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On Isotropic Minimal Surfaces in Euclidean Space

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Abstract.

We investigate a certain class of minimal surfaces in Euclidean space, which are constructed from a generalization of the Weierstrass formula. We also show a characterization of the catenoid.

§1. Introduction

Let f be a conformal minimal immersion from a Riemann surface M into Euclidean N-space $\mathbb{E}^N$. It is given (at least locally) by the real part of an isotropic holomorphic immersion F from M into complex Euclidean space $\mathbb{C}^N$. We say that f is *m-isotropic* if the derivatives $f^{(k)}$ of f of order k ($k = 1, 2, \ldots, m$) are isotropic vectors in $\mathbb{C}^N$. (Note that f is necessarily 1-isotropic, which is equivalent to the conformality of f.) In other words, an m-isotropic minimal surface is locally the projection from $\mathbb{C}^N$ of an m-isotropic curve to $\mathbb{E}^N$.

The m-isotropic curves fully immersed in $\mathbb{C}^{2m+1}$ have a remarkable representation formula (cf. [4]), which is a generalization of the integral-free form of the Weierstrass formula for minimal surfaces. In the first half of this paper, applying it, we present some examples of complete minimal surfaces in $\mathbb{E}^{2m+1}$. They are based on Enneper's surface and the catenoid.

In the latter half of this paper, we study the total curvature of m-isotropic complete minimal surfaces. Several interesting inequalities concerning the total curvature of complete minimal surfaces in $\mathbb{E}^N$ have been known (cf. [1], [5], [6]). Among those, we focus our attention on the following two inequalities.

Given an m-isotropic complete minimal immersion $f\colon M \to \mathbb{E}^N$, we denote the Gaussian curvature by K, the area element by dA, the genus by g, and the number of ends by r, respectively.

- **(Chern-Osserman's inequality)**

$$\int_M K dA \leq 4(1-g-r)\pi. \tag{1}$$

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- **(Ejiri's inequality)**
  If the immersion f is full and k-degenerate, then

$$\int_M K dA \leq 2(1-g-N+k)\pi. \tag{2}$$

Here, we say that an immersion $f\colon M \to \mathbb{E}^N$ is *full* if the image $f(M)$ is not contained in any hyperplanes of $\mathbb{E}^N$, and that f is *k-degenerate* if its Gauss image $\nu(M)$ is contained in an $(N-1-k)$-dimensional subspace of complex projective $(N-1)$-space $\mathbb{C}P^{N-1}$. (By definition, the Gauss map ν of f is given by $\nu = [\partial f/\partial z]\colon M \to \mathbb{C}P^{N-1}$, where $[\partial f/\partial z]$ denotes the complex line spanned by the vector $\partial f/\partial z \in \mathbb{C}$.

Recall that the catenoid is a complete minimal surface in $\mathbb{E}^3$, which is of genus zero, with two ends and of total curvature -4π. So it satisfies the equality in (1).

Jorge and Meeks [7] showed the formula

$$\int_M K dA = 2\left(2(1-g) - r - \sum_{j=1}^{r} I_j\right)\pi,$$

where $I_1, \ldots, I_r$ are positive integers that describe the behaviours of ends $p_1, \ldots, p_r$, respectively. In particular, they proved that an end p_j is embedded if and only if $I_j = 1$, and hence, that the equality in (1) holds if and only if all ends of M are embedded. Indeed, the catenoid has embedded ends. In [7], they also constructed examples with arbitrary number of embedded ends, which are now called Jorge-Meeks' n-noid (n is an integer greater than 1). Note that Jorge-Meeks' 2-noid is the catenoid.

On the other hand, the catenoid also satisfies the equality in (2). So the catenoid is an example satisfying the equality both in (1) and in (2). Then it is natural to ask if there are any other examples with the same property. We can answer this question for strictly m-isotropic complete minimal surfaces as follows:

Main Theorem (A characterization of the catenoid). *The catenoid in $\mathbb{E}^3$ is the only strictly m-isotropic complete minimal surface in $\mathbb{E}^{2m+1}$, which attains the equality both in Chern-Osserman's inequality and in Ejiri's inequality.*

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§2. Preliminaries

We denote by M a Riemann surface, and by $F\colon M \to \mathbb{C}^N$ a meromorphic curve. Let $\langle\ ,\ \rangle$ denote the standard inner product of $\mathbb{E}^N$, and the quadratic form on $\mathbb{C}^N$ which is the $\mathbb{C}$-linear extension of itself as well. A linear subspace V of $\mathbb{C}^N$ is said to be *isotropic* if $V \subset V^{\perp} := \{w \in \mathbb{C}^N |\ \langle v, w\rangle = 0 \text{ for all } v \in V\}$ holds. Note that if V is isotropic then $\overline{V}$ is also isotropic, $V \cap \overline{V} = \{0\}$ and $2\dim V \leq N$. Here, we denote by $\overline{V}$ the set of all complex conjugate vectors in V.

Definition 1. $F\colon M \to \mathbb{C}^N$ is called an *m-isotropic* curve if $\langle F^{(k)}, F^{(k)}\rangle = 0\,(1 \leq k \leq m)$ hold except at the poles. Here, $F^{(k)}$ denotes the derivative of F of order k with respect to a local coordinate z of M. For simplicity, a 1-isotropic curve is said to be isotropic. An m-isotropic curve that is not $(m+1)$-isotropic is called a *strictly m-isotropic* curve.

The following two lemmas can be easily checked.

Lemma 1. *If $F\colon M \to \mathbb{C}^N$ is m-isotropic, then the following equations hold except for poles of $F^{(k)}$:*

$$\langle F^{(i)}, F^{(j)}\rangle = 0, \quad i + j \leq 2m + 1.$$

Lemma 2. *$F\colon M \to \mathbb{C}^N$ is full if and only if at each point $p \in M$, the vectors $F', F'', \ldots, F^{(N)}$ are linearly independent except for isolated points.*

Note that Lemma 1 implies that Definition 1 is well-defined.

Proposition 1. *If $F\colon M \to \mathbb{C}^N$ is strictly m-isotropic, then $2m+1 \leq N$. Namely, $N = 2m + 1$ is the minimum dimension of $\mathbb{C}^N$ for which an m-isotropic curve exists.*

Proof. It is enough to prove this under the assumption that F is full. By Lemma 2, $F', \ldots, F^{(N)}$ are linearly independent almost everywhere on M. At such a point p, the subspace $V = \operatorname{Span}\{F'(p), \ldots, F^{(m)}(p)\}$ is an m-dimensional isotropic subspace of $\mathbb{C}^N$ by Lemma 1. Since F is strictly m-isotropic, $\langle F^{(m+1)}(p), F^{(m+1)}(p)\rangle \neq 0$. Hence, $F^{(m+1)}(p) \notin V \oplus \overline{V}$. In fact, suppose that $F^{(m+1)}(p) \in V \oplus \overline{V}$. Then we may write

$$F^{(m+1)}(p) = \sum \lambda_i F^{(i)}(p) + \sum \mu_i \overline{F^{(i)}(p)}. \tag{3}$$

The inner product of (3) and $F^{(j)}(p)$ then implies

$$\sum \mu_i \langle \overline{F^{(i)}(p)}, F^{(j)}(p)\rangle = 0. \tag{4}$$

Here, by the linearly independency of $F^{(k)}$, the matrix $(\langle \overline{F^{(i)}(p)}, F^{(j)}(p)\rangle)$ is nonsingular. Hence, each μ_i must be zero by (4). Substituting these into (3), we have $F^{(m+1)}(p) = \sum \lambda_i F^{(i)}(p)$. This contradicts to the linearly independency of $F^{(k)}$.

Therefore, $\mathbb{C}^N$ contains a $(2m+1)$-dimensional subspace $V \oplus \overline{V} \oplus \{F^{(m+1)}\}$. It implies that $2m+1 \leq N$. □

Lemma 3. *Let $F\colon M \to \mathbb{C}^{2m+1}$ be an m-isotropic curve. Then F is strictly m-isotropic if and only if F is full.*

Proof. It is obvious from Proposition 1 that the strictness implies the fullness.

Suppose now that the m-isotropicity of F is not strict. It implies that F is $(m+1)$-isotropic. By Lemma 1, the equations

$$\langle F^{(i)}, F^{(j)}\rangle = 0, \ i+j \leq 2m+3$$

holds, which implies that

$$\begin{pmatrix} {}^tF' \\ \vdots \\ {}^tF^{(2m+1)} \end{pmatrix} \begin{pmatrix} F' & \cdots & F^{(2m+1)} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & * & \\ 0 & & \end{pmatrix}.$$

(Here, we regard F as a column vector and denote by ${}^tF$ its transpose.) Hence, $\det(F' \cdots F^{(2m+1)}) = 0$. Therefore, F is not full. □

Now we are in a position to state an important and fundamental theorem concerning full m-isotropic curves in $\mathbb{C}^{2m+1}$, which will be needed in Section 3.

Theorem 1 (Weierstrass-Ejiri formula). *Let $G\colon M \to \mathbb{C}^{2m-1}$ be a full $(m-1)$-isotropic curve. Suppose that g is a meromorphic function on M which is not of the form $a\langle G, G\rangle + \langle B, G\rangle + c$, where a and c are complex numbers and B is a constant vector in $\mathbb{C}^{2m-1}$. Then the following system of equations*

$$\langle G^{(k)}, H\rangle = g^{(k)}, \quad k = 1, 2, \ldots, 2m-1,$$

has a unique solution $H\colon M \to \mathbb{C}^{2m-1}$.

Moreover, if we define a function h by $h = \langle G, H'\rangle / \langle G, G'\rangle$, then the curve defined by

$$\left(\frac{1}{2}\{1 - \langle G, G\rangle\}h + \langle H, G\rangle - g,\ \sqrt{-1}\left(\frac{1}{2}\{1 + \langle G, G\rangle\}h - \langle H, G\rangle + g\right),\right.$$
$$\left. hG - H\right)$$

is full and m-isotropic in $\mathbb{C}^{2m+1}$.

Conversely, any full m-isotropic curve in $\mathbb{C}^{2m+1}$ can be represented in this form.

In the case of $m = 1$, Theorem 1 is the integral-free version of the Weierstrass formula for minimal surfaces (cf. [2]). For general m, this formula was proved by Ejiri [4].

§3. Applications of Theorem 1

Recall that a minimal surface in $\mathbb{E}^n$ is given by the real part of an isotropic curve in $\mathbb{C}^n$ (at least locally). Namely, for a conformal minimal immersion $f\colon M \to \mathbb{E}^N$, there exists an isotropic curve $F\colon \tilde{M} \to \mathbb{C}^N$ such that $f \circ \pi = \mathrm{Re}F$. Here, $\pi\colon \tilde{M} \to M$ denotes the universal covering of M. In other words, there exists a multi-valued isotropic curve $F\colon M \to \mathbb{C}^N$ such that $f = \mathrm{Re}F$. We call F the *lift* of f.

First, let us recall well-known minimal surfaces in $\mathbb{E}^3$.

Example 1 (Enneper's surface). $M = \mathbb{C}$, and

$$f(z) = \mathrm{Re}\left(3z - z^3,\ \sqrt{-1}(3z + z^3),\ 3z^2\right). \tag{5}$$

Example 2 (the catenoid). $M = \mathbb{C} \setminus \{0\}$, and

$$f(z) = \mathrm{Re}\left(\frac{1}{2}\left(-\frac{1}{z} - z\right),\ \frac{\sqrt{-1}}{2}\left(-\frac{1}{z} + z\right),\ \log z\right). \tag{6}$$

Example 3 (Jorge-Meeks' n-noid). $M = (\mathbb{C} \cup \{\infty\}) \setminus \{z^n = 1\}$, and

(7)
$$f(z) = \mathrm{Re}\left(\int \frac{1 - z^{2n-2}}{2(z^n - 1)^2}dz,\ \int \frac{\sqrt{-1}(1 + z^{2n-2})}{2(z^n - 1)^2}dz,\ \int \frac{z^{n-1}}{(z^n - 1)^2}dz\right).$$

In the case of $n = 3$, integrating (7), we have

$$f(z) = \mathrm{Re}\begin{pmatrix} \dfrac{z}{6\,(1 + z + z^2)} - \dfrac{2\log(-1 + z)}{9} + \dfrac{\log(1 + z + z^2)}{9} \\ \sqrt{-1}\left\{\dfrac{z\,(1 + z)}{6 - 6\,z^3} + \dfrac{2\arctan((1 + 2z)/\sqrt{3}))}{3\sqrt{3}}\right\} \\ \dfrac{1}{3\,(z^3 - 1)} \end{pmatrix}. \tag{8}$$

We define the m-isotropicity for minimal surfaces in $\mathbb{E}^N$ as well as for curves in $\mathbb{C}^N$.

Definition 2 ([3]*). A conformal minimal immersion $f\colon M \to \mathbb{E}^N$ is said to be *m-isotropic* if it satisfies the condition that $\langle f^{(k)}, f^{(k)}\rangle = 0$ for $1 \leq k \leq m$. Here, $f^{(k)}$ denotes the partial derivative $\partial^k f/\partial z^k$ with respect to a local coordinate z of M. An m-isotropic minimal surface that is not $(m+1)$-isotropic is called a *strictly m-isotropic* minimal surface.

A conformal minimal immersion is necessarily 1-isotropic, because the conformality is nothing but the 1-isotropicity. Assume that F is a lift of f, that is, $f = \mathrm{Re}F$. It then follows from $2f^{(k)} = F^{(k)}$ that the (strictly) m-isotropicity of f is equivalent to that of F.

We will construct examples of strictly m-isotropic minimal surface in $\mathbb{E}^{2m+1}$ by making use of the Weierstrass-Ejiri formula. Our examples are based on Enneper's surface and the catenoid.

First, we recall Theorem 1, the Weierstrass-Ejiri formula. It asserts that a full m-isotropic curve $F\colon M \to \mathbb{C}^{2m+1}$ is constructed from a full $(m-1)$-isotropic curve $G\colon M \to \mathbb{C}^{2m-1}$ and a meromorphic function g. We denote the curve F constructed with these data by $\mathrm{WE}(M, G, g)$.

With this notation, Enneper's surface can be written as the real part of $F = \mathrm{WE}(\mathbb{C}, z, z^3)$. Namely, (5) is constructed from $G(z) = z$ and $g(z) = z^3$ through the Weierstrass-Ejiri formula. It is also easily verified that the catenoid (6) is given by the real part of $F = \mathrm{WE}(\mathbb{C} \setminus \{0\}, z, z\log z)$.

We note that the data of Enneper's surface are given by polynomials, and Enneper's surface is also given by polynomials.† We can construct a series of m-isotropic minimal surfaces $(m = 1, 2, \ldots)$ which are given by polynomials.

Proposition 2. *Consider the following recurrence formula:*

$$F_0(z) = z,\ F_m = \mathrm{WE}(\mathbb{C}, F_{m-1}, z^{2m+1})\ (m \geq 1)$$

Then it inductively defines strictly m-isotropic polynomials $F_m\colon \mathbb{C} \to \mathbb{C}^{2m+1}$ *of degree* $2m+1$. *The real part* $\mathrm{Re}F_m\colon \mathbb{C} \to \mathbb{E}^{2m+1}$ *is a simply-connected, complete minimal surface of total curvature* $-4m\pi$. *In particular,* $\mathrm{Re}F_1$ *is Enneper's surface.*

For the proof, we need the following lemma.

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*In [3], a full m-isotropic minimal surface in $\mathbb{E}^{2m+1}$ is simply called an *isotropic minimal surface.*

†We say that $F = (F_1, \ldots, F_N)$ is a polynomial if each component F_i is a polynomial. By the degree of F we mean the maximum of $\deg F_i$.

Lemma 4. *Let $F\colon \mathbb{C} \to \mathbb{C}^{2m+1}$ be an m-isotropic polynomial of degree $2m+1$. Then $\langle F, F\rangle$ is a polynomial of degree smaller than or equal to $2m+2$. Moreover, if F is full, then the degree of $\langle F, F\rangle$ is equal to $2m+2$.*

Proof. It follows from the m-isotropicity that

$$\langle F^{(i)}, F^{(j)}\rangle = 0, \quad i+j \leq 2m+1. \tag{9}$$

Since F is a polynomial of degree $2m+1$,

$$F^{(k)} = 0, \quad k \geq 2m+2. \tag{10}$$

Equations (9) and (10) then imply

$$\langle F, F\rangle^{(2m+2)} = 2\langle F^{(2m+1)}, F'\rangle, \tag{11}$$

$$\langle F, F\rangle^{(2m+3)} = 2\langle F^{(2m+1)}, F''\rangle. \tag{12}$$

In particular, we consider the case of $i+j = 2m+1$ in (9), that is,

$$\langle F^{(i)}, F^{(2m-i+1)}\rangle = 0, \quad i = 1, 2, \ldots, 2m. \tag{13}$$

Differentiating (13) twice, we have for i=1,2,...,2m,

(14)

$$\langle F^{(i+2)}, F^{(2m-i+1)}\rangle + 2\langle F^{(i+1)}, F^{(2m-i+2)}\rangle + \langle F^{(i)}, F^{(2m-i+3)}\rangle = 0.$$

We write the cases $i = 1, \ldots, m$ in (14) into the matrix form:

$$\begin{pmatrix} 2 & 1 & & \text{\Large 0} & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & \text{\Large 0} & \ddots & 2 & 1 \\ & & & 1 & 3 \end{pmatrix} \begin{pmatrix} \langle F^{(2)}, F^{(2m+1)}\rangle \\ \langle F^{(3)}, F^{(2m)}\rangle \\ \vdots \\ \langle F^{(m)}, F^{(m+3)}\rangle \\ \langle F^{(m+1)}, F^{(m+2)}\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{15}$$

Here, the $m \times m$ matrix on the left-hand side is nonsingular, in fact, its determinant is equal to $2m+1$. It then follows that $\langle F^{(2)}, F^{(2m+1)}\rangle = 0$. This implies that $\langle F, F\rangle^{(2m+3)} = 0$ by (12). So we can conclude that the degree of $\langle F, F\rangle$ is smaller than or equal to $2m+2$.

Suppose now that F is full. By Lemma 3 it is strictly m-isotropic. So $\langle F^{(m+1)}, F^{(m+1)}\rangle \neq 0$. Hence,

$$\begin{aligned} \langle F^{(2m+1)}, F'\rangle =& \langle F^{(2m)}, F'\rangle' - \langle F^{(2m)}, F''\rangle = -\langle F^{(2m)}, F''\rangle \\ &\vdots \\ =& (-1)^m \langle F^{(m+1)}, F^{(m+1)}\rangle \neq 0. \end{aligned}$$

Therefore, by (11), we may conclude that the degree of $\langle F, F\rangle$ is $2m+2$. □

Proof of Proposition 2. We prove this by an induction.

First, note that it is trivial in the case of $m=1$.

Assuming that the assertion is true up to $m-1$, we are going to show the case m.

The data for constructing F_m are $G=F_{m-1}$ and $g(z)=z^{2m+1}$. By our induction assumption and Lemma 4, the function $a\langle G, G\rangle+\langle B, G\rangle+c$ is a polynomial of degree $2m$, and hence, g is not identical with it. Therefore, it is assured that F_m can be constructed.

We show that F_m is a polynomial of degree $2m+1$. For this, it suffices to prove that H and h are also polynomials and that the following inequalities hold:

$$\deg H \leq 2m+1,\ \deg\langle H, G\rangle \leq 2m+1,\ \deg h \leq 1,$$

because $\deg G = 2m-1$ and $\deg g = 2m+1$.

First, we prove that H is a polynomial. Recall that H is determined by $\langle G^{(k)}, H\rangle = g^{(k)}$. Note that the determinant of the matrix $(G^{(k)})$ satisfies

$$|G' \cdots G^{(2m-1)}|' = |G' \cdots G^{(2m-2)}\ G^{(2m)}| = 0,$$

and hence, it is constant. This implies that the inverse of $(G^{(k)})$ has components consisting of polynomials. Therefore, H is also a polynomial.

In the following, we calculate the degree of H. Differentiating

$$\langle G^{(k)}, H\rangle = g^{(k)}, \quad k=1,\dots,2m-1 \tag{16}$$

we have

$$\langle G^{(k+1)}, H\rangle + \langle G^{(k)}, H'\rangle = g^{(k+1)}, \quad k=1,\dots,2m-1. \tag{17}$$

Substituting (17) into (16), we have

$$\begin{cases} \langle G^{(k)}, H'\rangle = 0, \quad k=1,\dots,2m-2, \\ \langle G^{(2m-1)}, H'\rangle = g^{(2m)}. \end{cases} \tag{18}$$

Moreover, if we differentiate (18) and carry out the calculation similar to the above, then we have

$$\begin{cases} \langle G^{(k)}, H''\rangle = 0, \quad k=1,\dots,2m-3 \\ g^{(2m)} + \langle G^{(2m-2)}, H''\rangle = 0, \\ \langle G^{(2m-1)}, H''\rangle = g^{(2m+1)}. \end{cases} \tag{19}$$

Similarly, it follows from (19) that

$$\begin{cases} \langle G^{(k)}, H'''\rangle = 0, \quad k = 1, \ldots, 2m-4, \\ -g^{(2m)} + \langle G^{(2m-3)}, H'''\rangle = 0, \\ 2g^{(2m+1)} + \langle G^{(2m-2)}, H'''\rangle = 0, \\ \langle G^{(2m-1)}, H'''\rangle = 0. \end{cases}$$

Proceeding successively, we obtain

$$\begin{cases} \langle G', H^{(2m)}\rangle = (2m-1)g^{(2m+1)} \neq 0, \\ \langle G^{(k)}, H^{(2m)}\rangle = 0, \quad k = 2, \ldots, 2m-1, \end{cases}$$

and

$$\langle G^{(k)}, H^{(2m+1)}\rangle = 0, \quad k = 1, \ldots, 2m-1.$$

Hence, we have $H^{(2m)} \neq 0$ and $H^{(2m+1)} = 0$, since $G', \ldots, G^{(2m-1)}$ are linearly independent. Hence the degree of H is $2m$.

Since $G', \ldots, G^{(2m-1)}$ form a basis of $\mathbb{C}^{2m-1}$ at every point $p \in \mathbb{C}$, we can write $H' = a_1 G' + \cdots + a_{2m-1} G^{(2m-1)}$. Hence, for $k = 1, \ldots, 2m-2$,

$$\begin{aligned} \langle G^{(k)}, H'\rangle &= \langle G^{(k)}, a_1 G' + \cdots + a_{2m-1} G^{(2m-1)}\rangle \\ &= a_1 \langle G^{(k)}, G'\rangle + \cdots + a_{2m-1} \langle G^{(k)}, G^{(2m-1)}\rangle. \end{aligned}$$

It follows from the isotropicity of G and (18) that $a_2 = \cdots = a_{2m-1} = 0$. So, $H' = a_1 G'$. It is easy to see that $a_1 = h$. Hence, $H' = hG'$. This implies that h is a rational function P_1/P_2 and $\deg P_1 - \deg P_2 = 1$. Furthermore, taking the inner product of $H' = hG'$ with $G^{(2m-1)}$, we conclude by (18) that $h\langle G', G^{(2m-1)}\rangle = g^{(2m)}$. Since $g^{(2m)}$ has degree 1, it follows from the above fact that $\deg h = 1$ and $\deg\langle G', G^{(2m-1)}\rangle = 0$.

Finally, it follows from

$$\langle H, G\rangle' = \langle H', G\rangle + \langle H, G'\rangle = h\langle G', G\rangle + g' = \frac{h}{2}\langle G, G\rangle' + g'$$

that $\deg\langle H, G\rangle'$ is at most $2m$. □

In Proposition 2, We have constructed examples F_m as a generalization of Enneper's surface. We also construct a generalized catenoid by applying the Weierstrass-Ejiri formula.

Let F_{m-1} be an $(m-1)$-isotropic curve obtained in Proposition 2. Then F_{m-1} and the multi-valued function $g(z) = z^m \log z$ on $\mathbb{C}\setminus\{0\}$ satisfy the assumption of Theorem 1. So, $C_m := \mathrm{WE}(\mathbb{C}\setminus\{0\}, F_{m-1}, z^m \log z)$

is a multi-valued strictly m-isotropic curve $C_m\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}^{2m+1}$. Explicit computations according to Theorem 1 shows that $\mathrm{Re}C_m\colon \mathbb{C}\setminus\{0\}\to\mathbb{E}^{2m+1}$ is single-valued for $m=1,2,3$. Indeed, they are given by

$$
\begin{aligned}
F_0(z) &= z,\\
C_1(z) &= \left(\frac{1}{2}\left(-\frac{1}{z}-z\right),\ \frac{\sqrt{-1}}{2}\left(-\frac{1}{z}+z\right),\ \log z\right),\\
F_1(z) &= \left(3z-z^3,\ \sqrt{-1}(3z+z^3),\ 3z^2\right),\\
C_2(z) &= \left(\frac{1}{72}\left(\frac{1}{z^2}-3z^2\right),\ \frac{\sqrt{-1}}{72}\left(\frac{1}{z^2}+3z^2\right),\ \frac{1}{18}\left(\frac{3}{z}+z\right),\right.\\
&\qquad \left.\frac{\sqrt{-1}}{18}\left(\frac{3}{z}-z\right),\ \frac{1}{12}(1-2\log z)\right),\\
F_2(z) &= \left(\frac{-1}{3}z(3z^4+5), \frac{\sqrt{-1}}{3}z(3z^4-5), \frac{5}{6}z^2(z^2-6),\right.\\
&\qquad \left.\frac{-5}{6}\sqrt{-1}z^2(z^2+6), \frac{-10}{3}z^3\right),\\
C_3(z) &= \left(\frac{1}{3600}\left(\frac{9}{z^3}+10z^3\right), \frac{\sqrt{-1}}{3600}\left(\frac{9}{z^3}-10z^3\right), -\frac{1}{400}\left(\frac{5}{z^2}-3z^2\right),\right.\\
&\qquad -\frac{\sqrt{-1}}{400}\left(\frac{5}{z^2}+3z^2\right),\ -\frac{1}{80}\left(\frac{6}{z}+z\right),\ -\frac{\sqrt{-1}}{80}\left(\frac{6}{z}-z\right),\\
&\qquad \left.\frac{1}{120}(6\log z-5)\right).
\end{aligned}
$$

Hence, $\mathrm{Re}C_1$, $\mathrm{Re}C_2$ and $\mathrm{Re}C_3$ are single-valued.‡

In the cases of $m=1,2,3$, explicit formulas of C_m also show that $\mathrm{Re}C_m$ is a complete minimal surface of genus zero, with two ends and of total curvature $-4m\pi$. In particular, $\mathrm{Re}C_1$ is the catenoid.

If we represent Jorge-Meeks' trinoid (8) by the Weierstrass-Ejiri Formula, then it is given by

$$
\begin{aligned}
G(z) &= z^2,\\
g(z) &= \frac{z^2}{6}+\frac{1}{3\sqrt{3}}\arctan\left(\frac{1+2z}{\sqrt{3}}\right)+\frac{1}{18}(z^4-1)\log\frac{z^2+z+1}{(z-1)^2}.
\end{aligned}
$$

Also, the following is an example similar to the trinoid.

‡For general m, it is still open whether $\mathrm{Re}C_m$ is single-valued or not.

Example 4. $M = (\mathbb{C} \cup \{\infty\}) \setminus \{z^3 = 1\}$, and

$$f(z) = \mathrm{Re} \int \left(\frac{3 - 3z^{10}}{(z^3-1)^4}, \frac{\sqrt{-1}\,(3+3z^{10})}{(z^3-1)^4}, \frac{z^2\,(5+5z^6)}{(z^3-1)^4}, \frac{\sqrt{-1}z^2\,(5-5z^6)}{(z^3-1)^4}, \frac{8\sqrt{-1}z^5}{(z^3-1)^4} \right) dz,$$

which is given by

$$G(z) = \left(\frac{5}{6} z^2(1+z^6), \frac{5}{6}\sqrt{-1}z^2(1-z^6), \frac{4}{3}\sqrt{-1}z^5 \right)$$

$$g(z) = \frac{4}{243} \left\{ 2z^2(10 - 41z^3 + 40z^6) + 30\sqrt{3}(z^{10}+1)\arctan\left(\frac{1+2\,z}{\sqrt{3}}\right) + 15(z^{10}-1)\log\frac{z^2+z+1}{(z-1)^2} \right\}$$

through the Weierstrass-Ejiri formula.

This is a strictly 2-isotropic complete minimal surface with three ends, of genus zero and of total curvature -20π.

§4. Total curvature

First, we recall some fundamental facts needed later (see [1], [5] etc.).

If a complete minimal surface $f\colon M \to \mathbb{E}^N$ has finite total curvature, then M is biholomorphic to a compact Riemann surface $\overline{M}$ punctured at a finite number of points $p_1, \dots, p_r$, i.e., $M \cong \overline{M} \setminus \{p_1, \dots, p_r\}$. A sufficiently small neighborhood of each p_s is called an *end* of M. The Gauss map $[\partial f/\partial z]\colon M \to \mathbb{C}P^{N-1}$ can extend to a holomorphic map from $\overline{M}$ to $\mathbb{C}P^{N-1}$. In other words, $\partial f/\partial z$ has a pole at each end. It is known that because of the completeness the order of pole at any end is greater than or equal to 2, that is, the Laurent expansion of $\partial f/\partial z$ centered at p_s $(s = 1, \dots, r)$

$$\text{(20)} \quad \frac{\partial f}{\partial z} = \frac{1}{z^{l_s}} a^s_{-l_s} + \cdots + \frac{1}{z} a^s_{-1} + \text{holomorphic part}, \quad a^s_{-l_s} \neq 0 \in \mathbb{C}^N$$

has the property that

$$\text{(21)} \qquad l_s \geq 2, \quad s = 1, \dots, r.$$

Note that Chern-Osserman's inequality is an immediate conclusion of (21).

We also have

$$a_{-1}^s \in \mathbb{R}^N, \quad s = 1, \dots, r, \tag{22}$$

since $\mathrm{Re} \int (\partial f / \partial z) dz$ is single-valued.

Let V be a complex vector subspace of $\mathbb{C}^N$ spanned by

$$a_{-l_s}^s, \dots, a_{-1}^s \quad (1 \le s \le r),$$

and $\tilde{V}$ a real vector subspace of $\mathbb{E}^N$ spanned by

$$\mathrm{Re} a_{-l_s}^s, \mathrm{Im} a_{-l_s}^s, \dots, \mathrm{Re} a_{-1}^s, \mathrm{Im} a_{-1}^s \quad (1 \le s \le r).$$

If f is a full immersion, then it holds that

$$\dim \tilde{V} = N. \tag{23}$$

On the other hand, it is known that the following equality, which is called the *Balancing formula* (see [5]), holds:

$$\sum_s a_{-1}^s = 0. \tag{24}$$

Hence, we have

$$\dim_{\mathbb{C}} V \le \sum_s l_s - 1. \tag{25}$$

Note that the inequality (25) is one of the reason why Ejiri's inequality holds.

In the following, we investigate what surface attains the equality both in Chern-Osserman's inequality and in Ejiri's inequality.

Recall that Jorge-Meeks' n-noid attains the equality in Chern-Osserman's inequality for any n and that Jorge-Meeks' 2-noid is the catenoid. On the other hand, the equality in Ejiri's inequality is attained by $\mathrm{Re}F_m$ ($m = 1, 2, \dots$) or $\mathrm{Re}C_m$ ($m = 1, 2, 3$) obtained in Section 3. This is verified by proving the following lemma.

Lemma 5. *A strictly m-isotropic minimal surface in $\mathbb{E}^{2m+1}$ is nondegenerate.*

Proof. Let $f\colon M \to \mathbb{E}^{2m+1}$ be a strictly m-isotropic minimal surface, and $F\colon M \to \mathbb{C}^{2m+1}$ its lift. Then F is also strictly m-isotropic, and hence is full by Lemma 3. The Gauss map $[f']$ is equal to $[F']$.

Assume that f is degenerate. Then there exists a constant vector $\xi \in \mathbb{C}^{2m+1}$ such that $\langle F', \xi \rangle = 0$. Hence, $\langle F, \xi \rangle =$ constant, which contradicts to the fullness of F. □

Recall that ReC_1 is also the catenoid. So, the catenoid is an example of complete minimal surfaces which attain the equality both in Chern-Osserman's inequality and in Ejiri's inequality. Conversely, Main Theorem in Section 1 asserts that the catenoid can be characterized as an m-isotropic surface in $\mathbb{E}^{2m+1}$ with these properties.

We now give a proof of Main Theorem in what follows.

Lemma 6. *If a strictly m-isotropic complete minimal surface M in $\mathbb{E}^{2m+1}$ attains the equality in Chern-Osserman's inequality, then the number of ends of M is greater than m.*

Proof. The equality implies that the order of pole at each end is exactly 2. Hence, the Laurent expansion (20) leads to

$$\frac{\partial f}{\partial z} = \frac{1}{z^2}a^s_{-2} + \frac{1}{z}a^s_{-1} + \text{holomorphic part}, \ a^s_{-2} \neq 0 \in \mathbb{C}^N. \tag{26}$$

If $m \geq 2$, then it is verified from the 2-isotropicity and (22) that

$$a^s_{-1} = 0 \tag{27}$$

in (26). Hence, $\dim \tilde{V} \leq 2r$. Therefore, $2m+1 \leq 2r$ by (23). Since m and r are integers, we conclude that $m+1 \leq r$.

If $m = 1$, then by the Balancing formula (24), we have

$$3 = \dim \tilde{V} \leq 2r + (r-1).$$

Hence, $4 \leq 3r$, which means that $2 \leq r$, since r is an integer. □

Lemma 7. *If a strictly m-isotropic complete minimal surface M in $\mathbb{E}^{2m+1}$ attains the equality both in Chern-Osserman's inequality and in Ejiri's inequality, then the genus of M is zero and the number of ends of M is $m+1$.*

Proof. By our assumption, we have

$$\int_M K dA = 4(1-g-r)\pi = 2(1-g-(2m+1))\pi,$$

which implies that

$$2(m+1-r) = g. \tag{28}$$

The left-hand side of (28) is smaller than or equal to 0 by Lemma 6 and the right-hand side is greater than or equal to 0. Therefore, both side must be 0. □

Proof of Main Theorem. The equality in Chern-Osserman's inequality implies that

$$\sum_s l_s = 2r = 2(m+1) \tag{29}$$

by Lemma 7. Moreover, the equality in Ejiri's inequality implies that the equality must hold in (25). It follows from (29) that

$$\dim_{\mathbb{C}} V = 2(m+1) - 1 = 2m+1. \tag{30}$$

On the other hand, if we assume $m \geq 2$, then $\dim_{\mathbb{C}} V \leq r = m+1$ holds because of (27), and hence the equality (30) cannot occur.

Therefore, we have $m = 1$. In this case, $g = 0$, $r = 2$ and the total curvature is -4π. So, it is the catenoid. □

Next, we consider only Chern-Osserman's inequality for strictly m-isotropic complete minimal surfaces in $\mathbb{E}^{2m+1}$.

Assume now that the equality is attained by a strictly m-isotropic complete minimal surface $f\colon M \to \mathbb{E}^{2m+1}$. By Lemma 6, the number of ends of M is greater than m. Hence, in the case of $m = 1$, the possibility of the number of ends is $2, 3, 4, \ldots$. Indeed, Jorge Meeks' n-noids realize these values. In the case of $m = 2$, the possibility of the number of ends is $3, 4, 5, \ldots$. However, this case is not quite similar to the case of $m = 1$.

Proposition 3. *A strictly 2-isotropic complete minimal surface of genus zero with three ends in* $\mathbb{E}^5$ *never attain the equality in Chern-Osserman's inequality.*

Proof. Assume that there exists a strictly 2-isotropic complete minimal surface of genus zero with three ends in $\mathbb{E}^5$ which attains the equality in Chern-Osserman's inequality.

By our assumption, the surface is biholomorphic to $\mathbb{C} \cup \{\infty\}$ punctured at three points. Without loss of generality, we may assume that these three points are cubic roots of 1, i.e., $\{z^3 = 1\}$ (if necessary, three punctured points can be mapped to $\{z^3 = 1\}$ by a linear transformation of $\mathbb{C} \cup \{\infty\}$).

Since the equality is attained in Chern-Osserman's inequality, the $\mathbb{C}^N$-valued one-form $(\partial f/\partial z)dz$ has a pole of order 2 at each end and the other points are regular. Hence, $\Omega := (z^3-1)^2(\partial f/\partial z)$ has a pole only at $z = \infty$. This implies that Ω is a polynomial of z. The degree of Ω is 4, since the induced metric

$$\left(\frac{\partial f}{\partial z}dz\right)\left(\overline{\frac{\partial f}{\partial z}dz}\right)$$

determines a positive definite inner product at $z = \infty$.

Now, we put

$$\frac{\partial f}{\partial z} = \frac{a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4}{(z^3-1)^2}, \quad a_j \in \mathbb{C}^5. \tag{31}$$

Since $\mathrm{Re}\int(\partial f/\partial z)dz$ is single-valued, the residue at each pole takes a value in $\mathbb{R}$.

Indeed,

$$\mathrm{Res}_{z=1} = \frac{1}{9}(2a_0 + a_1 - a_3 - 2a_4),$$

$$\mathrm{Res}_{z=(-1)^{2/3}} = -\frac{1}{18}(2a_0 + a_1 - a_3 - 2a_4) - \frac{\sqrt{3}}{18}i(2a_0 - a_1 - a_3 + 2a_4),$$

$$\mathrm{Res}_{z=(-1)^{4/3}} = -\frac{1}{18}(2a_0 + a_1 - a_3 - 2a_4) + \frac{\sqrt{3}}{18}i(2a_0 - a_1 - a_3 + 2a_4),$$

where i denotes the imaginary unit $\sqrt{-1}$. Hence, the following holds.

$$\begin{cases} 2a_0 + a_1 - a_3 - 2a_4 \in \mathbb{R}^5, \\ 2a_0 - a_1 - a_3 + 2a_4 \in \sqrt{-1}\mathbb{R}^5. \end{cases} \tag{32}$$

In the following, we show that the equation (32) contradicts the strictly 2-isotropicity of the surface.

By (31), the 1-isotropicity $\langle \partial f/\partial z, \partial f/\partial z\rangle = 0$ implies that

$$\langle a_0,a_0\rangle=0,\ \langle a_0,a_1\rangle=0,\ \langle a_1,a_1\rangle + 2\langle a_0,a_2\rangle=0,\ \langle a_1,a_2\rangle + \langle a_0,a_3\rangle=0,$$
$$\langle a_2,a_2\rangle + 2\langle a_1,a_3\rangle + 2\langle a_0,a_4\rangle = 0,\ \langle a_2,a_3\rangle + \langle a_1,a_4\rangle = 0,$$
$$\langle a_3,a_3\rangle + 2\langle a_2,a_4\rangle = 0,\ \langle a_3,a_4\rangle = 0,\ \langle a_4,a_4\rangle = 0,$$

and the condition $\langle \partial^2 f/\partial z^2, \partial^2 f/\partial z^2\rangle = 0$ implies that

$$\langle a_0,a_0\rangle = 0,\ \langle a_0,a_1\rangle = 0,\ 9\langle a_1,a_1\rangle + 16\langle a_0,a_2\rangle = 0,$$
$$3\langle a_1,a_2\rangle + 2\langle a_0,a_3\rangle = 0,\ 2\langle a_2,a_2\rangle + 3\langle a_1,a_3\rangle = 0,\ \langle a_2,a_3\rangle = 0,$$
$$\langle a_3,a_3\rangle = 0.$$

Summing up these, we have

$$\begin{aligned} \langle a_0,a_0\rangle &= \langle a_0,a_1\rangle = \langle a_0,a_2\rangle = \langle a_0,a_3\rangle = \langle a_1,a_1\rangle = \langle a_1,a_2\rangle \\ &= \langle a_1,a_4\rangle = \langle a_2,a_3\rangle = \langle a_2,a_4\rangle = \langle a_3,a_3\rangle = \langle a_3,a_4\rangle = \langle a_4,a_4\rangle = 0, \end{aligned} \tag{33}$$

$$\langle a_2,a_2\rangle + 2\langle a_1,a_3\rangle + 2\langle a_0,a_4\rangle = 0,\ 2\langle a_2,a_2\rangle + 3\langle a_1,a_3\rangle = 0. \tag{34}$$

Equations in (34) imply that

$$\langle a_1, a_3\rangle + 4\langle a_0, a_4\rangle = 0. \tag{35}$$

It then follows from (33) and (35) that $2a_0 + a_1 - a_3 - 2a_4$ and $2a_0 - a_1 - a_3 + 2a_4$ are both isotropic vectors. However, they are real-valued and purely imaginary-valued, respectively. Hence, they must be zero, i.e.,

$$2a_0 + a_1 - a_3 - 2a_4 = 0,\ 2a_0 - a_1 - a_3 + 2a_4 = 0.$$

Therefore, we have

$$2a_0 = a_3,\ a_1 = 2a_4.$$

It follows that

$$\langle a_1, a_3\rangle = 4\langle a_0, a_4\rangle,$$

which implies from (35) that

$$\langle a_1, a_3\rangle = \langle a_0, a_4\rangle = 0.$$

By (34), we also have $\langle a_2, a_2\rangle = 0$. Consequently, we have for all $j, k = 0, 1, 2, 3, 4$,

$$\langle a_j, a_k\rangle = 0.$$

Therefore, $\langle \partial^3 f/\partial z^3, \partial^3 f/\partial z^3\rangle = 0$, which is a contradiction to the strictness of the surface. □

Finally of this paper, we propose a problem related to Proposition 3.

Problem. Is there an inequality sharper than Chern-Osserman's inequality for strictly m-isotropic complete minimal surfaces in $\mathbb{E}^{2m+1}$ $(m \geq 2)$? Namely, is there a constant $C(m, g, r)$ depending only on m, g, r such that

$$\int_M K dA \leq C(m, g, r) \leq 4(1 - g - r)\pi$$

holds for all strictly m-isotropic complete minimal surfaces in $\mathbb{E}^{2m+1}$ of genus g and with r ends?

Proposition 3 means that

$$\int_M K dA < 4(1 - g - r)\pi$$

in the case of $m = 2$, $g = 0$ and $r = 3$. Hence, $C(2, 0, 3)$ is at most $4(1 - 0 - 3)\pi - 2\pi = -10\pi$, because the total curvature of complete minimal surface takes value in $2\pi\mathbb{Z}$. On the other hand, there exists a strictly 2-isotropic complete minimal surface with three ends, of genus zero and of total curvature -20π, which is stated in Example 4. Therefore we conclude that $-20\pi \leq C(2,0,3) \leq -10\pi$.

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