(\pi_K(a)K)$ . Finally, note that if  $\pi_K(a)$  is a finite rank operator then there exists some integer  $m \in \mathbb{N}$  such that  $\tilde{\pi}_m(a) = P_{\tilde{H}_m} \pi(a) P_{\tilde{H}_m} = 0$ . Hence

$$\begin{aligned} \pi(a) &= \tilde{\pi}_1(a) \oplus \dots \oplus \tilde{\pi}_{m-1}(a) \\ &\quad \oplus (1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}}) \pi(a) (1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}}) \\ &= \tilde{\pi}_1(a) \oplus \dots \oplus \tilde{\pi}_{m-1}(a) \oplus 0 \\ &= \pi_K(a), \end{aligned}$$

by condition 4 above. Hence  $\pi(a)$  is also a finite rank operator and clearly  $\dim(\pi(a)H) = \dim(\pi_K(a)K)$ . So we now show how to construct subspaces  $\tilde{H}_i$  as above.

Begin by letting  $\mathcal{F}(A) = \{a \in A : \dim(\pi(a)H) < \infty\}$  and choosing a countable dense subset  $\{a_i\}_{i \in \mathbb{N}} \subset \mathcal{F}(A)$ . For each  $i \in \mathbb{N}$  let  $L_i = \pi(a_i)H$  and define  $H_1$  to be the closure of

$$\text{span}\left\{\bigcup_{i=1}^{\infty} \pi(A)L_i\right\}.$$

By throwing in a countable number of vectors (as in the proof of Lemma 15.3) we can replace  $H_1$  with a larger  $\pi(A)$ -invariant subspace  $\tilde{H}_1$  such that the restriction of  $\pi(A)$  to  $\tilde{H}_1$  is also faithful. We claim that this  $\tilde{H}_1$  also satisfies condition 2 above. Indeed, if  $a \in \mathcal{F}(A)$  then we can find a subsequence  $a_{i_j} \rightarrow a$ . But since  $\pi(a_{i_j})H \subset \tilde{H}_1$  and  $\pi(a_{i_j}) \rightarrow \pi(a)$  it is clear that  $\pi(a)H \subset \tilde{H}_1$  as well. Hence we have constructed  $\tilde{H}_1$  with the desired properties.

Assume now that we have constructed  $\tilde{H}_1, \dots, \tilde{H}_{m-1}$  with the desired properties. To get  $\tilde{H}_m$  we simply consider the separable  $C^*$ -algebra

$$C = (1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}})\pi(A)(1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}}).$$

By the proof of Lemma 15.3 we can find a separable  $C$ -invariant subspace  $\tilde{H}_m \subset (1 - P_{\tilde{H}_1 \oplus \dots \oplus \tilde{H}_{m-1}})H$  such that the restriction of  $C$  to  $\tilde{H}_m$  is faithful. Evidently  $\tilde{H}_m$  is also  $\pi(A)$  invariant, perpendicular to  $\tilde{H}_1 \oplus \tilde{H}_2 \oplus \dots \oplus \tilde{H}_{m-1}$  and condition 4 above is nothing more than the statement that the map  $C \rightarrow P_{\tilde{H}_m} C P_{\tilde{H}_m}$  is faithful. Q.E.D.

As in section 3 we want to resolve the technical issue of nondegeneracy of representations.

**Lemma 15.5.** *Let  $A$  be a  $C^*$ -algebra and  $\pi : A \rightarrow B(H)$  be a faithful representation. Let  $L \subset H$  be the nondegeneracy subspace of  $\pi(A)$  and  $\pi_L : A \rightarrow B(L)$  denote the restriction. Then  $\pi(A)$  is a quasideagonal set of operators if and only if  $\pi_L(A)$  is a quasideagonal set of operators.*

*Proof.* The implication  $(\Leftarrow)$  is proved exactly as in Lemma 3.10. Also, if  $A$  is unital, the implication  $(\Rightarrow)$  is the same and so we only have to show  $(\Rightarrow)$  in the case that  $A$  is nonunital.

So assume that  $A$  is nonunital and  $\pi(A)$  is a quasideagonal set of operators. Note that we cannot apply Voiculescu's Theorem in this setting since the dimensions of  $H$  and  $L$  may be different. To resolve this problem we first note that since quasideagonality is defined via finite sets it suffices to show that  $\pi_L(B)$  is a quasideagonal set of operators for every separable  $C^*$ -subalgebra  $B \subset A$ .



Given a finite set of vectors  $\chi \subset L$ , by Lemma 15.4, we can find a separable subspace  $K \subset L$  with the property that  $\chi \subset K$ ,  $K$  is  $\pi_L(A)$ -invariant, the restriction to  $K$  is faithful,  $\pi_L(a)$  is finite rank if and only if  $\pi_K(a)$  is finite rank and in this case  $\text{rank}(\pi_L(a)) = \text{rank}(\pi_K(a))$ . As in the proof of Lemma 15.3 we can now enlarge  $K$  to a separable  $\pi(A)$ -invariant subspace  $\tilde{K} \subset H$  (we do not have  $\tilde{K} \subset L$ , of course) such that  $\pi_{\tilde{K}}(A)$  is a quasidiagonal set of operators. Since we have been careful about separability and preservation of rank it now follows from Voiculescu's theorem (version 2.6) that  $\pi_{\tilde{K}}$  and  $\pi_K$  are approximately unitarily equivalent and hence  $\pi_K(A)$  is a quasidiagonal set of operators. Q.E.D.

**Theorem 15.6.** *Let  $\pi : A \rightarrow B(H)$  be a faithful essential (cf. Definition 2.8) representation. Then  $A$  is QD if and only if  $\pi(A)$  is a quasidiagonal set of operators.*

*Proof.* Clearly we only have to prove the necessity. As in the proof of the previous lemma, it suffices to show that  $\pi(B)$  is a quasidiagonal set of operators for every separable subalgebra  $B \subset A$ .

Let  $\chi \subset H$  be an arbitrary finite set and use Lemma 15.4 to construct a separable  $\pi(B)$ -invariant subspace  $K \subset H$  such that  $\chi \subset K$  and the restriction to  $K$  is both faithful and essential. The remainder of the proof is now similar to that of Theorem 3.11. Q.E.D.

The next corollary shows that with care, one can usually just treat the separable case when dealing with quasidiagonality.

**Corollary 15.7.**  *$A$  is QD if and only if all of its finitely generated subalgebras are QD.*

*Proof.* The necessity is obvious from the definition. So assume all finitely generated subalgebras of  $A$  are QD and let  $\pi : A \rightarrow B(H)$  be a faithful essential representation. Then for each finitely generated subalgebra  $B \subset A$  the restriction  $\pi|_B$  is a faithful essential representation and hence (by Theorem 15.6)  $\pi(B)$  is a quasidiagonal set of operators. It then follows that  $\pi(A)$  is a quasidiagonal set of operators. Q.E.D.

Finally we observe the nonseparable version of Theorem 4.2.

**Corollary 15.8** (Voiculescu). *Let  $A$  be a  $C^*$ -algebra. Then  $A$  is QD if and only if  $A$  satisfies (\*).*

*Proof.* As in the proof of Theorem 4.2, we may assume that  $A$  is unital. From Arveson's Extension Theorem it follows that  $A$  satisfies (\*) if and only if every separable unital subalgebra of  $A$  satisfies (\*). Similarly, from Corollary 15.7 it follows that  $A$  is QD if and only if every separable unital subalgebra of  $A$  is QD. Hence this corollary follows from the separable case. Q.E.D.

**Acknowledgements.** These notes are based on a series of lectures given by the author at the University of Tokyo in the spring of 1999. Special thanks to the participants for enduring our lectures. We gratefully acknowledge the support of a NSF Dissertation Enhancement Award, which allowed us to spend one year in Tokyo, and a NSF Postdoctoral Fellowship which supported the final write-up of these notes. Finally, I must thank my thesis advisor, Marius Dădărlat, and Larry Brown for teaching me so much about QD  $C^*$ -algebras.

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