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The Moduli Space of Curves of Genus 4 and Deligne-Mostow's Complex Reflection Groups

Shigeyuki Kondō

§1. Introduction

In this note we shall show that the moduli space of curves of genus 4 is birational to an arithmetic quotient of the 9-dimensional complex ball and that the arithmetic subgroup is commensurable with one of Deligne-Mostow's complex reflection groups related to hypergeometric functions. Let C be a non-hyperelliptic curve of genus 4. Then its canonical model is the intersection of a quadric Q and a cubic S in $\mathbf{P}^3$. Let X be the minimal resolution of the triple cover of Q branced along C which is a $K3$ surface with an automorphism σ of order 3. The period domain of the pairs (X, σ) is a 9-dimensional complex ball $\mathcal{B}$. This gives an isomorphism between the moduli space of non-hyperelliptic curves of genus 4 and an arithmetic quotient $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ where $\mathcal{H}$ is the union of hyperplanes of $\mathcal{B}$ and Γ is an arithmetic subgroup of $Aut(\mathcal{B})$ (§2, Theorem 1). We remark that $\mathcal{H}$ consists of two components $\mathcal{H}_n$ and $\mathcal{H}_h$ so that a generic point of $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) corresponds to a curve of genus 4 with a node (resp. a hyperelliptic curve of genus 4 plus a point on the quotient of the hyperelliptic curve by the hyperelliptic involution) (§3, Theorem 2). The method works in some other cases, for example, the moduli space of universal curves of genus 2, 3 or del Pezzo surfaces of degree 1–4 (see Remarks 1–6).

The above $K3$ surface X has the structure of an elliptic fibration $\pi : X \longrightarrow \mathbf{P}^1$ which is induced from a ruling on Q. The automorphism σ acts on each fiber of π as an automorphism of order 3, and hence the functional invariant of π is constant. Moreover, for a generic X, this fibration has twelve singular fibers of type II in the sense of Kodaira [Ko],

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and hence this fibration gives twelve points on $\mathbf{P}^1$. This suggests a relation between Γ and Deligne-Mostow's complex reflection groups [DM], [M1], [M2]. In fact, in §4, Theorem 3, we shall show that our group Γ is commensurable with the largest Γ_μ in Deligne-Mostow's list where

$$\mu = (\mu_i) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$$

(No.1 in Deligne-Mostow's list [M2] and No.10 in Thurston's list [T]).

In [K2], we showed that the moduli space of curves of genus three is also birational to an arithmetic quotient of the 6-dimensional complex ball. In this case we take the 4-cyclic cover of $\mathbf{P}^2$ branched along a smooth plane quartic curve. Then we have a $K3$ surface with an automorphism of order 4. However this arithmetic subgroup does not appear in Deligne-Mostow's list (the corresponding $K3$ surface has no elliptic fibration invariant under the automorphism of order 4). We remark that the moduli space of curves of hyperelliptic curves of genus 3 or plane quartic curves with a node is birational to an arithmetic quotient of the 5-dimensional complex ball ([K2], §4, 5). Both of these arithmetic subgroups are commensurable with the group Γ_μ where

$$\mu = (\mu_i) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

(No.8 in Deligne-Mostow's list [M2] and No.3 in Thurston's list [T]).

For recent works related to this paper, we refer the reader to Allcock, Carlson, Toledo [ACT1], [ACT2], van Geemen, Izadi [vG], [vGI], Heckman, Looijenga [HL], Vakil [V].

In this paper we shall use the following notation: A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral non-degenerate symmetric bilinear form $\langle , \rangle$. A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$, and *unimodular* if its discriminant is ± 1. For a lattice L, we denote by L^* the dual of L, and by A_L the quotient group L^*/L. Let L be an even lattice. We extend the bilinear form on L to the one on L^* and define

$$q_L : A_L \longrightarrow \mathbf{Q}/2\mathbf{Z}, \quad q_L(x+L) = \langle x, x\rangle + 2\mathbf{Z}, \quad (x \in L^*)$$

which is called the *discriminant quadratic form.* We denote by A_m, D_n, E_k ($m \geq 1$, $n \geq 4$, $k = 6, 7, 8$) the negative definite lattice which is defined by the Cartan matrix of type A_m, D_n, E_k respectively. We also denote by U the lattice of signature $(1,1)$ defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For a lattice L and an integer m, $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m.

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§2. A ball quotient structure

In this section, we shall show that the moduli space of curves of genus 4 is birational to an arithmetic quotient of a 9-dimensional complex ball by using the periods of $K3$ surfaces with an automorphism of order 3.

Let C be a smooth non-hyperelliptic curve of genus 4. First we assume that C has no vanishing theta constants. Then its canonical model is the complete intersection of a smooth quadric Q and a cubic S in $\mathbf{P}^3$. Let X be the 3-cyclic cover of Q branched along C which is a $K3$ surface with an automorphism σ of order 3. We denote by L the second cohomology group $H^2(X, \mathbf{Z})$. Together with the cup product, L has the structure of a lattice which is even, unimodular and of signature $(3, 19)$. Let E (resp. F) be the inverse image of a general fiber of one of the rulings of Q (resp. another ruling of Q). Then E, F are smooth elliptic curves with $\langle E, F\rangle = 3$. Since σ has a fixed curve, σ acts on $H^0(X, \Omega^2_X)$ as a multiplication by a cube root ω of unity (Nikulin [N2], §5). We remark that $U(3) = H^2(X, \mathbf{Z})^{\langle\sigma\rangle}$. We have a 9-dimensional family of $K3$ surfaces with an automorphism of order 3. The transcendental lattice of a generic member of this family has rank 20 (see the definition of the period domain $\mathcal{B}$ in this section). Hence if C is generic, then the Picard lattice S_X of X is generated by E and F, and isometric to $U(3)$.

Next we consider the case that C has a vanishing theta constant. Then its canonical model is the complete intersection of a quadric cone Q' and a cubic S. By taking the minimal resolution of the triple covering of Q' branched along C, we have a $K3$ surface X' with an automorphism σ of order 3. Let R_1, R_2, R_3 be three smooth rational curves obtained by resolution of three rational double points of type A_1 of the triple cover of Q'. Let F be the pull back of a fiber of the ruling of Q'. Then F and $R_1 + R_2 + R_3$ generate the invariant part $H^2(X', \mathbf{Z})^{\langle\sigma\rangle}$ which is isomorphic to $U(3)$. For generic C with a vanishing theta constant, the Picard lattice of X' is generated by F, R_1, R_2, R_3 and isomorphic to $U \oplus A_2(2)$. By a result of Brieskorn [B], (X', σ) is a deformation of (X, σ). Thus the action of σ on the cohomology group does not depend on the condition whether C has a vanishing theta constant or not. It is known that σ^* fixes no non-zero vectors in $T \otimes \mathbf{Q}$ (Nikulin [N2], Theorem 3.1).

Let

$$T = U(3) \oplus U \oplus E_8 \oplus E_8$$

which is isometric to the orthogonal complement of $H^2(X, \mathbf{Z})^{\langle\sigma\rangle}$ ($\simeq U(3)$) in L. Let e, f (resp. e', f') be a basis of $U(3)$ (resp. U) with $e^2 = f^2 = 0, \langle e, f\rangle = 3$ (resp. $(e')^2 = (f')^2 = 0, \langle e', f'\rangle = 1$). Let ρ_1 be an isometry of $U(3) \oplus U$ defined by

$$\rho_1(e) = -2e + 3e', \quad \rho_1(f) = f + 3f',$$

$$\rho_1(e') = -e + e', \quad \rho_1(f') = -f - 2f'.$$

Note that ρ_1 is of order 3, has no non-zero fixed vectors in $U(3) \oplus U$ and acts on the discriminat of $U(3) \oplus U$ trivially. On the other hand, it is known that the root lattice E_8 can be regarded as a complex lattice defined over the Eisenstein integers (Allcock [A], §5). In other words, there exists an isometry ρ_2 on E_8 of order 3 which has no non-zero fixed vectors in E_8. We denote by ρ the isometry of $T = U(3) \oplus U \oplus E_8 \oplus E_8$ defined by

$$\rho = (\rho_1, \rho_2, \rho_2).$$

Note that ρ has order 3 and has no non-zero fixed vectors. Since ρ acts on T^*/T trivially, it can be extended to an isometry of $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$ which acts on the orthogonal complement $U(3)$ of T trivially. For simplicity, we also denote by the same ρ this isometry of L.

Now we shall define the period domain for the above $K3$ surfaces. Let X be a $K3$ surface with an automorphism σ of order 3 of above type. A *marking* of the pair (X, σ) is an isometry

$$\alpha_X : H^2(X, \mathbf{Z}) \longrightarrow L$$

with $\alpha_X \circ \sigma^* \circ \alpha_X^{-1} = \rho$ or ρ^{-1}. In the proof of Theorem 1, we shall show the existence of a marking for each pair. (Note that to prove the existence of a marking, it is enough to show this for some pair (X, σ) because our 9-dimensional family of $K3$ surfaces is irreducible). Let

$$T \otimes \mathbf{C} = T_\omega \oplus T_{\bar{\omega}}$$

be the decomposition into eigenspaces $T_\omega, T_{\bar{\omega}}$ of ρ with eigenvalues $\omega, \bar{\omega}$ respectively. Put

$$\mathcal{B} = \{z \in \mathbf{P}(T_\omega) : \langle z, \bar{z}\rangle > 0\}, \quad \Gamma = \{\phi \in O(T) : \phi \circ \rho = \rho \circ \phi\}.$$

Note that $\mathcal{B}$ is a bounded symmetric domian of type $I_{1,9}$, that is, a 9-dimensional complex ball. Also note that if $z \in \mathcal{B}$, then $\langle z, z \rangle = 0$. Hence $\mathcal{B}$ is contained in a 18-dimensional bounded symmetric domain $\mathcal{D}$ of type IV:

$$\mathcal{D} = \{z \in \mathbf{P}(T \otimes \mathbf{C}) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle > 0\}.$$

We call a vector r in T with $r^2 = -2$ a *root*. For a root r, we define

$$H_r = \{\omega \in \mathcal{B} : \langle r, \omega \rangle = 0\}, \quad \mathcal{H} = \bigcup_r H_r$$

where r varies over the set of all roots. It is known that for each X with an automorphism of order 3, $(H^2(X, \mathbf{Z})^{<\sigma^*>})^{\perp} \cap S_X$ contains no (-2)-vectors (Namikawa [Na], Theorem 3.10). Hence a marked $K3$ surface (X, σ, α_X) of above type determines the point $\alpha_X(\omega_X)$ in $\mathcal{B} \setminus \mathcal{H}$ where ω_X is a nowhere vanishing holomorphic 2-form on X. Thus $\mathcal{B} \setminus \mathcal{H}$ is the period domian of marked $K3$ surfaces of above type.

We remark that the natural map from $O(U(3))$ to $O(q_{U(3)})$ is surjective. In fact $O(q_{U(3)})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. The following isometries of $U(3)$ induce generators of $O(q_{U(3)})$: let e, f be a basis of $U(3)$ with $e^2 = f^2 = 0, \langle e, f \rangle = 3$. Then, with respect to this basis, the involutions

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

generate $O(q_{U(3)})$. This implies that the restriction map from $O(L)$ to $O(T)$ is surjective (Nikulin [N1] Proposition 1.6.1). Since $\rho \mid U(3) = 1$, we can easily see that the natural map from

$$\tilde{\Gamma} = \{\gamma \in O(L) : \gamma \circ \rho = \rho \circ \gamma\}$$

to Γ is surjective.

Theorem 1. $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ *is isomorphic to the moduli space of non-hyperelliptic curves of genus* 4.

Proof. Let $z \in \mathcal{B}$ $(\subset \mathcal{D})$. It follows from the surjectivity of the period map (Kulikov [Ku], Persson, Pinkham [PP]) that there exist a $K3$ surface X and an isometry

$$\alpha_X : H^2(X, \mathbf{Z}) \longrightarrow L$$

with $\alpha_X(\omega_X) = z$ where ω_X is a nowhere vanishing holomorphic 2-form on X.

In the following, we shall show that if $z \in \mathcal{B} \setminus \mathcal{H}$, then X has an automorphism σ with $\alpha_X \circ \sigma^* \circ \alpha_X^{-1} = \rho$, and X is obtained as a 3-cyclic cover of Q or Q' branched along a smooth curve C of genus 4 where Q is a smooth quadric and Q' is a quadric cone. We may identify $H^2(X, \mathbf{Z})$ with L by α_X.

First we remark that by the assumption $z \notin \mathcal{H}$, $(L^{\langle\rho\rangle})^{\perp} \cap z^{\perp}$ contains no roots. Hence ρ is induced from an automorphism σ of X of order 3 (Namikawa [Na], Theorem 3.10). Since σ acts on $H^0(X, \Omega_X^2)$ as a multiplication by ω, it acts on the tangent space of a fixed point as

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Hence the fixed locus of σ consists of the disjoint union of smooth curves and isolated fixed points. Note that the trace of ρ on L is -8. It follows from the topological Lefschetz fixed point formula that σ fixes a smooth curve C of genus $g(C) > 1$. Then by the Hodge index theorem, C is a unique fixed curve with $g(C) > 0$. On the other hand, $L^{\langle\rho\rangle} = U(3)$ contains no (-2)-vectors, σ has only one fixed curve C. Let k be the number of isolated fixed points of σ. Then by the topological Lefschetz fixed point formula, we have

$$k + 2 - 2g(C) = -6.$$

This implies $g(C) \geq 4$. Let $\{e, f\}$ be a basis of $L^{\langle\rho\rangle} = U(3)$ with $e^2 = f^2 = 0, \langle e, f\rangle = 3$. By the Riemann-Roch theorem, we may assume that both e and f are effective.

Claim. Either e or f is nef.

Proof of Claim. Assume that e is not nef. Then there exists a smooth rational curve R with $\langle R, e\rangle < 0$. Since $L^{\langle\rho\rangle}$ contains no (-2)-vectors, $R = r' + r''$ where $r' \in U(3)^*$ and $r'' \in (U(3)^{\perp})^*$, $r'' \neq 0$. Since $(L^{\langle\rho\rangle})^{\perp} \cap z^{\perp}$ is negative definite, $(r'')^2 < 0$. Put $r' = (me + nf)/3$ $(m, n \in \mathbf{Z})$. Since $\langle R, e\rangle < 0$, $n < 0$. If $m \leq 0$, then r' is not effective because we assume that e and f are effective. This contradicts the fact that $3r' = R + \sigma(R) + \sigma^2(R)$ is effective. Hence $m > 0$. By the equation $-2 = 2mn/3 + (r'')^2$ and $(r'')^2 < 0$, we have $(m, n) = (1, -1), (1, -2), (2, -1)$. In the last two cases, we have $(3r')^2 = (R + \sigma(R) + \sigma^2(R))^2 = -12$. On the other hand, $R^2 = -2$ and $R \neq \sigma(R)$, and hence $(R + \sigma(R) + \sigma^2(R))^2 \geq -6$. This is a contradiction. Thus $(m, n) = (1, -1)$ and $e - f = 3r'$ is effective. On the other hand, if f is not nef, the same argument shows that $f - e$ is effective. This is a contradiction. Hence we have proved the assertion. Q.E.D. for Claim.

Thus we may assume that e is nef , in other words, it gives an elliptic fibration

$$\pi : X \longrightarrow \mathbf{P}^1.$$

Let E be a general fiber of π. Let $C = ae + bf$. Then $6ab = C^2 = 2g(C) - 2 \geq 6$. If $b > 1$, then $C \cdot E \geq 6$. Since C is the fixed curve of σ, σ acts on the base of π trivially. Hence σ acts on E as an automorphism. Now by applying the Hurwitz formula to the pair (E, σ), we have a contradiction. Hence $b = 1$. Thus we have the following two cases:

Case 1. f is nef.

Case 2. f is not nef.

In Case 1, $a = b = 1$, $g(C) = 4$ and $k = 0$. By taking the quotient of X by σ, we have a smooth quadric surface $X/\langle\sigma\rangle$.

In Case 2, there exists a smooth rational curve R so that $\langle R, f\rangle < 0$. Since $U(3)$ contains no (-2)-vectors, R is not contained in $U(3)$. Let $R_1 = \sigma(R), R_2 = \sigma^2(R)$. Then $R + R_1 + R_2 \in U(3)$. By the same argument as in the proof of the claim, $R + R_1 + R_2 = -e + f$. Hence $\langle R, R_1\rangle = 0$. Since C is the fixed locus of σ, $\langle C, R\rangle = 0$. Hence $a = b = 1$, $g(C) = 4$ and $k = 0$. Now by taking the quotient by σ and contracting the (-2)-curve which is the image of R, R_1, R_2, we have a quadric cone.

Finally we shall show that the isomorphism class of C is uniquely determined by its image in $(\mathcal{B}\backslash\mathcal{H})/\Gamma$. Let C and C' be non-hyperelliptic curves of genus 4. Let (X, σ), (X', σ') be the corresponding $K3$ surfaces with automorphisms σ, σ' of order 3. Assume that their periods are the same in $(\mathcal{B} \setminus \mathcal{H})/\Gamma$. Since the natural map $\tilde{\Gamma} \longrightarrow \Gamma$ is surjective, there exists a Hodge isometry

$$\varphi : H^2(X, \mathbf{Z}) \longrightarrow H^2(X', \mathbf{Z})$$

with $\varphi \circ \sigma^* = (\sigma')^* \circ \varphi$. Since $(H^2(X, \mathbf{Z})^{<\sigma^*>})^\perp \cap S_X$ contains no (-2)-vectors, C is an ample class. Obviously $\varphi(C) = C'$. It now follows from the Torelli theorem for $K3$ surfaces (Piatetskii-Shapiro, Shafarevich [PS]) that there exists an isomorphism $f : X' \longrightarrow X$ with $f^* = \varphi$ and $f \circ \sigma' = \sigma \circ f$. By taking the quotient of X, X' by σ, σ' respectively, f induces an isomorphism between the canonical models of C and C'. Q.E.D.

Remark 1. The following was suggested by I. Dolgachev. In the same way as above, we can see that the moduli spaces $\mathcal{M}_{2,1}$, $\mathcal{M}_{3,1}$ of pointed curves of genus 2 and 3 have a ball quotient structure. Let (C, q) be a pair of a smooth curve of genus 2 (resp. genus 3) and $q \in C$. Then the linear system $\mid K_C + 2q \mid$ gives a plane quartic curve $\bar{C}$ with a cusp (resp. a curve $\bar{C}$ of bidegree (3,3) with a cusp in a smooth quadric Q).

By taking the 4-cyclic cover of $\mathbf{P}^2$ (resp. the triple cover of Q) branched along $\bar{C}$ and then by taking the minimal resolution of rational double point, we have a $K3$ surface with an automorphism of order 4 (resp. order 3). This correspondence gives a birational map from $\mathcal{M}_{2,1}$ (resp. $\mathcal{M}_{3,1}$) to an arithmetic quotient of 4-dimensional complex ball (resp. 7-dimensional complex ball). In case $\mathcal{M}_{2,1}$, the Picard lattice of a generic member X is isomorphic to $U \oplus D_4 \oplus D_4 \oplus A_1^2$. The pencil of lines on $\mathbf{P}^2$ through q induces an elliptic pencil of X with one singular fibers of type I_0^* and six singular fibers of type III. In case $\mathcal{M}_{3,1}$, the Picard lattice of a generic member Y is isomorphic to $U(3) \oplus D_4$, and a ruling of Q induces an elliptic pencil on Y which has one singular fiber of type I_0^* and 9 singular fibers of type II.

Remark 2. Let C be a plane quartic curve and let q be a flex. Then by considering the map $\mid K_C + 2q \mid$ as above, we can see that the moduli space $\mathcal{M}_{3,flex}$ of plane quartic curves with a flex is birational to an arithmetic quotient of the 6-dimensional complex ball. Let X be a generic $K3$ surface appearing in this family. Then its transcendental lattice, together with an automorphism of order 3, has the structure of a complex lattice defined over the Eisenstein integers $\mathbf{Z}[\omega]$. On the other hand, in [K2], we showed that the moduli space $\mathcal{M}_3$ of curves of genus 3 is birational to an arithmetic quotient of the 6-dimensional complex ball by taking the 4-cyclic cover of $\mathbf{P}^2$ branched along a plane quartic curve. In this case, the transcendental lattice, together with an automorphism of order 4, has the structure of a complex lattice defined over the Gaussian integers $\mathbf{Z}[\sqrt{-1}]$. By forgetting a flex we have a map

$$\mathcal{M}_{3,flex} \longrightarrow \mathcal{M}_3$$

of degree 24. The author does not know the relation between two complex ball quotient structures.

Remark 3. Let C be a general smooth curve of genus 6. It is known that the canonical model of C ($\subset \mathbf{P}^5$) lies on a unique del Pezzo surface R of degree 5 (Kollár, Schreyer [KS], Shepherd-Barron [SB]). By taking the double covering of R branched along C, we have a $K3$ surface X with the covering transformation σ. Let $p : R \longrightarrow \mathbf{P}^2$ be a blow up at 4 points. We can easily see that the Picard lattice S of a generic X is isomorphic to $\langle 2 \rangle \oplus A_1^4$ where $\langle 2 \rangle$ is generated by the pullback of the class of a line in $\mathbf{P}^2$ and A_1^4 correspond to 4 exceptional curves on R. It is known that $O(q_S) \simeq O^-(4, \mathbf{F}_2) \simeq S_5$ (Morrison-Saito [MS], Corollary 2.4, Lemma 2.5). Let T be the transcendental lattice of X. Let $\mathcal{D}$ be a bounded symmetric domain of type IV associated to T, and let $\Gamma = O(T)$. Then we can see that the moduli space of curves of genus 6 is birational to the arithmetic quotient $\mathcal{D}/\Gamma$. The author does not know whether the moduli of curves of genus 5 has a similar description as an arithmetic quotient or not.

§3. Discriminant locus

In this section we shall discuss the discriminant locus $\mathcal{H}$. Let $r \in T$ with $r^2 = -2$. By using the equation

$$\rho^2 + \rho + 1_T = 0,$$

we have $\langle r, \rho(r)\rangle = 1$. Let Λ_r be the lattice generated by r and $\rho(r)$. Obviously $\Lambda_r \simeq A_2$. Let $\Lambda_r^{\perp}$ be the orthogonal complement of Λ_r in T and let M be the orthogonal complement of $\Lambda_r^{\perp}$ in L. We remark here that ρ acts on T^*/T because $\rho \mid H^2(X, \mathbf{Z})^{\langle\rho\rangle} = 1$ (Nikulin [N1], Corollary 1.5.2). Also ρ acts on Λ_r^*/Λ_r trivially. This follows from the fact that $(r + 2\rho(r))/3$ is a generator of Λ_r^*/Λ_r and $\rho(r + 2\rho(r)) \equiv r + 2\rho(r)$ *mod* $3\Lambda_r$.

Lemma 1. *There are two possibilities:*

Case (i). $M \simeq U(3) \oplus A_2$ *and* $\Lambda_r^{\perp} \simeq U(3) \oplus U \oplus E_8 \oplus E_6$

Case (ii). $M \simeq U \oplus A_2$ *and* $\Lambda_r^{\perp} \simeq U \oplus U \oplus E_8 \oplus E_6$.

Proof. First note that M contains $S \oplus \Lambda_r$ as a sublattice of finite index where $S = L^{\langle\rho\rangle} \simeq U(3)$. M is determined by the isotropic subgroup $I = M/(S \oplus \Lambda_r)$ of $A_S \oplus A_{\Lambda_r}$ with respect to the discriminant quadratic form $q_S \oplus q_{\Lambda_r}$ (Nikulin [N1], Proposition 1.4.1). Since $A_S \oplus A_{\Lambda_r} \simeq (\mathbf{Z}/3\mathbf{Z})^3$, $I = \{0\}$ or $I = \mathbf{Z}/3\mathbf{Z}$. In the case $I = \mathbf{Z}/3\mathbf{Z}$, by using Nikulin [N1], Corollary 1.5.2, we can see that $q_M(a) = -2/3$ where a is a generator of $I = \mathbf{Z}/3\mathbf{Z}$. Now the assertion follows from Nikulin [N1], Theorem 1.14.2. Q.E.D.

It follows that $\mathcal{H}$ decomposes into two pieces $\mathcal{H}^n$ and $\mathcal{H}^h$ so that the first case $\mathcal{H}^n$ corresponds to the case (i) in Lemma 1. In the following, we shall study $K3$ surfaces whose periods are in $\mathcal{H}$.

Case (i): We shall show that the Case (i) in Lemma 1 corresponds to a curve in Q of bidegree $(3,3)$ with a node.

Example 1. Let C be a curve in a smooth quadric Q of bidegree $(3,3)$ with a node p. Let L_1, L_2 be the two lines through p. First blow up at p and denote by E the exceptional curve. Next blow up the two points in which E and the proper transform of C meet. Then take the 3-cyclic cover X' branched along the proper transforms of C and E. Then X' contains an exceptional curve of the first kind which is the pullback of the proper transform of E. By contracting this exceptional curve to a point q, we have a $K3$ surface X. On X, there are 4 smooth rational curves F_1, F_2, F_3, F_4 which are the inverse images of L_1, L_2 and the exceptional curves which appeared in the second blow up. They meet together at one point q. Each triple of F_j defines an elliptic pencil with singular fiber of type IV and a 3-section. For generic C as above, these 4 curves F_j generate the Picard lattice of X isometric to $U(3) \oplus A_2$, where $U(3)$ is generated by $F_1+F_2+F_3$, $F_1+F_2+F_4$ and A_2 is generated by F_1, F_2. It is known that X has a finite group of automorphisms and the F_i (i =1,2,3,4) are all the smooth rational curves on X (e.g., Nikulin [N3], §4, p.661).

Next we shall show that a generic point of $\mathcal{H}^n$ corresponds to a $K3$ surface mentioned in Example 1. Let z be a generic point in $\mathcal{H}^n$ which is orthogonal to a root $r \in T$ with $\Lambda_r^{\perp} \simeq U(3) \oplus U \oplus E_8 \oplus E_6$. Let Y be the $K3$ surface whose period is z. Then the Picard lattice of Y is isomorphic to $M \simeq U(3) \oplus A_2$. Since the dual graph of all smooth rational curves on Y depends only on the Picard lattice, Y contains exactly 4 smooth rational curves F_j' $(1 \leq j \leq 4)$ which form the same dual graph as that of F_j on X: $F_j' \cdot F_k' = 1, (j \neq k)$. Let ρ' be the isometry of L given by

$$\rho' = (1_M, \rho \mid \Lambda_r^{\perp}).$$

Obviously $(L^{\langle \rho' \rangle})^{\perp} \cap z^{\perp} = 0$, and hence it is induced from an automorphism σ' of order 3 (Namikawa [Na], Theorem 3.10). On the other hand, by the topological Lefschetz fixed point formula, σ' fixes a smooth curve C' of genus $g(C') > 1$. Now take any triple, for example, F_1', F_2', F_3' and consider the linear system $\mid F_1' + F_2' + F_3' \mid$ which defines an elliptic pencil $\pi : Y \longrightarrow \mathbf{P}^1$. By the Hodge index theorem, each fiber meets C', and hence σ' acts on the base of π trivially. Thus σ' acts on a general fiber as an automorphism of order 3, and hence the functional invariant of π is equal to 0. Hence the singular fiber $F_1' + F_2' + F_3'$ is of type IV. This implies that the four F_i' $(i = 1, 2, 3, 4)$ meet each other at one point q. Since σ' acts on M trivially, σ' preserves each F_i'. Since σ' acts on a fiber as an automorphism of order 3, it fixes 3 points on it. Hence C' meets a fiber at three points. Now we can easily conclude that C' meets each F_i' at one point, the fixed point set of σ' consists of $\{q\}$ and C', and $g(C') = 3$. Thus Y is obtained by the same way as X in Example 1.

Case (ii): We shall show that the Case (ii) in Lemma 1 corresponds to a smooth hyperelliptic curve of genus 4 plus a point.

Example 2. Let C be a hyperelliptic curve of genus 4. Then C is given by the equation

$$y^2 = \prod_{i=1}^{10}(x_0 - \lambda_i x_1)$$

which is unique up to automorphisms of $\mathbf{P}^1$. Let $(x_0 : x_1, y_0 : y_1)$ be a bi-homogeneous coordinate of $\mathbf{P}^1 \times \mathbf{P}^1$. Then C can be embeded in $\mathbf{P}^1 \times \mathbf{P}^1$ as follows:

$$y_0^2 \cdot \prod_{i=1}^{5}(x_0 - \lambda_i x_1) + y_1^2 \cdot \prod_{i=6}^{10}(x_0 - \lambda_i x_1) = 0.$$

Let E be the divisor defined by $y_0 = 0$. Let L be a general fiber of the ruling given by

$$p : (x_0 : x_1, y_0 : y_1) \longrightarrow (x_0 : x_1).$$

Note that the fiber given by $x_0 = \lambda_i x_1$ is tangent to C. By taking elementary transformations at the intersection of C and E, we have the Hirzebruch surface $\mathbf{F}_5$. Let C', E' be the proper transform of C, E respectively. Let R be a rational surface obtained by blowing up $\mathbf{F}_5$ at three points which are the intersection of C', E' and L. Let C'', E'', L'' be the proper transform of C', E', L respectively. Let Y' be the 3-cyclic cover of R branched along the divisor $C'' + E'' + L''$. The inverse image of L'' is a (-1)- curve. By contracting this we have a $K3$ surface Y. We can see that the ruling p induces a structure of an elliptic pencil

$$\pi : Y \longrightarrow \mathbf{P}^1$$

which has one singular fiber of type IV and 10 singular fibers of type II, and a section. Here the singular fiber of type IV corresponds to L, ten singular fibers of type II corresponds to fibers $x_0 + \lambda_i x_1$ and E corresponds to a section of π. For a generic C, the Picard lattice of Y is generated by three components of the singular fiber of type IV and a section, which is isomorphic to $U \oplus A_2$. In this case, $Aut(Y)$ is finite and Y contains exactly 4 smooth rational curves (e.g., Nikulin [N3], §4, p.661).

Next we shall show that a generic point of $\mathcal{H}^h$ corresponds to a $K3$ surface mentioned in Example 2. Let z be a generic point in $\mathcal{H}^h$ which is orthogonal to a root $r \in T$ with $\Lambda_r^\perp \simeq U \oplus U \oplus E_8 \oplus E_6$. Let Y' be the $K3$ surface whose period is z. Then the Picard lattice of Y' is isomorphic to $M \simeq U \oplus A_2$. Since the dual graph of all non-singular rational curves on Y' depends only on the Picard lattice, Y' contains exactly 4 smooth rational curves F_j $(0 \leq j \leq 3)$ which form the same dual graph as that of Example 2. Here we assume that F_1, F_2, F_3 form the dual graph of a singular fiber of type I_3 or of type IV, and F_0 meets F_1. Then the linear system $\mid F_1 + F_2 + F_3 \mid$ defines an elliptic pencil $\pi : Y \longrightarrow \mathbf{P}^1$ which has one singular fiber of type I_3 or of type IV, and a section F_0. Let ρ' be the isometry of L given by

$$\rho' = (1_M, \rho \mid \Lambda_r^\perp).$$

Obviously $(L^{\langle \rho' \rangle})^\perp \cap z^\perp = 0$, and hence ρ' is induced from an automorphism σ' of order 3 (Namikawa [Na], Theorem 3.10). On the other hand, by the topological Lefschetz fixed point formular, σ' fixes a smooth curve C' of genus $g(C') > 1$. By the Hodge index theorem, each fiber meets C', and hence σ' acts on the base of π trivially. Hence F_0 is a fixed curve of σ'. Also σ' acts on a general fiber as an automorphism of order 3, and hence the functional invariant is a constant. Hence the singular fiber $F_1 + F_2 + F_3$ is of type IV and all irreducible singular fibers are of type II. Thus π has one singular fiber of type IV and 10 singular fibers of type II. Since σ' acts on a general fiber as an automorphism of order 3, it fixes 3 points on a general fiber, i.e., C' meets a fiber at two points. Let q be the singular point of the fiber $F_1 + F_2 + F_3$. Then we can easily conclude that C' meets each F_2, F_3 at one point $(\neq q)$ and the fixed point set of σ' consists of $\{q\}$, F_0 and C'. It follows from the topological Lefschetz fixed point formula that the genus of C' is 4. Therefore Y is obtained by the same way as X in Example 2. Thus we have

Theorem 2. *A generic point in $\mathcal{H}^n$ (resp. in $\mathcal{H}^h$) corresponds to a curve in Q of bidegree $(3,3)$ with a node (resp. a pair of a hyperelliptic curve of genus 4 and a point on the quotient of the hyperelliptic curve by the hyperelliptic involution).*

Remark 4. (i) The above theorem 2 tells us that $\mathcal{B}/\Gamma$ looks like a blow up of the moduli space of curves of genus 4 along the hyperelliptic locus.

(ii) There is a family of codimension 1 in the moduli space of curves of genus 4 which consists of smooth curves with a vanishing theta null. In this case, the corresponding generic $K3$ surface contains smooth rational curves, but the covering transformation does not fix them (see the proof of Theorem 1, *Case* 2) and the periods of these (generic) $K3$ are contained in $\mathcal{D} \setminus \mathcal{H}$. The Picard lattice of a generic $K3$ surface in this family is isometric to

$$U \oplus A_2(2).$$

This lattice contains $U(3) \oplus A_2(2)$ as a sublattice of finite index and the factor $U(3)$ corresponds to the Picard lattice of a generic member of 9-dimensional family.

Remark 5. Recall that the anti-bicanonical map of a del Pezzo surface of degree 1 is the double cover of a quadric cone in $\mathbf{P}^3$ branched along the canonical curve C of genus 4 (Demazure [D]). Then C has a vanishing theta null. This gives a birational map from the moduli space of del Pezzo surfaces of degree 1 to the moduli space of curves of genus 4 with vanishing theta constant. Thus the moduli space of del Pezzo surfaces of degree 1 can be written as a ball quotient, too. Heckman, Looijenga [HL] and Vakil [V] studied this case from a different point of view. In case of del Pezzo surfaces of degree 2 or 3, van Geemen (unpublished), Kondo [K2], Allcock, Carlson, Toledo [ACT1],[ACT2] gave a ball quotient structure for this moduli space. In the case of del Pezzo surfaces of degree 4, the moduli space is also a ball quotient (e.g. Heckman, Looijenga [HL]): A del Pezzo surface of degree 4 is the complete intersection of two quadrics Q_1, Q_2 in $\mathbf{P}^4$. The discriminant locus of the pencil of quadrics defined by Q_1 and Q_2 gives five points on $\mathbf{P}^1$. This gives a correspondence between the moduli space of del Pezzo surfaces of degree 4 and a compact arithmetic quotient of 2-dimensional complex ball which appeared in Shimura [S], Terada [Te], Deligne-Mostow [DM].

Remark 6. Del Pezzo surfaces of degree 4 are also related to $K3$ surfaces with an automorphism of order 5. Let C be the plane quintic curve defined by

$$y^5 = \prod_{i=1}^{5}(x - \xi_i z)$$

which corresponds to five points $\{(\xi_i : 1)\}$ on $\mathbf{P}^1$ (see Remark 5). Let σ be an automorphism of $\mathbf{P}^2$ given by

$$\sigma(x, y, z) = (x, \zeta y, z)$$

where ζ is a primitive 5-th root of unity. Let L be the line defined by $y = 0$ which is fixed under the action of σ. Let X be the minimal resolution of the double cover branched along the sextic curve $C + L$. Then X is a $K3$ surface and σ can be lifted to an automorphism of X of order 5. We can see that the period domain of these $K3$ surfaces is a 2-dimensional complex ball. The pencil of lines through $(0 : 1 : 0)$ lifts to a pencil of curves of genus 2 on the $K3$ surface which has two base points. A general member of this pencil is a smooth curve of genus 2 with an automorphism of order 5 and this pencil contains five singular members corresponding to five lines

$$x - \xi_i z = 0, (1 \leq i \leq 5).$$

§4. Deligne-Mostow's complex reflection groups

In this section we shall show that Γ is commensurable with an arithmetic subgroup of $\mathbf{P}U(1,9)$ which appeared in Deligne-Mostow's list. The idea of the proof of Theorem 3 is due to T. Terasoma.

Let $\{\infty, 0, 1, x_2, ..., x_{d+1}\}$ be a set of $d+3$-distinct points in $\mathbf{P}^1$. For each integer i, with $0 \leq i \leq d+1$, and $i = \infty$, let μ_i be a real number such that the following equality holds:

$$\sum_i \mu_i = 2.$$

In [DM], for each $\mu = \{\mu_i\}$, Deligne and Mostow defined a subgroup Γ_μ of the automorphism group of a d-dimensional complex ball, which is the monodromy group of a hypergeometric equation.

Let $S_1 \subset S = \{\infty, 0, 1, ..., d+1\}$ and assume that $\mu_s = \mu_t$ for all $s, t \in S_1$. We assume that $\mu_s > 0$ for all $s \in S$ and $\{\mu_s\}$ satisfies the condition

(ΣINT): *For all $s \neq t$ such that $\mu_s + \mu_t < 1$, $(1 - \mu_s - \mu_t)^{-1}$ is an integer if s or t is not in S_1, and a half-integer if $s, t \in S_1$.*

Deligne and Mostow [DM],[M1] showed that this condition is a sufficient condition for which Γ_μ is a *lattice* in $\mathbf{P}U(1,d)$, i.e., Γ_μ is discrete and has cofinite volume. Conversely if Γ_μ is discrete and $d > 3$, then μ satisfies the condition (ΣINT) (Mostow [M2]). In [DM], [M1], Deligne and Mostow determined all such μ and listed them in case $d \geq 5$. Note that Thurston gave a correction of their list ([T]).

Now we shall show that our group Γ is commensurable with Γ_μ, where $\mu = \{\mu_i\}, \mu_i = 1/6$ for all $i = \infty, 0, 1, ..., 10$ (No.1 in Deligne-Mostow's list [M2] and No.10 in Thurston's list [T]).

Let C be a curve defined by

$$y^6 = \prod_{i=1}^{12} (x - \xi_i).$$

Let σ be the covering transformation of C over $\mathbf{P}^1$. Consider the action of σ on $H^1(C, \mathbf{C})$. Let $H^1(C, \mathbf{C})_{-\omega}$ be the eigenspace of σ with eigenvector $-\omega$ where ω is a primitive cube root of unity. Let $(\,,)$ be the symplectic form on $H^1(C, \mathbf{Z})$. Then $\psi(x, y) = \sqrt{-3}(x, \bar{y})$, $x, y \in H^1(C, \mathbf{C})_{-\omega}$, is a hermitian form on $H^1(C, \mathbf{C})_{-\omega}$. This space is defined over the Eisenstein integers $\mathbf{Z}[\omega]$. Deligne and Mostow showed that the signature of ψ is $(1, 9)$ and that Γ_μ is an arithmetic subgroup of $(\mathbf{P}U(H^1(C, \mathbf{C})_{-\omega}), \psi)$.

In the following we shall use the same notation as in §2. Recall that the $K3$ surface X has an elliptic fibration π induced from a ruling of the smooth quadric Q. For a generic X, π has twelve singular fibers of type II in the sense of Kodaira [Ko]. Note that π has no sections. Let $S = U(3)$ be the Picard lattice for a generic X and let $T = U(3) \oplus U \oplus E_8 \oplus E_8$ be the transcendental lattice. Since $\rho \mid S = 1$, ρ acts trivially on the discriminant group $S^*/S \simeq T^*/T$. Let $\alpha \in T^*/T$ be the non-zero isotropic vector corresponding to the class of a fiber of π under the canonical isomorphism $S^*/S \simeq T^*/T$. By adding vectors in T^* representing α to T, we have an even lattice T' which contains T as a sublattice of index 3. Hence T' is unimodular and isometric to $U \oplus U \oplus E_8 \oplus E_8$. Since ρ fixes α, ρ induces an isometry ρ'' of T' of order 3. We denote by ρ' the isometry $-\rho''$ of order 6. By the surjectivity of the period map for $K3$ surfaces, there exists a $K3$ surface Y whose transcendental lattice is isometric to T' and whose period is the same as that of X. Since T' is unimodular, ρ' can be extended to an isometry of $H^2(Y, \mathbf{Z})$ acting on $(T')^{\perp} (\simeq U)$ trivially. Hence it follows from the Torelli theorem that ρ' is represented by an automorphism of order 6. Since the Picard lattice of Y is isomorphic to U, Y has an elliptic fibration π' with a unique section (Kondo [K1], Lemma 2.1). If ρ' acts on the base of π' non trivially, then the set of fixed points of ρ' is contained in two fibers. On the other hand the Lefschetz number of ρ' is -6 (see the proof of Theorem 1). With this observation it follows from the topological Lefschetz fixed point formula that ρ' acts on the base trivially. Thus every smooth fiber has an automorphism of order 6 which fixes the intersection point with the section. Hence the functional invariant of this elliptic fibration is a constant. Since $Pic(Y) \simeq U$, every singular fiber is irreducible, and hence it is of type II. We remark that $\pi' : Y \longrightarrow \mathbf{P}^1$ is nothing but the Jacobian fibration of π.

By the theory of elliptic surfaces with a section (Kodaira [Ko]), we can easily see that there exists a Galois cover of Y with the Galois group $G \simeq \mathbf{Z}/6\mathbf{Z}$, which is birational to $C \times E$ where C is a $\mathbf{Z}/6\mathbf{Z}$-cover of $\mathbf{P}^1$ ramified at 12 points and E is an elliptic curve with a complex multiplication ω (a cube root of unity). We denote by σ (resp. τ) an automorphism of C (resp. E) of order 6. We may assume that (σ, τ) is a generator of G. Denote by f the rational map from $C \times E$ to Y. Then

$$f^*(T' \otimes \mathbf{Q}) \simeq (H^1(C, \mathbf{Q}) \otimes H^1(E, \mathbf{Q}))^G$$

and

$$(H^1(C, \mathbf{C}) \otimes H^1(E, \mathbf{C}))^G \simeq H^1(C)_{-\omega} \otimes H^1(E)_{-\bar{\omega}} \oplus H^1(C)_{-\bar{\omega}} \otimes H^1(E)_{-\omega}.$$

Moreover the action of ρ' on $T' \otimes \mathbf{Q}$ is compatible with that of τ on $f^*(T' \otimes \mathbf{Q})$. The above isomorphisms give an isomorphism between two hermitian spaces $T'_{-\omega} = \{x \in T' \otimes \mathbf{C} : \rho'(x) = -\omega x\}$ and $H^1(C, \mathbf{C})_{-\omega}$, which is defined over $\mathbf{Q}(\omega)$. Since both Γ_μ and

$$\Gamma' = \{g \in O(T') : g \circ \rho' = \rho' \circ g\}$$

are arithmetic, Γ_μ is commensurable with Γ'. On the other hand, by definition of T', Γ is commensurable with Γ', and hence with Γ_μ. Thus we have

Theorem 3. *Γ is commensurable with Γ_μ where*

$$\mu = (\mu_i) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}).$$

Remark 7. In [A], Allcock proved that Γ_μ in Theorem 3 is isomorphic to the hyperbolic reflection group of the complex lattice over the Eisenstein integers $\mathbf{Z}[\omega]$ whose real form is $U \oplus U \oplus E_8 \oplus E_8$. We also remark that van Geemen gave a similar correspondence between the curve C as above and some $K3$ surface (see [vG], Example 3.11).

Remark 8. In [K2], we showed that the moduli space of curves of genus 3 is birational to a ball quotient by taking the 4-cyclic cover of $\mathbf{P}^2$ branched along a plane quartic curve. The corresponding discrete group does not appear in Deligne-Mostow's list (the corresponding $K3$ surface has no elliptic fibration invariant under the action of the automorphism of order 4). However, for example, in case of hyperelliptic curves of genus 3 or plane quartic curves with a node, the corresponding generic $K3$ surface has an elliptic fibration with 8 singular fibers of type III (see [K2], §5). The same argument as above shows that the corresponding arithmetic subgroup is commensurable with Γ_μ where

$$\mu = (\mu_i) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

(No. 8 in Deligne-Mostow's list [M2] and No. 3 in Thurston's list [T]). Allcock informed the author that he showed the commensurability in this case by a different way. Shiga [Sh] suggested a relation between Deligne-Mostow's complex reflection groups and elliptic $K3$ surfaces with a section in some special cases. In the case of del Pezzo surfaces of degree 4 (see Remark 6), the pencil of curves of genus 2 with an order 5 automorphism on the $K3$ surface works like the elliptic pencil in the above cases.

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Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602
Japan