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On Semistable Extremal Neighborhoods

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Dedicated to Professor Heisuke HIRONAKA on the occasion of his seventieth birthday

Abstract.

We give an explicit description of an extremal nbd $X \supset C \simeq \mathbb{P}^1$ of type $k2A$. We also give a criterion for X to be a flipping contraction and an explicit description of the contraction and the flip.

§1. Introduction

In the three dimensional minimal model program, flips and divisorial contractions are the fundamental birational maps. Among them, flips are proved to exist [8]. This paper is concerned with the classification of flips. We give a brief background.

Let $f : X \to Y$ be a projective birational morphism from a threefold X with only terminal singularities to a normal threefold Y and $Q \in Y$ such that $C = f^{-1}(Q)$ is a curve and $-K_X$ is f-ample.

We note that, in the context of the minimal model program, we often assume that X is $\mathbb{Q}$-factorial and put the condition $\rho(X/Y) = 1$ on the relative Picard number. In this paper, we do not assume these conditions, because they are not preserved when we work on the associated formal scheme.

For an arbitrarily small open set $U \ni Q$, we call $f^{-1}(U) \supset C \to U \ni Q$ *an extremal neighborhood* (or, *an extremal nbd*, for short). It is said to be *flipping* (resp. *divisorial*) if the exceptional set is a curve (resp. a divisor). An extremal nbd is said to be *irreducible* if C is irreducible.

In [5], the irreducible extremal nbds $X \supset C \to Y \ni Q$ are studied as follows. A general member D of $|-K_X|$ is proved to have only Du Val singularities, and the irreducible extremal nbds are classified

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into 6 types $k1A$, $k2A$, $cD/3$, IIA, IC, kAD [5, (2.2)] according to the singularities of D. The first two (resp. the last four) cases are said to be *semistable* (resp. *exceptional*). For the exceptional irreducible flipping extremal nbds $X \supset C$, the singularities of the general member H of $|\mathcal{O}_X|$ containing C are computed in [5, Chapters 6–9] and the irreducible flipping $X \supset C$ is reconstructed as an essentially arbitrary one-parameter deformation space of H [5, Theorems 13.9–13.12] and the flip is described [5, Theorems 13.17 and 13.18].

However if we start with H of an irreducible semistable extremal nbd X, whether or not X is flipping depends not only on H but also on the individual one-parameter deformation, which is quite different from the exceptional cases.

In this paper, we treat the case of $k2A$. (The case of $k1A$ will be treated elsewhere.) In Section 2, we give an expression of an extremal nbd $X \supset C$ of type $k2A$ in terms of coordinates (Theorem 2.2) and graded rings (Definition 2.8, Theorem 2.9).

Section 3 is the core algorithm section of this paper, where we introduce a sequence $d(n)$ (Definitions 3.2 and 3.11) and present a series of divisions (Theorems 3.10–3.13) starting with the "graded equations" in Theorem 2.9.

Section 4 is the main section for applications, where we give a necessary and sufficient condition (Corollary 4.1) for $X \supset C$ to be flipping in terms of $d(n)$. Furthermore, the extremal contraction (Theorem 4.3) and the flip are explicitly constructed (Theorem 4.7).

In Section 5, we present the division in the case of a multi-parameter deformation space of H and comment on some of the further directions.

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§2. Good coordinates

2.1. Let $f : X \supset C\ (\simeq \mathbb{P}^1) \to Y \ni Q$ be an extremal nbd of type $k2A$ [5] with two terminal singular points P_1, P_2 of indices $m_1, m_2 > 1$ and axial multiplicities $\alpha_1, \alpha_2 \geq 1$, respectively.

Let $D \in |-K_X|$ be a Du Val member, whose minimal resolution has the dual configuration

$$\underbrace{\circ - \cdots - \circ}_{m_1\alpha_1 - 1} - C' - \underbrace{\circ - \cdots - \circ}_{m_2\alpha_2 - 1},$$

where C' is the proper transform of C and $\circ$ denotes an exceptional curve and all these curves are (-2)-curves [5, 2.2.4]. By adding two (non-compact) curves ℓ'_i at both ends

$$\ell'_1 - \underbrace{\circ - \cdots - \circ}_{m_1\alpha_1 - 1} - C' - \underbrace{\circ - \cdots - \circ}_{m_2\alpha_2 - 1} - \ell'_2,$$

we obtain a reduced curve on the minimal resolution such that the intersection numbers with C' and $\circ$'s are zero. Since $H^1(X, \mathcal{O}_X) = 0$ [8], we see that $\ell_1 + C + \ell_2 \sim 0$ on D, where $\ell_i \subset D$ denotes the image of ℓ'_i. Let

$$H_D := \ell_1 + C + \ell_2 \ (\sim 0 \text{ on } D).$$

By the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and the Grauert Riemenschneider vanishing $H^1(X, \mathcal{O}_X(K_X)) = 0$ [8], we obtain a trivial Cartier divisor $H = (u = 0)$ on X, which is normal and induces H_D on D.

Theorem 2.2. *Let U_i be the $\mathbb{Z}_{m_i}$-quotient of a "hypersurface" of $\mathbb{C}^4$,*

$$U_i := (\xi_i, \eta_i, \zeta_i, u; \xi_i\eta_i = g_i(\zeta_i^{m_i}, u))/\mathbb{Z}_{m_i}(1, -1, a_i, 0),$$

where a_i is an integer $\in [1, m_i]$ prime to m_i and $g_i(T, u) \in \mathbb{C}[[u]][T]$ is a monic polynomial in T of degree, say ρ_i such that $g_i(\zeta_i^{m_i}, u)$ is square-free. Let $P_i := 0$ and $C_i := \xi_i$-axis$/\mathbb{Z}_{m_i}$. U_i is defined to be a formal scheme along $C_i \simeq \mathbb{C}^1$ with only terminal singularities.

For a suitable choice of a_i and $g_i(T, u)$, we have

1. *these C_1 and C_2 are patched together to form $C \simeq \mathbb{P}^1$ and U_1 and U_2 are patched together to form the completion $\hat{X}$ of X along C by the identification on $U_1 \cap U_2$:*

$$\xi_1^{m_1} = (\xi_2^{m_2})^{-1},$$
$$\frac{\zeta_1}{\xi_1^{a_1}} = \xi_2^{m_2}\frac{\zeta_2}{\xi_2^{a_2}},$$

2. *$D = (\zeta_1 = 0)/\mathbb{Z}_{m_1} \cup (\zeta_2 = 0)/\mathbb{Z}_{m_2}$ and $H = (u = 0)$ under the identification.*

Remark 2.3. The assertions of Theorem 2.2 modulo the equation u of H, that is the corresponding assertion for H is easily seen as follows.

By the construction, $f(H_D)$ has an ordinary double point at Q. Since $K_H + H_D \sim 0$, we see that $(f(H), f(H_D))$ is lc ([6, (5.58)] or [12]). Since $f(H_D)$ has two analytic branches, this means that $(f(H), f(H_D))$ is the quotient of (xy-plane, $(xy = 0)$) by a diagonal action of some cyclic group [1] or [3]. In particular, it is toric.

Since C is a log crepant divisor of $(f(H), f(H_D))$ (that is, $K_H + H_D \sim 0$ and, and H_D is a reduced curve containing C), (H, H_D) is toric as well as $(f(H), f(H_D))$.

The index-1 cover of $(H \setminus \ell_i, H_D \setminus \ell_i)$ is toric. By the descrition of the terminal singularities P_i [7, 11], we obtain the isomorphisms

$$H \setminus \ell_{3-i} \simeq (\xi_i, \eta_i, \zeta_i; \xi_i\eta_i = \zeta_i^{m_i\rho_i})/\mathbb{Z}_{m_i}(1, -1, a_i)$$

with the properties

1. $H_D = (\zeta_i = 0)/\mathbb{Z}_{m_i}$, $m_i\rho_i\ell_i = (\xi_i = 0)/\mathbb{Z}_{m_i}$ and $C \setminus \ell_{3-i} = (\zeta_i = \eta_i = 0)/\mathbb{Z}_{m_i}$ on $H \setminus \ell_{3-i}$ under the identification, and
2. $\xi_1^{m_1} = \xi_2^{-m_2}$ on $H \setminus (\ell_1 \cup \ell_2)$,

for some $\rho_i \in \mathbb{Z}_{>0}$ and $a_i \in [1, m_i]$ such that $(a_i, m_i) = 1$. Once these properties are checked, it is easy to see the following.

3. $K_H \sim (m_1 - a_1) \cdot (\xi_1 = 0)/\mathbb{Z}_{m_i} - a_2 \cdot (\xi_2 = 0)/\mathbb{Z}_{m_2}$
4. $\xi_1^{-a_1}\zeta_1 = \xi_2^{m_2-a_2}\zeta_2$ on $H \setminus (\ell_1 \cup \ell_2)$

Indeed, the property 3 follows from $gr_X^0\omega \simeq \mathcal{O}_C(-1)$ [8, (1.14.(i))] and the assertion that

$$x_i^{m_i-a_i} \operatorname{Res} \frac{d\xi_i \wedge d\eta_i \wedge d\zeta_i}{\xi_i\eta_i - \zeta_i^{m_i\rho_i}} = -x_i^{m_i-a_i}(d\xi_i/\xi_i) \wedge d\zeta_i$$

is a generator of $gr_X^0\omega|_H$ on $H \setminus \ell_{3-i}$. The property 4 follows from the property 3 because $m_1 d\xi_1/\xi_1 = -m_2 d\xi_2/\xi_2$.

Proof of Theorem 2.2. We note that Theorem 2.2 is proved modulo (u), the equation of H (Remark 2.3). On the completion $\hat{X}$ of X along C, let U_i be the complement of P_{3-i}. Assume that Theorem 2.2 is proved modulo $(u)^N$ for some $N > 0$. We attach subscript N to the coordinates and the equations that are chosen to work for $(u)^N$.

From the $\mathbb{Z}_{m_i}$-invariant relation

$$\xi_{N,i}\eta_{N,i} = g_{N,i}(\zeta_{N,i}^{m_i}, u) \mod (u)^N,$$

we have

$$\xi_{N,i}\eta_{N,i} = g_i'(\zeta_{N,i}^{m_i}, u) + u^N\xi_{N,i}\alpha_i + u^N\eta_{N,i}\beta_i \mod (u)^{N+1}$$

for some $\alpha_i, \beta_i \in \mathbb{C}[\xi_{N,i}][[u, \eta_{N,i}, \zeta_{N,i}]]$ and $g_i'(T, u) \in \mathbb{C}[[u, T]]$ such that $g_i' \equiv g_{N,i} \mod (u)^N$. Then

$$(\xi_{N,i} - u^N\alpha_i)(\eta_{N,i} - u^N\beta_i) = g_i''(\zeta_{N,i}^{m_i}, u) \mod (u)^{N+1}$$

for some $g_i'' \in \mathbb{C}[[u, T]]$ such that $g_i'' \equiv g_{N,i} \mod (u)^N$.

The Cartier divisor

$$\Phi_i = (\xi_{N,i} - u^N \alpha_i)/\xi_i) = (\xi_{N,i} = u^N \alpha_i)/\mathbb{Z}_{m_i} - (\xi_{N,i} = 0)/\mathbb{Z}_{m_i}$$

on a neighborhood of P_i extends to a principal divisor on X, because Φ_i intersects properly with C and $(\Phi_i \cdot C) = 0$. By the exact sequence

$$0 \to \mathcal{O}_{\hat{X}}(\Phi_i) \xrightarrow{u^N} \mathcal{O}_{\hat{X}}(\Phi_i) \to \mathcal{O}_{N\hat{H}} \to 0$$

and $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$, there is a rational function φ_i on $\hat{X}$ such that $(\varphi_i) = \Phi_i$ and $\varphi_i|_{N\hat{H}} = 1$. We note that φ_i is invertible on U_{3-i}.

Let

$$\xi_{N+1,i} = \xi_{N,i} \cdot \varphi_i \cdot \varphi_{3-i}^{-m_{3-i}/m_i}.$$

Then we have

$$\begin{aligned} \xi_{N+1,i} &\equiv \xi_{N,i} \mod (u)^N, \\ \xi_{N+1,1}^{m_1} &= \xi_{N+1,2}^{-m_2} \end{aligned}$$

and $\xi_{N+1,i} = (\xi_{N,i} - u^N \alpha_i) \cdot (\text{unit on } U_i)$. Let

$$\zeta_{N+1,i} = \begin{cases} \zeta_{N,1}\varphi_2^{m_2-a_2}\varphi_2^{-a_1 m_2/m_1} & i = 1, \\ \zeta_{N,2}\varphi_1^{-a_1}\varphi_1^{-(m_2-a_2)m_1/m_2} & i = 2. \end{cases}$$

We note that $\zeta_{N+1,i}/\zeta_{N,i}$ is a unit on U_i and that

$$\begin{aligned} \xi_{N+1,1}^{-a_1}\zeta_{N+1,1} &= \xi_{N+1,2}^{m_2-a_2}\zeta_{N+1,2}, \\ \zeta_{N+1,i} &\equiv \zeta_{N,i} \mod (u)^N. \end{aligned}$$

By the Weierstrass preparation theorem, there exist a unit $\gamma_i \in \mathbb{C}[[u,T]]$ such that $\gamma_i \equiv 1 \mod (u)^N$ and $g_{N+1,i} := g_i''\gamma_i$ is a monic polynomial $\in \mathbb{C}[[u]][T]$. We then define $\eta_{N+1,i}$ such that $\eta_{N+1,i} = (\eta_{N,i} - u^N\beta_i) \cdot (\text{unit on } U_i)$ by the relation on U_i:

$$\xi_{N+1,i}\eta_{N+1,i} = (\xi_{N,i} - u^N\alpha_i)(\eta_{N,i} - u^N\beta_i)(\zeta_{N+1,i}/\zeta_{N,i})^{m_i\rho_i}\gamma_i.$$

By $g_{N+1,i}(T,0) = T^{\rho_i}$ and $\zeta_{N+1,i}/\zeta_{N,i} \equiv 1 \mod (u)^N$, we have

$$g_{N+1,i}(\zeta_{N+1,i}^{m_i}, u) \equiv g_{N+1,i}(\zeta_{N,i}^{m_i}, u) \cdot (\zeta_{N+1,i}/\zeta_{N,i})^{m_i\rho_i} \mod (u)^{N+1},$$

and hence

$$\xi_{N+1,i}\eta_{N+1,i} \equiv g_{N+1,i}(\zeta_{N+1,i}^{m_i}, u) \mod (u)^{N+1}.$$

Thus the theorem is proved modulo $(u)^{N+1}$. We can let $n \to \infty$.

Finally $g_i(\zeta_i^{m_i}, u)$ is square-free, because otherwise U_i has a non-isolated singularity. Q.E.D.

Remark 2.4. The numbers a_i and ρ_i in Theorem 2.2 can be easily read off from the information on $X \supset C$, D and H. Furthermore, α_i is related to g_i.

1. The numbers a_i are uniquely determined by $X \supset C \ni P_i$ because the $\mathbb{Z}_{m_i}$-action is normalized by its action on the ξ_i-axis.
2. By
$$H = (\xi_i\eta_i = g_i(\zeta_i^{m_i}, 0))/\mathbb{Z}_{m_i}(1, -1, a_i),$$
the index-one cover of H at P_i is an $A_{m_i\rho_i}$-point $(\xi_i\eta_i = g_i(\zeta_i^{m_i}, 0))$. Thus ρ_i is uniquely determined by H.
3. Similarly by
$$\begin{aligned} D &= (\xi_i\eta_i = g_i(0, u))/\mathbb{Z}_{m_i}(1, -1, 0), \\ &\simeq (xy = g_i(0, u)^{m_i}), \end{aligned}$$
we have $(g_i(0, u)) = (u)^{\alpha_i}$ in $\mathbb{C}[[u]]$.

Remark 2.5. Under the notation of Theorem 2.2, let $S_i = (\xi_i = 0)/\mathbb{Z}_{m_i}$. Then

1. $m_1S_1 \sim m_2S_2$, $S_1 \cap S_2 = \emptyset$,
2. $K_X \sim (m_1 - a_1)S_1 - a_2S_2$, and
3. $-m_iK_X \sim \delta S_{3-i}$, where $\delta := a_1m_2 + a_2m_1 - m_1m_2 > 0$.

Let

$$z \in \Gamma(\hat{X}, \mathcal{O}(a_1S_1 + (a_2 - m_2)S_2)), u \in \Gamma(\hat{X}, \mathcal{O}), x_i \in \Gamma(\hat{X}, \mathcal{O}(S_i))$$

be the sections defining D, H and S_i.

Let $G_i(T_1, T_2) = g_i(T_1/T_2, u)T_2^{\rho_i} \in \mathbb{C}[[u]][T_1, T_2]$, which is a homogeneous polynomial in T_1, T_2 of degree ρ_i. Since $z^{m_i}, x_{3-i}^{\delta}$ are both sections of $\mathcal{O}(\delta S_i)$, we can consider the section $G_i(z^{m_i}, x_{3-i}^{\delta})$ of $\mathcal{O}(\rho_i\delta S_{3-i})$. The section is divisible by x_i and the quotient y_i satisfies the condition

$$y_i \in \Gamma(\hat{X}, \mathcal{O}(\rho_i\delta S_{3-i} - S_i)),$$

which follows immediately from the local equation $\xi_i\eta_i = g_i(\zeta^{m_i}, u)$. We have thus two equations:

$$x_1y_1 - G_1(z^{m_1}, x_2^{\delta}) = 0, \quad x_2y_2 - G_2(z^{m_2}, x_1^{\delta}) = 0,$$

where $G_1(z^{m_1}, x_2^{\delta})$ and $G_1(z^{m_1}, x_2^{\delta})$ are square-free (Theorem 2.2).

The contractibility of C implies the following positivity result.

Proposition 2.6. *Under the notation and the assumptions of Theorem 2.2, we have*

$$\Delta := \rho_1 m_1^2 - \delta\rho_1\rho_2 m_1 m_2 + \rho_2 m_2^2 > 0.$$

Proof. By the property 1 of Remark 2.3, we have

$$(\ell_i \cdot C)_H = \frac{1}{m_i\rho_i}(S_i \cdot C) = \frac{1}{m_i^2\rho_i}.$$

Since $H \cap D = \ell_1 + C + \ell_2$, we have the following by Remark 2.5.

$$\begin{aligned}(\ell_1 + C + \ell_2 \cdot C)_H = (D \cdot C) &= \frac{1}{m_1}(-m_1 K_X \cdot C) \\ &= \frac{1}{m_1}(\delta S_2 \cdot C) = \frac{\delta}{m_1 m_2}.\end{aligned}$$

Thus we have

$$(C^2)_H = \frac{-\Delta}{\rho_1\rho_2 m_1^2 m_2^2}.$$

Since C is an exceptional curve on H, we have $(C^2)_H < 0$. Q.E.D.

Remark 2.7. The properties that

$$\begin{aligned} &z \in \Gamma(\hat{X}, \mathcal{O}(a_1 S_1 + (a_2 - m_2)S_2)), \\ &u \in \Gamma(\hat{X}, \mathcal{O}), \\ &x_i \in \Gamma(\hat{X}, \mathcal{O}(S_i)), \\ &y_i \in \Gamma(\hat{X}, \mathcal{O}(\rho_i\delta S_{3-i} - S_i)) \end{aligned}$$

in Remark 2.5 can be rephrased as follows. Let the group

$$\Gamma := \{(\gamma_1, \gamma_2) \in (\mathbb{C}^*)^2 | \gamma_1^{m_1} = \gamma_2^{m_2}\}$$

act on $H^0(\hat{X}, \mathcal{O}(\lambda_1 S_1 + \lambda_2 S_2))$ via the multiplication by $\gamma_1^{\lambda_1}\gamma_2^{\lambda_2}$. Then we have

$$\gamma(x_i, z, y_i, u) = (\gamma_i x_i, \gamma_1^{a_1}\gamma_2^{a_2 - m_2} z, \gamma_{3-i}^{\rho_i\delta}\gamma_i^{-1} y_i, u),$$

and $x_i y_i - G_i(z^{m_i}, x_{3-i}^{\delta})$ is semi-invariant under the Γ-action. The scheme $\hat{X}$ has an alternate description in terms of these data as follows.

Definition 2.8. Let $a_i, m_i, \alpha_i, \rho_i$ be positive integers (cf. Remark 2.10) and $G_i(T_1, T_2) \in \mathbb{C}[[u]][T_1, T_2]$ a homogeneous polynomial in T_1 and T_2 of degree ρ_i $(i = 1, 2)$ such that

1. $a_i \leq m_i$ and $(a_i, m_i) = 1$,
2. $\delta = a_1m_2 + a_2m_1 - m_1m_2 > 0$,
3. $G_i(1,0) = 1$, $G_i(0,1)\mathbb{C}[[u]] = u^{\alpha_i}\mathbb{C}[[u]]$,
4. $G_i(T_1^{m_i}, 1)$ is reduced, and
5. $\Delta = \rho_1 m_1^2 - \delta\rho_1\rho_2 m_1 m_2 + \rho_2 m_2^2 > 0$.

Let $R := \mathbb{C}[[u]][x_1, y_1, x_2, y_2, z]$ be the $\mathbb{C}[[u]]$-algebra with the Γ-action in Remark 2.7, and let $W = \text{Spec } R/I$ be the scheme with the Γ-action, where I is the ideal given by

$$I := (x_1y_1 - G_1(z^{m_1}, x_2^\delta), x_2y_2 - G_2(z^{m_2}, x_1^\delta)).$$

Set

$$X := (W \setminus V(x_1, x_2))/\Gamma \supset C := V(y_1, y_2, z)/\Gamma \simeq \mathbb{P}^1$$

and $\{P_i\} = V(x_i, y_1, y_2, z, u)$, where $V(I)$ denots the closed subset defined by all the equations in I.

Theorem 2.9. *With the above notation and the assumptions, we have the following.*

1. *X is a normal scheme of dimension 3 such that $X \setminus \{P_1, P_2\}$ is smooth and*

   $$P_i \in X \simeq (\xi_i, \eta_i, \zeta_i, u; \xi_i\eta_i = G_i(\zeta_i^{m_i}, 1))/\mathbb{Z}_{m_i}(1, -1, a_i, 0)$$

   is a terminal singularity with index m_i and $P_i \in C = \xi_i$-axis$/\mathbb{Z}_{m_i}$ under the identification,
2. *$S_i := (x_i = 0)/\Gamma$ is a $\mathbb{Q}$-Cartier Weil divisor on X, and a rational function ϕ on W such that $\phi/x_1^{-b_1}x_2^{-b_2}$ is Γ-invariant defines a $\mathbb{Q}$-Cartier Weil divisor $(\phi = 0)/\Gamma \sim b_1S_1 + b_2S_2$. In particular, $D := (z = 0)/\Gamma \in |-K_X|$ and $H = (u = 0)$ are as in 2.1,*
3. *the completion of X along C is isomorphic to $\hat{X}$ given in Theorem 2.2.*

Proof. On $U_i = \{x_{3-i} \neq 0\}$, we normalize $x_{3-i} = 1$ and set $\xi_i := x_i$, $\eta_i := y_i$ and $\zeta_i := z$ with the relation $\xi_i\eta_i = G_i(\zeta_i, 1)$. Note that η_{3-i} is not needed because $y_{3-i} = G_{3-i}(\zeta_i^{m_{3-i}}, \xi_i^\delta)$. The stabilizer $\Gamma_i \simeq \mathbb{Z}_{m_i}$ of x_{3-i} acts on $(\xi_i, \eta_i, \zeta_i, u)$ via the grading $(1, -1, a_i, 0) \mod (m_i)$, and the quotient is isomorphic to U_i. The rest of the assertion 1 follows from [7, 11]. The patching of the coordinates is obtained by

$$\gamma(\xi_1, 1, \zeta_1) = (1, \xi_2, \zeta_2).$$

Indeed we obtain $\gamma_1 = \xi_1^{-1}$, $\gamma_2 = \xi_2$ (whence $\xi_1^{m_1} = \xi_2^{-m_2}$) and the relation for ζ_i's: $\xi_1^{-a_1}\xi_2^{a_2-m_2}\zeta_1 = \zeta_2$. This proves the assertion 3, and the rest is obvious (cf. Remark 2.3). Q.E.D.

Remark 2.10. In 2.1, if we assume $m_1, m_2 \geq 1$ and that there is a Du Val member $C \subset D \in |-K_X|$ whose minimal resolution has the dual configuration

$$\circ - \cdots - \circ - C' - \circ - \cdots - \circ,$$

then Theorem 2.2 still holds. In this case, the axial multiplicity α_i is undefined and Remark 2.4.3 is irrelevant for i such that $m_i = 1$, and most importantly a general member of $|-K_X|$ does not contain C. That is, $X \supset C$ is an easy case of $k1A$.

In Definition 2.8, we assume $m_1, m_2 \geq 1$. This allows us to treat $k2A$ and some easy case of $k1A$ with no changes in our treatment.

§3. A division algorithm

3.1. We maintain the notation and the assumptions of Definition 2.8. We note that if $G_i(0,1) = u^{\alpha_i} v_i^{\rho_i \delta}$ for some unit $v_i \in \mathbb{C}[[u]]$ then replacing x_i, y_i by $x_i v_{3-i}^{-1}, y_i v_{3-i}$, we may assume $G_i(0,1) = u^{\alpha_i}$. In other words, we may further assume

$$G_i(T_1, T_2) = T_1^{\rho_i} + \cdots + u^{\alpha_i} T_2^{\rho_i}$$

without loss of generality.

We will study when $X \supset C \simeq \mathbb{P}^1$ is a flipping nbd.

Definition 3.2. In addition to the above $G_1(T_1, T_2)$, $G_2(T_1, T_2)$, we introduce $G_i(T_1, T_2)$ $(i = 3, 4)$ as follows:

$$G_i(T_1, T_2) := G_{i-2}(u^{\alpha_{i-2}} T_2, T_1)/u^{\alpha_{i-2}} \in \mathbb{C}[[u]][T_1, T_2] \quad (i = 3, 4).$$

We note that $G_i(T_1, T_2)$ is homogeneous of degree $\rho_i = \rho_{i-2}$ and is of the form

$$G_i(T_1, T_2) = T_1^{\rho_i} + \cdots + u^{\alpha_i} T_2^{\rho_i},$$

where $\alpha_i = \alpha_{i-2}(\rho_{i-2} - 1)$. We remark that $G_i \not\equiv T_1^{\rho_i} \mod (u)$ if $\rho_{i-2} = 1$.

For a positive integer a and an integer x, let x umod a be the integer $y \in [1, a]$ such that $y \equiv x \mod a$. For arbitrary $i \in \mathbb{Z}$, we use the following notation:

$$G_i := G_{i \text{ umod } 4}, \ \rho_i := \rho_{i \text{ umod } 4}, \ \alpha_i := \alpha_{i \text{ umod } 4}, \ \alpha_{i,2} := \alpha_{i \text{ umod } 2}.$$

We note then the obvious $\rho_i = \rho_{i-2}$ and the following formula

$$G_i(T_1, T_2) = G_{i-2}(u^{\alpha_{i-2,2}} T_2, T_1)/u^{\alpha_{i-2}} \quad (\forall i). \tag{3.1}$$

Let $d(n) \in \mathbb{Z}$ $(n \in \mathbb{Z})$ be a sequence determined by

$$d(1) = m_1, \; d(2) = m_2, \; d(n+1) + d(n-1) = \delta\rho_n d(n) \quad (\forall n).$$

Let $e(n) \in \mathbb{Z}$ $(n \in \mathbb{Z})$ be another sequence determined by

$$e(0) = 0, e(1) = -\alpha_1, e(2) = -\alpha_2, \; e(3) = 0,$$

$$e(n+1) + e(n-1) = \delta\rho_n e(n) + \delta\alpha_{n-2} - \alpha_{n-1,2} \; (n \neq 1, 2).$$

Let $\varepsilon := (\rho_1\rho_2\delta)^2 - 4\rho_1\rho_2$, the discriminant of the quadratic form $q(x_1, x_2) := \rho_1 x_1^2 - \rho_1\rho_2\delta x_1 x_2 + \rho_2 x_2^2$.

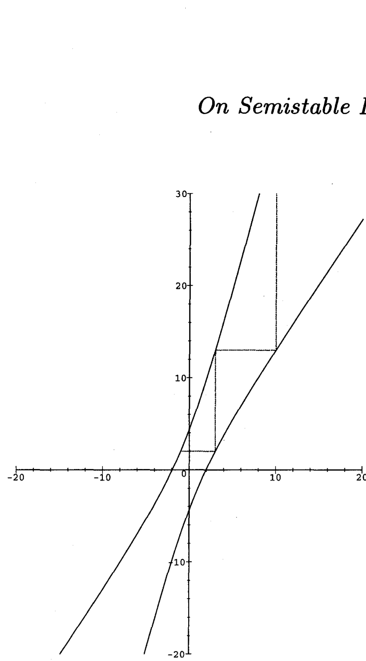
Lemma 3.3. *Let δ, ρ_n be as above and let $\varepsilon := (\rho_1\rho_2\delta)^2 - 4\rho_1\rho_2$. Let $x(n) \in \mathbb{Q}$ be a sequence for $n \in \mathbb{Z}$ such that*

$$x(n) = \delta\rho_{n-1}x(n-1) - x(n-2).$$

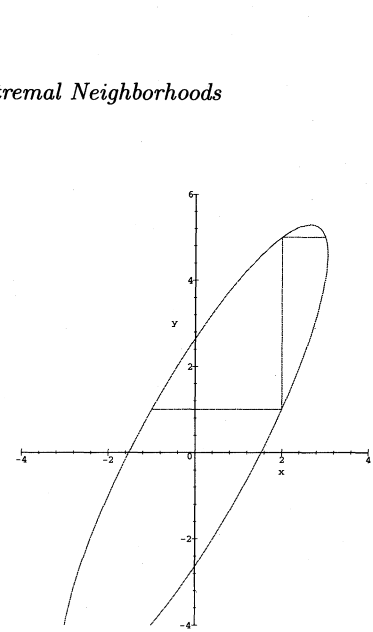
Then we have the following.

1. *If $x(n_0 - 1) > x(n_0 + 1)$ for some n_0 such that $0 \leq x(n_0)$, then $x(n-1) > x(n+1)$ for every $n \geq n_0$ such that $0 \leq x(n_0), \cdots, x(n)$.*
2. *If $x(n_0 - 1) = x(n_0 + 1)$ for some n_0, then $x(n_0 - n) = x(n_0 + n)$ for every n. If furthermore $x(n_0) = x(n_0 + 2)$ and $(x(n_0), x(n_0 + 1)) \neq (0, 0)$, then $\varepsilon = 0$ and $q(x(1), x(2)) = 0$.*
3. *Assume that $\varepsilon \geq 0$. If $x(n_0 - 1) < x(n_0 + 1)$ (resp. $x(n_0 - 1) > x(n_0 + 1)$) for some n_0, then $x(n-1) < x(n+1)$ (resp. $x(n-1) > x(n+1)$) for every n.*

Proof. Draw the graph of the conic $C := \{(x_1, x_2) \mid q(x_1, x_2) = A\}$, with some constant A so that $(x(2i - 1), x(2i)) \in C$ for some i.



$\varepsilon \geq 0$



$\varepsilon < 0$

The induction formula for $x(n)$ implies that

$$(x(2i+1), x(2i)) \in C \Leftrightarrow (x(2i-1), x(2i)) \in C \quad (\forall i),$$
$$(x(2i+1), x(2i)) \in C \Leftrightarrow (x(2i+1), x(2i+2)) \in C \quad (\forall i).$$

So $(x(2i+1), x(2i)), (x(2i-1), x(2i))$ all lie on C. Except for the second half of the assertion (2), the assertions are obvious from the geometric considerations.

For the second half of the assertion (2), assume that $x(n_0 - 1) = x(n_0 + 1), x(n_0) = x(n_0 + 2)$ and $(x(n_0), x(n_0 + 1)) \neq (0, 0)$. By the first half of the assertion (2), we have $x(n) = x(n+2)$ for all n. By $x(i-1) = x(i+1)$, we see that the line $x_i = x(i)$ is tangent to the conic at the point $P = (x(1), x(2)) \neq (0, 0)$ for $i = 1, 2$. This means that P is a singular point of C, whence C is a double line. Hence $\varepsilon = 0$ and $A = 0$. Q.E.D.

Corollary 3.4. *If we switch* (a_1, m_1, α_1, ρ_1, x_1, y_1, G_1) *and* (a_2, m_2, α_2, ρ_2, x_2, y_2, G_2), *then* (α_n, $d(n)$, $e(n)$) *and* (α_{3-n}, $d(3-n)$, $e(3-n)$) *are switched for all* n. *Modulo this switching, we may assume that* $d(1) > d(3)$.

Proof. The first assertion is obvious. We note that $d(1) = m_1 > 0$ and $d(2) = m_2 > 0$. Thus we are also done if $d(1) > d(3)$ or if $d(0) > d(2)$ by Lemma 3.3.1. So we may assume that $d(1) \leq d(3)$ and $d(0) \leq d(2)$.

By $\Delta > 0$, we have $(d(0), d(1)) \neq (d(2), d(3))$ by Lemma 3.3.2. Hence we have either $d(1) < d(3)$ or $d(0) < d(2)$.

If we switch the two sets as above, we have either $d(2) < d(0)$ or $d(3) < d(1)$ after the switch. Thus $d(1) > d(3)$ by Lemma 3.3.1. Q.E.D.

Lemma 3.5. *Assume that* $d(1) > d(3)$ (*cf. Corollary* 3.4) *and that* $\varepsilon < 0$. *Then* $d(k) < 0$ *for some* $k \leq 5$.

Proof. By $\varepsilon = \rho_1\rho_2(\delta^2\rho_1\rho_2 - 4) < 0$, we have $\delta = 1$ and $\rho_1\rho_2 \leq 3$.

Assume first that $\rho_1 = 1$ and $\rho_2 \leq 3$. By $d(3) = \rho_2 m_2 - m_1 < m_1$, we have $\rho_2 m_2 < 2m_1$. Thus the lemma follows from

$$\begin{aligned} d(5) &= \rho_2(\rho_2 - 2)m_2 - (\rho_2 - 1)m_1 \\ &< (\rho_2 - 2)2m_1 - (\rho_2 - 1)m_1 \\ &= (\rho_2 - 3)m_1 \leq 0. \end{aligned}$$

Assume next that $\rho_2 = 1$ and $\rho_1 \leq 3$. By $d(3) = m_2 - m_1 < m_1$, we have $m_2 < 2m_1$, and we are done by

$$\begin{aligned} d(5) &= (\rho_1 - 2)m_2 - (\rho_1 - 1)m_1 \\ &< (\rho_1 - 2)2m_1 - (\rho_1 - 1)m_1 \\ &= (\rho_1 - 3)m_1 \leq 0. \end{aligned}$$

Q.E.D.

Remark 3.6.

1. We note that $e(4) = \delta\alpha_1 > 0$, $e(5) = (\delta^2\rho_2 - 1)\alpha_1 + \delta\alpha_2 > 0$ and

$$e(6) = (\delta^2\rho_2 + \rho_1 - 3)\delta\rho_1\alpha_1 + (\delta^2\rho_1 - 1)\alpha_2.$$

In particular, $e(6) \leq 0$ implies that $\delta = 1, \rho_1 = 1$ and $\rho_2 = 1, 2$.

2. If we set

$$e_0(n) := -\alpha_{n-2}/\rho_n \ (\forall n),$$

then $e_0(n)$'s satisfy the conditions for $e(n)$ except for the values of $e(0), \cdots, e(3)$. Therefore if we put $e_1(n) = e(n) - e_0(n)$, then

$$e_1(0) = \alpha_2/\rho_2, e_1(1) = -\alpha_1/\rho_1, e_1(2) = -\alpha_2/\rho_2, e_1(3) = \alpha_1/\rho_1$$

and the following induction formula holds.

$$e_1(n+1) + e_1(n-1) = \delta\rho_n e_1(n) \quad (n \neq 1, 2).$$

Corollary 3.7. *Assume that* $\varepsilon \geq 0$. *Then* $e_1(n) > e_1(n-2)$ *for all* n; $e(n) > 0$ *for all* $n \geq 4$; $e(n) \geq \alpha_1 + \alpha_2$ *for all* $n \geq 7$.

Proof. We have $e_1(2) = -\alpha_2/\rho_2$ and $e_1(4) = \delta\alpha_1 + \alpha_2/\rho_2$. If we temporarily mean the coefficient of α_i with the subscript α_i, then we have $e_1(4)_{\alpha_i} > e_1(2)_{\alpha_i}$ for $i = 1, 2$. By $\varepsilon \geq 0$, we can apply Lemma 3.3 to $e_1(n)_{\alpha_i}$ and obtain $e_1(n)_{\alpha_i} > e_1(n-2)_{\alpha_i}$ for all $n \geq 4$ and $i = 1, 2$. Since $e_0(n)$ depends only on $n \mod (4)$, we have $e(n) \geq e(n-4) + \alpha_1 + \alpha_2$ for all $n \geq 6$. Also by $e_0(n) \in \alpha_{n,2} \cdot [-1, 0]$, we have $e(n)_{\alpha_i} - e(n-2)_{\alpha_i} > 0$ (resp. ≥ 0) for $i \not\equiv n$ (resp. $i \equiv n$) $\mod (2)$ if $n \geq 4$. In other words, we have $e(n) \geq e(n-2) + \alpha_{n+1,2}$ for all $n \geq 4$.

By $e(3) = 0$ and $e(4) = \delta\alpha_1$, we have $e(5) \geq \alpha_2$, $e(6) \geq \alpha_1$ and $e(n) \geq \alpha_1 + \alpha_2$ $(n \geq 7)$. Q.E.D.

Corollary 3.8. *Assume that $d(1) > d(3)$ (cf. Corollary 3.4). Let k be the smallest integer ≥ 3 such that $d(k) \leq 0$. (The integer k exists by Lemma 3.3.) Then $e(n) > 0$ if $4 \leq n \leq k+1$.*

Proof. We note that $e(4), e(5) > 0$ by Remark 3.6.1. Thus we are done if $k \leq 4$. If $\varepsilon \geq 0$, then $e(n) > 0$ for all $n \geq 4$ by Corollary 3.7. Thus we are also done if $\varepsilon \geq 0$.

Thus we may assume that $\varepsilon < 0$ and $d(4) > 0$. Hence $k = 5$ by Lemma 3.5. It is enough to derive a contradiction assuming $e(6) \leq 0$. By Remark 3.6.1, we have $\delta = \rho_1 = 1$ and $\rho_2 = 1, 2$. We have

$$m_1 > d(3) = \rho_2 m_2 - m_1 > 0,$$

$$d(4) = d(3) - m_2 = (\rho_2 - 1)m_2 - m_1 > 0.$$

From the second equation, we have $\rho_2 = 2$. We have $m_1 > m_2$ from the first and $m_2 > m_1$ from the second. This is a contradiciton. Q.E.D.

Definition 3.9. Let $S_i = (x_i = 0)/\Gamma$ and $D = (z = 0)/\Gamma$ be $\mathbb{Q}$-Cartier Weil divisors on X by Theorem 2.9.2, then we have $-a_1 S_1 + (m_2 - a_2)S_2 + D \sim 0$, $m_1 D \sim \delta S_2$ and $m_2 D \sim \delta S_1$.

Then we introduce the following sections and divisors.

$$\begin{array}{ll} F_0 := y_1 \in H^0(X, \mathcal{O}(L_0)), \text{where} & L_0 := \delta\rho_1 L_1 - L_2, \\ F_1 := x_2 \in H^0(X, \mathcal{O}(L_1)), \text{where} & L_1 := S_2, \\ F_2 := x_1 \in H^0(X, \mathcal{O}(L_2)), \text{where} & L_2 := S_1, \\ F_3 := y_2 \in H^0(X, \mathcal{O}(L_3)), \text{where} & L_3 := \delta\rho_2 L_2 - L_1. \end{array}$$

We note that the formulas

$$F_{n-1}F_{n+1} = G_n(z^{d(n)}, F_n^{\delta}) \qquad (n = 1, 2)$$

can be rewritten in the form

$$F_{n-1}F_{n+1} = G_{n-2}(F_n^{\delta}, z^{d(n)}u^{e(n)})u^{\alpha_n} \qquad (n = 1, 2).$$

by the formula (3.1).

These L_i and F_i are extended as follows.

Theorem 3.10. *Let $n_0, n_1 \in \mathbb{Z}$ be such that $n_0 \leq 1$, $2 \leq n_1$ and*

$$\begin{aligned} d(n) > 0 &\quad \text{if } n \in [n_0, n_1], \\ e(n) > 0 &\quad \text{if } n \in [n_0, n_1] \setminus [0, 3]. \end{aligned}$$

Then $L_0, \cdots, L_3$ and $F_0, \cdots, F_3$ can be extended to divisors L_n and $F_n \in H^0(X, \mathcal{O}(L_n))$ for $n \in [n_0 - 1, n_1 + 1]$ such that the following hold.

0_n. *$L_{n-1} + L_{n+1} = \delta\rho_n L_n$, if $n \in [n_0, n_1]$.*

1_n. *F_n, F_{n-1} are relatively prime on X (that is, $\{F_n = F_{n-1} = 0\}$ contains no divisors on X), if $n \in [n_0, n_1 + 1]$.*

2_n. *F_n, zu are relatively prime on X, if $n \in [n_0 - 1, n_1 + 1]$.*

3_n. $$F_{n-1}F_{n+1} = \begin{cases} G_n(z^{d(n)}, F_n^\delta) = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} \\ (n = 1, 2), \\ G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)}) \\ (n \neq 1, 2), \end{cases}$$

if $n \in [n_0, n_1]$.

Proof. By Corollary 3.4, we only need to consider $n \geq 2$. Thus we set $n_0 = 1$ and use induction on n_1.

The theorem is obvious if $n_1 = 2$. Assume that $n_1 \geq 3$ and let $n = n_1 \geq 3$. By the induction hypotheses, it is enough to define L_{n+1} by the assertion 0_n, construct F_{n+1} satisfying the assertion 3_n and prove the assertions 1_{n+1} and 2_{n+1}.

We will construct F_{n+1} satisfying 3_n using 1_{n-1}, 2_{n-1}, 3_{n-2} and 3_{n-1}. During this proof, $\equiv$ denotes the congruence modulo the ideal $F_{n-1}\mathbb{C}[[u]][F_{n-3}, F_{n-2}, F_{n-1}, F_n, z]$ unless otherwise mentioned. We first claim that

$$F_{n-2}^\delta F_n^\delta \equiv z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}} \cdot z^{d(n)}u^{e(n)}.$$

We note that the claim is obvious for $n = 3$ by $F_1^\delta F_3^\delta \equiv z^{\delta\rho_2 d(2)}$. If $n \geq 4$, the claim follows from

$$\begin{aligned} F_{n-2}^\delta F_n^\delta &= G_{n-3}(F_{n-1}^\delta, z^{d(n-1)}u^{e(n-1)})^\delta \\ &\equiv z^{\delta\rho_{n-3}d(n-1)}u^{\delta\rho_{n-3}e(n-1)+\delta\alpha_{n-3}} \\ &= z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}} \cdot z^{d(n)}u^{e(n)}. \end{aligned}$$

In the following, we use a temporary notation that $\#$ denotes any sufficiently large integer. Let

$$M := F_{n-2}^{\delta\rho_{n-2}}G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})z^\# u^\#.$$

Then by the above claim, we have the following.

$$\begin{aligned}
M &= G_{n-2}(F_{n-2}^\delta F_n^\delta, F_{n-2}^\delta z^{d(n)}u^{e(n)})z^\# u^\# \\
&\equiv G_{n-2}(z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}}, F_{n-2}^\delta)z^\# u^\# \\
&= G_{n-4}(F_{n-2}^\delta, z^{d(n-2)}u^{e(n-2)})z^\# u^\# \\
&= F_{n-1}F_{n-3}z^\# u^\# \\
&\equiv 0.
\end{aligned}$$

Thus $G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})$ (or $G_n(z^{d(n)}, F_n^\delta)$ if $n = 1,2$) vanishes on the divisor $(F_{n-1} = 0)$ by 1_{n-1} and 2_{n-1}. Hence we obtain $F_{n+1} \in H^0(X, \mathcal{O}(L_{n+1}))$ as claimed.

We will prove 1_{n+1} and 2_{n+1} using 2_n and 3_n. We see that $F_{n+1} = zu = 0$ implies $F_n = 0$ by the formula 3_n. Indeed one can use $d(n), e(n) > 0$ if $n \geq 4$ and $d(3) > 0$ and $G_1(T_1, T_2) \equiv T_1^{\rho_1} \mod (u)$ if $n = 3$. Thus by 2_n, F_{n+1} and zu are relatively prime on X, which is 2_{n+1}. $F_n = F_{n+1} = 0$ implies $zu = 0$. So by 2_n, F_n, F_{n+1} are relatively prime on X, which is 1_{n+1}. Q.E.D.

Definition 3.11. By Corollary 3.4, we will assume that $d(1) > d(3)$. Let $k \geq 3$ be the smallest integer such that $d(k) \leq 0$, which exists by Corollary 3.3.1. Then we have

$$\begin{aligned}
d(1), d(2), \cdots, d(k-1) &> 0, \\
e(4), \cdots, e(k+1) &> 0,
\end{aligned}$$

by Corollary 3.8.

By Theorem 3.10, $\mathbb{Q}$-Cartier Weil divisors L_i and sections $F_i \in H^0(X, \mathcal{O}_X(L_i))$ $(i = 0, \cdots, k)$ satisfy the following.

0. $L_{n-1} + L_{n+1} = \delta\rho_n L_n$, if $1 \leq n \leq k-1$.
1. F_n, F_{n-1} are relatively prime on X if $1 \leq n \leq k$.
2. F_n, zu are relatively prime on X, if $0 \leq n \leq k$.
3. $F_{n-1}F_{n+1} = \begin{cases} G_n(z^{d(n)}, F_n^\delta) = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} \\ (n = 1,2), \\ G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)}) \\ (n \neq 1,2), \end{cases}$

   if $1 \leq n \leq k-1$.

We then introduce the modified sequence $d^*(n)$ for the uniform treatment of F_n as follows.

$$d^*(n) = \begin{cases} d(n) & (n \leq k), \\ -d(n-2) & (n \geq k+1). \end{cases}$$

The following is one of the key results that the exceptional locus C of X is a set-theoretic complete intersection of two divisors: $F_k = F_{k+1} = 0$ (cf. [4, 20.11]).

Theorem 3.12. *Under the notation and the assumptions of Definition* 3.11, *we have*

$$F_{k+1} := \frac{G_{k-2}(F_k^\delta z^{-d(k)}, u^{e(k)})}{F_{k-1}} = \frac{G_{k-2}(F_k^\delta, z^{d(k)}u^{e(k)})z^{-\rho_{k-2}d(k)}}{F_{k-1}}$$

belongs to $H^0(X, \mathcal{O}(L_{k+1}))$, *where* $L_{k+1} := -L_{k-1}$.

Furthermore, F_k *and* F_{k+1} *satisfy the following.*

1. $C = \{F_{k+1} = F_k = 0\}$ *as a set.*
2. $C = \{F_{k+1} = u = 0\}$ *as a set.*

Proof. The proof that F_{k+1} is a regular section of $\mathcal{O}(L_{k+1})$ is similar to the one for F_{n+1} in Theorem 3.10, and we omit it.

The assertion 1 is immediately reduced to 2. Indeed $F_{k+1} = F_k = 0$ implies $u = 0$ by the definition of F_{k+1} (note that $e(k) > 0$ if $k \geq 4$). It remains to prove the assertion 2.

Let $F_n|_H$ denote the restriction of F_n to H and $(F_n|_H)$ the divisor defined by $F_n|_H = 0$. We note

$$\begin{array}{rclccc} D \cap H & = & \ell_1 & +\ell_2 & +C \\ (F_1|_H) & = & & m_2\rho_2\ell_2 & \\ (F_2|_H) & = & m_1\rho_1\ell_1 & & \\ (F_3|_H) & = & m_2\rho_2\ell_1 & & +C. \end{array}$$

We claim

$$(F_n|_H) \equiv \rho_{n-1}d(n-1)\ell_1 \mod \mathbb{Z}C \quad \text{for } n \in [2, k]. \tag{3.2}$$

We prove the claim by induction on n, where the cases $n = 2, 3$ are checked. Assume that the claim is proved up to n ($\leq k-1$). By Definition 3.11, we have

$$\begin{aligned} (F_{n+1}|_H) &\equiv (\delta\rho_{n-2}\rho_{n-1}d(n-1) - \rho_{n-2}d(n-2))\ell, \\ &\equiv \rho_n d(n)\ell_1 \mod \mathbb{Z}C. \end{aligned}$$

Thus the claim is proved. We then have

$$\begin{aligned} (F_{k+1}|_H) &\equiv \rho_{k-2}(\delta \cdot (F_k|_H) - d(k)D) - (F_{k-1}|_H) \\ &\equiv 0 \mod \mathbb{Z}C. \end{aligned}$$

Hence $C = \{F_{k+1} = u = 0\}$, and we are done. Q.E.D.

There is another important division.

Theorem 3.13. *Under the notation and the assumptions of Theorem 3.12, we have*

$$\begin{aligned} F_{k+2} &:= \frac{G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)})}{F_k} \\ &= \frac{G_{k-1}(F_{k+1}^{\delta}, z^{d^*(k+1)} u^{e(k+1)}) z^{-\rho_{k-1} d^*(k+1)}}{F_k} \end{aligned}$$

belongs to $H^0(X, \mathcal{O}(L_{k+2}))$, *where* $L_{k+2} := -L_k$.
Furthermore, $C = \{F_{k-1} = F_{k+2} = u = 0\}$.

Proof. We will closely follow the proof for Theorem 3.10. In this proof, $\equiv$ denotes the congruence modulo $F_k \mathbb{C}[[u]][F_{k-2}, F_{k-1}, F_k, F_{k+1}, z]$.
By the induction formula of $e(n)$, we have:

$$\begin{aligned} F_{k-1}^{\delta} F_{k+1}^{\delta} z^{d(k-1)} &= G_{k-2}(F_k^{\delta} z^{-d(k)}, u^{e(k)})^{\delta} z^{d(k-1)} \\ &\equiv z^{d(k-1)} u^{\delta\rho_{k-2} e(k) + \delta\alpha_{k-2}} \\ &= u^{e(k+1)} \cdot z^{d(k-1)} u^{\alpha_{k-1,2} + e(k-1)}. \end{aligned}$$

Thus for

$$M := F_{k-1}^{\delta\rho_{k-1}} G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)}) u^{\#},$$

we have the following.

$$\begin{aligned} M &= G_{k-1}(F_{k-1}^{\delta} F_{k+1}^{\delta} z^{d(k-1)}, F_{k-1}^{\delta} u^{e(k+1)}) u^{\#} \\ &\equiv G_{k-1}(z^{d(k-1)} u^{\alpha_{k-1,2} + e(k-1)}, F_{k-1}^{\delta}) u^{\#} \\ &= G_{k-3}(F_{k-1}^{\delta}, z^{d(k-1)} u^{e(k-1)}) u^{\#} \\ &= F_k F_{k-2} u^{\#} \\ &\equiv 0. \end{aligned}$$

Since $F_{k-1}u, F_k$ are relatively prime on X by Theorem 3.10, we see that $G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)})$ is divisible by F_k and that F_{k+2} is a regular section of L_{k+2}.

For the last assertion, we borrow the notation and the argument in the proof of the previous theorem 3.12. We saw $(F_{k-1}|_H) \subset C \cup \ell_1$ there. By the formula (3.2), we have

$$\begin{aligned} (F_{k+2}|_H) &\equiv \rho_{k-1}(\delta(F_{k+1}|_H) + d(k-1)(D \cap H)) - (F_k|_H) \\ &\equiv \rho_{k-1} d(k-1) - \rho_{k-1} d(k-1) \equiv 0 \mod \mathbb{Z}C. \end{aligned}$$

This means $(F_{k+2}|_H) \subset C \cup \ell_2$, which proves the claim. Q.E.D.

The following are elementary properties which are immediate to check.

Proposition 3.14. *We have*

1. $d^*(n)L_{n+1} - d^*(n+1)L_n \sim 0$ *is a generating relation for* L_n, L_{n+1} *in* Pic X *if* $1 \le n \le k+1$,
2. $(L_n \cdot C) = d^*(n)/(m_1 m_2)$ *if* $1 \le n \le k+2$.

Proof. The case $n = 1$ of the first assertion is on $m_1 S_1 \sim m_2 S_2$. If $iS_1 - jS_2 \sim 0$ then it is Cartier at P_1. Hence $m_1 | i$ and $iS_1 - jS_2$ is a multiple of $m_1 S_1 - m_2 S_2$. If $2 \le n \le k-1$, then we are done by the change of the basis, $L_{n+1} \sim \delta\rho_n L_n - L_{n-1}$:

$$\begin{aligned} d(n)L_{n+1} - d(n+1)L_n &\sim d(n)(\delta\rho_n L_n - l_{n-1}) - d(n+1)L_n \\ &\sim d(n-1)L_n - d(n)L_n. \end{aligned}$$

The cases $n = k, k+1$ follow from the case $n = k-1$. The first assertion is thus proved. The second assertion follows from the first because $(L_1 \cdot C) = 1/m_2$ and $(L_2 \cdot C) = 1/m_1$. Q.E.D.

Proposition 3.15. *Let* $c(i)$ $(i \in [1, k+2])$ *be a sequence determined by*

i. $c(1) = a_1$, $c(2) = m_2 - a_2$,
ii. $c(n+1) = \delta\rho_n c(n) - c(n-1)$ *for* $n \in [2, k-1]$,
iii. $c(k+1) = -c(k-1)$, $c(k+2) = -c(k)$.

Then we have

1. $-K_X \sim c(n)L_{n+1} - c(n+1)L_n$ *if* $1 \le n \le k+1$,
2. $c(n)d^*(n+1) - c(n+1)d^*(n) = \delta$ *for all* $n \in [1, k+1]$,
3. $c(n)$ *and* $d^*(n)$ *are relatively prime for all* $n \in [1, k+2]$, *and*
4. $-d^*(n)K_X \sim \delta L_n$ *for all* $n \in [1, k+1]$.

Proof. The case $n = 1$ of the assertion 1 follows from $-K_X \sim a_1 L_2 - (m_2 - a_2)L_1$. One can check the case $n \in [2, k-1]$ inductively by using $L_{n+1} \sim \delta\rho_n L_n - L_{n-1}$ $(n \in [2, k-1])$ as follows:

$$\begin{aligned} c(n)L_{n+1} - c(n+1)L_n &\sim c(n)(\delta\rho_n L_n - L_{n-1}) - c(n+1)L_n \\ &\sim c(n-1)L_n - c(n)L_{n-1} \sim -K_X. \end{aligned}$$

The cases $n = k, k+1$ are equivalent to the case $n = k-1$ because $L_{k+1} \sim -L_{k-1}$ and $L_{k+2} \sim -L_k$. This proves the assertion 1.

By the induction formula, we immediately see that the value

$$c(n)d(n+1) - c(n+1)d(n) = c(n-1)d(n) - c(n)d(n-1)$$

does not depend on n and hence equal to $\delta = c(1)d(2) - c(2)d(1)$, which proves the assertion 2.

Let $\gcd(n) = (c(n), d(n))$, then $\gcd(n)$ divides δ. By the induction formula, we see that $d(i) \equiv -d(i-2)$ and $c(i) \equiv -c(i-2)$ modulo $\gcd(n)$ for all i. This implies that $\gcd(n)$ divides $\gcd(1)$ or $\gcd(2)$ depending on the parity of n. Since $\gcd(1) = (a_1, m_1) = 1$ and $\gcd(2) = (a_2, m_2) = 1$, we get $\gcd(n) = 1$, the assertion 3.

The assertion 4 follows from Proposition 3.14.

$$\begin{aligned} -d^*(n)K_X &\sim c(n)d^*(n)L_{n+1} - c(n+1)d^*(n)L_n \\ &\sim (c(n)d^*(n+1) - c(n+1)d^*(n))L_n = \delta L_n. \end{aligned}$$

Q.E.D.

§4. Contractions and flips

In this section, we give an explicit description of the contractions and the flips using the divisions in Section 3.

Although we work on the specific model X, our description can treat arbitrary extremal nbd of type $k2A$ by passing to the formal completion or the associated analytic space by Theorems 2.2 and 2.9.

First by Theorem 3.12 alone (without the further division), we can decide exactly when $X \supset C$ is a flipping nbd as follows.

Corollary 4.1. *Let $X \supset C \simeq \mathbb{P}^1$ be the scheme introduced in* 3.1. *Under the notation and the assumptions of Theorem* 3.12, *we have*

1. *If $d(k) < 0$ then the formal completion $\hat{X}$ and the associated algebraic space of $X \supset C$ are flipping nbds.*
2. *If $d(k) = 0$ then $\hat{X} \supset C$ is not a flipping nbd. Indeed, C is a fiber of a divisorial contraction of $\hat{X}$ (or the algebraic space X) and $\{F_{k+1} = 0\}$ is the exceptional divisor for the contraction.*

Proof. By Theorem 3.12, C is a set-theoretic complete intersection of two Cartier divisors $N_1 := a(F_k = 0) \sim aL_k$ and $N_2 := a(F_{k+1} = 0) \sim aL_{k+1}$ for some integer $a > 0$.

Assume that $d(k) < 0$. Then $-N_1$ and $-N_2$ are ample on C by $(L_k \cdot C), (L_{k+1} \cdot C) < 0$. Then the defining ideal J of $N_1 \cap N_2$ has the property that J/J^2 is ample on $C = \operatorname{Supp}(\mathcal{O}_X/J)$. Thus $C \subset \hat{X}$ can be contracted by [2, 6.2] and the associated algebraic space can be contracted by [2, 3.1]. Since $(K_X \cdot C) < 0$, these are flipping contractions.

Assume thet $d(k) = 0$. Then $aL_k \sim 0$ and $(L_{k+1} \cdot C) < 0$. Then $F_k^a : X \to \mathbb{A}^1$ induces, on the divisor N_2, a morphism $g : N_2 \to \mathbb{A}^1$ such that $C = g^{-1}(0)$ as a set. We note that $\mathcal{O}_{N_2}(-N_2)$ is g-ample by

$(L_{k+1} \cdot C) < 0$. Thus we can similarly see that N_2 can be contracted to a curve such that C is one of its set-theoretic fiber by a birational contraction of the formal completion (and also the associated algebraic space) of $X \supset C$. We note that $-K_X$ is relatively ample and N_2 is exceptional with respect to the contraction. Q.E.D.

The extremal contraction of $\hat{X}$ is expressed as Spec $H^0(\hat{X}, \mathcal{O}_{\hat{X}})$ (or its formal scheme version). We give here an explicit construction using the further division Theorems 3.12 and 3.13.

Definition 4.2. Let

$$\begin{array}{lll} y_1' & := F_{k+2} & \in H^0(X, \mathcal{O}(-L_k)), \\ x_2' & := F_{k+1} & \in H^0(X, \mathcal{O}(L_{k+1})), \\ x_1' & := F_k & \in H^0(X, \mathcal{O}(L_k)), \\ y_2' & := F_{k-1} & \in H^0(X, \mathcal{O}(-L_{k+1})), \\ z & & \in H^0(X, \mathcal{O}(c(k)L_{k+1} + c(k-1)L_k)), \end{array}$$

on which we have the Γ-action defined in Remark 2.7. We rewrite the action as follows.

By $\mathbb{Z}L_1 + \mathbb{Z}L_2 = \mathbb{Z}L_k + \mathbb{Z}L_{k+1} \subset \text{Pic } X$ (Proposition 3.14.1), we set

$$\Gamma' := \text{Hom } (\mathbb{Z}L_k + \mathbb{Z}L_{k+1}, \mathbb{C}^*) = \text{Hom } (\mathbb{Z}L_1 + \mathbb{Z}L_2, \mathbb{C}^*) = \Gamma,$$

and $\gamma' \in \Gamma'$ acts on $H^0(X, \mathcal{O}(L_i))$ as the multiplication by $\gamma'(L_i) \in \mathbb{C}^*$. Let $m_1' := d(k-1) > 0$, $m_2' := -d(k) \geq 0$. Then we have $m_2' L_{k+1} \sim m_1' L_k$ (Proposition 3.14) and hence

$$\Gamma' = \{\gamma' = (\gamma_1', \gamma_2') \in (\mathbb{C}^*)^2 \mid (\gamma_1')^{m_1'} = (\gamma_2')^{m_2'}\}.$$

The Γ-action is equivalent to the Γ'-action given by

$$\gamma'(x_i', z, y_i', u) = ((\gamma_i')x_i', (\gamma_1')^{c(k-1)}(\gamma_2')^{c(k)}z, (\gamma_i')^{-1}y_i', u).$$

Γ' acts on the ring $R' := \mathbb{C}[[u]][x_1', y_1', x_2', y_2', z]$, the ideal

$$I := (x_1'y_1' - G_{k-1}((x_2')^{\delta} z^{m_1'}, u^{e(k+1)}), x_2'y_2' - G_{k-2}((x_1')^{\delta} z^{m_2'}, u^{e(k)}))$$

and the scheme $W' := \text{Spec } R'/I'$. We note that it is easy to check that W' is a complete intersection and is an integral domain by Proposition 4.8 and that W' is normal by the Jacobian criterion. Let

$$(R'/I')^{\Gamma'} := \{r \in R'/I' \mid \gamma' r = r\}$$

and $Y := \text{Spec } (R'/I')^{\Gamma'}$ with the origin 0. Because of the construction, we have a natural morphism $\pi : X \to Y$.

Theorem 4.3. *There is an open subset $U \ni 0$ of Y such that $\pi : \pi^{-1}(U) \to U$ is either a flipping contraction with C the only flipping curve (the case $m'_2 > 0$), or a divisorial contraction with $(F_{k+1} = 0)$ the only exceptional divisor (the case $m'_2 = 0$).*

Proof. First of all, W (cf. Definition 2.8) and W' are birationally equivalent because of the inductive formulas in Defintion 3.11 and Theorems 3.12 and 3.13. The birational map is Γ-equivariant as explained in Definition 4.2. Because of these, it is easy to see that π is birational.

Next, we claim that $\pi^{-1}(0) = C$ as a set, and prove it in two cases.

Case 1 $(m'_2 > 0)$. For arbitrary i, j, we have $u, (x'_i)^a(y'_j)^b \in (R'/I')^{\Gamma'}$ for some positive integers a, b depending on x'_i, y'_j. Thus

$$C \subset \pi^{-1}(0) \subset \{x'_1 = x'_2 = 0\} \cup \{y'_1 = y'_2 = u = 0\}.$$

Thus by Theorems 3.12 and 3.13, we have $\pi^{-1}(0) = C$ as a set.

Case 2 $(m'_2 = 0)$. We have $(y'_1)^a, (x'_1)^a, x'_2y'_2, u \in (R'/I')^{\Gamma'}$ for some positive integer a. Thus

$$C \subset \pi^{-1}(0) \subset \{x'_1 = x'_2 = 0\} \cup \{y'_1 = y'_2 = u = 0\}.$$

The rest is the same as Case 1. This proves the claim.

By [9], π can be extended to a proper birational morphism $\bar{\pi} : \bar{X} \to Y$. Then we have $\pi^{-1}(0) = C$. Indeed, by the normality of Y, $\pi^{-1}(0)$ is a connected set containing C as a connected component.

Thus it is enough to set $U = Y \setminus \bar{\pi}(\bar{X} \setminus X)$ to make π proper above it. Shrink U further so that L_{k-1} is π-ample over U.

Assume that $m'_2 > 0$. Then $C = \pi^{-1}(0)$ is a set-theoretic complete intersection of two π-negative divisors $(F_k = 0)$ and $(F_{k+1} = 0)$. Every π-exceptional curve $\subset \pi^{-1}(U)$ is contained in these divisors. Thus C is the only π-exceptional curve $\subset \pi^{-1}(U)$.

Assume next that $m'_2 = 0$. In this case, the arguments are similar to those in the proof of Corolary 4.1.2. $C = (F_k = 0) \cap (F_{k+1} = 0)$ being π-exceptional and $F_k \sim 0$ imply that F_{k+1} is contracted by π. Then, $-F_{k+1}$ being π-ample implies that F_{k+1} contains all the curves contracted by π. Q.E.D.

We will closely study the divisorial contraction or the flip as follows.

Definition 4.4. Let $a'_1 := c(k-1)$ umod (m'_1) (cf. Definition 3.2), and if $m'_2 > 0$ then we also let $a'_2 := c(k)$ umod (m'_2). Since $(c(i), d(i)) = 1$ by Proposition 3.15, we have $(m'_i, a'_i) = 1$ and $0 < a'_i \leq m'_i$ if $m'_i > 0$.

Theorem 4.5. *With the above notation and assumptions, assume further that $d(k) = 0$. Then $m'_1 = d(k-1) = \delta = \gcd(m_1, m_2)$, $c(k) = -1$ and we have a terminal singularity of index m'_1,*

$$0 \in Y \simeq (\xi, \eta, \zeta, u; \xi\eta - G_{k-1}(\zeta^{m'_1}, u^{e(k+1)}))/\mathbb{Z}_{m'_1}(1, -1, a'_1, 0),$$

where the π-fundamental set is the curve $\{\zeta = G_{k-2}(\xi^{m'_1}, u^{e(k)}) = 0\}/\mathbb{Z}_{m'_1}$ under the identification.

Proof. By the induction formula, we see $\gcd(m_1, m_2) = \gcd(d(i), d(i+1))$ for all i. In particular, we have $d(k-1) = \gcd(m_1, m_2)$. By Proposition 3.15, we have $c(k) = \pm 1$ and $-c(k)d(k-1) = \delta$. Thus $c(k) = -1$ and $m'_1 = d(k-1) = \delta$.

By $d(k) = 0$, we have $\Gamma' = \mathbb{Z}_{m'_1} \times \mathbb{C}^*$ and can obtain the isomorphism by taking the invariants in two steps. We note that $\xi = x'_1$, $\eta = y'_1$ and $\zeta = x'_1 z$. Since the fundamental set on Y is defined by $x'_2 y'_2 = x'_2 z = 0$, we are done by $x'_2 y'_2 = G_{k-2}(\xi^{m'_1}, u^{e(k)})$. Q.E.D.

Definition 4.6. Let

$$X' := (W' \setminus \{x'_1 = x'_2 = 0\})/\Gamma' \supset C' := \{y'_1 = y'_2 = z = u = 0\}/\Gamma'$$

and P'_i the point, $x'_i = y'_1 = y'_2 = z = u = 0$. We note that $C' \simeq \mathbb{P}^1$.

Let $\pi' : X' \to Y$ be the induced morphism.

Theorem 4.7. *With the above notation and the assumptions, assume further that $d(k) < 0$. Then we have*

1. *X' is a normal scheme of dimension 3 such that $X' \setminus \{P'_i, P'_2\}$ is smooth and the germ*

$$P'_i \in X' \simeq (\xi'_i, \eta'_i, \zeta'_i, u; \xi'_i\eta'_i = G'_i(\zeta_i'^{m'_i}, 1))/\mathbb{Z}_{m'_i}(1, -1, a'_i, 0)$$

is a terminal singularity of index m'_i and $P_i \in C' = \xi'_i$-axis$/\mathbb{Z}_{m'_i}$ under the identification, where

$$G'_i(T_1, T_2) := G_{k-i}(T_1, u^{e(k+2-i)}T_2) \quad (i = 1, 2),$$

2. *X' is proper and is the flip of X over some open set $\ni 0$ of Y.*

Proof. The proof of the first assertion is similar to the one for Theorem 2.9, and we omit it.

As in the proof of Theorem 4.3, we see that π' is a birational morphism. Although X' is proper over X, we only claim it over an open set $\ni 0$. For this we only need to show $(\pi')^{-1}(0) = C'$ as in the proof of Theorem 4.3.

For arbitrary i, j, we have $(x_i')^a(y_j')^b, (x_i')^a z^c \in (R'/I')^{\Gamma'}$ for some positive integers a, b, c depending on x_i', y_j', z. Thus

$$(\pi')^{-1}(0) \subset \{x_1' = x_2' = 0\} \cup \{y_1' = y_2' = z = u = 0\} = C',$$

and the properness is settled.

By the construction of W' and by $C = \{F_k = F_{k+1} = 0\}$ (Theorem 3.12), we have a natural birational morphism $X \setminus C \to X'$. By Theorem 4.3, $X \setminus C \simeq X' \setminus C'$ over an open set $U \ni 0$ of Y. It only remains to show that $(K_{X'} \cdot C') > 0$.

Let $S_1' := (x_1' = 0)/\Gamma'$ be the $\mathbb{Q}$-Cartier divisor on X'. Then $(S_1' \cdot C') > 0$. Since $X \simeq X'$ in codimension 1, we can pull back S_1' and $K_{X'}$ on X' to $L_k \sim (F_k = 0)$ and K_X on X. By $-d(k)K_X \sim \delta L_k$ on X (Proposition 3.15), we have $-d(k)K_{X'} \sim \delta S_1'$. Hence $(K_{X'} \cdot C') > 0$ as required. Q.E.D.

We used the following elementary result in this section. We give a proof for the readers' convenience.

Proposition 4.8. *Let A be an integral domain, and x_1, x_2, u_1, $u_2 \in A$. Assume that x_1, x_2 are prime elements (that is, x_1A, x_2A are non-zero prime ideals) and that (x_1, x_2) is a prime ideal $\neq x_1A, x_2A$. Then*

1. *If $u_1 \notin x_1A$, then $A[y_1]/(x_1y_1 - u_1)$ is an integral domain.*
2. *If $u_1 \notin (x_1, x_2)$ and $u_2 \notin x_2A$, then $A[y_1, y_2]/(x_1y_1 - u_1, x_2y_2 - u_2)$ is an integral domain.*

Proof. Let $P := \{f(y_1) \in A[y_1] \mid f(u_1/x_1) = 0\}$.

We claim that if $f(y_1) = a_ny_1^n + \cdots + a_0 \in P \setminus \{0\}$, then $n > 0$ and $a_n \in x_1A$. Indeed $n \geq \deg f \geq 1$ is obvious, and from

$$a_nu_1^n + x_1(a_{n-1}u_1^{n-1} \cdots + a_0x_1^{n-1}) = 0,$$

we get $a_n \in x_1A$ by $u_1 \notin x_1A$. This proves the claim.

For the assertion 1, it is enough to prove that $P = (x_1y_1 - u_1)$.

Let $f \in P\setminus\{0\}$. By the claim, we can lower $\deg f$ modulo $(x_1y_1 - u_1)$. Hence the assertion 1 is proved by induction on $\deg f$.

For the assertion 2, let $S = A[y_1]/(x_1y_1 - u_1)$, which is an integral domain. We note that $S/x_2S \simeq (A/x_2A)[y_1]/(x_1y_1 - u_1)$ is an integral domain by $u_1 \notin (x_1, x_2)$. We claim that $u_2 \mod x_2S \neq 0$. Indeed $u_2 \mod x_2A$ is a non-zero constant of the integral domain $(A/x_2A)[y_1]$. Hence $u_2 \mod x_2A \notin (A/x_2A)[y_1](x_1y_1 - u_1)$, which proves the claim. Finally applying the assertion 1 on S, we obtain the assertion 2. Q.E.D.

§5. Further discussions

In this section, we consider the case of a base ring which is more general than $\mathbb{C}[[u]]$ in Definition 3.2. We note that, over Spec $\mathbb{Z}$, our finite group action $\mathbb{Z}_m$ is actually the finite multiplicative group scheme action $\mu_n \subset \mathbb{G}_m$, which is linearly reductive over Spec $\mathbb{Z}$. Hence no changes are needed for characteristic ≥ 0.

Definition 5.1. Let (Λ, m_Λ) be a regular local ring and let $u_1, u_2 \in m_\Lambda$ be non-zero elements. Let a_i, m_i, ρ_i be positive integers and $G_i(T_1, T_2) \in \Lambda[T_1, T_2]$ a homogeneous polynomial in T_1 and T_2 of degree ρ_i $(i = 1, 2)$ such that

1. $a_i \leq m_i$ and $(a_i, m_i) = 1$,
2. $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2 > 0$,
3. the coefficient of $T_1^{\rho_i}$ (resp. $T_2^{\rho_i}$) in G_i is 1 (resp. u_1),
4. $\Delta := \rho_1 m_1^2 - \delta \rho_1 \rho_2 m_1 m_2 + \rho_2 m_2^2 > 0$.

By formally writing $\alpha_i = log_u u_i$ (or $u^{\alpha_i} = u_i$) for $i = 1, 2$, Definition 3.2 applies to our case. By Corollary 3.4, we may assume that

5. $d(1) > d(3)$.

Corollary 3.8 implies that

6. $u^{e(n)} \in (u_1, u_2)\Lambda$ if $4 \leq n \leq k + 1$.

Let $R := \Lambda[x_1, y_1, x_2, y_2, z]$ be the Λ-algebra with the Γ-action in Remark 2.7, and let $W =$ Spec R/I be the scheme with the Γ-action, where I is the ideal given by

$$I := (x_1 y_1 - G_1(z^{m_1}, x_2^\delta), x_2 y_2 - G_2(z^{m_1}, x_1^\delta)).$$

As in Definition 2.8, we set

$$X := (W \setminus V(x_1, x_2))/\Gamma \supset C := V(y_1, y_2, z, m_\Lambda)/\Gamma \simeq \mathbb{P}^1_{\text{Spec } \Lambda/m_\Lambda}$$

and $P_i = V(x_i, y_1, y_2, z, m_\Lambda)/\Gamma \simeq$ Spec Λ/m_Λ. Let L_i be the $\mathbb{Q}$-Cartier divisor classes and $F_i \in H^0(X, \mathcal{O}(L_i))$ be the sections as in Definition 3.9.

Theorem 5.2. *$L_0, \cdots, L_3$ and $F_0, \cdots, F_3$ can be extended to $\mathbb{Q}$-Cartier divisor classes L_i and sections $F_i \in H^0(X, \mathcal{O}(L_i))$ for $i \in [0, k+2]$ such that the following hold.*

0_n. $L_{n-1} + L_{n+1} = \begin{cases} \delta \rho_n L_n & (n \leq k-1) \\ 0 & (n = k, k+1), \end{cases}$ *if* $n \in [1, k+1]$.

1_n. *F_n, F_{n-1} are relatively prime on X if $n \in [1, k+2]$.*

2_n. *$F_n, z u_1 u_2$ are relatively prime on X, if $n \in [0, k+2]$.*

$$3_n.\ F_{n-1}F_{n+1} = \begin{cases} G_{n-2}(F_n^\delta z^{-d^*(n)}, u^{e(n)}) \\ = G_{n-2}(F_n^\delta, z^{d^*(n)}u^{e(n)})z^{-\rho_n d^*(n)} & (n = k, k+1), \\ G_{n-2}(F_n^\delta, z^{d^*(n)}u^{e(n)}) & (2 < n < k), \\ G_n(z^{d(n)}, F_n^\delta) \\ = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} & (n = 1, 2), \end{cases}$$

if $n \in [1, k+1]$.

Our argument here is slightly stronger than those for Theorems 3.10, 3.12 and 3.13, since we introduce the intermediate schemes X^i and study them closely.

Lemma 5.3. *Let the notation and the assumptions be as in Theorem* 5.2. *For* $i \in [1, k]$, *let* $R^i := \Lambda[F_{i-1}, \cdots, F_{i+2}, z]$ *be the polynomial ring with* 5 *variables and* $I^i \subset R^i$ *the ideal generated by the relations* 3_i *and* 3_{i+1}.

As in Definition 2.8, *let* $X^i := (\operatorname{Spec} R^i/I^i \setminus \{F_i = F_{i+1} = 0\})/\Gamma$, *and let* L^i_j $(j \in [i-1, i+2])$ *be the* $\mathbb{Q}$*-Cartier divisor class on* X^i *induced by* F_j. *By the condition corresponding to* 0_n, *we define* L^i_j *for all* $j \in [0, k+2]$. *Let* B^i_1 *(resp.* B^i_2*) be the closed subset of* X^i *defined by* $F_{i-1} = F_i = 0$ *(resp.* $F_{i+1} = F_{i+2} = 0$*).*

Then for every $i \in [1, k]$, *we have the following.*

1. R^i/I^i *is a normal domain of complete intersection,*
2. $\operatorname{codim}_{X^i}(B^i_j) \geq 2$ *for every* $j = 1, 2$.

By the relations $3_1, \cdots, 3_n$, $R^1/I^1, \cdots, R^{n-1}/I^{n-1}$ *are all birational to each other. For every* $i \in [2, k]$, *the birational map* $X^{i-1} \dashrightarrow X^i$ *induces*

3. *an isomorphism* $X^{i-1} \setminus B^{i-1}_2 \simeq X^i \setminus B^i_1$,
4. *the identification* $L^{i-1}_j = L^i_j$, *which is simply denoted by* L_j, *and*
5. $H^0(X^{i-1}, \mathcal{O}(L_j)) = H^0(X^i, \mathcal{O}(L_j))$ *for all* j.

Proof. It is easy to check that R^i/I^i is a complete intersection integral domain by Proposition 4.8 and $u_1u_2 \neq 0 \in \Lambda$. The normality can be checked by the Jacobian criterion at codimension 1 points, which is the assertion 1. Again using $u_1u_2 \neq 0 \in \Lambda$, one can easily check the assertion 2.

We now regard F_{i+2} as a rational function in $F_{i-2}, \cdots, F_{i+1}, z$. We can see that the regular section

$$F_{i+2}F_i \in H^0(X^{i-1}, \mathcal{O}(L^{i-1}_{i+2} + L^{i-1}_i))$$

satisfies the condition

$$F_{i+2}F_i(F_{i-1}zu_1u_2)^\# \in F_i\Lambda[F_{i-2}, F_{i-1}, F_i, F_{i+1}, z]$$

from the relations 3_{i+1}, 3_i and 3_{i-1} by pure computation, where $\#$ denotes an arbitrarily large positive integer. Indeed the computation was carried out in the proof of Theorem 3.10 with $n = i+1 \le k-1$. Since the computation is similar in other cases, we omit the computation. This means that the regular section $F_{i+2}F_i$ vanishes on the divisor $(F_i = 0) \sim L_i^{i-1}$, whence $F_{i+1} \in H^0(X^{i-1}, \mathcal{O}(L_{i+1}^{i-1}))$. Hence $(F_{i-1}, F_i, F_{i+1}, F_{i+2}, z)$ induce a morphism $X^{i-1} \setminus B_2^{i-1} \to X^i \setminus B_1^i$.

The inverse $X^i \setminus B_1^i \to X^{i-1} \setminus B_2^{i-1}$ can be constructed similarly from the assertion:

$$F_{i-2}F_i(F_{i+1}zu_1u_2)^{\#} \in F_i\Lambda[F_{i-1}, F_i, F_{i+1}, F_{i+2}, z].$$

Indeed, we can prove this using 3_{i-1}, 3_i and 3_{i+1} by the computation similar to the above. The rest are obvious. Q.E.D.

Proof of Theorem 5.2. By 0_n, we define L_j's. By Lemma 5.3, we have the extension F_j $(j \in [0, k+2])$ satisfying 3_n $(n \in [1, k+1])$ by $F_j \in H^0(X, \mathcal{O}(L_j)) = H^0(X^i, \mathcal{O}(L_j))$ for some $i \in [1, k]$ such that $j \in [i-1, i+2]$.

By Lemma 5.3, the assertions 1_n and 2_m can be examined on X^i such that $n, m \in [i-1, i+2]$.

On X^i, we know that $B_1^i = \{F_{i-1} = F_i = 0\}$, $\emptyset = \{F_i = F_{i+1} = 0\}$, $B_2^i = \{F_{i+1} = F_{i+2} = 0\}$ are of codimension ≥ 2 on X^i. This proves 1_n. The computation of 2_n can be done through a simple but tedious computation, which we omit. Q.E.D.

Remark 5.4. For the family of surfaces $\pi : X \to \operatorname{Spec} \Lambda$, we have a divisorial contraction or a flipping contraction depending on whether $d(k) = 0$ or not (Corollary 4.1). It is not difficult to obtain an analogue of Theorem 2.2 for a multi-parameter analytic deformation space of H and analogues of Theorems 3.12 and 3.13 for Λ, and furthermore to carry out a detailed computation as in Sections 2 and 4.

For instance, C need not be the only contractible curve over $[m_\Lambda]$ because we do not assume $G_i \equiv T_1^{\rho_i} \mod m_\Lambda$ in Definition 5.1.3. The contractible curves over $[m_\Lambda]$ are contained in $F_4 = 0$, which follows from $F_k = F_{k+1} = u_1 = u_2 = 0$ through the relations in Theorem 5.2.3.

Using such G_i, we can systematically construct reducible flipping curves.

Remark 5.5. An interesting problem in order to understand flips is to find the generators of the graded ring $\oplus_{\nu \in Z} H^0(X, \mathcal{O}(\nu K_X))$ or some of its variants. We note that our z, $F_0, \cdots, F_{k+2}$ are a part of the key generators.

It is possible to carry out further divisions to get F_i for $i < 0$ and $i > k+2$. The former case was treated in Theorem 3.10. The latter case corresponds to the case $i < 0$ for X' in Theorem 4.7, or we can continue the division imitating the arguments in Theorem 3.10.

However this immediate generalization does not give the right homogeneous elements as pointed out by M. Reid. He has been proposing a more general division [10] using pfaffians.

Our standpoint is that, with our easier divisions, we can determine many of the structures of the flips.

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