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Floer Homology and Mirror Symmetry II

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Abstract.

This is the second part of a series of articles explaining applications of Floer homology to mirror symmetry and D-brane. This article is independent of part I [Fu9]. We will associate an A_∞ category to a symplectic manifold. This is an improved version of previous ones [Fu1], [Fu4] in which there were some flaw. The correction is based on a book [FOOO] written jointly with Oh, Ohta, Ono. While correcting the flaw, we find various interesting new phenomena which are related to mirror symmetry.

We also discuss homological algebra of A_∞ category in this article.

This article is a survey article. So most of the material written here are minor modifications of the results which are already known to somebody. However it is rather hard to find a reference of them.

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Key words and phrases. Symplectic geometry, Floer homology, mirror symmetry, D-brane, A_∞ algebra.

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§0. Introduction

This is the second part of a series of articles, describing a project in progress to study homological mirror symmetry [Ko1], [Ko2] and D-brane using Floer homology of Lagrangian submanifolds. See [Fu5], [Fu6], [Fu7], [FOOO], [Ko1], [Ko2], [KoS1], [Ot], [PZ], [Se2], [Se3], [ST], [SYZ] for related or different aspects of homological mirror symmetry. Though this is the second part, it is independent of the first part [Fu9]. In this second part, we focus the A_∞ category constructed by using Floer homology of Lagrangian submanifolds, and homological algebra of A_∞ categories. In this sense, this article is an updated version of author's previous papers [Fu1], [Fu4]. Time has passed after [Fu1], [Fu4] were written. During those period, we made several progress, some of which are explained in this article.

Among the points where the contents of this article overlaps with [Fu4], there are three points where we improve the constructions there. One is that we removed an assumption in [Fu4], that all Lagrangian submanifolds are monotone and have minimum Maslov index ≥ 3. (See §2 for the definition of Maslov index.) We assumed it in [Fu4] because we used results by Oh [Oh1] to define Floer homology of Lagrangian submanifolds. This assumption is not extremely restrictive for the purpose of [Fu4], that is to study relative Floer homology of 3 manifolds with boundary. However, for the purpose of this article, that is to study mirror symmetry, this assumption is rather restrictive. To study mirror symmetry, the case of Lagrangian submanifolds in Calabi-Yau 3 fold with Maslov index 0 is the most important. Such Lagrangian submanifolds are monotone but do not satisfy the condition that minimal Maslov index ≥ 3.

To define Floer homology of non monotone Lagrangian submanifolds or to remove the condition that minimal Maslov index ≥ 3, we need the obstruction theory developed in [FOOO]. The summary of a part of it is included in this article. (See [Ot] for another summary of [FOOO].) We generalize the construction of A_∞ category explained in [Fu1], [Fu4] to more general situation using the idea of [FOOO].

The second point is that we put precise sign in each formula. In the side of geometry, this requires us to describe orientations of the moduli spaces of pseudoholomorphic discs involved in the construction. Actually the argument we need to do so is already developed in detail in [FOOO] Chapter 6[1]. There, the case when only one or two Lagrangian submanifolds appear, are studied. But the method there can be generalized easily to the present situation where more than two Lagrangian submanifolds appear at the same time.

We need to study sign also in the algebraic construction involved. Various constructions on homological (homotopical) algebra of A_∞ category is developed in [Fu4] Part II. There we worked over $\mathbb{Z}_2$ coefficient. In this article, we give precise sign to the discussion there. The sign in the study of A_∞ category is rather hard and is not at all a trivial matter. Actually the author was unable to work over $\mathbb{Z}$ coefficient in [Fu4] Part II, because he could not find correct sign when he was writing [Fu4]. To fix the sign we develop more systematic way to write the formulas in [Fu4] Part II. As a consequence, the description of this article is considerably simplified compared to one in [Fu4] Part II.

Thirdly we solved in [FOOO] the trouble related to the existence of the identity element in the A_∞ category of Lagrangian submanifold, which was discussed in [Fu4] §13 in unsatisfactory way. In [FOOO] §20 we discussed the problem of identity in the A_∞ algebra we associate to a Lagrangian submanifold. As a consequence of [FOOO] §20 and §5 of present article, we have an A_∞ category with identity. The identity element plays a central role in the proof of Yoneda's lemma which we prove in §9 of present article, refining the proof given in [Fu4] §12.

Now the outline of each sections are in order. In §1, we introduce the notions of A_∞ category and filtered A_∞ category. The other sections of Chapter 1 are devoted to the construction of its example. Namely we associate a filtered A_∞ category to each symplectic manifold.

In §2, we describe an objects of this filtered A_∞ category. The object is roughly speaking a pair of a Lagrangian submanifold and a flat $U(1)$ bundle on it. But we need to add some additional topological data. The main point of this section is to describe precisely the additional data we add and explain the reason we need it. The additional data are relative spin structure (which is related to the orientation or sign), and the grading, (which is related to the degree). We follow [FOOO]

[1]When we quote [FOOO] in this article, we refer its preprint version which was completed in 2000 December and can be obtained from the author's home page http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html at the time of writing this article. We are now adding more materials to it and there will be some change of the order of the chapters in the final version.

Chapter 6 on the first point and follow Kontsevich, Seidel [Se1] on the second point.

§3, §4 are devoted to the definition of the module of morphisms and operations to define our filtered A_∞ category. To define an operator we need to consider two cases separately. The first case is when the Lagrangian submanifolds involved are mutually transversal. This case was discussed in [Fu1], [Fu4]. We discuss this case in §3.

In §4 we discuss the general case, namely the case when we do not assume transversality. Especially we need to study the case when two Lagrangian submanifolds involved coincide to each other. In a similar way, we can discuss more general case when they are of clean intersection. But we do not do it in this article. (See [Po] and [FOOO] §16.) The construction in §4 is a natural generalization of one in [FOOO] where we associate a filtered A_∞ algebra to a Lagrangian submanifold.

In §5 we discuss the problem of unit. Namely we define a notion, homotopy unit (which was first introduced in [FOOO] in the case of A_∞ algebra) and sketch the idea how to construct the homotopy unit in the case of the A_∞ category of Lagrangian submanifolds.

In Chapter 2 we discuss homological algebra of A_∞ category. After writing [Fu4], the author leaned that there are many works done in this direction, in the case of A_∞ algebra or differential graded category, especially by Russian mathematicians. However the reference on it is rather scattered and it is rather hard to find a good reference where we find an appropriate description (and its proofs) of the results we need, especially in the case of A_∞ category. So the author include it in this article. But he does not assert so much credit on it. Namely they are rather minor modification of the results already known to specialists in closely related situations, though some of the results in Chapter 2 are new in the sense they are not proved in the references. The author tries to quote appropriate reference in case he found it. However, since the author is far from being a specialist of homological algebra there should be many works which is closely related to Chapter 2 but is not known to the author.

In §6, we describe the notion of twisted complex and derived A_∞ category. Twisted complex is a natural analogue of chain complex. Namely in the case of abelian category we study chain complex and use it to construct derived category. In a similar way we use twisted complex in the case of A_∞ category. Twisted complex was introduced by Bondal-Kapranov [BoK] in the case of differential graded category. Kontsevitch [Ko1] proposed to use it in mirror symmetry. (It was also applied in [Fu7] to mirror symmetry.)

The author, in [Fu1], [Fu4], proposed to use the A_∞ category of A_∞ functors in the study of Floer homology of 3 manifolds with boundary. For each twisted complex of an A_∞ category $\mathcal{C}$, we can associate an A_∞ functor from $\mathcal{C}$ to $\mathcal{CH}$. Here $\mathcal{CH}$ is the A_∞ category whose object is a chain complex. In this sense, A_∞ functor to $\mathcal{CH}$ is a natural generalization of twisted complex. Also the idea of A_∞ functor is important to understand Floer homology of family of Lagrangian submanifolds (see [Fu5]) and its application to mirror symmetry. Moreover, to clarify the dependence (or independence) of the A_∞ category described in Chapter 1, we need to define a notion of homotopy equivalence of A_∞ categories and hence we use the notion of A_∞ functors. So we introduce the notion of A_∞ functors in section 7. There we construct a representable A_∞ functor and an A_∞ category whose objects are A_∞ functors.

In §8 we define homotopy equivalence of A_∞ category. We prove in §8 that an A_∞ functor which induces isomorphismes on cohomologies is a homotopy equivalence. This result is an algebraic analogue of J. H. C. Whitehead theorem in topology and is proved in [FOOO] §A5 in the case of A_∞ algebra. (A similar results should had been proved in various cases in the reference. The author was unable to locate the first place where this kinds of results appeared.)

In §9, we prove another main result of the homological algebra of A_∞ category, an A_∞ analogue of Yoneda's lemma. This result implies that we can embed any A_∞ category $\mathcal{C}$ to the A_∞ category of A_∞ functors from $\mathcal{C}$ to $\mathcal{CH}$. (Namely we identify an object of $\mathcal{C}$ to a functor represented by it.) As mentioned above, this point will be important to the further study of mirror symmetry and of Floer homology of 3 manifolds with boundary. An A_∞ analogue of Yoneda's lemma was proved in [Fu4] except signs. We give a proof with sign here and also we simplify the proof in [Fu4].

Those results on homological algebra of A_∞ category will be used in future to further study the filtered A_∞ category constructed in Chapter 1 of this article. For example we will prove that the filtered A_∞ category constructed in Chapter 1 of this article is independent of the various choices involved up to homotopy equivalence. This results together with other results are not included in this Part II and is postponed to Part III etc. So two chapters of this article are yet rather independent in this article but will be unified in future.

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Chapter 1: Floer homology and A_∞ category

§1. A_∞ category and filtered A_∞ category

To study the part of homological mirror symmetry conjecture we concern with in this article, we need to use the notion of A_∞ category appeared in [Fu1] in the study of topological field theory. (A_∞ structure had been studied in homotopy theory for a long time, especially in [St].) Actually we need to include the obstruction theory we developed in [FOOO], and to modify the definition of A_∞ category a bit, in order to correct the flaw pointed out in [Ko2], [Oh1]. We call this modified version *filtered A_∞ category.* We start with the usual definition of A_∞ category. Most of the constructions in this section are straight forward generalizations of the definition of filtered A_∞ algebra in [FOOO] Chapter 4.

We fix R, a commutative ring with unit. In our main application, R will be $\mathbb{C}$ or $\mathbb{Q}$.

Definition 1.1. An *A_∞ category*, $\mathcal{C}$, is a collection of a set $\mathfrak{Ob}(\mathcal{C})$, the set of objects, a graded free R module $\mathcal{C}(c_1, c_2)$ for each $c_1, c_2 \in \mathfrak{Ob}(\mathcal{C})$, the operations

$$\mathfrak{m}_k : \mathcal{C}[1](c_0, c_1) \otimes \cdots \otimes \mathcal{C}[1](c_{k-1}, c_k) \to \mathcal{C}[1](c_0, c_k),$$

of degree $+1$ for $k = 1, 2, \dots$ and $c_i \in \mathfrak{Ob}(\mathcal{C})$. Here $\mathcal{C}[1](c_0, c_1)$ is $\mathcal{C}(c_0, c_1)$ with degree shifted. ($\mathcal{C}[1]^m(c_0, c_1) = \mathcal{C}^{m+1}(c_0, c_1)$). They are assumed to satisfy the A_∞ formula (1.3) described below.

To describe the A_∞ formula, we introduce notations. Let $a, b \in \mathfrak{Ob}(\mathcal{C})$, we put

$$(1.2) \quad B_k\mathcal{C}[1](a, b) = \bigoplus_{a=c_0, c_1, \dots, c_{k-1}, c_k = b} \mathcal{C}[1](c_0, c_1) \otimes \cdots \otimes \mathcal{C}[1](c_{k-1}, c_k).$$

We define, in case $k = 0$,

$$B_0\mathcal{C}[1](a, a) = R, \qquad B_0\mathcal{C}[1](a, b) = 0 \quad \text{if } a \neq b.$$

We put

$$BC[1](a,b) = \bigoplus_k B_k\mathcal{C}[1](a,b), \quad B_k\mathcal{C}[1] = \bigoplus_{a,b} B_k\mathcal{C}[1](a,b),$$
$$BC[1] = \bigoplus_{a,b} BC[1](a,b).$$

We define a homomorphism

$$\Delta : B_k\mathcal{C}[1](a,b) \to \bigoplus_{k_1+k_2=k} \bigoplus_c B_{k_1}\mathcal{C}[1](a,c) \otimes B_{k_2}\mathcal{C}[1](c,b)$$

by

$$\Delta(x_1 \otimes \cdots \otimes x_k) = (x_1 \otimes \cdots \otimes x_{k_1}) \otimes (x_{k_1+1} \otimes \cdots \otimes x_k).$$

It induces maps $\Delta : B_k\mathcal{C}[1] \to \oplus_{k_1+k_2=k} B_{k_1}\mathcal{C}[1] \otimes B_{k_2}\mathcal{C}[1]$, $\Delta : B\mathcal{C}[1] \to B\mathcal{C}[1] \otimes B\mathcal{C}[1]$. $(B\mathcal{C}[1](a,a), \Delta)$ and $(B\mathcal{C}[1], \Delta)$ are graded coalgebras. (They are coassociative but not cocomutative.)

Operations $\mathfrak{m}_k$ define homomorphisms: $B_k\mathcal{C}[1](a,b) \to \mathcal{C}[1](a,b)$. It can be extended uniquely to coderivations

$$\hat{d}_k : B\mathcal{C}[1] \to B\mathcal{C}[1], \quad \hat{d}_k : B\mathcal{C}[1](a,b) \to B\mathcal{C}[1](a,b)$$

by

$$\hat{d}_k(x_1 \otimes \cdots \otimes x_n) = \sum_\ell (-1)^{(\deg x_1+1)+\cdots+(\deg x_{\ell-1}+1)} x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_\ell, \ldots, x_{\ell+k-1}) \otimes \cdots \otimes x_n.$$

We put

$$\hat{d} = \sum_k \hat{d}_k.$$

Now the A_∞ formula is

$$\hat{d} \circ \hat{d} = 0. \tag{1.3}$$

We can expand it and rewrite it using $\mathfrak{m}_k$. We thus obtain, for example,

$$0 = \mathfrak{m}_1\mathfrak{m}_1,$$
$$0 = \mathfrak{m}_1\mathfrak{m}_2(x \otimes y) + \mathfrak{m}_2(\mathfrak{m}_1(x) \otimes y) + (-1)^{\deg x+1}\mathfrak{m}_2(x \otimes \mathfrak{m}_1(y)).$$

(See [GJ], [FOOO] Chapter 4.) Namely $(\mathcal{C}(a,b), \mathfrak{m}_1)$ is a chain complex and $\mathfrak{m}_2$ is a derivation up to sign.

Remark 1.4. Here we follow the sign convention of [FOOO] and not of [Fu7].

Definition 1.5. Let $c \in \mathfrak{Ob}(\mathcal{C})$. We say an element $\mathbf{e}_c \in \mathcal{C}^0(c,c) = \mathcal{C}^1[1](c,c)$ is a *unit* if

$$\mathfrak{m}_2(\mathbf{e}_c, x_1) = x_1, \quad \mathfrak{m}_2(x_2, \mathbf{e}_c) = (-1)^{\deg x_2} x_2 \tag{1.5.1}$$

for $x_1 \in \mathcal{C}(c, c')$, $x_2 \in \mathcal{C}(c', c)$ and

$$\mathfrak{m}_{k+\ell+1}(x_1, \dots, x_\ell, \mathbf{e}_c, y_1, \dots, y_k) = 0 \tag{1.5.2}$$

for $k + \ell \neq 1$.

Definition 1.6. A_∞ category with one object is called an A_∞ *algebra.*

Example-Lemma 1.7. *Let $(A, d, \cdot)$ be a differential graded algebra. Namely $d : A^k \to A^{k+1}$, $\wedge : A^k \otimes A^\ell \to A^{k+\ell}$ with*

$$d \circ d = 0, \tag{1.8.1}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{1.8.2}$$
$$d(x \cdot y) = dx \cdot y + (-1)^{\deg x} x \cdot dy. \tag{1.8.3}$$

We put

$$\mathfrak{m}_1(x) = (-1)^{\deg x} dx, \tag{1.9.1}$$
$$\mathfrak{m}_2(x, y) = (-1)^{\deg x(\deg y + 1)} x \cdot y, \tag{1.9.2}$$
$$\mathfrak{m}_k = 0 \quad \textit{for } k > 2. \tag{1.9.3}$$

Then $(A, \mathfrak{m})$ is an A_∞ algebra.

Proof. $\mathfrak{m}_1\mathfrak{m}_1 = 0$ follows from (1.8.1). We calculate using (1.8.3) and (1.9.1), (1.9.2) that

$$\begin{aligned} \mathfrak{m}_1(\mathfrak{m}_2(x, y)) &= (-1)^{\deg x + \deg y + \deg x(\deg y + 1)} d(x \cdot y) \\ &= -\mathfrak{m}_2(\mathfrak{m}_1(x), y) - (-1)^{\deg x + 1} \mathfrak{m}_2(x, \mathfrak{m}_1(y)). \end{aligned}$$

This is A^∞ formula on $B_2 A[1]$. We can also check

$$\begin{aligned} &\mathfrak{m}_2(\mathfrak{m}_2(x, y), z) + \mathfrak{m}_2(x, \mathfrak{m}_2(y, z)) \\ &= (-1)^{\deg x \deg y + \deg x \deg z + \deg x \deg z + \deg y}((x \cdot y) \cdot z - x \cdot (y \cdot z)), \end{aligned}$$

which is zero by (1.8.2). □

Definition-Example 1.10 ([BoK]). A *differential graded category* $\mathcal{C}$ is a collection of a set $\mathfrak{Ob}(\mathcal{C})$, the set of objects, a differential graded R module $\mathcal{C}(c_1, c_2)$ for each $c_1, c_2 \in \mathfrak{Ob}(\mathcal{C})$, the operations

$$\circ : \mathcal{C}(c_1, c_2) \otimes \mathcal{C}(c_2, c_3) \to \mathcal{C}(c_1, c_3),$$

which is a chain map and is associative in the sense of (1.8.2). We then define $\mathfrak{m}_k$ by (1.9). We obtain an A_∞ category.

For later use, we introduce some other notations. Let $\mathcal{C}$ be an A_∞ category and c be its object. We put

$$B\mathcal{C}[1](c) = \bigoplus_k B_k\mathcal{C}[1](c) = \bigoplus_k \mathcal{C}[1](c,c)^{\otimes k}.$$

(We remark that $B_k\mathcal{C}[1](c) \neq B_k\mathcal{C}[1](c,c)$.) $B\mathcal{C}[1](c)$ is a coalgebra and $\mathfrak{m}_k$ defines a structure of A_∞ algebra on it.

Actually our main example is a filtered A_∞ category rather than A_∞ category. We are going to define it. We first define a universal Novikov ring [No], [FOOO].

Definition 1.11. Let T be a formal parameter. We consider a formal power series

$$\sum_{i=1}^{\infty} a_i T^{\lambda_i} \tag{1.12}$$

where $a_i \in R$, $\lambda_i \in \mathbb{R}$. We assume $\lambda_i < \lambda_{i+1}$ and $\lim_{i\to\infty} \lambda_i = \infty$. We denote by $\Lambda_{R,\mathrm{nov}}$ the set of all such series. It has an obvious ring structure.

We consider its subring consisting of (1.12) such that $\lambda_i \geq 0$ in addition and denote it by $\Lambda_{R,0,\mathrm{nov}}$. The ring $\Lambda_{R,0,\mathrm{nov}}$ has a maximal ideal consisting of all series (1.12) such that $\lambda_i > 0$ in addition. We denote it by $\Lambda_{+,\mathrm{nov}}$.

We write Λ_{nov}, $\Lambda_{0,\mathrm{nov}}$, $\Lambda_{+,\mathrm{nov}}$ in place of $\Lambda_{R,\mathrm{nov}}$, $\Lambda_{R,0,\mathrm{nov}}$, $\Lambda_{R,+,\mathrm{nov}}$ in case no confusion can occur.

For each λ, we define $F^\lambda\Lambda_{\mathrm{nov}}$ so that it consists of the elements (1.12) satisfying $\lambda_i \geq \lambda$ in addition. It induces a filtration on Λ_{nov}. Namely each $F^\lambda\Lambda_{\mathrm{nov}}$ is an additive subgroup and $F^{\lambda_1}\Lambda_{\mathrm{nov}} \cdot F^{\lambda_2}\Lambda_{\mathrm{nov}} \subseteq F^{\lambda_1+\lambda_2}\Lambda_{\mathrm{nov}}$. Filtration on Λ_{nov} induces ones on $\Lambda_{0,\mathrm{nov}}$ and $\Lambda_{+,\mathrm{nov}}$.

Our filtration induces a uniform structure on Λ_{nov}, $\Lambda_{0,\mathrm{nov}}$ and $\Lambda_{+,\mathrm{nov}}$ in a usual way. Then these rings are complete with respect to it.

Remark 1.13. In [FOOO], we considered the set of series $\sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{n_i}$ where n_i are integers, and denote by Λ_{nov} the set of all such series. However for the present purpose (that is to discuss Mirror symmetry where the case $c^1(M) = 0$ is important) it is more convenient to use Definition 1.11.

Now we define filtered A_∞ category.

Definition 1.14. A filtered A_∞ category $\mathcal{C}$ is a correction of a set $\mathfrak{Ob}(\mathcal{C})$, the set of objects, a graded torsion free filtered $\Lambda_{0,nov}$ module $\mathcal{C}(c_1, c_2)$ for each $c_1, c_2 \in \mathfrak{Ob}(\mathcal{C})$, the operations

$$\mathfrak{m}_k : \mathcal{C}[1](c_0, c_1) \otimes \cdots \otimes \mathcal{C}[1](c_{k-1}, c_k) \to \mathcal{C}[1](c_0, c_k),$$

of degree $+1$ for $k = 0, 1, 2, \ldots$, and $c_i \in \mathfrak{Ob}(\mathcal{C})$. Note that $k = 0$ is included in the case of filtered A_∞ category. Here $\mathfrak{m}_0$ is a map

$$\mathfrak{m}_0 : \Lambda_{0,nov} \to \bigoplus_c \mathcal{C}[1](c, c).$$

($\mathfrak{m}_0$ is not included in the case of A_∞ category.) We assume that $\mathfrak{m}_k$ respects the filtration in the sense that

$$\begin{aligned} \mathfrak{m}_k(F^{\lambda_1}\mathcal{C}[1](c_0, c_1) \otimes \cdots \otimes F^{\lambda_k}\mathcal{C}[1](c_{k-1}, c_k)) \\ \subseteq F^{\sum \lambda_i}\mathcal{C}[1](c_0, c_k). \end{aligned} \tag{1.15}$$

They induce coderivations

$$\hat{d}_k : B\mathcal{C}[1] \to B\mathcal{C}[1], \quad \hat{d}_k : B\mathcal{C}[1](a, b) \to B\mathcal{C}[1](a, b)$$

in the same way as before. Our filtrations on $\mathcal{C}$ induces one on $B\mathcal{C}[1]$ and we let $\hat{B}\mathcal{C}[1]$ be the completion. $\hat{d}_k$ induces a map: $\hat{B}\mathcal{C}[1] \to \hat{B}\mathcal{C}[1]$, which we denote by the same symbol. The sum

$$\hat{d} = \sum_{k=0}^{\infty} \hat{d}_k$$

converges by virtue of (1.15). Now we assume

$$\hat{d} \circ \hat{d} = 0. \tag{1.16}$$

We assume also

$$\mathfrak{m}_0 \equiv 0 \mod \Lambda_{+,nov}. \tag{1.17}$$

We define a unit of filtered A_∞ category in the same way as Definition 1.5.

A filtered A_∞ category with one object is called a *filtered A_∞ algebra.*

For each filtered A_∞ category $\mathcal{C}$ and $c \in \mathfrak{Ob}(\mathcal{C})$, the operations $\mathfrak{m}_k : B_k\mathcal{C}(c) \to \mathcal{C}(c,c)$ define a structure of filtered A_∞ algebra on $\mathcal{C}(c,c)$.

We can construct an A_∞ category (of $\Lambda_{0,\mathrm{nov}}$ module) from a filtered A_∞ category $\mathcal{C}$ in the following way. Let $c \in \mathfrak{Ob}(\mathcal{C})$. We define

Definition 1.18. An element b of $F^+\mathcal{C}^1(c,c)$ is said to be a *bounding chain* if

$$\hat{d}(e^b) = 0.$$

Here $F^+\mathcal{C}^1(c,c) = \cup_{\lambda>0} F^\lambda \mathcal{C}^1(c,c)$ and

$$e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \cdots \in \hat{B}\mathcal{C}(c,c).$$

We define $\tilde{\mathcal{M}}(c)$ be the set of all bounding chains of c.

Definition 1.19. Let $b_i \in \tilde{\mathcal{M}}(c_i)$, $i = 0, \ldots, k$, $k > 0$. We define

$$\mathfrak{m}_k^{(b_0,\ldots,b_k)} : \mathcal{C}[1](c_0,c_1) \otimes \cdots \otimes \mathcal{C}[1](c_{k-1},c_k) \to \mathcal{C}[1](c_0,c_k)$$

by

$$\begin{aligned} &\mathfrak{m}_k^{(b_0,\ldots,b_k)}(x_1,\ldots,x_k) \\ &= \sum_{\ell_0,\ldots,\ell_k} \mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(b_0^{\ell_0}, x_1, b_1^{\ell_1}, \ldots, b_{k-1}^{\ell_{k-1}}, x_k, b_k^{\ell_k}). \end{aligned}$$

Here

$$b^\ell = \overbrace{b \otimes \cdots \otimes b}^{\ell \text{ times}}.$$

Proposition 1.20. *We put*

$$\mathfrak{Ob}(\mathcal{C}') = \bigcup_{c \in \mathfrak{Ob}(\mathcal{C})} \tilde{\mathcal{M}}(c) \times \{c\},$$

$\mathcal{C}'((c,b),(c',b')) = \mathcal{C}(c,c')$ *and let* $\mathfrak{m}_k^{(b_0,\ldots,b_k)}$ *be the operations. Then* $\mathcal{C}'$ *is an* A_∞ *category.*

The proof (which is easy) goes in exactly the same was as the proof of [FOOO] Lemma 13.37. Hence we omit it.

Actually $\tilde{\mathcal{M}}(c)$ is too big and it is more reasonable to define an equivalence relation $\sim$ on it and divide $\tilde{\mathcal{M}}(c)$ by $\sim$. See [FOOO], [Fu5], [Ot] on it.

We can continue and study A_∞ functor, natural transformations etc. We will do it later. We give our main example before continuing the discussion on algebraic formalism. If the reader is mainly interested in algebraic formalism then he can skip §2 $\sim$ §5 and proceed to Chapter 2.

§2. Floer homology and A_∞ category I – the set of objects –

The idea of this section is rather old. Inspired by S. Donaldson's lecture [Do], the author [Fu1] found that Floer homology and counting of holomorphic polygons will define an A_∞ category. However there was a trouble in defining Lagrangian intersection Floer homology as was mentioned in [Ko2], [Oh1]. We could overcome this trouble in [FOOO] and the construction is now presented here in a modified way. The notion of filtered A_∞ category introduced in the last section is defined for this purpose.

We explain, in this section, the definition of the objects of our filtered A_∞ category $\mathcal{LAG}$ and define a graded $\Lambda_{0,\mathrm{nov}}$ module $\mathcal{LAG}(c,c')$ for two objects when $c \neq c'$. Basically an object of $\mathcal{LAG}$ is a Lagrangian submanifold. However we need to add some topological data to it. One of the topological data to be added is related to the orientation problem of the moduli space of pseudoholomorhic discs. Another data to be added is related to the way to fix the degree of elements of $\mathcal{LAG}(c,c')$. To motivate those data we mention some comments on how they will be used. During those comments we assume that the reader is familiar to the Floer homology and pseudoholomorphic curves. (Please skip them otherwise.) Let (M,ω) be a symplectic manifolds. Let B be a closed 2 form which we call the B-field. We put $\Omega = \omega + 2\pi\sqrt{-1}B$.

Definition 2.1. Let $\mathfrak{Ob}_1(\mathcal{LAG}(M,\Omega))$ be the set of all pairs $(L,\mathcal{L})$ such that:

(2.1.1) L is a Lagrangian submanifold. Namely $\dim L = \dim M/2$ and $\omega|_L = 0$.

(2.1.2) $\mathcal{L}$ is a complex line bundle equipped with a unitary connection ∇ such that its curvature F_∇ coincides with the restriction of $2\pi\sqrt{-1}B$ to L. (Here we identify the Lie algebra of $U(1)$ with $\sqrt{-1}\mathbb{R}$.)

Remark 2.2. One may consider more general objects than $\mathfrak{Ob}_1(\mathcal{LAG})(M,\Omega)$. There are at least two generalizations.

(2.2.1) One may relax the condition on L so that $L \to M$ is a Lagrangian immersion.

(2.2.2) One may consider the vector bundle $\mathcal{L}$ together with its unitary connection ∇ whose curvature is $2\pi\sqrt{-1}B$ times the unit matrix.

The modification of the construction to include these cases will be discussed elsewhere. (See [Ak] on (2.2.1).)

It seems impossible to define an A_∞ category whose objects are all elements of $\mathfrak{Ob}_1(\mathcal{LAG}(M,\Omega))$ because of the transversality problem. (We use Bair's category theorem to achieve transversality quite frequently.) So instead we take and fix a countable set of Lagrangian submanifolds and let $\mathfrak{Ob}_2(\mathcal{LAG}(M,\Omega))$ be the set of all elements $(L,\mathfrak{L}) \in \mathfrak{Ob}_1(\mathcal{LAG}(M,\Omega))$ such that L is in this countable set.

The module of morphisms $\mathcal{LAG}((L_1,\mathfrak{L}_1),(L_2,\mathfrak{L}_2))$ of our filtered A_∞ category is Floer's chain complex, which is the free $\Lambda_{0,\mathrm{nov}}$ module generated by the intersection points $\in L_1 \cap L_2$. More precisely

$$\text{(2.3)} \quad \mathcal{LAG}((L_1,\mathfrak{L}_1),(L_2,\mathfrak{L}_2)) = \bigoplus_{p\in L_1\cap L_2} \mathrm{Hom}((\mathfrak{L}_1)_p,(\mathfrak{L}_2)_p) \otimes_{\mathbb{C}} \Lambda_{0,\mathrm{nov}}.$$

However there are two delicate points which will soon come to the story. The first of them is sign or orientation of the moduli space of pseudo-holomorphic discs which we will use to define operations $\mathfrak{m}_k$ (see §3, §4), and the other is the degree in the Floer homology.

We start with the first point. We refer [FOOO] Chapter 6 for the thorough argument on the orientation and present only a sketch of it in this article. We first fix an element $st \in H^2(M;\mathbb{Z}_2)$. We take a 3-skeleton $M^{(3)}$ of M. Then there exists a unique real rank 2 vector bundles $V(st)$ on $M^{(3)}$ such that $w^1(V(st)) = 0$, $w^2(V(st)) = st$, here w is the Stiefel-Whitney class.

Definition 2.4 ([FOOO]). L is said to be *relatively spin* in (M,st) if it is oriented and if the second Stiefel-Whitney class of (the tangent bundle of) L coincides with the restriction of st.

Let L be relatively spin in (M,st), and let $L^{(2)}$ be the two skeleton of L. Then $V \oplus TL$ is trivial on $L^{(2)}$.

A (M,st)-*relative spin structure* of L is by definition a spin structure of the restriction of $V\oplus TL$ to $L^{(2)}$.

We remark that the two spin structures are equivalent if it is equivalent on the first skeleton. Moreover oriented vector bundle is trivial on two skeleton if it is spin. Therefore the set of (M,st)-relative spin structures of L corresponds one to one to the set of trivializations of the restriction of $V \oplus TL$ to $L^{(1)}$ which can be extended to $L^{(2)}$. We use this remark to show the following:

Lemma 2.5. *The group $H^1(L;\mathbb{Z}_2)$ acts simple transitively on the set of all (M,st)-relative spin structures of L if it is nonempty.*

Proof. Let $\psi \in C^1(L;\mathbb{Z}_2)$ be a cocycle defining an element of $H^1(L;\mathbb{Z}_2)$. Let $\Psi : V \oplus TL|_{L^{(2)}} \to L^{(2)} \times \mathbb{R}^{n+2}$ be isomorphism of bundles whose restrictions to $V \oplus TL|_{L^{(1)}}$ define a relative spin structures of L. For each one cell Δ^1 of L we define a map

$$g_{\psi,\Delta^1} : (\Delta^1,\partial\Delta^1) \to (SO(n+2), I)$$

representing $\psi(\Delta^1) \in \pi_1(SO(n+2)) = \mathbb{Z}_2$. ($I \in SO(n+1)$ is the unit matrix.) We put

$$\Psi'(x,v) = g_{\psi,\Delta^1}(x)(x,v)$$

for $(x,v) \in V \oplus TL|_{L^{(1)}}$, $x \in \Delta^1$. Since ψ is a cocycle it follows that $\Psi' : V \oplus TL|_{L^{(1)}} \to L^{(1)} \times \mathbb{R}^{n+2}$ can be extended to $L^{(2)}$. It is easy to see that the relative spin structure determined by Ψ' depends only on the cohomology class of ψ and the relative spin structure Ψ. We put $[\psi] \cdot [\Psi] = [\Psi']$.

Conversely, let $\Psi_i : V \oplus TL|_{L^{(2)}} \to L^{(2)} \times \mathbb{R}^{n+2}$ be isomorphism of bundles whose restrictions to $V \oplus TL|_{L^{(1)}}$ define two relative spin structures of L. There exists a map $g : L^{(2)} \to SO(n+2)$ such that

$$\Psi_2(x,v) = g(x)(\Psi_1(x,v)).$$

Since $\pi_0(SO(n+2))$ is trivial we may modify Ψ_i so that $g(x) = 1$ for $x \in L^{(1)}$. Then for each 1 cell Δ^1 of L we have

$$[g|_{\Delta^1}] \in \pi_1(SO(n+2)) = \mathbb{Z}_2.$$

We regard it as a cochain. It is a cocycle since g can be extended to $L^{(2)}$. We thus obtain $\psi = g$ such that $[\psi] \cdot [\Psi_1] = [\Psi_2]$. □

We denote by $\mathfrak{Ob}_3(\mathcal{LAG}(M,\Omega,st))$ the set of all pairs of $(L,\mathcal{L}) \in \mathfrak{Ob}_2(\mathcal{LAG}(M,\Omega))$ and an (M,st)-relative spin structure on L. The reason we add relative spin structure is that it induces orientations of various moduli spaces we use in a canonical way. (See the next section and [FOOO] Chapter 6.)

Next we consider the degree problem. We use the notion of graded Lagrangian submanifold due to M. Kontsevich and P. Seidel [Se1] for this purpose. Let $(\mathbb{R}^{2n},\omega)$ be a symplectic vector space of dimension $2n$. We denote by $\mathrm{Lag}_n = \mathrm{Lag}(\mathbb{R}^{2n},\omega)$ the set of all oriented linear subspaces V of $\mathbb{R}^{2n}$ of dimension n such that $\omega|_V \equiv 0$. We call it the oriented Lagrangian Grassmanian. It is well-known that $\pi_1(\mathrm{Lag}_n) \cong \mathbb{Z}$ and it has a generator called the (universal) Maslov class. (See [AG]). Let $\widetilde{\mathrm{Lag}}_n$ be the universal covering space of Lag_n.

Let (M,ω) be a symplectic manifold. Then we have a fiber bundle $\mathrm{Lag}(M) \to M$ whose fiber at $p \in M$ is identified with $\mathrm{Lag}(T_pM)$. We remark that we can find an almost complex structure compatible with its symplectic structure, and it is unique up to homotopy. Hence the Chern classes of the tangent bundle of a symplectic manifolds are well defined.

Lemma 2.6. *The following two conditions are equivalent.*
(2.7.1) $c^1(M) = 0$ *where* $c^1(M)$ *is the first Chern class of* TM.
(2.7.2) *There exists a covering space* $\widetilde{\mathrm{Lag}}(M)$ *of* $\mathrm{Lag}(M)$ *such that its restriction to each fiber is identified with* $\widetilde{\mathrm{Lag}}_n \to \mathrm{Lag}_n$.

Proof. Since $\widetilde{\mathrm{Lag}} \to \mathrm{Lag}$ is a covering space, the obstruction to construct $\widetilde{\mathrm{Lag}}(M)$ lies in the second cohomology. Hence it is easy to see that (2.7.2) is equivalent to the triviality of the bundle $\mathrm{Lag}(M) \to M$ at the two skeleton $M^{(2)}$ of M.

Now let us assume (2.7.1). Then the complex vector bundle TM is trivial on two skeleton. Since Symplectic group $\mathrm{Symp}(n)$ is homotopy equivalent to $U(n)$, it follows that $\mathrm{Lag}(M) \to M$ is trivial on two skeleton. The proof of the converse is similar. □

From now on, we assume $c^1(M) = 0$. (In the case when the compatible almost complex structure is integrable, this condition implies that M has a Ricci flat Kähler metric, due to Yau's proof of Calabi conjecture.) We also *fix* a covering space $\widetilde{\mathrm{Lag}}(M)$ of $\mathrm{Lag}(M)$ as in (2.7.2).

Remark 2.8. In a way similar to the proof of Lemma 2.5, we can show that the set of such lifts $\widetilde{\mathrm{Lag}}(M)$ is an affine space over $H^1(M;\mathbb{Z})$ if it is nonempty.

Let L be an oriented Lagrangian submanifold. We have a canonical section s of the restriction of $\mathrm{Lag}(M)$ to L. Namely

$$s(p) = T_pL \subseteq T_pM.$$

Definition 2.9. A *graded Lagrangian submanifold* of $(M, \widetilde{\mathrm{Lag}}(M))$ is a pair of oriented Lagrangian submanifold L and a lift of s to $\tilde{s} : L \to \widetilde{\mathrm{Lag}}(M)$. We call $\tilde{s}$, the *grading* of L.

Definition 2.10. We denote by $\mathfrak{Ob}_4(\mathcal{LAG}(M,\Omega,st,\widetilde{\mathrm{Lag}}(M)))$ the set of all triples $(L,\mathcal{L},\tilde{s})$ such that $(L,\mathcal{L}) \in \mathfrak{Ob}_3(\mathcal{LAG}(M,\Omega,st))$ and that $(L,\tilde{s})$ is a graded Lagrangian submanifold.

Example 2.11. Let $T^*N \to N$ be a cotangent bundle of an oriented manifold N with a canonical symplectic structure. The tangent bundle TT^*N is isomorphic to the complexification of the pull back π^*TN.

Hence its structure group is reduced to $U(n) \cap GL(n;\mathbb{R}) = O(n)$. Moreover, since N is oriented, it follows that the structure group is reduced to $SO(n)$. Since $\mathrm{Lag}_n = U(n)/SO(n)$ it follows that the bundle $\mathrm{Lag}_n(T^*N)$ is trivial. We thus obtain $\widetilde{\mathrm{Lag}}(T^*N)$ as in (2.5.2).

We remark that the zero section is a Lagrangian submanifold. Hence we have a section s_0 of $\mathrm{Lag}(T^*N)$ on the zero section. Since zero section $\cong N$ is homotopy equivalent to T^*N it induces a section s_0 of $\mathrm{Lag}(T^*N)$. It then induces a trivialization of $\mathrm{Lag}(T^*N)$ and hence a section $\tilde{s}_0$: $T^*N \to \widetilde{\mathrm{Lag}}(T^*N)$. We may choose $\tilde{s}_0$ such that $\tilde{s}_0(p)$ is transversal to the tangent space of the fibers of $T^*N \to N$.

Now let L be a Lagrangian submanifold of T^*N transversal to the fiber. Then, for each $p \in L$, there exists a path ℓ_p which joints T_pL to $s_0(p)$ in $\mathrm{Lag}(T_pT^*N)$ such that $\ell_p(t)$ is transversal to the tangent space of the fiber for each $t \neq 0$. The homotopy class of such ℓ_p is unique. We lift it so that $s_0(p)$ will be lifted to $\tilde{s}_0(p)$. In this way we obtain a lift $\tilde{s}(p) \in \widetilde{\mathrm{Lag}}(T_pT^*N)$. It is easy to see that this lift is independent of various choices.

We thus obtain a graded Lagrangian submanifold $(L, \tilde{s})$.

In a similar way, we can consider the case when (M, ω) has a singular Lagrangian fibration as follows. Let $\pi : M \to N$ be a smooth map with the following properties.

(2.12.1) There exists a subcomplex $X \subset M$ of codimension > 2 such that π is a submersion on $M - X$.
(2.12.2) The kernel of the differential of $d_p\pi$ at $p \in M - X$ is a Lagrangian vector subspace of T_pM.
(2.12.3) π is proper.

Now we define a section s' of $\mathrm{Lag}(M)$ on $M - X$ by putting $s'(p) = \mathrm{Ker}\, d_p\pi$. It defines a real vector bundle on $M-X$ whose fiber at p is $s'(p)$. This vector bundle tensored with $\mathbb{C}$ is isomorphic to the tangent bundle of $M - X$. Thus, in the same way as above, we have a trivialization of $\mathrm{Lag}(M)$ on $M-X$. It induces a bundle $\widetilde{\mathrm{Lag}}(M-X)$ on $M-X$. We can extend it to $\widetilde{\mathrm{Lag}}(M) \to M$ uniquely since the codimension of X in M is > 2. (Note the trivialization of the restriction of $\mathrm{Lag}(M)$ to $M - X$ may not be extended.)

Actually the condition for a Lagrangian submanifold to be graded is related to the (absolute) Maslov index $\eta : \pi_2(M, L) \to \mathbb{Z}$ as follows.

Let us first review Maslov index. Let $\varphi : (D^2, \partial D^2) \to (M, L)$ be a map representing an element of $\pi_2(M, L)$. The pullback bundle $\varphi^*(TM)$ has a trivialization since D^2 is contractible. We restrict this trivialization to ∂D^2. On the other hand, for each $t \in \partial D^2$ we have a Lagrangian subspace $T_{\varphi(p)}L \subset T_{\varphi(p)}M$. Hence we have $S^1 \to \mathrm{Lag}_n$. It determines an element of $\pi_1(\mathrm{Lag}_n) \cong \mathbb{Z}$. We call it *Maslov index* and write $\eta([\varphi])$.

We consider the composition $\pi_2(M) \to \pi_2(M,L) \to \mathbb{Z}$. One can verify easily that the composition coincides with the twice of the first Chern class $c^1 : \pi_2(M) \to \mathbb{Z}$.

Therefore, in the case when $c^1(M) = 0$, the homomorphism η induces a homomorphism : $\mathrm{Im}(\pi_2(M,L) \to \pi_1(L)) \to \mathbb{Z}$. We can extend it to $\pi_1(L)$ as follows. Using $c^1(M) = 0$ there exists a lift $\widetilde{\mathrm{Lag}}(M) \to M$. Let $\ell : S^1 \to L$ be a loop representing an element of $\pi_1(L)$. We define a map

$$\ell^+ : S^1 \to \mathrm{Lag}(M)$$

by

$$\ell^+(t) = T_{\ell(t)}L \in \mathrm{Lag}(T_pM).$$

Since $\widetilde{\mathrm{Lag}}(M) \to \mathrm{Lag}(M)$ is a covering space we have a lift

$$\tilde{\ell}^+(t) : [0,1] \to \widetilde{\mathrm{Lag}}(M)$$

of ℓ^+. Since $\widetilde{\mathrm{Lag}}/\mathbb{Z} = \mathrm{Lag}$ there exists $\overline{\eta}(\ell) \in \mathbb{Z}$ such that

$$\overline{\eta}(\ell) \cdot \tilde{\ell}^+(0) = \tilde{\ell}^+(1).$$

It is easy to see that $\overline{\eta}$ defines a homomorphism

$$\overline{\eta} : \pi_1(L) \to \mathbb{Z}.$$

Now it is easy to show the following two lemmata.

Lemma 2.13. *The composition of $\pi_2(M,L) \to \pi_1(L)$ with $\overline{\eta}$ is η.*

Lemma 2.14. *There exists a lift $\tilde{s}$ of $s : L \to \mathrm{Lag}(M)$ if and only if $\overline{\eta} : \pi_1(L) \to \mathbb{Z}$ is 0.*

We remark that $\overline{\eta}$ depends on the choice of $\widetilde{\mathrm{Lag}}(M)$, while η does not depend on it. Note the set of all choices of $\widetilde{\mathrm{Lag}}(M)$ is an affine space over $H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. In view of the exact sequence

$$\pi_2(M;L) \to \pi_1(L) \to \pi_1(M),$$

the group $\mathrm{Hom}(\pi_1(M),\mathbb{Z})$ controls the way to extend η (which is defined on the image of $\pi_2(M;L) \to \pi_1(L)$) to $\overline{\eta}$ which is defined on $\pi_1(L)$.

Now let $(L_1,\tilde{s}_1)$ and $(L_2,\tilde{s}_2)$ be graded Lagrangian submanifolds which intersect transversally each other. Let $p \in L_1 \cap L_2$. We are going to define an index $\eta_{L_1,L_2}(p)$.

We first consider a pair of family of Lagrangian submanifolds $\ell_0(\tau)$, $\ell_1(\tau) \in \mathrm{Lag}_n$, $\tau \in \mathbb{R}$ so that it is constant for $|\tau|$ large. We assume that

$$\ell_0(-\infty) \cap \ell_1(-\infty) = \ell_0(\infty) \cap \ell_1(\infty) = \{0\}. \tag{2.15}$$

We want to associate an integer, Maslov-Viterbo index $\eta(\ell_0, \ell_1)$, for such a pair. We need its relation to the index of Cauchy-Riemann operator also. Let us now describe it. We consider the product $\mathbb{R} \times [0,1] \subset \mathbb{C}$ and identify τ to the first coordinate. We consider the operator

$$\overline{\partial} : W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n) \to W^{0,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n \otimes \Lambda^{0,1}),$$

where $W^{k,p}$ denotes the set of sections of Sobolev (k,p) class, that is the linear space of sections whose derivatives up to order k are of L^p class. (We take $p > 2$. So $W^{1,p} \subseteq C^0$.) For an element $u \in W^{k+1,2}(\mathbb{R} \times [0,1]; \mathbb{C}^n)$, we put the boundary condition

$$u(\tau, 0) \in \ell_0(\tau), \quad u(\tau, 1) \in \ell_1(\tau). \tag{2.16}$$

We denote by $W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_0, \ell_1)$ the subset of $W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n)$ satisfying the boundary condition (2.16). Then the $\overline{\partial}$ operator induces an operator:

$$\overline{\partial} : W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_0, \ell_1) \to W^{0,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n \otimes \Lambda^{0,1}). \tag{2.17}$$

Lemma 2.18. (2.17) *is a Fredholm operator.*

The lemma is a consequence of (2.15). The proof is omitted.

We can calculate the index of (2.17) in the following way. We first take $g(\tau)$ such that $g(\tau)\ell_0(\tau) = \ell_0(-\infty)$. Then we consider the pair $\ell_0'(\tau) \equiv \ell_0(-\infty)$, $\ell_1'(\tau) \equiv g(\tau)\ell_1(\tau)$. It is easy to see that the index of (2.17) does not change if we replace $\ell_0(\tau)$, $\ell_1(\tau)$ by $\ell_0'(\tau), \ell_1'(\tau)$. Hence we may assume that $\ell_0(\tau)$ is independent of τ. We put $V = \ell_0(\tau)$.

For $V \in \mathrm{Lag}_n$, let $X(V)$ be the set of all $V' \in \mathrm{Lag}_n$ such that V' is not transversal to V. $X(V)$ is a (real) codimension one subcomplex such that it is a smooth submanifold outside a set of (real) codimension > 1 in $X(V)$.

Proposition 2.19. *There exists an orientation of the regular part of $X(V)$ such that the index of* (2.17) *is equal to the intersection number $\ell_1 \cdot X(V)$ when $V \equiv \ell_0(\tau)$.*

Sketch of a proof. We explain an idea of the proof of Proposition 2.19 by using the notion of spectral flow ([APS]). Let us consider the operator (2.17). It is an operator

$$\overline{\partial} = \frac{\partial}{\partial \tau} + J_{t,\tau} \circ \frac{\partial}{\partial t}. \tag{2.20}$$

$J_{t,\tau} \circ (\partial/\partial t)$ is a family of elliptic operators on $[0,1]$ and is parametrized by τ. Also the boundary condition (2.16) is τ dependent. We choose an appropriate bundle automorphism of φ^*TM on $\mathbb{R} \times [0,1]$ and use it to regard $J_{t,\tau} \circ (\partial/\partial t)$ as a family of elliptic operators whose boundary condition is τ *independent.* We thus obtain an operator

$$\frac{\partial}{\partial \tau} + P_\tau \tag{2.21}$$

where

$$P_\tau : W^{1,p}([0,1];\mathbb{C}^n;\Xi) \to W^{0,p}([0,1];\mathbb{C}^n).$$

Here Ξ is a boundary condition independent of τ. Then, as in [APS], the index of (2.21) is calculated by the spectral flow of the family of elliptic operators P_τ. We remark that the operator P_τ will have a kernel in a neighborhood of the points τ where $\ell_1(\tau)$ are not transversal to ℓ_0.

Thus the index of the spectral flow coincides with the intersection number $\ell_1 \cdot X(V)$. □

For our purpose, we need to relax the condition (2.15) and consider the case

$$\ell_0(-\infty) = \ell_1(-\infty), \quad \ell_0(\infty) \cap \ell_1(\infty) = \{0\}. \tag{2.22}$$

(We still assume $\ell_0(\tau) \equiv \ell_0(0)$.) We perturb ℓ so that it is transversal to $X(\ell(0))$ on $(0,1]$. Then we will put

$$\eta(\ell_0,\ell_1) \text{ `` = '' } \ell_1|_{(0,1]} \cdot X(\ell_0(0)) \tag{2.23}$$

where $\cdot$ is the intersection number. However, since $\ell_0(0) \in X(\ell_0(0))$, we need to be very careful to define the intersection pairing in (2.23). Actually $\ell_0(0) \in X(\ell_0(0))$ is a singular point of $X(\ell_0(0))$ where n components of codimension one meet each other. To define right hand side of (2.23) rigorously, we modify ℓ_1 in a neighborhood of 0 as follows. Let $U(\ell_0(0))$ be a small neighborhood of $\ell_0(0)$ in Lag_n. We will define components $U_i(\ell_0(0))$ such that

$$U(\ell_0(0)) - X(\ell_0(0)) = \bigcup_{i=0}^{n} U_i(\ell_0(0)), \tag{2.24}$$

as follows.

We take a symplectic isomorphism $\mathbb{C}^n \simeq T^*\ell_0(0)$. For each

$$V \in U(\ell_0(0)) - X(\ell_0(0)),$$

there exists a non degenerate quadratic function f on $\ell_0(0)$ such that V is a graph of df. We define $U_i(\ell_0(0))$ so that $V \in U_i(\ell_0(0))$ if f is a quadratic function of index i.

We may choose $U(\ell_0(0))$ so that $U_i(\ell_0(0))$ is contractible. Note that $\overline{U}_i(\ell_0(0)) \cap \overline{U}_{i+1}(\ell_0(0))$ is of codimension one in $X(\ell(0))$ and they are all of the codimension 1 components of $X(\ell(0)) \cap U(\ell_0(0))$.

Now we modify ℓ_1 so that $\ell_1(\epsilon)$ is in $U_n(\ell_0(0))$ for small ϵ. Now we define $\eta(\ell_0, \ell_1)$ by

$$\eta(\ell_0, \ell_1) = \ell_1|_{[\epsilon,1]} \cdot X(\ell_0(0)). \tag{2.25}$$

If we take another perturbation ℓ such that $\ell_1'(\epsilon)$ is in $U_k(\ell_0(0))$ for small k, then

$$\ell_1|_{[\epsilon,1]} \cdot X(\ell_0(0)) + n = \ell_1'|_{[\epsilon,1]} \cdot X(\ell_0(0)) + k. \tag{2.26}$$

Let $-\ell$ be the path such that $-\ell(t) = \ell(1-t)$ then we have

Lemma 2.27. *$\eta(\ell_0, \ell_1) + \eta(\ell_1, \ell_0) = -n$, if (2.22) holds.*

Proof. Let us perturb ℓ_1 so that $\ell_1(\epsilon)$ is in $U_n(\ell_0(0))$ for small ϵ. Then we have

$$\ell_1|_{(0,1)} \cdot X(\ell_0(0)) + \ell_0|_{(0,1)} \cdot X(\ell_1(0)) = 0.$$

Let us change the pair ℓ_1, ℓ_0 by the one parameter family of authomorphisms of $\mathbb{C}^n$ as follows: Let us denote the modified pair by ℓ_1', ℓ_0': we take it so that ℓ_1' is constant.

We then find that $\ell_0'(\epsilon) \in U_0(\ell_1'(0))$ for ϵ small. Lemma 2.27 now follows from (2.26). □

Now we are going to define an index of the intersection point of two graded Lagrangian submanifolds $(L_1, \tilde{s}_1)$, $(L_2, \tilde{s}_2)$. Let $p \in L_1 \cap L_2$. We take a pair $(\tilde{\ell}_0, \tilde{\ell}_1)$ of path in $\widetilde{\mathrm{Lag}}(T_pM)$ such that $\tilde{\ell}_0(0) = \tilde{\ell}_1(0)$, and $\tilde{\ell}_0(1) = \tilde{s}_1(p)$, $\tilde{\ell}_1(1) = \tilde{s}_2(p)$. We put $\ell_i = \pi \circ \tilde{\ell}_i$. We then put

$$\eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p) = \eta(\ell_0, \ell_1). \tag{2.28}$$

It is easy to see that the right hand side is independent of the path $(\tilde{\ell}_0, \tilde{\ell}_1)$.

Definition 2.29. Let

$$(L_1, \tilde{s}_1, \mathfrak{L}_1),\ (L_2, \tilde{s}_2, \mathfrak{L}_2) \in \mathfrak{Ob}_4(\mathcal{LAG}(M, \Omega, st, \widetilde{\mathrm{Lag}}(M))).$$

We assume $L_1 \neq L_2$. We define:

$$\text{(2.30)} \quad \begin{aligned} &\mathcal{LAG}^k((L_1, \tilde{s}_1, \mathfrak{L}_1), (L_2, \tilde{s}_2, \mathfrak{L}_2)) \\ &= \bigoplus_{\substack{p \in L_1 \cap L_2 \\ \eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p)=k}} \operatorname{Hom}((\mathfrak{L}_1)_p, (\mathfrak{L}_2)_p) \otimes_{\mathbb{C}} \Lambda_{0,\text{nov}}. \end{aligned}$$

We discuss the case $L_1 = L_2$ later. (We actually need one more small modification to (2.30) for the orientation problem. We will discuss it later. See (3.16).)

Lemma 2.27 implies:

$$\text{(2.31)} \qquad \eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p) = -n - \eta_{(L_2,\tilde{s}_2),(L_1,\tilde{s}_1)}(p)$$

Example 2.32. Let $u = df$ be an exact 1 form on N and $L_2 \subset T^*N$ be its graph. Let L_1 be a zero section. We assume that f is a Morse function. If $p \in L_1 \cap L_2$ then $p = (x, 0)$ and $df(x) = 0$. We define $\tilde{s}_1$, $\tilde{s}_2$ as in Example 2.11. We find

$$\eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p) = \operatorname{Index}(\nabla_x^2 f) - n.$$

Here $\nabla_x^2 f$ is the Hessian of f at x. Namely in this case the Maslov index reduces to the Morse index.

Remark 2.33. Let $(L_1, \tilde{s}_1)$ be a graded Lagrangian submanifold. We assume that L_1 is connected. When we change the lift $\tilde{s}_1$, then the degree η changes only by an overall integer. Let us state it more precisely. We recall that $\widetilde{\operatorname{Lag}}(M) \to \operatorname{Lag}(M)$ is a normal covering whose deck transformation group is $\mathbb{Z}$. Let $\tilde{s}_1'$ be another lift of $s_1(p) = T_p(L)$. Then there exists an integer k such that $\tilde{s}_1 = k \cdot \tilde{s}_1'$. We have:

$$\text{(2.34.1)} \qquad \eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p) = \eta_{(L_1,\tilde{s}_1'),(L_2,\tilde{s}_2)}(p) + k$$

and

$$\text{(2.34.2)} \qquad \eta_{(L_1,\tilde{s}_2),(L_2,\tilde{s}_1)}(p) = \eta_{(L_1,\tilde{s}_2),(L_2,\tilde{s}_1')}(p) - k$$

for every $(L_1, \tilde{s}_2)$ and $p \in L_1 \cap L_2$. More precisely the identification of the deck transformation group of $\widetilde{\operatorname{Lag}}(M) \to \operatorname{Lag}(M)$ and $\mathbb{Z}$ has an ambiguity $\operatorname{Aut}(\mathbb{Z}) = \{\pm 1\}$. We can fix it by requiring (2.34).

So far, we restricted ourselves to a symplectic manifold with $c^1 = 0$ and Lagrangian submanifold with Maslov index 0. This allows us to define Floer homology with $\mathbb{Z}$ grading. However it is useful for some other purposes to relax the condition c^1 = Maslov index = 0. Let us discuss the general case here, following [Se1]. Let us define $N \in \mathbb{Z}_{\geq 0}$ by

$$\mathrm{Im}(c^1 : \pi_1(M) \to \mathbb{Z}) = N\mathbb{Z}.$$

N is called the *minimal Chern number* of (M, ω). In a way similar to the proof of Lemma 2.6, we can construct a covering space

$$\pi : \widetilde{\mathrm{Lag}}^N(M) \to \mathrm{Lag}(M) \tag{2.35}$$

such that π induces $2N$ hold covering on each fiber. (We remark that two hold fiberwise covering of $\mathrm{Lag}(M)$ always exists because M is orientable.)

For any N' dividing N, we put

$$\widetilde{\mathrm{Lag}}^{N'}(M) = \widetilde{\mathrm{Lag}}^N(M)/\mathbb{Z}_{N/N'}$$

which is a $2N'$ hold fiberwise covering of $\mathrm{Lag}(M)$.

Let $L \subset M$ be a Lagrangian submanifold. We can define

$$\overline{\eta} : \pi_1(L) \to \mathbb{Z}_{2N}$$

using (2.35). Moreover the composition of $\pi_2(M; L) \to \pi_1(L)$ and $\overline{\eta}$ coincides with the mod $2N$ reduction of $\eta : \pi_2(M; L) \to \mathbb{Z}$. (Note that the Maslov index $\eta : \pi_2(M; L) \to \mathbb{Z}$ is defined for any pair of symplectic manifold M and its Lagrangian submanifold L.) Now we have:

Lemma 2.36. *A lift*

$$\tilde{s} : L \to \tilde{\mathrm{Lag}}^{N'}(M)|_L$$

of the map $s : L \to \mathrm{Lag}(M)|_L$ *exists if and only if the image of* $\overline{\eta} : \pi_1(L) \to \mathbb{Z}_{2N}$ *is contained in* $2N'\mathbb{Z}_{2N}$.

Such a lift $\tilde{s}$ is called an N' *grading* of L'. Let L_i $(i = 1, 2)$ be a pair of Lagrangian submanifolds with N' grading. We assume they are transversal to each other. Let $p \in L_1 \cap L_2$. Then we can define its index $\eta(p) \in \mathbb{Z}_{2N'}$ in a similar way to Definition 2.25. Thus we have a $\mathbb{Z}_{2N'}$ graded Floer homology.

§3. Floer homology and A_∞ category II – the operator $\mathfrak{m}_k$ – the transversal case –

We next define

$$\mathfrak{m}_k : \mathcal{LAG}[1](c_0, c_1) \otimes \cdots \otimes \mathcal{LAG}[1](c_{k-1}, c_k) \to \mathcal{LAG}[1](c_0, c_k)$$

where $c_i = (L_i, \tilde{s}_i, \mathfrak{L}_i) \in \mathfrak{Ob}_4(\mathcal{LAG}(M, \Omega, st, \widetilde{\mathrm{Lag}}(M)))$, under the additional asumption that $L_i \neq L_{i+1}$. (Here $L_{k+1} = L_0$ by convention.) It follows from our assumption that L_i is transversal to L_{i+1}. Again the idea presented here is not new and is discussed already in [Fu1], [Fu4]. However we here give a more precise argument especially on degree and orientation. In fact there were several errors on those points in [Fu1].

We use moduli spaces of pseudoholomorphic discs. Let us consider a pair $(D^2, \vec{z})$ of 2 disc D^2 with the canonical complex structure and $\vec{z} = (z_0, \dots, z_k)$, the ordered set of $k+1$ points on its boundary. We assume that $(z_0, \dots, z_k)$ respects the cyclic order. We denote by $\tilde{\mathcal{M}}_{k+1}$ the space of all such pairs. The group $PSL(2;\mathbb{R}) = \mathrm{Aut}(D^2, J)$ acts on it, and let $\mathcal{M}_{k+1}$ be the quotient space. It is well known that $\mathcal{M}_{k+1}$ is diffeomorphic to $\mathbb{R}^{k-2}$ and carries a natural orientation. We can compactify it to $\mathcal{CM}_{k+1}$ in a way similar to the Deligne-Mumford compactification of the moduli space of marked closed Riemann surfaces. (See [FOh], [FOOO] §3.)

Let $(D^2, \vec{z}) \in \tilde{\mathcal{M}}_{k+1}$. Let $\partial_i D^2$ be the part of ∂D^2 between z_{i-1} and z_i. (Here $z_{-1} = z_{k+1}$ by notation.)

Let $p_i \in L_i \cap L_{i+1}$. ($L_{k+1} = L_0$ by notation.) We fix a compatible almost complex structure J on M. Namely we assume

$$\omega(JX, JY) = \omega(X, Y), \quad \omega(X, JX) \geq 0.$$

We consider the set of all $((D^2, \vec{z}), \varphi)$ such that

(3.1.1) $(D^2, \vec{z}) \in \mathcal{M}_{k+1}$, $\varphi : D^2 \to M$,
(3.1.2) φ is pseudoholomorphic,
(3.1.3) $\varphi(\partial_i D^2) \subset L_i$,
(3.1.4) $\varphi(z_i) = p_i$.

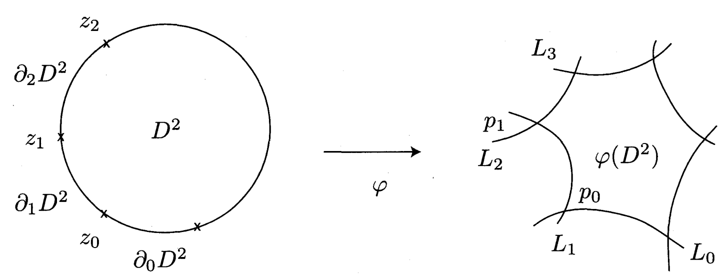

Figure 3.1.

We denote by $\tilde{\mathcal{M}}_{k+1}(L_0, \dots, L_k; p_1, \dots, p_k)$ the set of all such

$((D^2,\vec{z}),\varphi)$.

The group $PSL(2;\mathbb{R})$ acts on $\tilde{\mathcal{M}}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ by

$$u\cdot((D^2,\vec{z}),\varphi)=((D^2,u(\vec{z})),\varphi\circ u^{-1}).$$

Let $\mathcal{M}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ be the quotient space. We can compactify our moduli space $\mathcal{M}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ by using a notion of stable map from open Riemann surface. (See [FOOO] §3.) Let $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ be the compactification.

Theorem 3.2. *There exists a Kuranishi structure (with corners) of dimension* $n+(k+1)-\sum\eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i)-3$ *on* $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$.

The notion of Kuranishi structure is defined in [FOn2], to handle transversality problem appeared in the construction of fundamental chains of various moduli spaces in a uniform way. We do not try to define it here. Roughly speaking it is a way to restate the following imprecise statement in a rigorous way:

"Theorem 3.2′"[2]. *By a " generic perturbation ",* $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ *will become a manifold with corner. Its dimension is* $n+(k+1)-\sum\eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i)-3$.

The word "generic perturbation" in "Theorem3.2′" should be made precise. Kuranishi structure is a way to include as the most general perturbation as possible. Usually the reader do not have to be bothered with the detail of the study of a space with Kuranishi structure. The frame work of Kuranishi structure is designed so that desired fundamental chain (usually over $\mathbb{Q}$) which has the properties expected from naive guess can be constructed in the situation we are interested in (that is the moduli space of pseudoholomorphic discs). However if the reader is interested in the most delicate parts of the proofs, he needs to investigate Kuranishi structures. (For example the discussion of Remark 3.3 requires a detail of the frame work of Kuranishi structure.) We pretend as if a statement like "Theorem 3.2′" is correct usually and give remarks on Kuranishi structure when necessary.

Remark 3.3. In our case, the fundamental chain of $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ can be defined over integer. In fact, by Lemma 2.10, the Maslov index is 0 in our case. Moreover by Lemma

[2]We write Theorem in the quote since the statement as is stated is not correct.

2.6, $c^1(M) = 0$. It follows that our Lagrangian submanifold is semi-positive in the sense of [FOOO] Chapter 3. Therefore, we can apply the method of [FOOO] §9, §A3 to obtain a perturbation of our moduli space so that it is a simplicial complex with fundamental chain over $\mathbb{Z}$. (It is actually a space with Whitney stratification.) Using this remark, our filtered A_∞ category is defined over $\Lambda_{\mathbb{Z},0,\mathrm{nov}}$. Also we can work over $\Lambda_{\mathbb{Z}_2,0,\mathrm{nov}}$ and forget all the orientation problems (as we did in [Fu4]). However to work over $\mathbb{C}$ (or $\Lambda_{\mathbb{C},\mathrm{nov}}$) seems to be natural when our main interest is in mirror symmetry and not in applications of Floer homology to symplectic topology.

The proof of Theorem 3.2 consists of two parts. One is to use implicit function theorem and Taubes' type gluing argument to construct Kuranishi structure. This part is in fact a straightforward analogue of the argument presented in [FOn2] Chapter 3, [FOh] and [FOOO] Chapter 5. So we do not repeat it. The second part is a calculation of the (virtual) dimension, which is related to the study of Maslov index in the last section. We discuss this second point now. The argument here is also an analogy of one in [FOOO] Chapter 6.

First we remark that the number $k+1-3$ is the dimension of the moduli space $\mathcal{CM}_{k+1}$. There is a natural projection

$$\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k) \to \mathcal{CM}_{k+1}.$$

Hence we are only to show that the virtual dimension of the fiber of the projection is $n - \sum \eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i)$. (In the case of the space with Kuranishi structure, its virtual dimension is its dimension by definition. So it suffices to calculate the virtual dimension.)

Namely we fix $(D^2,\vec{z}) \in \mathcal{M}_{k+1}$ and study the moduli space of pseudoholomorphic maps φ. (Actually we need to study the case when the domain of φ is singular also. However since the study of it is similar to and is written in detail in [FOOO] Chapter 5, we omit it.)

The study of virtual dimension is a problem calculating the index. We first choose a metric on the domain and fix a function space to work with. For our purpose, it is convenient to use an alternative representative of $(D^2,\vec{z})$. Namely we take a one dimensional Kähler manifold Σ with the following properties:

(3.4.1) There exists a compact subset Σ_0 such that the complement $\Sigma - \Sigma_0$ is isometric to the disjoint union of $k+1$ copies of $(-\infty,0]\times[0,1]$.
(3.4.2) Σ is conformaly equivalent to $D^2 - \{z_1,\dots,z_{k+1}\}$.

By using [FOh] Theorem 10.4, such Σ with a singular Riemannian metric is given in a canonical way. Also Σ and metric in [FOh] Theorem 10.4 depends smoothly on $(D^2,\vec{z})$. This fact is essential to work out the

Taubes' type gluing construction and the analytic detail of the proof of Theorem 3.2, as in [FOOO] Chapter 5. For our purpose, that is to calculate the index, we do not need it.

Note that $k+1$ copies of $(-\infty,0]\times[0,1]$ in (3.4.1) corresponds to the marked points $z_0,\dots,z_{k+1}$ by (3.4.2). We let $\mathrm{End}_i\,\Sigma \equiv (-\infty,0]\times[0,1]$ be the copy corresponding to z_i.

Let $\varphi : D^2 \to M$ be a smooth map satisfying (3.1.1), (3.1.3), (3.1.4). It induces a map $\Sigma \to M$, which we denote by the same symbol. We consider the operator

$$D_\varphi\overline{\partial} : W^{1,p}(\Sigma,\varphi^*TM;\varphi^*TL) \to W^{0,p}(\Sigma,\varphi^*TM\otimes\Lambda^{0,1}\Sigma). \tag{3.5}$$

Here (3.5) is a linearization operator of the pseudoholomorphic curve equation:

$$\overline{\partial}\varphi = 0,$$

$W^{k,p}$ in (3.5) denotes the space of all sections whose derivative up to order k is of L^p class, and

$$\begin{aligned} &W^{1,p}(\Sigma,\varphi^*TM;\varphi^*TL)\\ &= \quad \{u\in W^{1,p}(\Sigma,\varphi^*TM) \mid u(x)\in T_{\varphi(x)}L_i \text{ if } x\in\partial_i\Sigma\}\end{aligned}$$

where $\partial_i\Sigma$ is a part of $\partial\Sigma$ which corresponds to $\partial_i D^2$. (We remark that elements of $W^{1,p}$ is continuous since we assumed $p>2$.)

Note that ends of Σ are of product type. We also assumed that L_i is transversal to L_{i+1} at p_i. Hence the operator (3.5) is nondegenerate at the end. Therefore, by a standard argument, we can prove that (3.5) is Fredholm. We are going to calculate its index.

We first trivialize φ^*TM on Σ. The trivialization is unique up to homotoy since Σ is contractible. We next take a path $s_i : \partial_i\Sigma \to \mathrm{Lag}_n$. s_i is defied by $s_i(x) = T_{\varphi(x)}TL \subset T_{\varphi(x)}TM = (\varphi^*TM)_x \cong \mathbb{C}^n$.

We thus reduced the problem calculating the virtual dimension of the moduli space $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_1,\dots,p_k)$ to the problem calculating the index of the operator

$$\overline{\partial} : W^{1,p}(\Sigma,\mathbb{C}^n;s_0,\dots,s_k) \to W^{0,p}(\Sigma,\mathbb{C}^n\otimes\Lambda^{0,1}\Sigma), \tag{3.6}$$

where

$$W^{1,p}(\Sigma,\mathbb{C}^n;s_0,\dots,s_k) = \{u\in W^{1,p}(\Sigma,\mathbb{C}^n) \mid u(x)\in s_i(x) \text{ if } x\in\partial_i\Sigma\}.$$

The index of (3.6) does not change if we change s_i in a homotopy class. Hence we may assume:

$$\begin{cases} s_i(\tau,1) = T_{p_i}L & \text{if } \tau<-T,\ (\tau,1)\in\mathrm{End}_{i+1}\Sigma,\\ s_i(\tau,0) = T_{p_i}L & \text{if } \tau<-T,\ (\tau,0)\in\mathrm{End}_i\Sigma. \end{cases} \tag{3.7}$$

We next construct an elliptic complex whose index is $\eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p)$. We put:

$$Y = D^2 \cup \{x + \sqrt{-1}y \mid x \geq 0, y \in [-1,1]\} \subset \mathbb{C}.$$

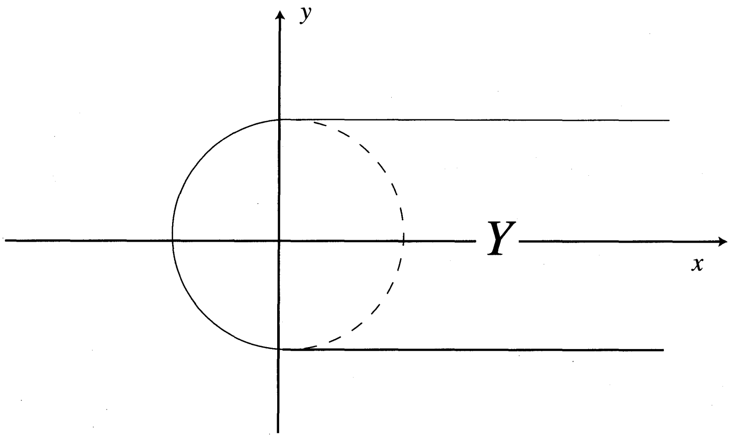

Figure 3.2.

We remark $\partial Y \cong \mathbb{R}$ where $-\infty$ corresponds to $\infty - \sqrt{-1}$ and ∞ corresponds $\infty + \sqrt{-1}$. We also define a path $\tilde{\ell}_i : \mathbb{R} \to \widetilde{\mathrm{Lag}}(T_{p_i}M)$ such that $\tilde{\ell}_i(-\infty) = \tilde{s}_i(p)$ and $\tilde{\ell}_i(\infty) = \tilde{s}_{i+1}(p)$, where $\tilde{s}_i$ is the grading of the Lagrangian submanifold of L_i we have chosen. We may also assume that $\ell_i(t)$ is locally constant if $|t| > T$. We thus obtain a Fredholm operator

$$(3.8.i) \qquad \overline{\partial} : W^{1,p}(Y, T_{p_i}M; \ell_i) \to W^{0,p}(Y, T_{p_i}M \otimes \Lambda^{0,1}\Sigma).$$

Here $\ell_i = \pi \circ \tilde{\ell}_i$ and $W^{1,p}(Y, T_{p_i}M; \ell_i)$ is defined by

$$W^{1,p}(Y, T_{p_i}M; \ell_i) = \{u \in W^{1,p}(Y, T_{p_i}M) \mid u(x) \in \ell_i(x) \text{ if } x \in \partial Y \cong \mathbb{R}\}.$$

Now we have the following two Lemmata 3.9 and 3.13.

Lemma 3.9. *The index of* (3.8.i) *is* $\eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p)$.

Proof. Take a biholomorphic map $\psi : \mathbb{R} \times [0,1] \to Y - \{-1\}$ such that (∞, t) corresponds $\infty + t\sqrt{-1}$ and $(-\infty, t)$ corresponds $-1 \in \partial \overline{Y}$. We pull back the boundary condition ℓ_i to $\mathbb{R} \times \{0,1\}$ and obtain $(\ell_{i,0}, \ell_{i,1})$. We then consider the operator

$$(3.10) \quad \overline{\partial} : W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_{i,0}, \ell_{i,1}) \to W^{0,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n \otimes \Lambda^{0,1}).$$

The operator (3.10) is similar to (2.17). However (3.10) is not a Fredholm operator since the operator is degenerate when $\tau \in \mathbb{R}$ goes to $-\infty$. In fact $\ell_{i,0}(-\infty) = \ell_{i,1}(-\infty)$. So to obtain a Fredholm operator, we need to use a weighted Sobolev norm. We consider the weighted Sobolev norm

$$\|u\|_{1,p;\delta}^p = \|e^{\delta|\tau|}u\|_p^p + \|e^{\delta|\tau|}\nabla u\|_p^p$$

where $\| \ \|_p$ is the usual L^p norm and ∇ is a covariant derivative. We define $W^{1,p;\delta}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_{i,0}, \ell_{i,1})$ using this weighted Sobolev norm and the same boundary condition as $W^{1,p}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_{i,0}, \ell_{i,1})$. We then consider

$$\begin{aligned} \overline{\partial} : W^{1,p;\delta}(\mathbb{R} \times [0,1]; \mathbb{C}^n; \ell_{i,0}, \ell_{i,1}) \\ \to W^{0,p;\delta}(\mathbb{R} \times [0,1]; \mathbb{C}^n \otimes \Lambda^{0,1}). \end{aligned} \tag{3.11.δ}$$

(3.11.δ) is a Fredholm operator for nonzero small δ. We can prove the following easily.

Sublemma 3.12. *The index of* (3.11.δ) *is* $\eta(\ell_{i,0}, \ell_{i,1}) - n$ *if* $\delta > 0$ *and if* $|\delta|$ *is small, and is* $\eta(\ell_{i,0}, \ell_{i,1})$ *if* $\delta < 0$ *and if* $|\delta|$ *is small.*

In fact, if we study (3.10) using spectral flow as in the sketch of the proof of Proposition 2.19, then we find that n is the number of eigenvalues converging to 0 as $\tau \to -\infty$. Hence the index changes by n when we move δ from negative to positive. This difference corresponds to the way to perturb $\ell_{i,1}(\tau)$ for τ close to $-\infty$. Namely if we perturb so that $\ell_{i,0}(-\infty)$ is transversal to $\ell_{i,1}(-\infty)$. (Then the operator (3.11.δ) will become Fredholm.) We can prove the sublemma using this observation.

Now we consider the case when $\delta > 0$. This means that we consider the solution of $\overline{\partial} u = 0$ with u converges to 0 in the exponential order as $\tau \to -\infty$. Then when we transform u by ψ, it will corresponds to an element of the kernel of (3.8.i) such that its value at -1 is zero. Therefore we find that the index of (3.11.δ) for $\delta > 0$ is the index of (3.8.i) minus n. Lemma 3.9 follows from Sublemma 3.12. □

Lemma 3.13. *The index of* (3.6) *plus the sum of the indices of* (3.8.i) *for* $i = 0, \ldots, k$ *is* n.

Proof. We glue the elliptic operator (3.6) with the elliptic operators (3.8.i) on their boundaries. Then we obtain an elliptic operator

$$\overline{\partial} : W^{1,p}(D^2; \mathbb{C}^n; \ell) \to W^{0,p}(D^2; \mathbb{C}^n \otimes \Lambda^{0,1}). \tag{3.14}$$

Here notations in (3.14) are defined as follows.

$\ell : S^1 \to \mathbb{C}^n$ is a path obtained by joining $s_0, \ell_0, s_1, \ell_1, s_2, \ell_2, \dots, \ell_k, s_k$ in this order. $W^{1,p}(D^2; \mathbb{C}^n; \ell)$ is the set of $W^{1,p}$ sections u of $\mathbb{C}^n$ on D^2 such that $u(t) \in \ell(t)$ for $t \in S^1 = \partial D^2$.

Now the index of (3.14) is equal to the index of (3.6) plus the sum of the indices of (3.8.i) for $i = 0, \dots, k$ by excision property of indices.

On the other hand, the homotopy invariance of index implies that the index of (3.14) depends only on the homotopy type of ℓ. Moreover, in fact, ℓ is lifted to $\widetilde{\mathrm{Lag}}_n$ since $s_0, \ell_0, s_1, \ell_1, s_2, \ell_2, \dots, \ell_k, s_k$ are all lifted. Hence we may assume that ℓ is constant. Then it is easy to see that the index of (3.14) is n. The proof of Lemma 3.13 is complete. □

From Lemma 3.9 and Lemma 3.13 we find that the virtual dimension of the moduli space $\mathcal{CM}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$ is as asserted in Theorem 3.2. □

We next discuss the orientation. To define an orientation of the moduli space $\mathcal{CM}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$, we need one extra data, which we discuss now. We consider the complex (3.8.i). Its index as a virtual vector space depends only on the homotopy class of the path ℓ_i. Since $\ell_i = \pi \circ \tilde{\ell}_i$, and $\tilde{\ell}_i$ is a path joining $\tilde{s}_i(p)$ and $\tilde{s}_{i+1}(p)$ which is unique up to homotopy, it follows that the homotopy class of ℓ_i is defined in a canonical way for *graded* Lagrangian submanifold. Therefore the virtual vector space, which is an index of (3.8.i), is well-defined. Now we *choose* the orientation of this virtual vector space, the index of (3.8.i).

Remark 3.15. In a similar context of finite dimensional Morse theory, we need to fix an orientation of the stable (or unstable) manifold of each critical point in order to fix an orientation of the moduli space of connecting orbit. (See for example [FOn2] §21.)

The orientation of the index virtual vector space of (3.8.i) corresponds to the choice of orientation of stable manifold in the finite dimensional Morse theory.

To describe the choice of the orientation of the index of (3.8.i) in more canonical way, we proceed as follows. We modify the definition in (2.30) and put

$$\begin{aligned}
&\mathcal{LAG}^k((L_1, \tilde{s}_1, \mathfrak{L}_1), (L_2, \tilde{s}_2, \mathfrak{L}_2)) \\
&\quad = \bigoplus_{\substack{p \in L_1 \cap L_2 \\ \eta_{(L_1,\tilde{s}_1),(L_2,\tilde{s}_2)}(p) = k}} \mathrm{Hom}((\mathfrak{L}_1)_p, (\mathfrak{L}_2)_p) \\
&\qquad\qquad \otimes_{\mathbb{C}} \Lambda_{0,\mathrm{nov}} \otimes_{\mathbb{R}} \Lambda^{top}(\mathrm{Index}\,(3.8.i)).
\end{aligned} \tag{3.16}$$

(3.16) is isomorphic to (2.30). The choice of isomorphism corresponds one to one to the choice of orientations of the index of (3.8.i). In case we fix a choice of the index virtual vector space of (3.8.i) for each p, we use (2.30) in place of (3.16).

Remark 3.17. In (3.16) we include the determinant of (3.8.i) in Floer's chain complex. The necessity of it becomes more apparent if we consider more general situation. Namely let us consider the case when L_1 may not be transversal to L_2 but is of clean intersection. (It means that the $L_1 \cap L_2$ is a submanifold and the $T_pL_1 \cap T_pL_2 \cong T_p(L_1 \cap L_2)$ for $p \in L_1 \cap L_2$ is of constant dimension.) In that case the right hand side of (3.16) will become

$$\begin{aligned}&\mathcal{LAG}((L_1, \tilde{s}_1, \mathfrak{L}_1), (L_2, \tilde{s}_2, \mathfrak{L}_2)) \\ &\quad = \Gamma(L_1 \cap L_2; \mathrm{Hom}(\mathfrak{L}_1|_{L_1 \cap L_2}, \mathfrak{L}_2|_{L_1 \cap L_2}) \\ &\qquad\qquad \otimes_{\mathbb{C}} \Lambda_{0,\mathrm{nov}} \otimes_{\mathbb{R}} \Lambda^{top}(\mathrm{Index}(3.8.i)).\end{aligned}$$

Here $\Lambda^{top}(\mathrm{Index}(3.8.i))$ is one dimensional real vector bundle on $L_1 \cap L_2$, which corresponds to a local system $\pi_1(L_1 \cap L_2) \to \{\pm 1\}$. See [FOOO] Chapter 6 §25.6 for detail.

Theorem 3.18. *A choice of (M, st)-relative spin structures on L_i and a choice of an orientation of the index virtual vector space of* (3.8.i) *induce an orientation of $\mathcal{CM}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$, in a canonical way.*

Here orientation of $\mathcal{CM}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$ means orientation in the sense of Kuranishi structure [FOn2].

Proof. The idea of the proof is a combination of ones in [FOOO] Chapter 6 and [FOn] §21, (the later is suggested already in Floer's paper [Fl3]). Let E_i be the index virtual vector space of (3.8.i). For each $\varphi : (D^2, \partial D^2) \to (M, L)$, let $E(\varphi)$ be the index virtual vector space of (3.6). φ induces paths $s_i : \partial_i \Sigma \to \mathrm{Lag}_n$ as explained during the proof of Theorem 3.2. Joining them with ℓ_i we obtain a loop $\ell : \Sigma \to \mathrm{Lag}_n$ as in the proof of Theorem 3.2. ℓ depends continuously on φ, so we write $\ell(\varphi)$. We thus find a family of elliptic operators

$$\overline{\partial} : W^{1,p}(D^2; \mathbb{C}^n; \ell(\varphi)) \to W^{0,p}(D^2; \mathbb{C}^n \otimes \Lambda^{0,1}) \tag{3.19}$$

parametrized by φ. The following result is proved in [FOOO] Chapter 6 Theorem 21.1 (see also D. Silva [Sil]).

Theorem 3.20. *The choice of (M, st)-relative spin structures on L_i induces an orientation of the index bundle of* (3.19).

Let $E'(\varphi)$ be the virtual vector space which is the index of (3.19). Let $E(\varphi)$ be the virtual vector space which is the index of (3.5). Let E_i

be the index of (3.8.i). Then by family of index gluing theorem (see for example [Fu8] §4) we have an isomorphism

$$E'(\varphi) \oplus \bigoplus_{i=0}^{k} E_i \cong E(\varphi). \tag{3.21}$$

Here $E'(\varphi)$, $E(\varphi)$ are virtual vector bundles and E_i are virtual vector spaces. Theorem 3.18 follows from Theorem 3.20 and (3.21). □

Remark 3.22. We may take an orientation of $\mathcal{CM}_{k+1}(L_0, \ldots, L_k; p_0, \ldots, p_k)$ such that (3.21) holds as an identify of oriented vector space. However in order to define an operator $\mathfrak{m}_k$ satisfying A_∞ formula, we need to change this orientation in a way depending on the $\dim E_i$ and n in a similar way to [FOOO] Chapter 6. Namely we put

$$E'(\varphi) \oplus (-1)^{\epsilon} \bigoplus_{i=0}^{k} E_i \cong E(\varphi),$$

$$\epsilon = \sum_{j=0}^{k} \sum_{\ell=1}^{j} \dim E_i.$$

(Compare the above formula to [FOOO] Remark 25.2 (1).)

The problem of degree and orientation being understood, we are ready to define $\mathfrak{m}_k$ in the case $L_i \neq L_{i+1}$.

We consider the case when the dimension

$$\begin{aligned} &\dim \mathcal{CM}_{k+1}(L_0, \ldots, L_k; p_0, \ldots, p_k) \\ &\qquad = n + (k+1) - \sum_{i=0}^{k} \eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i) - 3 = 0. \end{aligned}$$

This (together with (2.37)) implies

$$1 + \sum_{i=0}^{k-1} (\eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i) - 1) = \eta_{(L_0,\tilde{s}_0),(L_k,\tilde{s}_k)}(p_k) - 1. \tag{3.23}$$

Let us put $\deg p_i = \eta_{(L_i,\tilde{s}_i),(L_{i+1},\tilde{s}_{i+1})}(p_i)$ for $i = 0, \ldots, k-1$, and $\deg p_k = \eta_{(L_0,\tilde{s}_0),(L_k,\tilde{s}_k)}(p_k)$. Then (3.23) implies that

$$1 + \sum (\deg p_i - 1) = \deg p_k - 1.$$

We are going to define the matrix elements

$$(3.24)\qquad \langle \mathfrak{m}_k(p_0,\dots,p_{k-1}),p_k\rangle \\ \in \mathrm{Hom}\left(\bigotimes_{i=0}^{k-1}\mathrm{Hom}((\mathfrak{L}_i)_{p_i},(\mathfrak{L}_{i+1})_{p_i}),\mathrm{Hom}((\mathfrak{L}_0)_{p_k},(\mathfrak{L}_k)_{p_k})\right)\otimes_{\mathbb{C}}\Lambda_{\mathrm{nov}}$$

when (3.23) is satisfied.

Before defining (3.24), we explain how (3.24) defines $\mathfrak{m}_k$. We recall that an element of $\mathcal{LAG}((L_i,\tilde{s}_i,\mathfrak{L}_i),(L_{i+1},\tilde{s}_{i+1},\mathfrak{L}_{i+1}))$ is, by definition, a formal sum

$$\sum_{j=0}^{\infty} T^{\lambda_i^{(j)}}[v_i^{(j)}]$$

where

$$v_i^{(j)}\in \mathrm{Hom}((\mathfrak{L}_i)_{p_i^{(j)}},(\mathfrak{L}_{i+1})_{p_i^{(j)}}),\quad p_i^{(j)}\in L_i\cap L_{i+1}.$$

Then we define

$$\mathfrak{m}_k(v_1^{(j_1)},\dots,v_{k-1}^{(j_{k-1})}),$$

by

$$(3.25)\qquad \mathfrak{m}_k(v_1^{(j_1)},\dots,v_{k-1}^{(j_{k-1})}) \\ =\sum_{j_1,\dots,j_k}\sum_{p_k\in L_0\cap L_k} T^{\lambda_1^{(j_1)}+\dots+\lambda_{k-1}^{(j_{k-1})}} \\ \langle \mathfrak{m}_k(p_0^{(j_1)},\dots,p_{k-1}^{(j_{k-1})}),p_k\rangle(v_1^{(j_1)}\otimes\dots\otimes v_{k-1}^{(j_{k-1})}),$$

using (3.24). Now we define (3.24).

Definition 3.26. We define (3.24) by

$$(3.27)\qquad \sum_{\varphi\in\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_0,\dots,p_k)} T^{\mathrm{Re}\,E(\varphi)}e^{\sqrt{-1}\,\mathrm{Im}\,E(\varphi)}h_{\nabla}(\varphi(\partial D^2))\epsilon_{\varphi}$$

where E,h,ϵ are defined below. First we define E by

$$E(\varphi)=\int_{D^2}\varphi^*\Omega.$$

$\mathrm{Re}\,E$ and $\mathrm{Im}\,E$ are its real and imaginary parts, respectively.

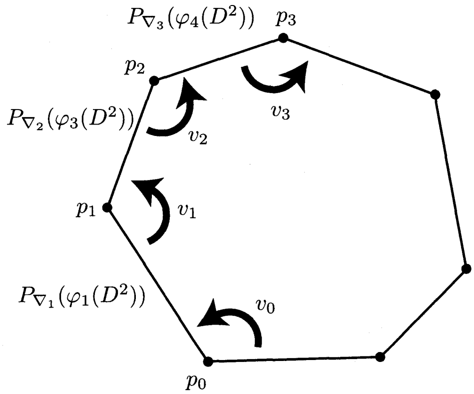

Figure 3.3.

Next we define h. Let $v_i \in \mathrm{Hom}((\mathfrak{L}_i)_{p_i}, (\mathfrak{L}_{i+1})_{p_i})$. Then

$$(3.28) \quad h_\nabla(\varphi(\partial D^2))(v_0 \otimes \cdots \otimes v_{k-1}) \\ = P_{\nabla_k}(\varphi(\partial_k D^2)) \circ v_{k-1} \circ \cdots \circ P_{\nabla_1}(\varphi(\partial_1 D^2)) \circ v_0.$$

Here $P_{\nabla_i}(\varphi(\partial_i D^2)) : (\mathfrak{L}_i)_{p_i} \to (\mathfrak{L}_i)_{p_{i+1}}$ is the parallel transport along the path $\varphi(\partial_i D^2)$ of the bundle $\mathfrak{L}_i$ with respect to the connection ∇_i.

Finally $\epsilon_\varphi \in \{\pm 1\}$ is determined by the orientation of $\mathcal{CM}_{k+1}(L_0, \ldots, L_k; p_0, \ldots, p_k)$ defined by Theorem 3.20. (See also Remark 3.22.)

Lemma 3.29. (3.27) *is an element of* $\mathrm{Hom}((\mathfrak{L}_0)_{p_k}, (\mathfrak{L}_k)_{p_k}) \otimes \Lambda_{+,\mathrm{nov}}$.

Lemma 3.27 is a consequence of Gromov compactness theorem. (See [FOOO] Proposition 5.8 for the proof of a similar statement.) The following lemma is used in the proof of A_∞ formula.

Lemma 3.30. $T^{\mathrm{Re}\,E(\varphi)} e^{\sqrt{-1}\,\mathrm{Im}\,E(\varphi)} h_\nabla(\varphi(\partial D^2))$ *depends only on the homotopy class of* φ.

Proof. We can prove that $\mathrm{Re}\,E(\varphi)$ depends only on the homotopy class of φ by using Stokes' theorem and the fact that L_i is a Lagrangian submanifold. Homotopy independence of $e^{\sqrt{-1}\,\mathrm{Im}\,E(\varphi)} h_\nabla(\varphi(\partial D^2))$ follows from the condition that the curvature of ∇_i is $2\pi\sqrt{-1}B$ (Condition (2.1.2)). □

Remark 3.31. We remark that we are working under the hypothesis $L_i \neq L_{i+1}$ in this section. In this case we put $\mathfrak{m}_0 = 0$.

We also remark that in this case the coefficient of the operator belongs to the maximal ideal $\Lambda_{+,\mathrm{nov}}$, since the energy of nonconstant pseudo-holomorphic map is always positive.

Remark 3.32. Theorem 3.2 implies that $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_0,\dots,p_k)$ has a Kuranishi structure but it is in general not true that we can find a generic perturbation using J so that it is a manifold. So in order to make sense of (3.27) we need to take multivalued perturbation (multisection) as in [FOn2] and use it to define a fundamental chain over $\mathbb{Q}$ of our moduli space $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_0,\dots,p_k)$. In the situation of (3.27) where the moduli space is of dimension 0, it means that we need to put multiplicity $=\pm 1/\text{integer}$ in place of $\epsilon_\varphi \in \{\pm 1\}$.

However, in our situation, all of our Lagrangian submanifolds are semipositive in the sense defined in [FOOO] Chapter 3. Hence by using normally polynomial sections described in [FOOO] §A3, we can prove that $\mathcal{CM}_{k+1}(L_0,\dots,L_k;p_0,\dots,p_k)$ has a fundamental chain over $\mathbb{Z}$. In other words our operation $\mathfrak{m}_k$ is defined over $\Lambda_{\mathbb{Z},0,\mathrm{nov}}$.

The next task to be carried out might be the proof of A_∞ formula $\hat{d}\circ\hat{d}=0$. However the formula

$$\sum \pm\mathfrak{m}_*(x_0,\dots,x_{\ell-1},\mathfrak{m}_*(x_\ell,\dots,x_m),x_{m+1},\dots,x_k)=0 \tag{3.33}$$

does *not* holds if we consider only $\mathfrak{m}_*$ with $*>0$. In other words we can not prove A_∞ formula when we consider only L_i's with $L_i \ne L_{i+1}$. We need to include the case $L_i = L_{i+1}$ and $\mathfrak{m}_0$, which is nonzero in the general case. Namely, for example, we are *not* able to prove

$$\begin{aligned}
0 = \;& \mathfrak{m}_1(\mathfrak{m}_3(x_0,x_1,x_2)) \\
& +\mathfrak{m}_2(\mathfrak{m}_2(x_0,x_1),x_2) + (-1)^{\deg x_0+1}\mathfrak{m}_2(x_0,\mathfrak{m}_2(x_1,x_2)) \\
& +\mathfrak{m}_3(\mathfrak{m}_1(x_0),x_1,x_2) + (-1)^{\deg x_0+1}\mathfrak{m}_3(x_0,\mathfrak{m}_1(x_1),x_2) \\
& +(-1)^{\deg x_0+1+\deg x_1+1}\mathfrak{m}_3(x_0,x_1,\mathfrak{m}_1(x_2)),
\end{aligned} \tag{3.34}$$

which is (3.33) in case $k=2$, if we consider only $\mathfrak{m}_*$ with $*>0$. In place of (3.34), we will prove

$$\begin{aligned}
0 = \;& \mathfrak{m}_1(\mathfrak{m}_3(x_0,x_1,x_2)) \\
& +\mathfrak{m}_2(\mathfrak{m}_2(x_0,x_1),x_2) + (-1)^{\deg x_0+1}\mathfrak{m}_2(x_0,\mathfrak{m}_2(x_1,x_2)) \\
& +\mathfrak{m}_3(\mathfrak{m}_1(x_0),x_1,x_2) + (-1)^{\deg x_0+1}\mathfrak{m}_3(x_0,\mathfrak{m}_1(x_1),x_2) \\
& +(-1)^{\deg x_0+1+\deg x_1+1}\mathfrak{m}_3(x_0,x_1,\mathfrak{m}_1(x_2)) \\
& +\mathfrak{m}_4(\mathfrak{m}_0(1),x_0,x_1,x_2) + (-1)^{\deg x_0+1}\mathfrak{m}_4(x_0,\mathfrak{m}_0(1),x_1,x_2) \\
& +(-1)^{\deg x_0+1+\deg x_1+1}\mathfrak{m}_4(x_0,x_1,\mathfrak{m}_0(1),x_2) \\
& +(-1)^{\deg x_0+1+\deg x_1+1+\deg x_2+1}\mathfrak{m}_4(x_0,x_1,x_2,\mathfrak{m}_0(1)).
\end{aligned} \tag{3.35}$$

Here, for example, the element $\mathfrak{m}_0(1)$ which appeared in $\mathfrak{m}_4(\mathfrak{m}_0(1), x_0, x_1, x_2)$ is an element of the $\Lambda_{0,nov}$ module $\mathcal{LAG}((L_0, \tilde{s}_0, \mathfrak{L}_0), (L_0, \tilde{s}_0, \mathfrak{L}_0))$. Thus $\mathfrak{m}_0$ and $\mathfrak{m}_4$ in (3.35) are not defined in this section. So we generalize the definition of this section and discuss the case when $L_i = L_{i+1}$ for some i, in the next section.

§4. Floer homology and A_∞ category III – the operator $\mathfrak{m}_k$ – the general case –

The discussion of this section is a combination of the argument of [FOOO] (especially its Chapter 4) and one of the last section. Namely, in the case when $c_0 = \cdots = c_{k+1} = c = (L, \tilde{s}, \mathfrak{L})$, the operator

$$\mathfrak{m}_k : \mathcal{LAG}[1](c,c) \otimes \cdots \otimes \mathcal{LAG}[1](c,c) \to \mathcal{LAG}[1](c,c)$$

$(k = 0, 1, \ldots)$ is the operator of the filtered A_∞ algebra structure constructed in [FOOO] Theorem 13.22.

We consider the following situation. Let $L_{(j)}$, $j = 1, \ldots, m$ be Lagrangian submanifolds such that $L_{(j)} \neq L_{(j+1)}$. Let $\ell_j \in \mathbb{Z}_{>0}$. We put $L_0 = \cdots = L_{\ell_1 - 1} = L_{(0)}$, $L_{\ell_1} = \cdots = L_{\ell_1+\ell_2-1} = L_{(1)}$, $\ldots$, $L_{\sum_{j=0}^{m-1}\ell_j} = \cdots = L_{\sum_{j=0}^{m}\ell_j - 1} = L_{(m)}$. We take $\tilde{s}_i$, $\mathfrak{L}_i$ so that

$$c_i = (L_i, \tilde{s}_i, \mathfrak{L}_i) \in \mathfrak{Ob}_4(\mathcal{LAG}(M, \Omega, st, \widetilde{\mathrm{Lag}}(M))).$$

We write also

$$s_i^{(j)} = s_{\ell_0+\cdots+\ell_{i-1}+j}, \quad \mathfrak{L}_i^{(j)} = \mathfrak{L}_{\ell_0+\cdots+\ell_{i-1}+j},$$

sometimes. We are going to define:

$$\mathfrak{m}_k : \mathcal{LAG}[1](c_0, c_1) \otimes \cdots \otimes \mathcal{LAG}[1](c_{k-1}, c_k) \to \mathcal{LAG}[1](c_0, c_k).$$

(Here $k = 0, 1 \ldots.$) Note that we defined the graded $\Lambda_{0,nov}$ module $\mathcal{LAG}[1](c_i, c_{i+1})$ if $L_i \neq L_{i+1}$ in Definition 2.29. But in case $L_i = L_{i+1}$ we need to start with the definition of $\mathcal{LAG}[1](c_i, c_{i+1})$. Let us put $L = L_i = L_{i+1}$. We have $(\tilde{s}_i, \mathfrak{L}_i)$, $(\tilde{s}_{i+1}, \mathfrak{L}_{i+1})$ on L_i, L_{i+1}.

Roughly speaking the graded $\Lambda_{0,nov}$ module $\mathcal{LAG}((L, \mathfrak{L}_i), (L, \mathfrak{L}_{i+1}))$ we are going to define is a singular chain complex with local coefficient $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})$. (We remark that the curvature of $\mathfrak{L}_i$ coincides with the curvature of $\mathfrak{L}_{i+1}$ by (2.2.2). Hence $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})$ is a flat $U(1)$ bundle.) But we need to be a bit careful in defining it, so that the definition of $\mathfrak{m}_k$, which is based on the smooth correspondence, works. (See also [FOOO] §A1.)

Let $\sigma : \Delta^q \to L$ be a smooth (singular) simplex. Let $(1, 0, \ldots, 0) \in \Delta^q$ be the base point. We take a trivialization $\sigma^* \mathfrak{L}_i \cong \Delta^q \times (\mathfrak{L}_i)_{\sigma(1,0,\ldots,0)}$ by using a parallel transport along the σ image of the straight line joining $(1, 0, \ldots, 0)$ with a given point on Δ^q. A *smooth singular simplex with local coefficient* $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})$ is a pair (σ, u) such that $\sigma : \Delta^q \to L$ and $u \in \mathrm{Hom}((\mathfrak{L}_i)_{\sigma(1,0,\ldots,0)}, (\mathfrak{L}_{i+1})_{\sigma(1,0,\ldots,0)})$. We now put:

$$S_q(L; \mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})) = \left\{ \sum a_i(\sigma_i, u_i) \Big| \, a_i \in \mathbb{C} \right\} \Big/ \sim_1 . \tag{4.1}$$

Here (σ_i, u_i) are smooth singular chain complexes with local coefficient, $\sum$ is a finite sum, and $\sim_1$ is defined by $a_i(\sigma_i, u_i) + a_i'(\sigma_i, u_i') \sim_1 (\sigma_i, a_i u_i + a_i' u_i')$. We can define a boundary operator ∂ on it using a trivialization $\sigma^* \mathfrak{L}_i \cong \Delta^q \times (\mathfrak{L}_i)_{\sigma(1,0,\ldots,0)}$.

We need to divide it further by an appropriate equivalence relation and then take a countably generated subcomplex, in order to obtain the chain complex we use. Let us now describe this process.

Let $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})^*$ be the dual bundle of $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})$. We consider the vector bundle $\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1}) \otimes \Lambda^q(L)$ on L and let

$$W^{-\infty}(\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1}) \otimes \Lambda^q(L))$$

be the set of all distribution valued sections of it. Note an element of it can be identified with a linear map

$$C^\infty(\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})^* \otimes \Lambda^{n-q}(L)) \to \mathbb{C}$$

by

$$T \mapsto \left(w \mapsto \int_L T \wedge w \right).$$

(We assume that L is oriented.) We use this identification to define a map

$$T : S_q(L; \mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})) \to W^{-\infty}(\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1}) \otimes \Lambda^{n-q}(L))$$

by putting

$$\int_L T(\sigma_i, u_i) \wedge w = \int_{\Delta^q} \langle u_i, \sigma^* w \rangle, \tag{4.2}$$

and extending it to a complex linear map in an obvious way. Note that we used the trivialization $\sigma^* \mathfrak{L}_i \cong \Delta^q \times (\mathfrak{L}_i)_{\sigma(1,0,\ldots,0)}$ mentioned before to define the right hand side of (4.2).

We define an equivalence relation $S_q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))$ such that $x \sim_2 y$ if and only if $T(x) = T(y)$. We now put

$$\overline{S}_q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1})) = S_q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))/\sim_2 .$$

Let P be a simplicial complex of dimension q and $f : P \to L$ be a piecewise smooth map. We assume that P has a base point. We assume that the pull back $f^*\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1})$ is trivial and we fix a trivialization. Let u be an element of the fiber of $f^*\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1})$ at the base point. Then, from f and u, we obtain an element of $\overline{S}_q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))$ in an obvious way. This element is independent of the subdivision of the simplicial complex P (as a chain). We denote it by $[P,f,u]$. Every element of $\overline{S}_q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))$ is realized in this way. We remark that, if we do not divide by the equivalence relation $\sim_2$, then the element we obtain will depend on the subdivision of the simplicial complex P.

We use cohomology rather than homology (since the product structure is the main issue here). We write

$$\overline{S}^q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1})) = \overline{S}_{n-q}(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1})).$$

Next we need to take a countably generated (over $\mathbb{C}$) subcomplex of $\overline{S}^q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))$. We recall that we have fixed a countable set of Lagrangian submanifolds and we assumed that the Lagrangian submanifold part of the objects of our category is always in this set. We next choose a countably generated subcomplex of $\overline{S}^*(L)$ for each member L of the countable set of Lagrangian submanifolds we have chosen.

The condition that this subcomplex is assumed to satisfy, is rather delicate and is not mentioned here. (See [FOOO] §A1, §A5.) The reason we need to choose a countably generated subcomplex is that we need to use frequently Bair's category theorem to achieve transversality in various situations and in Bair's category theorem countability is an essential issue. The transversality here is not only a technical problem but also is related to many essential points of the story. It is related to the fact that, for example, the square of delta function is ill-defined, and hence is also related to the problem of infinity in quantum field theory.

We denote by $C^q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))$ the countably generated subcomplex we have chosen. We put

$$\mathcal{LAG}^q((L,\mathfrak{L}_i),(L,\mathfrak{L}_{i+1})) = C^q(L;\mathrm{Hom}(\mathfrak{L}_i,\mathfrak{L}_{i+1}))\hat{\otimes}_{\mathbb{C}}\Lambda_{0,\mathrm{nov}}.$$

Here $\hat{\otimes}_{\mathbb{C}}$ means that we take a completion by using topology induced by the filtration of $\Lambda_{0,\mathrm{nov}}$. (Note we assumed $L_i = L_{i+1} = L$.) In other words an element of $\mathcal{LAG}^q((L,\mathfrak{L}_i),(L,\mathfrak{L}_{i+1}))$ is realized by countable sum

$$\sum T^{\lambda_i}[P_i, f_i, u_i]$$

where $\lambda_i \to \infty$. Using the grading of our Lagrangian submanifold, we shift the degree as follows. Let $\tilde{s}_i$ and $\tilde{s}_{i+1}$ be two gradings of our Lagrangian submanifold L. Then, as mentioned in section 2, there exists a unique integer k such that $\tilde{s}_{i+1} = k \cdot \tilde{s}_i$. Here $\mathbb{Z}$ acts on $\widetilde{\mathrm{Lag}}_n$ as a deck transformation group. We put

$$k = \tilde{s}_{i+1} - \tilde{s}_i \in \mathbb{Z}. \tag{4.3}$$

Now we define:

Definition 4.4.

$$\begin{aligned}&\mathcal{LAG}^q((L, \tilde{s}_i, \mathfrak{L}_i), (L, \tilde{s}_{i+1}, \mathfrak{L}_{i+1})) \\ = \ & C^{q+(\tilde{s}_{i+1}-\tilde{s}_i)}(L; \mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1}))\hat{\otimes}_{\mathbb{C}}\Lambda_{0,\mathrm{nov}}.\end{aligned}$$

We thus defined $\mathcal{LAG}(c, c')$ in our general situation. We turn to the definition of our operators $\mathfrak{m}_k$.

Let $L_{(j)}$, ℓ_j, $(j = 1, \dots, m)$, $c_i = (L_i, \tilde{s}_i, \mathfrak{L}_i)$ $(i = 0, \dots, \sum_{j=0}^m \ell_j)$ be as in the beginning of this section. For i with $L_{(j)} = L_i \neq L_{i+1} = L_{(j+1)}$, we take $p_i = p_{(j)} \in L_{(j)} \cap L_{(j+1)}$ and

$$v_i \in \mathrm{Hom}((\mathfrak{L}_i)_{p_{(j)}}, (\mathfrak{L}_{i+1})_{p_{(j)}}).$$

For i with $L_{(j)} = L_i = L_{i+1}$, we take

$$x_i = [P_i, f_i, u_i] \in C^{g_i+(\tilde{s}_{i+1}-\tilde{s}_i)}(L_{(j)}; \mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})).$$

We are going to define

$$\begin{aligned}\mathfrak{m}_{\sum_{j=0}^{m-1}\ell_j}(x_0, \dots, \ & x_{\ell_0-1}, v_{\ell_0}, x_{\ell_0+1}, \dots, x_{\ell_0+\ell_1-1}, v_{\ell_0+\ell_1}, \\ & x_{\ell_0+\ell_1-1}, \dots, x_{\ell_0+\dots+\ell_{m-1}-1}, \\ & v_{\ell_0+\dots+\ell_{m-1}}, x_{\ell_0+\dots+\ell_{m-1}+1}, \dots, x_{\ell_0+\dots+\ell_m-1}).\end{aligned} \tag{4.5}$$

Hereafter we write (4.5) as $\mathfrak{m}(x_0, \dots, x_{\ell_0+\dots+\ell_m-1})$, for simplicity. There are two cases:

(4.6.1) $L_{(m)} \neq L_{(0)}$.
(4.6.2) $L_{(m)} = L_{(0)}$.

Case (4.6.1): In this case we take $p_{(m)} \in L_{(m)} \cap L_{(0)}$ and are going to define the matrix element

$$\begin{aligned}&\langle \mathfrak{m}(x_0, \dots, x_{\ell_0+\dots+\ell_m-1}), p_{(m)}\rangle \\ & \qquad \in \mathrm{Hom}((\mathfrak{L}_{\ell_0+\dots+\ell_m-1})_{p_{(m)}}, (\mathfrak{L}_0)_{p_{(m)}}).\end{aligned} \tag{4.7}$$

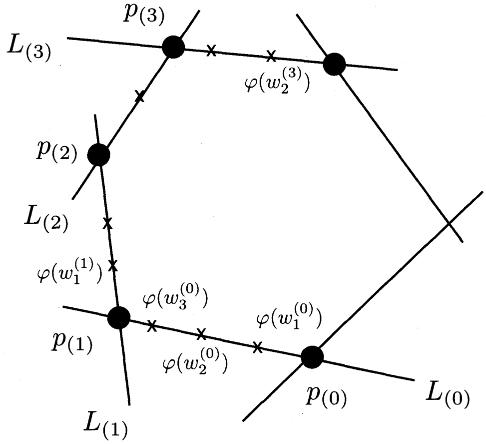

Figure 4.1.

We need to define (4.7) only in case the degree is correct. Namely in case

$$\sum(g_i+1)+\sum(\eta(p_{(j)})+1)=\eta(p_{(m)})+1. \tag{4.8}$$

To define (4.7) in case (4.8) is satisfied, we use a moduli space which is similar to but a bit more complicated than one we used in the last section. Let us define it now. Let us consider the system $((D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi)$ such that

(4.9.1) $((D^2,\vec{z}),\varphi)\in\tilde{\mathcal{M}}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)})$, where the right hand side is as in (3.1).
(4.9.2) $\vec{w}^{(j)}=(w_1^{(j)},\dots,w_{\ell_j-1}^{(j)})$. $w_i^{(j)}\in\partial_jD^2$. Here ∂_jD^2 is as in the last section.
(4.9.3) If $i\neq i'$ then $w_i^{(j)}\neq w_{i'}^{(j)}$. $w_1^{(j)},\dots,w_{\ell_j-1}^{(j)}$ respects the order of ∂_jD^2.

The totality of such $((D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi)$ is denoted by

$$\tilde{\mathcal{M}}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)};\ell_0,\dots,\ell_m).$$

We divide it by an obvious action of $PSL(2;\mathbb{R})$ and denote the quotient space by

$$\mathcal{M}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)};\ell_0,\dots,\ell_m).$$

We can compactify it by using the notion of stable maps. (See [FOOO] §3 for its definition in the case Riemann surface has a boundary.) We denote the compactification by

$$\mathcal{CM}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)};\ell_0,\dots,\ell_m).$$

We define evaluation maps

$$\vec{ev}=(\vec{ev}^{(0)},\dots,\vec{ev}^{(m)}),\quad \vec{ev}^{(j)}=(ev_1^{(j)},\dots,ev_{\ell_j-1}^{(j)}),$$

such that

$$\vec{ev}^{(j)}:\mathcal{CM}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)};\ell_0,\dots,\ell_m)\to L_{(j)}^{\ell_j-1}$$

is

$$\vec{ev}_i^{(j)}[(D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi]=\varphi(w_i^{(j)}).$$

We consider the fiber product

$$\mathcal{CM}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m)};\ell_0,\dots,\ell_m) \;{}_{\vec{ev}}\times_{f_*}\prod_j\prod_{i=1}^{\ell_j-1}P_{\ell_0+\cdots+\ell_{j-1}+i}. \tag{4.10}$$

Now we have:

Lemma 4.11. *If* (4.8) *is satisfied then* (4.10) *has a Kuranishi structure of dimension* 0. *The choice of relative spin structure on* $L_{(j)}$ *and the choices of the orientations of the index virtual bundle of* (3.8.i), *determine an orientation of* (4.10) *in a canonical way.*

Proof. The proof of Lemma 4.11 is a straight forward combination of the argument of [FOOO] Chapters 5, 6 and one in the last section. We remark that to prove Lemma 4.11 we need to restrict the chains P_i so that only countably many of them are studied, since we need to use Bair's category theorem to prove Lemma 4.11.

We need to choose the orientation by modifying the fiber product orientation in a way combining [FOOO] Chapter 6 and (3.21) as follows:

We remark that each P_i is regarded as a cochain rather than a chain. This means that we coorient it. Namely for each $x\in P_i$ such that $f_i:P_i\to L$ (f_i is the map defining P_i as a differential form valued distribution) is an immersion at x, we have an orientation of the normal bundle $N_{f_i(x)}f_i(P_i)$.

For each marked point z_i we define E_i as follows. If $L_i\neq L_{i+1}$ then we define E_i as in (3.21) namely it is the index of (3.8.i). In case $L_i\neq$

L_{i+1}, then z_i corresponds to some $P_{i'}$. Then we put $E_i = N_{f_j(x)} f_j(P_j)$ here $f_j(x) = \varphi(z_i)$. Then (3.21) holds. So we define an orientation by (3.22). □

Now to define the matrix element (4.7), we need to define a weight. We put

$$E((D^2, \vec{z}; \vec{w}^{(0)}, \dots, \vec{w}^{(m)}), \varphi) = \int_{D^2} \varphi^* \omega.$$

Then the "absolute value" of the weight we put is $T^{E((D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi)}$. We next are going to define the "phase factor"

$$\begin{aligned} &H(((D^2, \vec{z}; \vec{w}^{(0)}, \dots, \vec{w}^{(m)}), \varphi); \vec{u}, \vec{v}) \\ &\in \ \mathrm{Hom}((\mathfrak{L}_{\ell_0+\cdots+\ell_m-1})_{p_{(m)}}, (\mathfrak{L}_0)_{p_{(m)}}). \end{aligned}$$

Here $\vec{u}$ denotes the totality of $u_i \in \mathrm{Hom}((\mathfrak{L}_i)_{q_i}, (\mathfrak{L}_{i+1})_{q_i})$ here q_i is the image by f_i of the base point of P_i, and $\vec{v}$ denotes the totality of $v_i \in \mathrm{Hom}((\mathfrak{L}_i)_{p_{(j)}}, (\mathfrak{L}_{i+1})_{p_{(j)}})$. We put

$$\alpha_i = \begin{cases} u_i & \text{if } L_i = L_{i+1}, \\ v_i & \text{if } L_i \neq L_{i+1}. \end{cases}$$

Now we put

$$\begin{aligned} &P_\nabla(\varphi(\partial D^2)) \\ &= \ P_{\nabla_{\sum_{j=0}^m \ell_j}}(\varphi(\partial_{\sum_{j=0}^m \ell_j} D^2)) \circ \alpha_{\sum_{j=0}^m \ell_j - 1} \circ \cdots \circ \alpha_1 \circ P_{\nabla_1}(\varphi(\partial_1 D^2)). \end{aligned}$$

We remark here that we use the trivialization of $f_i^* \,\mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})$ to regard u_i as an element of $\mathrm{Hom}((\mathfrak{L}_i)_{w_{i'}^{(j)}}, (\mathfrak{L}_{i+1})_{w_{i'}^{(j)}})$ where $\ell_0 + \cdots + \ell_{j-1} + i' = i$. (A priori, u_i is a homomorphism between the fibers at $q_i = f_i(1, 0, \dots, 0)$.)

Now we define

$$\begin{aligned} &H(((D^2, \vec{z}; \vec{w}^{(0)}, \dots, \vec{w}^{(m)}), \varphi); \vec{u}, \vec{v}) \\ &= \ \exp\left(2\pi\sqrt{-1} \int_{D^2} \varphi^* B\right) P_\nabla(\varphi(\partial D^2)). \end{aligned}$$

Definition 4.12. We assume that (4.6.1) and (4.8) are satisfied. Then we define the matrix element (4.7) by

$$\sum_{[(D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi] \in (4.10)} \epsilon_\varphi T^{E((D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)}),\varphi)} H(((D^2, \vec{z}; \vec{w}^{(0)}, \dots, \vec{w}^{(m)}), \varphi); \vec{u}, \vec{v}). \tag{4.13}$$

Here $\epsilon_\varphi = \pm 1$ is determined by the orientation of the moduli space (4.10).

It is a consequence of Gromov compactness that (4.13) is an element of

$$\mathrm{Hom}((\mathfrak{L}_{\ell_0+\cdots+\ell_m-1})_{p_{(m)}}, (\mathfrak{L}_0)_{p_{(m)}})\hat{\otimes}\Lambda_{\mathrm{nov}}.$$

We thus defined the operator $\mathfrak{m}_k$ in Case (4.6.1).

Case (4.6.2): This case is similar to the case of (4.6.1) and we proceed as follows.

Let us define a moduli space

$$\mathcal{M}_{m+1}(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m-1)};\ell_0,\dots,\ell_m)$$

as follows. We have $p_{(j)} \in L_{(j)} \cap L_{(j+1)}$ for $j = 0,\dots,m-1$. We consider the system $((D^2,\vec{z};\vec{w}^{(0)},\dots,\vec{w}^{(m)},u),\varphi)$ such that

(4.14.1) Put $\vec{z} = (z_0,\dots,z_m)$, $\vec{z}_- = (z_0,\dots,z_{m-1})$. Then $((D^2,(z_-)),\varphi) \in \tilde{\mathcal{M}}_m(L_{(0)},\dots,L_{(m)};p_{(1)},\dots,p_{(m-1)})$. Where the right hand side is as in (3.1).

(4.14.2) $\vec{w}^{(j)} = (w_1^{(j)},\dots,w_{\ell_j-1}^{(j)})$. $w_i^{(j)} \in \partial_j D^2$. Here $\partial_j D^2$ is as in the last section.

(4.14.3) If $i \neq i'$ then $w_i^{(j)} \neq w_{i'}^{(j)}$. $w_1^{(j)},\dots,w_{\ell_j-1}^{(j)}$ respects the order of $\partial_j D^2$.

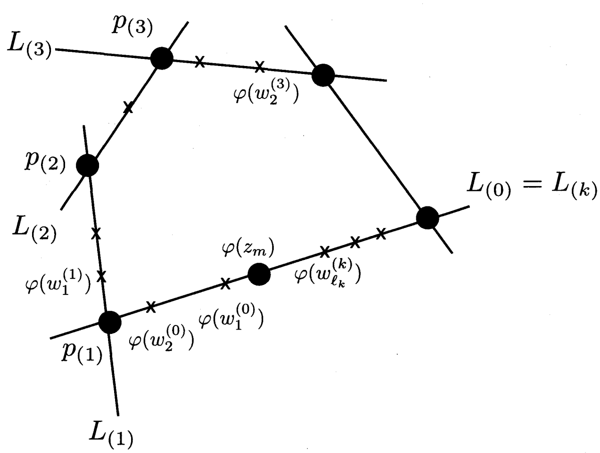

Figure 4.2.

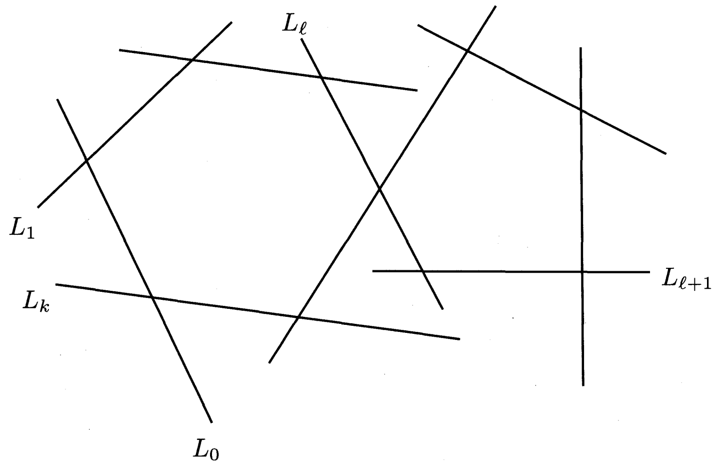

Figure 4.3.

We split this moduli space according to the homotopy type of the map φ. We denote a homotopy class by β and let

$$\mathcal{M}_{m+1}(\beta; L_{(0)}, \dots, L_{(m)}; p_{(1)}, \dots, p_{(m-1)}; \ell_0, \dots, \ell_m)$$

be the corresponding component of the moduli space. We define an evaluation map

$$\vec{ev}^{(j)} : \mathcal{CM}_{m+1}(\beta; L_{(0)}, \dots, L_{(m)}; p_{(1)}, \dots, p_{(m-1)}; \ell_0, \dots, \ell_m) \to L_{(j)}^{\ell_j - 1},$$

in a way similar to the case (4.6.1). Using it we define the fiber product.

$$\begin{aligned}(4.15)\quad &\mathfrak{M}_\beta(p_{(1)}, \dots, p_{(m-1)}; P_1, \dots, P_{\sum \ell_j}) \\ &= \mathcal{CM}_{m+1}(\beta; L_{(0)}, \dots, L_{(m)}; p_{(1)}, \dots, p_{(m-1)}; \ell_0, \dots, \ell_m) \\ &\qquad\qquad {}_{\vec{ev}} \times_{f_*} \prod_j \prod_{i=1}^{\ell_j - 1} P_{\ell_0 + \cdots + \ell_{j-1} + i}.\end{aligned}$$

We also define another evaluation map at the remaining mark point z_m. Namely we define

$$\begin{aligned} ev : \mathcal{CM}_{m+1}(\beta; L_{(0)}, \dots, L_{(m)}; p_{(1)}, \dots, p_{(m-1)}; \ell_0, \dots, \ell_m) &\to L_{(m)} \\ &= L_{(0)} \end{aligned}$$

by

$$ev((D^2, \vec{z}; \vec{w}^{(0)}, \dots, \vec{w}^{(m)}, u), \varphi) = \varphi(z_m).$$

It induces

$$ev : \mathfrak{M}_\beta(p_{(1)}, \ldots, p_{(m-1)}; P_1, \ldots, P_{\sum \ell_j}) \to L_{(0)}.$$

Now we put

$$\begin{aligned}(4.16)\qquad &\sum_\beta T^{E((D^2,\vec z;\vec w^{(0)},\ldots,\vec w^{(m)}),\varphi_\beta)} \\ &\qquad H(((D^2, \vec z; \vec w^{(0)}, \ldots, \vec w^{(m)}), \varphi_\beta); \vec u, \vec v) \\ &\qquad ev_*[\mathfrak{M}_\beta(p_{(1)}, \ldots, p_{(m-1)}; P_1, \ldots, P_{\sum \ell_j})],\end{aligned}$$

where φ_β is a map with homotopy class β. (4.16) is an element of

$$C^{g+(\tilde s_k - \tilde s_0)}(L; \mathrm{Hom}(\mathfrak{L}_{(0)}, \mathfrak{L}_{(m)})).$$

($g = \ell_0 + \cdots + \ell_m - 1$.) We define

$$\mathfrak{m}(x_0, \ldots, x_{\ell_0+\cdots+\ell_m-1}) = (4.16).$$

Now we have:

Theorem 4.17. *The operation $\mathfrak{m}_k$ defined above satisfies A_∞ relations.*

Since the detail of the definition of module of morphisms and operations are already discussed, the proof of Theorem 4.17 is in fact a straight forward generalization of the argument of [FOOO] and one of [Fu1], [Fu4]. So we do not repeat it. The main idea is that the degeneration of elements of $\mathcal{M}_{k+1}$ can be described as in the Figure 4.3 and they correspond to the terms in the A_∞ formula.

§5. Unit and Homotopy unit

We have thus constructed operations $\mathfrak{m}_k$ which satisfy the A_∞ relations. To complete the construction of our filtered A_∞ category $\mathcal{LAG}(M, \omega)$ we need to construct a unit. There is a delicate problem related to transversality to construct a unit of our A_∞ category $\mathcal{LAG}(M, \omega)$. This problem is solved in [FOOO] §20. We discuss an outline of it here (together with its slight generalization. Namely we generalize from the case A_∞ algebra discussed in [FOOO] to A_∞ category).

Let $(L, \tilde s, \mathfrak{L})$ be an object of $\mathcal{LAG}(M, \omega)$. Then, by definition, the module of morphisms, $\mathcal{LAG}((L, \tilde s, \mathfrak{L}), (L, \tilde s, \mathfrak{L}))$ is a subcomplex of the de-Rham complex $W^{-\infty}(L; \mathbb{C} \otimes \Lambda^*(L))$ of distribution valued forms, since $\mathrm{Hom}(\mathfrak{L}, \mathfrak{L})$ together with its connection is trivial. Then $1 \in W^{-\infty}(L; \mathbb{C}\otimes$

$\Lambda^0(L))$ is an element of $\mathcal{LAG}^0((L, \tilde{s}, \mathfrak{L}), (L, \tilde{s}, \mathfrak{L}))$. Actually 1 is the element corresponding to the fundamental chain $[L]$ by our identification (4.2), the Poincaré duality. So we write $[L]$ rather than 1.

"Theorem 5.1"[3]. *$[L]$ is a unit of our filtered A_∞ category $\mathcal{LAG}(M, \omega)$.*

"Proof"[4]. We need to show

$$\mathfrak{m}_k(x_1, \ldots, x_{i-1}, [L], x_{i+1}, \ldots, x_k) = 0 \tag{5.2}$$

for $k > 2$. We also need to show

$$\mathfrak{m}_1([L]) = 0, \quad (-1)^{\deg x}\mathfrak{m}_2([L], x) = \mathfrak{m}_2(x, [L]) = x. \tag{5.3}$$

(5.3) can be "proved" in a way similar to (5.2). So we explain the idea of the "proof" of (5.2). There are two cases to study. Namely the transversal case (which corresponds §3) and the nontransversal case (which corresponds to §4). Actually the general case is a mixture. In the case when $L_0 = \cdots = L_k$ the argument is explained in detail in [FOOO] Chapter 5 (and is outlined in [FOOO] Chapter 2 section 7). So we restrict ourselves to the following special case. Suppose we have mutually transversal three Lagrangian submanifolds L_0, L_1, L_2. We put $L_3 = L_2$. We also assume $\mathfrak{L}_3 = \mathfrak{L}_2$. Let $p_i \in L_i \cap L_{i+1}$, $i = 0, 1$ and let $p_2 \in L_1 \cap L_2$. Let $u_i \in \mathrm{Hom}(\mathfrak{L}_i, \mathfrak{L}_{i+1})_{p_i}$, $i = 0$, 1 and $u_2 \in \mathrm{Hom}(\mathfrak{L}_0, \mathfrak{L}_2)_{p_2}$. We want to show that the matrix element

$$\langle \mathfrak{m}_3((p_0, u_0), (p_1, u_1), [L_2]), (p_2, u_2)\rangle \tag{5.4}$$

is zero. (This is a part of Formula (5.2) to be shown.) We recall that the matrix element (5.4) is defined as follows. We consider the set of $(\varphi, (z_0, z_1, z_2), w)$ such that the following holds:

(5.5.1) $\varphi : D^2 \to M$ is a pseudoholomorphic map.

(5.5.2) z_0, z_1, w, $z_2 \in \partial D^2$. And they are in this order, with respect to the usual cyclic orientation on ∂D^2.

(5.5.3) $\varphi(\overline{z_0, z_1}) \in L_1$, $\varphi(\overline{z_1, z_2}) \in L_2$, $\varphi(\overline{z_2, z_0}) \in L_0$.

We divide the set of all such $(\varphi, (z_0, z_1, z_2), w)$ by an obvious $PSL(2; \mathbb{R})$ action and let

$$\mathcal{M}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1)$$

[3]We put this statement in the quote since it is not correct as it is stated. We will state a precise theorem later (Theorem 5.13).

[4]We put proof in the quote since it contains a gap. The correct proof of the theorem will be explained later. We need to go into the detail of the framework of Kuranishi structure to make it precise.

be the quotient space. (This definition is a special case of the definition in the last section.) We compactify it to $\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1)$. We define an evaluation map

$$ev : \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1) \to L_2$$

by

$$ev(\varphi, (z_0, z_1, z_2), w) = \varphi(w).$$

Now, by definition, the matrix element (5.4) is given by the order counted with sign of the (finite) set

$$\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1) \times_{L_2} L_2, \tag{5.6}$$

in the case when the moduli space and the fiber product is transversal and the virtual dimension of (5.6) is zero. We need to show that this number is zero.

To "prove" it we consider the moduli space

$$\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2) = \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 0)$$

defined in section 3. This moduli space consists of isomorphism classes of $(\varphi, (z_0, z_1, z_2))$ satisfying the same conditions as (5.5). So we can define a map

$$\pi : \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1) \to \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2) \tag{5.7}$$

by

$$\pi(\varphi, (z_0, z_1, z_2), w) = (\varphi, (z_0, z_1, z_2)).$$

It is easy to see that the fiber of the map (5.7) is one dimensional and is parametrized by the position of $w \in \overline{p_2 p_0}$.

It follows that

$$\begin{aligned} &\operatorname{Virdim} \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1) \\ = \quad &\operatorname{Virdim} \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2) + 1. \end{aligned}$$

One the other hand, by the definition of fiber product and virtual dimension, we have, in general

$$\operatorname{Virdim} A \times_B C = \operatorname{Virdim} A + \operatorname{Virdim} C - \operatorname{Virdim} B.$$

Hence

$$\begin{aligned} &\operatorname{Virdim} \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1) \times_{L_2} L_2 \\ &= \operatorname{Virdim} \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1). \end{aligned}$$

We are studying the case when the virtual dimension of (5.6) is zero. Therefore we have

$$\text{(5.8)} \qquad \operatorname{Virdim} \mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2) = -1.$$

(5.8) implies that (if everything is transversal then)

$$\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2)$$

is empty. Hence $\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2; 0, 0, 1)$ is also empty.

Thus we are done ? □

Actually there is a gap in the above argument. (5.8) implies that the space $\mathcal{CM}_3(L_0, L_1, L_2; p_0, p_1, p_2)$ is empty, only in case it is transversal. Using the theory of Kuranishi structure and multivalued perturbation, we can always make it transversal. However the trouble is whether we can make this perturbation to be compatible with the map π in (5.7). Namely if we take a perturbation compatible with (5.7), then it is not consistent with other maps we mention below.

The boundary of moduli spaces

$$\mathcal{CM}_m(L_0, \ldots, L_m; p_{(0)}, \ldots, p_{(m)}; \ell_1, \ldots, \ell_m)$$

can be described by products of similar moduli spaces (but with smaller m or ℓ_i). This consistency is essential to show A_∞ formula, Theorem 4.17. It is possible to make the perturbations consistent with this identification of the boundary to other moduli spaces. However, we cannot find, in general, a perturbation which is compatible to both. Namely in general there is no perturbation which is compatible with π and the identification of the boundary of the moduli space with the product of the other moduli spaces.

This problem looks rather technical. But it is quite delicate and essential point. It is explained in detail in [FOOO] Chapter 5 and is solved there. In [FOOO], the case we have only one Lagrangian submanifold is discussed. But the general case is completely parallel. So we only give a statement here.

To state the precise version of "Theorem 5.1", we need to introduce a notion of homotopy unit. To explain it, let us start with a geometric model of this notion. We first review the following well know notion in topology.

Definition 5.9. An *H-space* is a space X with a map $p : X \times X \to X$ and a base point $e \in X$ such that $p(x, e) = p(e, x) = x$.

Homotopy H space is (X, p, e) such that the restrictions of p to $\{e\} \times X \cong X$ and to $X \times \{e\} \cong X$ are homotopic to identity.

Lemma 5.10. *Homotopy H space is homotopy equivalent to an H space.*

Proof. Let us consider $X \cup [0,1]$ and identify 0 with e. We get a space X_+ which is obviously homotopy equivalent to X. By assumption there exists a homotopies

$$H_l : X \times [0,1] \to X, \quad H_r : X \times [0,1] \to X$$

such that

$$H_l(x,0) = p(e,x), \quad H_r(x,0) = p(x,e), \quad H_l(x,1) = H_r(x,1) = x.$$

We extend p to $X_+ \times X_+$ by putting

$$p(t,x) = H_l(x,t), \quad p(x,t) = H_r(x,t).$$

Then $(X_+, p, 1)$ is an H space. □

Now we translate the definition of X_+ in the proof above into the algebraic language and define homotopy unit as follows. Let $\mathcal{C}$ be an A_∞ category without unit. Let

$$\mathbf{e}_c \in \mathcal{C}^0(c,c)$$

be elements. We define

$$\mathcal{C}(c,c)_+ = \mathcal{C}(c,c) \oplus R \cdot \mathbf{e}_{c+} \oplus R \cdot \mathbf{f}_c$$

such that $\deg \mathbf{e}_{c+} = 0$, $\deg \mathbf{f}_c = -1$. We extend $\mathfrak{m}_1$ to $\mathcal{C}(c,c)_+$ by

$$\begin{aligned} \mathfrak{m}_1(\mathbf{f}_c) &= \mathbf{e}_{c+} - \mathbf{e}_c, \\ \mathfrak{m}_1(\mathbf{e}_{c+}) &= 0. \end{aligned}$$

We put $\mathcal{C}(c,c')_+ = \mathcal{C}(c,c')$. We then define $B\mathcal{C}_+$ as in §1.

Definition 5.11. We say that $\mathbf{e}_c$ is a *homotopy unit* if we can extend $\mathfrak{m}_k$ to $B\mathcal{C}_+$ so that it will become an A_∞ category with unit $\mathbf{e}_{c+}$.

We remark that, by the definition of unit, the extension of $\mathfrak{m}_k$ to $B\mathcal{C}_+$ is automatically determined in the case $\mathbf{e}_{c+}$ is included in the formula. So to extend $\mathfrak{m}_k$ we only need to define

$$\begin{aligned} \mathfrak{m}_k(x_{1,1}, \ldots, x_{1,\ell_1}, \mathbf{f}_c, x_{2,1}, \ldots, x_{2,\ell_2}, \\ , \mathbf{f}_c, x_{3,1}, \ldots, x_{m-1,\ell_{m-1}}, \mathbf{f}_c, x_{m,1}, \ldots, x_{m,\ell_m}), \end{aligned} \tag{5.12}$$

where $k = \sum_{i=1}^m (\ell_i + 1) - 1$. We can write down the equation they are supposed to satisfy by rewriting A_∞ formula. See [FOOO] §20 for such a formula.

We can define a notion of homotopy unit of filtered A_∞ category in the same way. Now we have:

Theorem 5.13. *$[L]$ is a homotopy unit of $\mathcal{LAG}(M,\omega)$.*

The proof is a straight forward generalization of the argument in [FOOO] §20. So we discuss it only very briefly. As we mentioned above, the reason that $[L]$ fails to be a unit (or in other words the reason (5.2) can be nonzero) is that the perturbation is not compatible with maps (5.7) (and its analogues). Let us restrict ourselves to the case of (5.4).

On the other hand, we can find another perturbation so that it is compatible with π but is not compatible with other operations. We now choose these two perturbations and take a homotopy between them. We now have a moduli space using this one parameter family of perturbations. Taking its fundamental chain (or counting its order with sign) we obtain a matrix element

$$\left\langle \mathfrak{m}_3((p_0,u_0),(p_1,u_1),\mathbf{f}_{[L_2]}),(p_2,u_2)\right\rangle .$$

The other operations in (5.12) can be defined in a similar way.

Using Theorem 5.13 we can modify our filtered A_∞ category $\mathcal{LAG}(M,\omega)$, (which has only a homotopy unit) so that it has an (exact) unit. From now one, we write $\mathcal{LAG}(M,\omega)$ for this modified one.

Chapter 2: Homological algebra of A_∞ category

§6. Twisted complex and derived A_∞ category

We have constructed our main example in §2 $\sim$ §5. So we go back to the continuation of Section 1 and further study algebraic formalism. In this section we follow Bondal-Kapranov [BoK] and Kontsevich [Ko1] (see also [Fu7] §16), to define twisted complex and derived A_∞ category.

In the case of abelian category, its derived category $\mathcal{C}$ is constructed in the following way. First, we consider the category $\mathcal{CC}$ of chain complex of objects of $\mathcal{C}$. We then consider the weak equivalence between objects of $\mathcal{CC}$. (Weak equivalence is a chain map which induces an isomorphism in cohomology.) We divide our category $\mathcal{CC}$ by weak equivalence and obtain $\mathbb{D}\mathcal{C}$. The category $\mathbb{D}\mathcal{C}$ is not an abelian category, but has an operation which is an algebraic version of the construction of the mapping cone (in topology). This construction of mapping cone gives a notion of distinguished triple on $\mathbb{D}\mathcal{C}$. Then $\mathbb{D}\mathcal{C}$ will become a triangulated category. (See [GM], [KaS], [Ha] etc. for detail.)

To generalize this construction to A_∞ category, we need to generalize the notion of chain complex (of elements of $\mathcal{C}$) to twisted complex.

Remark 6.1. The twisted complex is defined by Bondal-Kapranov [BoK] in the case when $\mathfrak{m}_k = 0$ for $k \geq 3$. Kontsevitch [Ko1] mentioned its generalization to A_∞ category and suggested its application to mirror symmetry. The twisted complex was also applied in [Fu7] to mirror symmetry. (The author learned some part of the contents of this section from P. Seidel's talks and papers [Se2], [Se3].)

Let $\mathcal{C}$ be an A_∞ category. We first increase the objects of it a bit in the following way. Let c be an object of $\mathcal{C}$ and k be an integer. We add an object $c[k]$ and put

$$\mathcal{C}(c'[\ell], c[k]) = \mathcal{C}(c', c)[\ell - k]. \tag{6.2}$$

We add all of $c[k]$ and define $\mathfrak{m}_k$ as follows. Let

$$x_i \in \mathcal{C}^{\deg x_i - k_{i-1} + k_i}(c_{i-1}, c_i). \cong \mathcal{C}^{\deg x_i}(c_{i-1}[k_{i-1}], c_i[k_i]).$$

We write $s^* x_i$ when we regard it as an element of $\mathcal{C}^{\deg x_i}(c_{i-1}[k_{i-1}], c_i[k_i])$, and write x_i when we regard it as an element of $\mathcal{C}^{\deg x_i + k_i - k_{i-1}}(c_{i-1}, c_i)$. Now we put

$$\mathfrak{m}_k(s^* x_1, \dots, s^* x_k) = (-1)^{k_0} s^* \mathfrak{m}_k(x_1, \dots, x_k). \tag{6.3}$$

Then we have

$$\begin{aligned}
& \sum_{1 \leq \ell < m \leq k} (-1)^{\deg s^* x_1 + \cdots + \deg s^* x_{\ell-1} + \ell - 1} \\
& \qquad \mathfrak{m}_{k-m+\ell+1}(s^* x_1, \dots, \mathfrak{m}_{m-\ell+1}(s^* x_\ell, \dots, s^* x_m), \dots, s^* x_k) \\
= & \sum_{1 \leq \ell < m \leq k} (-1)^{\deg x_1 + \cdots + \deg x_{\ell-1} + \ell - k_0 + k_{\ell-1} - 1} \\
& \qquad \mathfrak{m}_{k-m+\ell+1}(s^* x_1, \dots, \mathfrak{m}_{m-\ell+1}(s^* x_\ell, \dots, s^* x_m), \dots, s^* x_k) \\
= & \sum_{1 \leq \ell < m \leq k} (-1)^{\deg x_1 + \cdots + \deg x_{\ell-1} + \ell - 1} \\
& \qquad s^*(\mathfrak{m}_{k-m+\ell+1}(x_1, \dots, \mathfrak{m}_{m-\ell+1}(x_\ell, \dots, x_m), \dots, x_k)) \\
= & \ 0.
\end{aligned}$$

Thus A_∞ formula holds.

We add more objects to $\mathcal{C}$ as follows. Let us consider formally the direct sum $c_1 \oplus \cdots \oplus c_k$ and regard it as an object of our A_∞ category. We then define

$$\mathcal{C}(c_1 \oplus \cdots \oplus c_k, c'_1 \oplus \cdots \oplus c'_m) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{m} \mathcal{C}(c_i, c'_j).$$

We define operations $\mathfrak{m}_k$ in an obvious way.

From now on, we always extend the set of objects of A_∞ category in this way. Then, for any objects c, c', we have $c[k]$ and $c \oplus c'$. The object $c \oplus c'$ is the direct sum in the sense of additive category.

Actually in our main example, the object $c[k]$ exists already since it corresponds to the same Lagrangian submanifold, $U(1)$ bundle, etc. as c but with different grading $\tilde{s}$.

On the other hand, the direct sum $c \oplus c'$ (already) exists if c and c' correspond to Lagrangian submanifolds L, L' such that $L \cap L' = \emptyset$. In this case, $c \oplus c'$ corresponds $L \cup L'$ together with line bundle, framing, relative spin structure induced ones from c and c'.

However, in case $L \cap L' \neq \emptyset$, the disjoint union $L \cup L'$ is not an embedded Lagrangian submanifold and hence is not an objects of our A_∞ category. (If we include immersed Lagrangian submanifold as an objects, then again $c \oplus c'$ will be included (geometrically).) Here we simply add $c \oplus c'$ formally.

Now we start the construction of twisted complex and derived A_∞ category. Let $k_1 < k_2$ be integers. We consider a finitely many objects $c_i \in \mathfrak{Ob}(\mathcal{C})$, $(i = k_1, \dots, k_2)$ and elements

$$x_{i,j} \in \mathcal{C}[1]^0(c_i[i], c_j[j]) \cong \mathcal{C}^{1+i-j}(c_i, c_j)$$

for each $i < j$.

Definition 6.4. We say $(c_{k_1}, \dots, c_{k_2}; (x_{i,j}))$ is a *twisted complex* if for each $i < j$, the equation:

$$(6.5) \qquad \sum_{m \geq 1} \sum_{i=\ell_0 < \cdots < \ell_m = j} \mathfrak{m}_m(x_{\ell_0,\ell_1}, \dots, x_{\ell_{m-1},\ell_m}) = 0$$

is satisfied. The set of all twisted complex is denoted by $\mathfrak{Ob}(\mathcal{DC})$.

Example 6.6. If $x_{i,j} = 0$ for $j \neq i+1$ then (6.5) is

$$(6.7.1) \qquad \mathfrak{m}_2(x_{i,i+1}, x_{i+1,i+2}) = 0,$$

$$(6.7.2) \qquad \mathfrak{m}_1(x_{i,i+1}) = 0.$$

(6.7.2) implies that $x_{i,i+1}$ is a cocycle. Since $\mathfrak{m}_2$ is the composition of morphisms (upto sign), (6.7.1) implies that

$$0 \to c_{k_1} \xrightarrow{x_{k_1,k_1+1}} \cdots \xrightarrow{x_{k_2-1,k_2}} c_{k_2} \to 0$$

is a chain complex in our additive category $\mathcal{C}$. (Note $\deg x_{i,i+1} = 0$ if we regard it as an element of $\mathcal{C}(c_i, c_{i+1})$ (and not of $\mathcal{C}(c_i[i], c_{i+1}[i+1])$).

We are going to show that twisted complex consist a triangulated category. (See [GM], [Ha], [KaS] for the definition of triangulated category.)

We first define an additive category whose object is a twisted complex. Let $\mathbf{c}^{(1)} = (c^{(1)}_{k_1}, \ldots, c^{(1)}_{k_2}; (x^{(1)}_{i,j}))$, $\mathbf{c}^{(2)} = (c^{(2)}_{k_1}, \ldots, c^{(2)}_{k_2}; (x^{(2)}_{i,j}))$ be twisted complexes. We first define a morphism between them. We put

$$C^k(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}) = \bigoplus_{i,j} \mathcal{C}[1]^k(c^{(1)}_i[i], c^{(2)}_j[j]). \tag{6.8}$$

We define a boundary operator $\hat{\mathfrak{m}}_1$ on it by

$$\hat{\mathfrak{m}}_1(y_{i,j}) = (z_{i,j})$$

where

$$z_{i,j} = \sum_{i \le a < b \le j} \sum_{m_1 \ge 1, m_2 \ge 1} \sum_{i = \ell^{(1)}_0 < \cdots < \ell^{(1)}_{m_1} = a < b = \ell^{(2)}_0 < \cdots < \ell^{(2)}_{m_2} = j}$$
$$\mathfrak{m}_m\left(x^{(1)}_{\ell^{(1)}_0, \ell^{(1)}_1}, \ldots, x^{(1)}_{\ell^{(1)}_{m_1-1}, \ell^{(1)}_{m_1}}, y_{a,b}, x^{(2)}_{\ell^{(2)}_0, \ell^{(2)}_1}, \ldots, x^{(2)}_{\ell^{(2)}_{m_2-1}, \ell^{(2)}_{m_2}}\right).$$

Lemma 6.9. $\hat{\mathfrak{m}}_1 \circ \hat{\mathfrak{m}}_1 = 0$.

Since the formulas which we need to prove Lemma 6.9 are rather long, we develop some notations before starting calculation. We define

$$\mathbf{x}^{(1)}_{i,j} \in B\mathcal{C}[1]^0(c^{(1)}_i[i], c^{(1)}_j[j])$$

by

$$\mathbf{x}^{(1)}_{i,j} = \sum_{m \ge 1} \sum_{i = \ell_0 < \cdots < \ell_m = j} x^{(1)}_{\ell_0, \ell_1} \otimes \cdots \otimes x^{(1)}_{\ell_{m-1}, \ell_m}.$$

Then (6.5) is

$$\mathfrak{m}(\mathbf{x}^{(1)}_{i,j}) = \mathfrak{m}(\mathbf{x}^{(2)}_{i,j}) = 0. \tag{6.10}$$

Here and hereafter $\mathfrak{m} : B\mathcal{C}[1] \to B_1\mathcal{C}[1]$ is the operation which is $\mathfrak{m}_k$ on $B_k\mathcal{C}[1]$. The definition of $\hat{\mathfrak{m}}_1$ is rewritten as

$$z_{i,j} = \sum_{i \le a < b \le j} \mathfrak{m}(\mathbf{x}^{(1)}_{i,a} \otimes y_{a,b} \otimes \mathbf{x}^{(2)}_{b,j}). \tag{6.11}$$

We may further simplify the notation as follows. We consider $B\mathcal{C}[1]$ and define a product $\bullet$ on it as follows. Let $\mathbf{x} \in B\mathcal{C}[1](a, b)$, $\mathbf{y} \in B\mathcal{C}[1](c, d)$, Then:

$$\mathbf{x} \bullet \mathbf{y} = \begin{cases} \mathbf{x} \otimes \mathbf{y} & \text{if } b = c, \\ 0 & \text{if not.} \end{cases}$$

We put

$$\mathbf{x}_{i,*} = \sum_j \mathbf{x}_{i,j}, \quad \mathbf{x}_{*,j} = \sum_i \mathbf{x}_{i,j}, \quad \mathbf{x}_{*,*} = \sum_{i,j} \mathbf{x}_{i,j}.$$

We define $y_{i,*}, y_{*,j}, y_{*,*}$, $z_{i,*}, z_{*,j}, z_{*,*}$ in a similar way. Then (6.10) is equivalent to

$$\hat{d}(\mathbf{x}^{(0)}_{*,*}) = \hat{d}(\mathbf{x}^{(1)}_{*,*}) = 0. \tag{6.12}$$

(6.11) can be written as

$$z_{*,*} = \mathfrak{m}(\mathbf{x}^{(1)}_{*,*} \bullet y_{*,*} \bullet \mathbf{x}^{(2)}_{*,*}). \tag{6.13}$$

Proof of Lemma 6.9. We have

$$\hat{\mathfrak{m}}_1(\hat{\mathfrak{m}}_1(y_{i,j})) = (w_{i,j})$$

where

$$w_{*,*} = \mathfrak{m}(\mathbf{x}^{(1)}_{*,*} \bullet \mathfrak{m}(\mathbf{x}^{(1)}_{*,*} \bullet y_{*,*} \bullet \mathbf{x}^{(2)}_{*,*}) \bullet \mathbf{x}^{(2)}_{*,*}).$$

(Here we assumed $c_i \neq c_j$ for $i \neq j$ for simplicity.) Note the degree of $x_{i,j}$ are all zero (after shifted). Hence A_∞ formula implies

$$\begin{aligned}\mathfrak{m}(\mathbf{x}^{(1)}_{*,*} &\bullet \mathfrak{m}(\mathbf{x}^{(1)}_{*,*} \bullet y_{*,*} \bullet \mathbf{x}^{(2)}_{*,*}) \bullet \mathbf{x}^{(2)}_{*,*}) \\ &+\mathfrak{m}(\hat{d}(\mathbf{x}^{(1)}_{*,*}) \bullet y_{*,*} \bullet \mathbf{x}^{(2)}_{*,*}) + (-1)^{\deg y_{*,*}+1}\mathfrak{m}(\mathbf{x}^{(1)}_{*,*} \bullet y_{*,*} \bullet \hat{d}(\mathbf{x}^{(2)}_{*,*})) = 0.\end{aligned}$$

The second and the third terms vanish by (6.10). The lemma follows. □

Definition 6.14. We say an element of $\mathcal{C}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)})$ a *morphism in $\mathcal{DC}$*. It is said to be a *closed morphism* if it is $\hat{\mathfrak{m}}_1$ closed. We also put:

$$\mathbb{D}\mathcal{C}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}) = H^*(\mathcal{C}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}); \hat{\mathfrak{m}}_1).$$

We next define (higher) compositions:

$$\hat{\mathfrak{m}}_k : C(\mathbf{c}^{(0)}, \mathbf{c}^{(1)}) \otimes \cdots \otimes C(\mathbf{c}^{(k-1)}, \mathbf{c}^{(k)}) \to C(\mathbf{c}^{(0)}, \mathbf{c}^{(k)}) \tag{6.15}$$

by

$$\hat{\mathfrak{m}}_k(y^{(1)}_{*,*}, \ldots, y^{(k)}_{*,*}) = \mathfrak{m}(\mathbf{x}^{(0)}_{*,*} \bullet y^{(1)}_{*,*} \bullet \mathbf{x}^{(1)}_{*,*} \bullet \cdots y^{(k)}_{*,*} \bullet \mathbf{x}^{(k)}_{*,*}). \tag{6.16}$$

Theorem 6.17. *We have an A_∞ category $\mathcal{DC}$ such that the set of objects is $\mathfrak{Ob}(\mathcal{DC})$, the set of morphisms is given by* (6.8), *and the operations are given by* (6.16).

Proof. We are only to check the A_∞ formula. The proof of it is quite similar to the proof of Lemma 6.9 and is left to the reader. $\square$

We next are going to define the mapping cone in our category $\mathcal{DC}$. For this purpose, we introduce a systematic way to associate an object of $\mathcal{DC}$ to each object of $\mathcal{DDC}$. (More precisely we can define an A_∞ functor $\mathcal{DDC} \to \mathcal{DC}$. See the next section for the definition of A_∞ functor.)

Let $\mathbf{c}^{(k)} = ((c_i^{(k)}; i \in I^k), (x_{i,j}^{(k)}))$ be a twisted complex, where $k \in I$. (I, I^k be subsets of $\mathbb{Z}$ of the form $\{a, a+1, a+2, \dots, b\}$.) Let

$$\mathbf{y}^{(k,n)} \in (\mathcal{DC})[1]^0(\mathbf{c}^{(k)}[k], \mathbf{c}^{(n)}[n]) = \bigoplus_{\ell,m} \mathcal{C}[1]^0(c_\ell^{(k)}[k+\ell], c_m^{(n)}[n+m]).$$

We write its $\mathcal{C}[1]^0(c_\ell^{(k)}[k+\ell], c_m^{(n)}[n+m])$ component by $y_{\ell,m}^{(k,n)}$.

We assume $\mathfrak{c} = ((\mathbf{c}^{(k)}; k \in I), (\mathbf{y}^{(k,n)}))$ is a twisted complex of the A_∞ category $\mathcal{DC}$. We are going to construct an object $|\mathfrak{c}|$ of $\mathcal{DC}$. The construction is an analogy of the construction of double complex.

We put

$$|\mathfrak{c}|_i = \bigoplus_{k+\ell=i} c_\ell^{(k)}. \tag{6.18}$$

Let $i < j$. We are going to define $z_{i,j} \in \mathcal{C}^0[1](|\mathfrak{c}|_i[i], |\mathfrak{c}|_j[j])$. Let k, ℓ, m, n be integers such that $k + \ell = i$, $m + n = j$. We define $z_{i,j}^{(k,\ell,m,n)}$, the $\mathcal{C}(c_\ell^{(k)}, c_m^{(n)})$ component of $z_{i,j}$ by:

$$z_{i,j}^{(k,\ell,m,n)} = \begin{cases} y_{\ell,m}^{(k,n)} & \text{if } k < n \\ x_{\ell,m} & \text{if } k = n, \ell < m \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

Lemma 6.20. $|\mathfrak{c}| = (|\mathfrak{c}|_i, z_{i,j})$ *is a twisted complex.*

Proof. The condition (6.5) for $|\mathfrak{c}|$ reduces to the condition (6.5) for $\mathfrak{c}$ and ones for $\mathbf{c}_i$. $\square$

Suppose we have two twisted complexes $\mathbf{c}$, $\mathbf{c}'$ and $\mathbf{y} \in \mathcal{C}^0(\mathbf{c}, \mathbf{c}') = \mathcal{C}[1]^0(\mathbf{c}[-1], \mathbf{c}')$, be a closed morphism. We put $\mathbf{c}_{-1} = \mathbf{c}$ and $\mathbf{c}_0 = \mathbf{c}'$. Then $(\mathbf{c}_{-1}, \mathbf{c}_0; \mathbf{y})$ is an object of $\mathcal{DDC}$. Hence $|\mathbf{c}_{-1}, \mathbf{c}_0; \mathbf{y}|$ is an object of $\mathcal{DC}$. We call it the *mapping cone* of $\mathbf{y} : \mathbf{c} \to \mathbf{c}'$ and write it as $\mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})$.

Lemma 6.21. *There exists* $\mathcal{I} : \mathbf{c}' \to \mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})$, $\mathcal{J} : \mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y}) \to \mathbf{c}[1]$ *such that, for any twisted complex* $\mathbf{b}$, *we have a long exact sequence*

$$\begin{aligned} H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}), \mathfrak{m}_1) &\overset{R_{\mathbf{y}}}{\to} H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}'), \mathfrak{m}_1) \\ &\overset{\mathcal{I}}{\to} H^*(\mathcal{DC}(\mathbf{b}, \mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})), \mathfrak{m}_1) \\ &\overset{\mathcal{J}}{\to} H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}[1]), \mathfrak{m}_1) \\ &\overset{R_{\mathbf{y}}}{\to} H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}'[1]), \mathfrak{m}_1) \to \cdots \end{aligned}$$

Here $R_{\mathbf{y}}$ *is induced by the right multplication by* $\mathbf{y}$.

Proof. We remark that

$$\mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})_i = c_{i+1} \oplus c'_i.$$

Hence we can define $\mathcal{I}$ as an inclusion $c'_i \to c_{i+1} \oplus c'_i \cong \mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})_i$. And we define $\mathcal{J}$ as a projection $\mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})_i \cong c_{i+1} \oplus c'_i \to c_{i+1}$. It is easy to see that $\mathcal{I}$ and $\mathcal{J}$ are morphisms in $\mathcal{DC}$. By definition we have an exact sequence of chain complex

$$\begin{aligned} 0 \to (\mathcal{DC}(\mathbf{b}, \mathbf{c}'), \mathfrak{m}_1) &\overset{\mathcal{I}}{\to} (\mathcal{DC}(\mathbf{b}, \mathrm{Cone}(\mathbf{c}, \mathbf{c}'; \mathbf{y})), \mathfrak{m}_1) \\ &\overset{\mathcal{J}}{\to} (\mathcal{DC}(\mathbf{b}, \mathbf{c}[1]), \mathfrak{m}_1) \to 0. \end{aligned}$$

By definition, the operator $H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}), \mathfrak{m}_1) \to H^*(\mathcal{DC}(\mathbf{b}, \mathbf{c}'), \mathfrak{m}_1)$ of the associated long exact sequence is $R_{\mathbf{y}}$. □

So far we defined a notion corresponding to chain complex and to mapping cone, in the case of A_∞ category. Usually to construct the derived category from the category of chain complex, we need to divide it by weak equivalence. We define a similar notion, homotopy equivalence between two objects of A_∞ category.

Definition 6.22. Let $\mathcal{C}$ be an A_∞ category and $c, c' \in \mathfrak{Ob}(\mathcal{C})$. Let $x \in \mathcal{C}^0(c, c')$. We say that x is a *homotopy equivalence* if there exists $y \in \mathcal{C}^0(c', c)$ such that

(6.23.1) $\mathfrak{m}_1(x) = \mathfrak{m}_1(y) = 0$.

(6.23.2) $\mathfrak{m}_2(y, x) - \mathbf{e}_c \in \mathrm{Im}\, \mathfrak{m}_1$, $\mathfrak{m}_2(x, y) - \mathbf{e}_{c'} \in \mathrm{Im}\, \mathfrak{m}_1$.

Two objects $c, c' \in \mathfrak{Ob}(\mathcal{C})$ are said to be *homotopy equivalent* to each other if there exists a homotopy equivalence between them.

Lemma 6.24. *Let* $c, c' \in \mathfrak{Ob}(\mathcal{C})$, $x \in \mathcal{C}^0(c, c')$ *with* $\mathfrak{m}_1(x) = 0$. *Then the following five conditions are equivalent to each other.*

(6.25.1) *x is a homotopy equivalence.*

(6.25.2) *For each $b \in \mathfrak{Ob}(\mathcal{C})$ the map $R_x : \mathcal{C}(b,c) \to \mathcal{C}(b,c')$, $z \mapsto \mathfrak{m}_2(z,x)$ induces an isomorphism on homology.*
(6.25.3) *The map $R_x : \mathcal{C}(c',c) \to \mathcal{C}(c',c')$ induces a surjection on homology and $R_x : \mathcal{C}(c,c) \to \mathcal{C}(c,c')$ induces an injection on homology.*
(6.25.4) *For each $b \in \mathfrak{Ob}(\mathcal{C})$ the map $L_x : \mathcal{C}(c',b) \to \mathcal{C}(c,b)$, $z \mapsto \mathfrak{m}_2(x,z)$ induces an isomorphism on homology.*
(6.25.5) *The map $L_x : \mathcal{C}(c',c) \to \mathcal{C}(c,c)$ induces a surjection on homology and $L_x : \mathcal{C}(c',c') \to \mathcal{C}(c,c')$ induces an injection on homology.*

Proof. (6.25.1)$\Rightarrow$(6.25.2): $\mathfrak{m}_1(x) = 0$ together with A_∞ formulae implies that R_x is a chain map. Let y be as in (6.23). Then it is easy to see that $R_x \circ R_y$, $R_y \circ R_x$ induces identity in homology. (6.25.2) follows.
(6.25.2)$\Rightarrow$(6.25.3): Obvious.
(6.25.3)$\Rightarrow$(6.25.1): By (6.25.3) there exists y such that $\mathfrak{m}_2(y,x) - \mathbf{e}_{c'} \in \operatorname{Im}\mathfrak{m}_1$. Hence

$$\begin{aligned}
\mathfrak{m}_2(\mathfrak{m}_2(x,y),x) &\equiv -(-1)^{\deg' x}\mathfrak{m}_2(x,\mathfrak{m}_2(y,x)) \mod \operatorname{Im}\mathfrak{m}_1 \\
&\equiv (-1)^{\deg x}\mathfrak{m}_2(x,\mathbf{e}_{c'}) \mod \operatorname{Im}\mathfrak{m}_1 \\
&\equiv x \mod \operatorname{Im}\mathfrak{m}_1 \\
&\equiv \mathfrak{m}_2(\mathbf{e}_c,x) \mod \operatorname{Im}\mathfrak{m}_1.
\end{aligned}$$

Hence $\mathfrak{m}_2(x,y) \equiv \mathbf{e}_c \mod \operatorname{Im}\mathfrak{m}_1$.

The proof of equivalence between (6.25.1), (6.25.4), (6.25.5) is similar. □

Lemma 6.24 implies that the composition of homotopy equivalences is a homotopy equivalence.

We define a category $\mathbb{D}\mathcal{C}$ as follows. (It is a category in the usual sense and is an additive category.) Its object is a homotopy equivalence class of the objects of A_∞ category $\mathcal{DC}$. Morphism between them is defined by Definition 6.14. By (6.25.2), (6.25.4), a homotopy equivalence induces an isomorphism on $\mathrm{Hom}(\mathbf{c}^{(1)},\mathbf{c}^{(2)})$ in Definition 6.14 hence the set of morphisms is well defined. The composition of the morphisms is induced by $\mathfrak{m}_2$. The A_∞ formula implies that the composition in the homology level is (exactly) associative.

(Actually to perform this construction in a rigorous way we need to define and use a notion of quotient category. Since it is standard we do not discuss it. See for example [GM], [KaS].)

We next define distinguished triangles in $\mathbb{D}\mathcal{C}$ and prove that $\mathbb{D}\mathcal{C}$ will be a triangulated category. We use the notion of mapping cone for this purpose. To do so, we need to show that the homotopy equivalence classes of $\mathrm{Cone}(\mathbf{c}^{(1)},\mathbf{c}^{(2)};\mathbf{y})$ depends only of the homotopy class of $(\mathbf{c}^{(1)},\mathbf{c}^{(2)};\mathbf{y})$. More precisely we prove the following.

Lemma 6.26. *Let* $\mathbf{u}_1 \in \mathcal{DC}(\mathbf{c}^{(1)}, \mathbf{c}'^{(1)})$, $\mathbf{u}_2 \in \mathcal{DC}(\mathbf{c}^{(2)}, \mathbf{c}'^{(2)})$ *be homotopy equivalences. We assume* $\mathfrak{m}_2(\mathbf{u}_1, \mathbf{y}') - \mathfrak{m}_2(\mathbf{y}, \mathbf{u}_2) \in \operatorname{Im} \mathfrak{m}_1$. *Then* $\operatorname{Cone}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y})$ *is homotopy equivalent to* $\operatorname{Cone}(\mathbf{c}'^{(1)}, \mathbf{c}'^{(2)}; \mathbf{y}')$ *in the* A_∞ *category* $\mathcal{DC}$.

Proof. We put

$$\mathfrak{m}_2(\mathbf{u}_1, \mathbf{y}') - \mathfrak{m}_2(\mathbf{y}, \mathbf{u}_2) = \mathfrak{m}_1(\mathbf{z}).$$

Then $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{z})$ defines a closed morphism

$$\mathcal{DC}((\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y}), (\mathbf{c}'^{(1)}, \mathbf{c}'^{(2)}; \mathbf{y}')).$$

Hence we have a closed morphism

$$\operatorname{Cone}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y}) \to \operatorname{Cone}(\mathbf{c}'^{(1)}, \mathbf{c}'^{(2)}; \mathbf{y}'). \tag{6.27}$$

Then we obtain a commutative diagram:

$$\begin{tikzcd}
\mathbf{c}^{(1)} \arrow[r, "\mathbf{y}"] \arrow[d] & \mathbf{c}^{(2)} \arrow[r] \arrow[d] & \mathit{Cone}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y}) \arrow[d] \\
\mathbf{c}'^{(1)} \arrow[r, "\mathbf{y}'"] & \mathbf{c}'^{(2)} \arrow[r] & \mathit{Cone}(\mathbf{c}'^{(1)}, \mathbf{c}'^{(2)}; \mathbf{y}')
\end{tikzcd}$$

Diagram 6.1.

comparing exact sequences in Lemma 6.21. Then Lemma 6.21 and five lemma implies that (6.27) induces an isomorphism

$$\mathcal{DC}(\mathbf{b}, \operatorname{Cone}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y})) \cong \mathcal{DC}(\mathbf{b}, \operatorname{Cone}(\mathbf{c}'^{(1)}, \mathbf{c}'^{(2)}; \mathbf{y}'))$$

for any $\mathbf{b}$. The lemma now follows from Lemma 6.24. □

Using Lemma 6.26 we can define a notion of distinguished triangle in $\mathbb{D}\mathcal{C}$ as follows.

Definition 6.28.

$$[\mathbf{c}^{(1)}] \to [\mathbf{c}^{(2)}] \to [\operatorname{Cone}(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}; \mathbf{y})] \to [\mathbf{c}^{(1)}[1]]$$

is said to be a *distinguished triangle*.

Theorem 6.29. $\mathbb{D}\mathcal{C}$ *is a triangulated category.*

Figure 6.1.

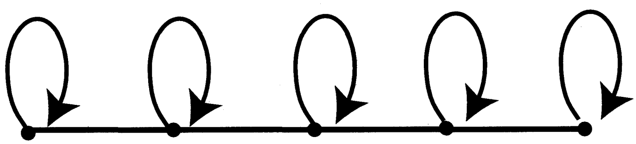

Figure 6.2.

We omit the proof, which is rather a straightforward check of the axiom.

Definition 6.30. $\mathbb{D}\mathcal{C}$ is called the *derived A_∞ category* of our A_∞ category $\mathcal{C}$.

In case we start with filtered A_∞ category $\mathcal{C}$ we first construct an A_∞ category as in Definition 1.14 and then construct derived A_∞ category. We call it also the derived A_∞ category of $\mathcal{C}$ and write it also as $\mathbb{D}\mathcal{C}$.

Note that the objects of derived A_∞ category of a filtered A_∞ category $\mathcal{C}$ can be regarded as $\mathbf{c} = ((c_i; i \in \{k_1, k_1+1, \dots, k_2\}), (x_{i,j}^{(k)}))$. But, in this case, $x_{i,i}$ may be nonzero, ($x_{i,i}$ should be in $\mathcal{C}[1]^0(c_i, c_i)$ and moreover $x_{i,i} \in \cup_{\lambda>0} F^\lambda \mathcal{C}[1]^0(c_i, c_i)$.) $x_{i,j}$ is supposed to satisfy an equation

$$\sum_m \sum_{\ell_0,\dots,\ell_m} \mathfrak{m}_m(x_{\ell_0,\ell_1}, \dots, x_{\ell_{m-1},\ell_m}) = 0, \tag{6.31}$$

which is similar to (6.5). However, in (6.31), $m = 0, 1, \dots$, $\ell_0 \le \ell_1 \le \cdots \le \ell_m$. (Namely the case $\ell_i = \ell_{i+1}$ is included.) Actually, in the case $\ell_0 = \cdots = \ell_m = \ell$, (6.31) is

$$\mathfrak{m}(e^{x_{\ell,\ell}}) = 0,$$

that is the definition $x_{\ell,\ell}$ to be a bounding chain.

An element $\mathbf{c} = ((c_i; i \in \{k_1, k_1+1, \dots, k_2\}), (x_{i,j}^{(k)}; i \le j))$ can be described by Figure 6.1 above. On the other hand, an element $((c_i; i \in \{k_1, k_1+1, \dots, k_2\}), (x_{i,j}^{(k)}; i < j))$ can be described by Figure 6.2 above.

§7. A_∞ functor and natural transformation

In sections 6, we consider more general objects than the original objects (that is the twisted complex) of our A_∞ category $\mathcal{C}$. In this section, we further generalize it and define a notion, A_∞ functor $\mathcal{C} \to \mathcal{CH}$, which was defined in [Fu4] over $\mathbb{Z}_2$ coefficient. We simplify the description of [Fu4] using Bar complex and also we put here the precise sign, which was not discussed in [Fu4].

Definition 7.1. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be A_∞ categories. An A_∞ *functor* $\mathcal{F}$ from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a collection of $\mathcal{F}_k$, $k = 1, 2, \dots$ such that

(7.2.1) $\mathcal{F}_0 : \mathfrak{Ob}(\mathcal{C}_1) \to \mathfrak{Ob}(\mathcal{C}_2)$ is a map.
(7.2.2) For $c_1, c_2 \in \mathfrak{Ob}(\mathcal{C}_1)$,

$$\mathcal{F}_k(c_1, c_2) : B_k\mathcal{C}_1[1](c_1, c_2) \to \mathcal{C}_2[1](\mathcal{F}_0(c_1), \mathcal{F}_0(c_2))$$

is a homomorphism of degree 0.
(7.2.3) We extend $\mathcal{F}_k(c_1, c_2)$ to a coalgebra homomorphism

$$\hat{\mathcal{F}}(c_1, c_2) : B\mathcal{C}_1[1](c_1, c_2) \to B\mathcal{C}_2[1](\mathcal{F}_0(c_1), \mathcal{F}_0(c_2)). \tag{7.3.1}$$

Then $\hat{\mathcal{F}}(c_1, c_2)$ is a chain map with respect to the boundary operator d in Definition 1.1.

Note

$$\begin{aligned}
&\hat{\mathcal{F}}(x_1 \otimes \cdots \otimes x_k) \\
&= \sum_m \sum_{0=\ell_1<\ell_2<\cdots<\ell_m=k} \mathcal{F}_{\ell_2-\ell_1-1}(x_{\ell_1+1} \otimes \cdots \otimes x_{\ell_2}) \otimes \cdots \\
&\qquad \cdots \otimes \mathcal{F}_{\ell_m-\ell_{m-1}-1}(x_{\ell_{m-1}+1} \otimes \cdots \otimes x_{\ell_m}).
\end{aligned}$$

Our homomorphism $\hat{\mathcal{F}}$ on $B_0\mathcal{C}_1[1]$ is defined as follows. We remark

$$B_0\mathcal{C}_1[1](c_1, c_2) \cong \begin{cases} R & c_1 = c_2, \\ 0 & c_1 \neq c_2. \end{cases}$$

We put

$$\hat{\mathcal{F}}(x) = \begin{cases} x & \text{if } x \in B_0\mathcal{C}_1[1](c, c), \\ 0 & \text{if } x \in B_0\mathcal{C}_1[1](c_1, c_2), c_1 \neq c_2. \end{cases} \tag{7.3.2}$$

We next give an example of A_∞ functor, that is a representable functor. We first define an A_∞ category $\mathcal{CH}$ for this purpose.

Definition 7.4. $\mathfrak{Ob}(\mathcal{CH})$ is the set of (all) chain complexes of free R nodules[5]. Let $(C,d),(C',d)\in\mathfrak{Ob}(\mathcal{CH})$. Then

$$\mathcal{CH}^k((C,d),(C',d)) = \bigoplus_{\ell}\mathrm{Hom}_R(C^{\ell},C'^{\ell+k}).$$

We define

$$\begin{aligned}
&(7.5.1) &\quad \mathfrak{m}_1(x) &= d\circ x + (-1)^{\deg x+1}x\circ d,\\
&(7.5.2) &\quad \mathfrak{m}_2(x,y) &= (-1)^{\deg x(\deg y+1)}y\circ x.
\end{aligned}$$

We put $\mathfrak{m}_k = 0$ for $k\geq 3$.

Remark 7.6. The sign in (7.5.2) is the same as one we need to regard differential graded algebra as an A_∞ algebra. (See Example-Lemma 1.7.)

Proposition 7.7. *$\mathcal{CH}$ is an A_∞ category.*

Proof. It is easy to check $\mathfrak{m}_1\circ\mathfrak{m}_1 = 0$. We calculate

$$\begin{aligned}
(\mathfrak{m}_1\circ\mathfrak{m}_2)(x,y) &= (-1)^{\deg x(\deg y+1)}d\circ y\circ x\\
&\quad +(-1)^{\deg x+\deg y+1}(-1)^{\deg x(\deg y+1)}y\circ x\circ d\\
&= (-1)^{\deg x\deg y+\deg x}d\circ y\circ x\\
&\quad +(-1)^{\deg x(\deg y+1)+\deg y+1}y\circ d\circ x\\
&\quad -(-1)^{(\deg x+1)(\deg y+1)}y\circ d\circ x\\
&\quad -(-1)^{\deg x+\deg y}(-1)^{\deg x(\deg y+1)}y\circ x\circ d\\
&= -(-1)^{\deg x+1}\mathfrak{m}_2(x,\mathfrak{m}_1(y)) - \mathfrak{m}_2(\mathfrak{m}_1(x),y).
\end{aligned}$$

This implies $\mathfrak{m}\circ\hat{d} = 0$ on B_2. We next calculate

$$\begin{aligned}
&\mathfrak{m}_2(\mathfrak{m}_2(x,y),z) + (-1)^{\deg x+1}\mathfrak{m}_2(x,\mathfrak{m}_2(y,z))\\
&\quad = (-1)^{\deg x(\deg y+1)+(\deg x+\deg y)(\deg z+1)}z\circ(y\circ x)\\
&\quad\quad +(-1)^{\deg x+1+\deg y(\deg z+1)+\deg x(\deg y+\deg z+1)}(z\circ y)\circ x\\
&\quad = 0,
\end{aligned}$$

which is the third and the last part of the A_∞ formulae to be checked. □

We next define:

[5]To avoid Russell paradox in set theory, we fix a sufficiently large set (a universe) and consider only free R modules contained in this set.

Definition 7.8. Let $\mathcal{C}$ be an A_∞ category. We define its *opposite* A_∞ *category* $\mathcal{C}^o$ as follows.
(7.9.1) $\mathfrak{Ob}(\mathcal{C}^o) = \mathfrak{Ob}(\mathcal{C})$.
(7.9.2) Let $c, c' \in \mathfrak{Ob}(\mathcal{C}^o) = \mathfrak{Ob}(\mathcal{C})$. We put $\mathcal{C}^o(c,c') = \mathcal{C}(c',c)$.
(7.9.3) We define (higher) composition operators $\mathfrak{m}_k^o$ of $\mathcal{C}^o$ by:

$$\mathfrak{m}_k^o(x_1,\dots,x_k) = (-1)^\epsilon \mathfrak{m}_k(x_k,\dots,x_1),$$

where

$$\epsilon = \sum_{1\le i<j\le k} (\deg x_i + 1)(\deg x_j + 1) + 1.$$

Lemma 7.10. *$\mathcal{C}^o$ is an A_∞ category.*

Proof. First we introduce some notations to simplify the formula. We put $\mathbf{x} = x_1 \otimes \cdots \otimes x_k \in B_k\mathcal{C}$ and

$$\Delta^{m-1}\mathbf{x} = \sum_a \mathbf{x}_a^{(1)} \otimes \cdots \otimes \mathbf{x}_a^{(m)}. \tag{7.11}$$

Here

$$\Delta^{m-1} = \cdots \circ (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1) \circ \Delta.$$

Let

$$\deg \mathbf{x} = \deg x_1 + \cdots + \deg x_k \tag{7.12.1}$$

be the degree of $\mathbf{x}$ and

$$\deg' \mathbf{x} = \deg x_1 + \cdots + \deg x_k + k \tag{7.12.2}$$

be its degree after shifted. We use notations (7.11), (7.12.1), (7.12.2) frequently for the rest of this article. We put

$$\mathbf{x}^{op} = x_k \otimes \cdots \otimes x_1, \tag{7.13.1}$$

and

$$\epsilon(\mathbf{x}) = \sum_{1\le i<j\le k} (\deg x_i + 1)(\deg x_j + 1). \tag{7.13.2}$$

The A_∞ formula for $\mathfrak{m}$ can be written as

$$\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \mathfrak{m}(\mathbf{x}_a^{(1)}, \mathfrak{m}(\mathbf{x}_a^{(2)}), \mathbf{x}_a^{(3)}) = 0. \tag{7.14}$$

We have

$$\text{(7.15)}\quad \begin{aligned}\mathfrak{m}(\mathbf{x}_a^{(1)},\mathfrak{m}(\mathbf{x}_a^{(2)}),\mathbf{x}_a^{(3)}) &= (-1)^{\epsilon(\mathbf{x}_a^{(1)})+\epsilon(\mathbf{x}_a^{(2)})+\epsilon(\mathbf{x}_a^{(3)})+\epsilon_1(a)} \\ &\qquad \mathfrak{m}^o(\mathbf{x}_a^{(3)op},\mathfrak{m}^o(\mathbf{x}_a^{(2)op}),\mathbf{x}_a^{(1)op}),\end{aligned}$$

where $\epsilon(\mathbf{x}_a^{(j)})$ are as in (7.13.2) and

$$\epsilon_1(a) = (\deg'\mathbf{x}_a^{(1)} + \deg'\mathbf{x}_a^{(3)})(\deg'\mathbf{x}_a^{(2)}+1) + \deg'\mathbf{x}_a^{(1)}\deg'\mathbf{x}_a^{(3)}.$$

We remark that

$$\begin{aligned}\epsilon(\mathbf{x}_a^{(1)}) + \epsilon(\mathbf{x}_a^{(2)}) + \epsilon(\mathbf{x}_a^{(3)}) + (\deg'\mathbf{x}_a^{(1)} + \deg'\mathbf{x}_a^{(3)})\deg'\mathbf{x}_a^{(2)}& \\ + \deg'\mathbf{x}_a^{(1)}\deg'\mathbf{x}_a^{(3)} &= \epsilon(\mathbf{x}),\end{aligned}$$

and is independent of a. Hence (7.14) and (7.15) imply

$$\sum_a (-1)^{\deg'\mathbf{x}_a^{(3)op}}\mathfrak{m}^o(\mathbf{x}_a^{(3)op},\mathfrak{m}^o(\mathbf{x}_a^{(2)op}),\mathbf{x}_a^{(1)op}) = 0.$$

This is the A_∞ formula of $\mathfrak{m}^o$ to be checked. □

Definition 7.16. Let $\mathcal{C}$ be an A_∞ category and $c \in \mathfrak{Ob}(\mathcal{C})$. We construct an A_∞ functor $\mathcal{F}^c = \mathrm{Hom}(\cdot, c) : \mathcal{C} \to \mathcal{CH}^o$ as follows.
(7.17.1) $\mathcal{F}^c_0(b) = \mathcal{C}(b,c)$. (We take $\mathfrak{m}_1$ as the boundary operator.)
(7.17.2) $\mathcal{F}^c_k(x_1,\dots,x_k)(y) = \mathfrak{m}_{k+1}(x_1,\dots,x_k,y)$. Here

$$y \in \mathcal{C}(b,c), b_1,\dots,b_{k+1} = b \in \mathfrak{Ob}(\mathcal{C}), x_i \in \mathcal{C}(b_i,b_{i+1}).$$

Proposition 7.18. *$\mathcal{F}^c$ is an A_∞ functor.*

Proof. We calculate

$$\begin{aligned}&\mathfrak{m}_2^o(\mathcal{F}(\mathbf{x}_a^{(1)}),\mathcal{F}(\mathbf{x}_a^{(2)})) \\ = &-(-1)^{(\deg\mathcal{F}(\mathbf{x}_a^{(1)})+1)(\deg\mathcal{F}(\mathbf{x}_a^{(2)})+1)}\mathfrak{m}_2(\mathcal{F}(\mathbf{x}_a^{(2)}),\mathcal{F}(\mathbf{x}_a^{(1)})) \\ = &-(-1)^{(\deg\mathcal{F}(\mathbf{x}_a^{(1)})+1)(\deg\mathcal{F}(\mathbf{x}_a^{(2)})+1)+(\deg\mathcal{F}(\mathbf{x}_a^{(1)})+1)\deg\mathcal{F}(\mathbf{x}_a^{(2)})} \\ &\mathcal{F}(\mathbf{x}_a^{(1)})\circ\mathcal{F}(\mathbf{x}_a^{(2)}) \\ = &-(-1)^{\deg'\mathbf{x}_a^{(1)}}\mathcal{F}(\mathbf{x}_a^{(1)})\circ\mathcal{F}(\mathbf{x}_a^{(2)}).\end{aligned}$$

We recall $\mathfrak{m}_k = 0$ for $k \ge 3$ in $\mathcal{CH}$. Hence the condition that $\hat{\mathcal{F}}$ is a chain map is

$$\text{(7.19)}\quad \begin{aligned}0 = &\sum_a (-1)^{\deg'\mathbf{x}_a^{(1)}}\mathcal{F}(\mathbf{x}_a^{(1)})\circ\mathcal{F}(\mathbf{x}_a^{(2)}) \\ &+(-1)^{\deg'\mathbf{x}}\mathcal{F}(\mathbf{x})\circ\mathfrak{m}_1 + \mathfrak{m}_1\circ\mathcal{F}(\mathbf{x}) \\ &+\sum_a (-1)^{\deg'\mathbf{x}_a^{(1)}}\mathcal{F}(\mathbf{x}_a^{(1)},\mathfrak{m}(\mathbf{x}_a^{(2)}),\mathbf{x}_a^{(3)}).\end{aligned}$$

(We use also $\mathfrak{m}_1^o = -\mathfrak{m}_1$ to deduce (7.19).) We plug in y to the right hand side of (7.19). Then the first term of the right hand side will be

$$\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \mathfrak{m}(\mathbf{x}_a^{(1)}, \mathfrak{m}(\mathbf{x}_a^{(2)}, y)). \tag{7.20}$$

The second and the third terms of the right hand side will be

$$(-1)^{\deg' \mathbf{x}} \mathfrak{m}(\mathbf{x}, \mathfrak{m}_1(y)) + \mathfrak{m}_1(\mathfrak{m}(\mathbf{x}, y)). \tag{7.21}$$

The fourth term will be

$$\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \mathfrak{m}(\mathbf{x}_a^{(1)}, \mathfrak{m}(\mathbf{x}_a^{(2)}), \mathbf{x}_a^{(3)}, y). \tag{7.22}$$

(7.20) + (7.21) + (7.22) = 0 is the A_∞ formula for $\mathfrak{m}$. We thus proved (7.19). □

We next define a similar but a bit different A_∞ functor $\mathfrak{Rep}_0(c) : \mathcal{C}^o \to \mathcal{CH}$. (At this stage $\mathfrak{Rep}_0(c)$ is just a symbol. We will define $\mathfrak{Rep}_k$ in section 9.) For this purpose, we prove the following.

Definition-Lemma 7.23. *For each A_∞ functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, we can construct its opposite A_∞ functor $\mathcal{F}^o : \mathcal{C}_1^o \to \mathcal{C}_2^o$ as follows.*
(7.24.1) $\mathcal{F}_0^o = \mathcal{F}_0$.
(7.24.2) $\mathcal{F}_k^o(\mathbf{x}) = (-1)^{\epsilon(\mathbf{x})}\mathcal{F}_k(\mathbf{x}^{op})$. *Here we use notation* (7.13).

Proof. We need to check

$$\begin{aligned}&\textstyle\sum_\ell \sum_a \mathfrak{m}_\ell^o(\mathcal{F}^o(\mathbf{x}_a^{(1)}), \dots, \mathcal{F}^o(\mathbf{x}_a^{(\ell)}))\\ &= \textstyle\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \mathcal{F}^o(\mathbf{x}_a^{(1)}, \mathfrak{m}^o(\mathbf{x}_a^{(2)}), \mathbf{x}_a^{(3)}).\end{aligned} \tag{7.25}$$

The left hand side of (7.25) is

$$\sum_\ell \sum_a (-1)^{\epsilon(\mathbf{x})+1} \mathfrak{m}_\ell(\mathcal{F}(\mathbf{x}_a^{(\ell)op}), \dots, \mathcal{F}(\mathbf{x}_a^{(1)op})). \tag{7.26}$$

In a way similar to the proof of Lemma 6.8, we can check that the right hand side of (7.25) is

$$\sum_a (-1)^{\deg' \mathbf{x}_a^{(3)}} \mathcal{F}(\mathbf{x}_a^{(3)op}, \mathfrak{m}(\mathbf{x}_a^{(2)op}), \mathbf{x}_a^{(1)op}). \tag{7.27}$$

(7.26) = (7.27) is the condition that $\mathcal{F}$ is an A_∞ functor. □

In view of (7.24) and (7.17), we can define an A_∞ functor $\mathfrak{Rep}_0(c) : \mathcal{C}^o \to \mathcal{CH}$ as follows.

Definition 7.28.

$$\mathfrak{Rep}_0(c)_0(b_0) = \mathcal{C}(b_0, c). \tag{7.29.1}$$

Let $\mathbf{x} \in B_k\mathcal{C}^o(b_0, b_k) = B_k\mathcal{C}(b_k, b_0)$, $y \in \mathfrak{Rep}_0(c)_0(b_0) = \mathcal{C}(c, b_0)$. Then

$$\mathfrak{Rep}_0(c)_k(\mathbf{x})(y) = (-1)^{\epsilon(\mathbf{x})}\mathcal{F}^c(\mathbf{x}^{op})(y) = (-1)^{\epsilon(\mathbf{x})}\mathfrak{m}(\mathbf{x}^{op}, y). \tag{7.29.2}$$

We next apply the construction of $\mathcal{F}^c$ and $\mathfrak{Rep}_0(c)$ to the opposite A_∞ category $\mathcal{C}^o$ and define ${}^c\mathcal{F}$, $\mathfrak{OpRep}_0(c)$ as follows.

Definition 7.30. ${}^c\mathcal{F} : \mathcal{C} \to \mathcal{CH}$ is defined by

(7.31.1) ${}^c\mathcal{F}_0(b_0) = \mathcal{C}(c, b_0)$.

(7.31.2) ${}^c\mathcal{F}_k(\mathbf{x})(y) = \mathfrak{m}^o_{k+1}(\mathbf{x}, y) = -(-1)^{\epsilon(\mathbf{x})+\deg' y \deg' \mathbf{x}}\mathfrak{m}_{k+1}(y, \mathbf{x}^{op})$,

where $\mathbf{x} \in B_k\mathcal{C}^o(b_0, b_k) = B_k\mathcal{C}(b_k, b_0)$, $y \in {}^c\mathcal{F}_0(b_0) \in \mathcal{C}(c, b_0)$.

$\mathfrak{OpRep}_0(c) : \mathcal{C}^o \to \mathcal{CH}^o$ is defined by

(7.32.1) $\mathfrak{OpRep}_0(c)(b_0) = \mathcal{C}(c, b_0)$.

(7.32.2) $$\begin{aligned}\mathfrak{OpRep}_k(\mathbf{x})(y) &= (-1)^{\epsilon(\mathbf{x})}\, {}^c\mathcal{F}_k(\mathbf{x})(y) \\ &= -(-1)^{\deg' y \deg' \mathbf{x}}\mathfrak{m}_{k+1}(y, \mathbf{x}^{op}).\end{aligned}$$

It follows from construction that ${}^c\mathcal{F}$ and $\mathfrak{Rep}_0(c)$ are A_∞ functors.

Definition 7.33. We say an A_∞ functor : $\mathcal{C} \to \mathcal{CH}^o$, $\mathcal{C}^o \to \mathcal{CH}$, $\mathcal{C} \to \mathcal{CH}$, $\mathcal{C}^o \to \mathcal{CH}^o$ to be *representable* if it is homotopic to $\mathcal{F}^c$, $\mathfrak{Rep}_0(c)$, ${}^c\mathcal{F}$ and $\mathfrak{OpRep}_0(c)$, respectively. (Homotpy between A_∞ functors will be defined in the next section.)

We next generalize the constructions above to the case when c is a twisted complex. For this purpose we first define a composition of two A_∞ functors. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be A_∞ categories and $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ be A_∞ functors.

Definition 7.34. The composition $\mathcal{G} \circ \mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_3$ is defined as follows.

$$(\mathcal{G} \circ \mathcal{F})_0 = \mathcal{G}_0 \circ \mathcal{F}_0 \tag{7.35.1}$$

$$\begin{aligned}\widehat{\mathcal{G} \circ \mathcal{F}}(c_1, c_2) &= \widehat{\mathcal{G}}(\mathcal{F}_0(c_1), \mathcal{F}_0(c_2)) \circ \widehat{\mathcal{F}}(c_1, c_2) \\ &: B\mathcal{C}_1(c_1, c_2) \to B\mathcal{C}_3(\mathcal{G}_0(\mathcal{F}_0(c_1)), \mathcal{G}_0(\mathcal{F}_0(c_2))).\end{aligned} \tag{7.35.2}$$

It is easy to see that the composition is an A_∞ functor.

We remark that there is an obvious A_∞ functor $\mathcal{C} \to \mathcal{DC}$.

Definition 7.36. Let $\mathbf{c}$ be an twisted complex of A_∞ category $\mathcal{C}$. We then consider the composition

$$\mathcal{C} \to \mathcal{DC} \xrightarrow{\mathcal{F}^{\mathbf{c}}} \mathcal{CH}^o \tag{7.37}$$

where $\mathcal{F}^{\mathbf{c}}$ is defined by applying Definition 6.15 to A_∞ category $\mathcal{DC}$. We write also $\mathcal{F}^{\mathbf{c}}$ the composition (7.37). Similarly we define ${}^{\mathbf{c}}\mathcal{F}$ as the composition

$$\mathcal{C}^o \to (\mathcal{DC})^o \stackrel{{}^{\mathbf{c}}\mathcal{F}}{\to} \mathcal{CH}^o,$$

where $\mathbf{c}$ is a twisted complex of $\mathcal{C}^o$. (Actually the twisted complex of $\mathcal{C}^o$ can be constructed from one on $\mathcal{C}$ by changing the sign of the maps $x_{i,j}$ appropriately. We leave the reader the problem to find the correct sign.) We define $\mathfrak{Rep}_0(\mathbf{c})$, $\mathfrak{OpRep}_0(\mathbf{c})$ in a similar way.

Definition 7.38. An A_∞ functor : $\mathcal{C} \to \mathcal{CH}^o$, $\mathcal{C}^o \to \mathcal{CH}$, $\mathcal{C} \to \mathcal{CH}$, $\mathcal{C}^o \to \mathcal{CH}^o$ is said to be *derived representable* if it is homotopic to $\mathcal{F}^{\mathbf{c}}$, $\mathfrak{Rep}_0(\mathbf{c})$, ${}^{\mathbf{c}}\mathcal{F}$, $\mathfrak{OpRep}_0(\mathbf{c})$, respectively.

We next explain that A_∞ functors are natural generalization of the notion of differential graded modules. Let $(\mathcal{A}, d, \wedge)$ be a differential graded algebra. It defines an A_∞ algebra as in Example-Lemma 1.7. We write it $(\mathcal{A}, \mathfrak{m}_1, \mathfrak{m}_2)$. ($\mathfrak{m}_3 = \cdots = 0$.) We may regard it as an A_∞ category $\mathcal{A}$ with one object.

Lemma 7.39. *Homotopy classes of A_∞ functors $\mathcal{F} : \mathcal{A} \to \mathcal{CH}^0$ such that $\mathcal{F}_k = 0$ for $k \geq 2$, correspond one to one to the homotopy equivalence classes of graded differential left module over $(\mathcal{A}, d, \wedge)$.*

Proof. Let $(D, d, \cdot)$ be a graded differential module over $(\mathcal{A}, d, \wedge)$. We define

$$\mathcal{F}_0^d(c_0) = (D, d).$$

Here c_0 is the unique object of $\mathcal{A}$. we define

$$\mathcal{F}_1^d : \mathcal{A} \to \mathrm{Hom}((D, d), (D, d))$$

by

$$\mathcal{F}_1^d(x)(v) = (-1)^{\deg x(\deg v + 1)} x \cdot v.$$

As in Example-Lemma 1.7, we can easily check that $\mathcal{F}_1^d$ is a chain map and

$$\mathcal{F}_1^d(\mathfrak{m}_2(x, x')) = \mathcal{F}_1^d(x') \circ \mathcal{F}_1^d(x).$$

Hence by putting $\mathcal{F}_k^d = 0$, $k > 1$ we find an A^∞ functor $\mathcal{F} : \mathcal{A} \to \mathcal{CH}^0$.

The converse can be proved in a similar way. □

We remark that a representable functor $\mathcal{F} : \mathcal{A} \to \mathcal{CH}^0$ corresponds to $\mathcal{A}$ itself (regarded as an $\mathcal{A}$ module) by Lemma 7.39.

Definition 7.40. A *left A_∞ module* of an A_∞ algebra $\mathcal{C}$ is an A_∞ functor $\mathcal{C} \to \mathcal{CH}^0$. A *right A_∞ module* of an A_∞ algebra $\mathcal{C}$ is an A_∞ functor $\mathcal{C} \to \mathcal{CH}$.

In a way similar to the proof of Lemma 7.39, we can check that this definition coincides with one given in [FOOO] §14.

Let $\mathcal{C}$ be an A_∞ category and c, c' be objects of it. There is an obvious A_∞ functor $\mathcal{C}(c) \to \mathcal{C}$. (Here we regard $\mathcal{C}(c)$ an A_∞ algebra, that is an A_∞ category with single object c.) We compose it with $\mathcal{F}^{c'}$: $\mathcal{C} \to \mathcal{CH}$ and we obtain a left A_∞ $\mathcal{C}(c)$ module, which is $\mathcal{C}(c, c')$ as an R module. In a similar way, $\mathcal{C}(c, c')$ has a structure of right $\mathcal{C}(c')$ module. In other words, $\mathcal{C}(c, c')$ is a left $\mathcal{C}(c)$ and right $\mathcal{C}(c')$ bimodule in the sense defined in [FOOO] §14.

In the case of $\mathcal{LAG}(M, \omega)$, this implies that we have a $\mathcal{LAG}(L, b)$, $\mathcal{LAG}(L', b')$ bimodule $\mathcal{LAG}((L, b), (L', b'))$. (Here b and b' are bounding chains.) This is what is constructed in [FOOO] §14. The homology of $\mathcal{LAG}((L, b), (L', b'))$ is the Floer homology between two Lagrangian submanifolds.

Remark 7.41. Let X be a scheme over R. We can associate the following category $\mathcal{C}(X)$, which is an A_∞ category. (Note that it satisfies $\mathfrak{m}_k = 0$ for $k \neq 2$, and that $\mathfrak{m}_2$ is commutative.)

The object of $\mathcal{C}(X)$ is an affine open subsets $U_A = \mathrm{Spec}(A) \subset X$. The set of morphisms from U_A to U_B is $\{0\}$ unless $U_B \subseteq U_A$. In the case $U_B \subseteq U_A$, the set of the morphisms $\mathcal{C}(X)(U_A, U_B)$ is the ring A. If $U_C \subseteq U_B \subseteq U_A$ then $A \subseteq B \subseteq C$. $\mathfrak{m}_2$ is defined as $A \otimes B \to B$ (the product of ring B).

A functor from $\mathcal{C}(X)$ to the category of R modules can be identified with a presheaf on X.

We further study relations of A_∞ functor to twisted complex.

Lemma 7.42. *Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ be an A^∞ functor. Then there exists an A^∞ functor $\mathcal{DF} : \mathcal{DC}_1 \to \mathcal{DC}_2$ such that the following diagram commutes.*

Sketch of the proof. Let $(c_{k_1}, \ldots, c_{k_2}; (x_{i,j}))$ to be a twisted complex of $\mathcal{C}_1$. We put

$$y_{a,b} = \sum_k \sum_{a=i_1<\cdots<i_k=b} \mathcal{F}_k(x_{i_1}, \ldots, x_{i_k}).$$

It is easy to see that $(\mathcal{F}_0(c_{k_1}), \ldots, \mathcal{F}_0(c_{k_2}); (y_{i,j}))$ is a twisted complex of $\mathcal{C}_2$. We put

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{C}_2 \arrow[d] \\
\mathcal{D}\mathcal{C}_1 \arrow[r, "\mathcal{D}\mathcal{F}"] & \mathcal{D}\mathcal{C}_2
\end{tikzcd}
$$

Diagram 7.1.

$$(\mathcal{D}\mathcal{F})_0(c_{k_1}, \dots, c_{k_2}; (x_{i,j})) = (\mathcal{F}_0(c_{k_1}), \dots, \mathcal{F}_0(c_{k_2}); (y_{i,j})).$$

We omit the definition of $(\mathcal{D}\mathcal{F})_k$, $k \geq 1$. □

Lemma 7.43. *There exists an A^∞ functor $\mathcal{P} : \mathcal{D}\mathcal{D}\mathcal{F} \to \mathcal{D}\mathcal{F}$ such that $\mathcal{P}_0(\mathbf{c}) = |\mathbf{c}|$. Here $|\mathbf{c}|$ is defined in Lemma* 6.20.

The proof is straightforward and is left to the reader.

We now proceed to the definition of an A_∞ category $\mathfrak{Funk}(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are A_∞ functors. (Here $\mathcal{C}_1$, $\mathcal{C}_2$ are A_∞ categories.) This section is almost the same as [Fu4] §10. However we put signs to every formula and check the formula with signs. (In [Fu4] we worked over $\mathbb{Z}_2$.) The presentation of the proof is improved also.

Let $\mathcal{F}^1$, $\mathcal{F}^2$ be A_∞ functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. Let $a, b \in \mathfrak{Ob}(\mathcal{C}_1)$. Let ℓ be an integer. We consider a family of operators

$$T_k(a,b) : B_k\mathcal{C}_1[1](a,b) \to \mathcal{C}_2[1](\mathcal{F}_0^1(a), \mathcal{F}_0^2(b)) \tag{7.44}$$

of degree t. (Here $k = 1, 2, \dots$ for $a \neq b$ and $k = 0, 1, 2, \dots$ for $a = b$.) We write $t - 1 = \mathfrak{deg}T$, $t = \mathfrak{deg}'T$. We use $\mathfrak{deg}$ in place of deg here to avoid confusion. $\mathfrak{deg}T$ will be a degree of T as a pre natural transformation as we will define below (Definition 7.49). $\deg T_k(a,b)$ is a degree as a homomorphism between graded modules. We remark that $\mathfrak{deg}'T = \deg T_k(a,b)$. We also remark that, for $\mathbf{x} \in B_k\mathcal{C}[1](a,b)$, we have

$$\deg' T_k(a,b)(\mathbf{x}) = \mathfrak{deg}'T + \deg' \mathbf{x} = \mathfrak{deg}T + \deg' \mathbf{x} + 1.$$

For $a', b' \in \mathfrak{Ob}(\mathcal{C}_2)$, let

$$\pi_{a',b'} : B\mathcal{C}_2[1](a',b') \to \mathcal{C}_2[1](a',b')$$

be the projection.

Lemma 7.45. *For each family $T_k(a,b)$ there exists uniquely a family*

$$\hat{T}(a,b) : B\mathcal{C}_1[1](a,b) \to B\mathcal{C}_2[1](\mathcal{F}_0^1(a), \mathcal{F}_0^2(b)),$$

of homomorphisms with the following properties.

$$\pi_{\mathcal{F}_0^1(a),\mathcal{F}_0^2(b)} \circ \hat{T}(a,b) = T_k(a,b) \quad \text{on } B_k\mathcal{C}_1[1](a,b). \tag{7.46.1}$$

$$\Delta \circ \hat{T}(a,b) = \sum_c (\hat{\mathcal{F}}^1 \hat{\otimes} \hat{T}(c,b) + \hat{T}(a,c) \hat{\otimes} \hat{\mathcal{F}}^2) \circ \Delta. \tag{7.46.2}$$

Here $\hat{\otimes}$ is defined by $(A\hat{\otimes}B)(\mathbf{x},\mathbf{y}) = (-1)^{\deg B \deg' \mathbf{x}} A(\mathbf{x}) \otimes B(\mathbf{y})$. (Note $\deg B = \mathfrak{deg}' B$ and $\deg' \mathbf{x}$ is the degrees after shifted.)

Proof. Let $\mathbf{x} \in B\mathcal{C}_1[1](a,b)$. We use notation (7.11). We define

$$\hat{T}(\mathbf{x}) = \sum_a (-1)^{\deg T \deg' \mathbf{x}_a^{(1)}} \hat{\mathcal{F}}^1(\mathbf{x}_a^{(1)}) \otimes T(\mathbf{x}_a^{(2)}) \otimes \hat{\mathcal{F}}^3(\mathbf{x}_a^{(3)}). \tag{7.47}$$

It is easy to check (7.46). Uniqueness is also easy to show. □

Hereafter we write T etc. in place of $T_k(a,b)$ etc. in case no confusion can occur.

Lemma 7.48. *For each family $T_k(a,b)$ with $t = \mathfrak{deg}' T_k(a,b) = \deg T$, there exists a family $(\delta T)_k(a,b)$ such that*

$$\widehat{\delta T} = \hat{d} \circ \hat{T} + (-1)^{t+1} \hat{T} \circ \hat{d}.$$

Proof. We calculate

$$\begin{aligned}
&\Delta \circ (\hat{d} \circ \hat{T} + (-1)^{t+1} \hat{T} \circ \hat{d}) \\
&= (\hat{d} \hat{\otimes} 1 + 1 \hat{\otimes} \hat{d}) \circ \Delta \circ \hat{T} + (-1)^t \Delta \circ \hat{T} \circ \hat{d} \\
&= (\hat{d} \hat{\otimes} 1 + 1 \hat{\otimes} \hat{d}) \circ (\hat{\mathcal{F}}^1 \hat{\otimes} \hat{T} + \hat{T} \hat{\otimes} \hat{\mathcal{F}}^2) \circ \Delta \\
&\qquad + (-1)^{t+1} (\hat{\mathcal{F}}^1 \hat{\otimes} \hat{T} + \hat{T} \hat{\otimes} \hat{\mathcal{F}}^2) \circ (\hat{d} \hat{\otimes} 1 + 1 \hat{\otimes} \hat{d}) \circ \Delta \\
&= (\hat{\mathcal{F}}^1 \hat{\otimes} (\hat{d} \circ \hat{T} + (-1)^{t+1} \hat{T} \circ \hat{d}) + (\hat{d} \circ \hat{T} + (-1)^{t+1} \hat{T} \circ \hat{d}) \hat{\otimes} \hat{\mathcal{F}}^2) \circ \Delta.
\end{aligned}$$

Lemma 7.48 then follows from Lemma 7.45. □

Definition 7.49. We say a family T as in (7.44), a *pre natural transformation* from $\mathcal{F}^1$ to $\mathcal{F}^2$. We let $\mathfrak{Funk}(\mathcal{F}^1, \mathcal{F}^2)$ be the set of all pre natural transformations. (It is a graded module over the commutative ring R we work on.)

We define a boundary operator $\mathfrak{M}_1 = -\delta$ on it by Lemma 7.48.

We say that T is a *natural transformation* or A_∞ *transformation* if it is δ closed.

We remark that $\mathfrak{deg}(T) = \deg(T_0(1))$, where $T_0(1) \in \mathcal{C}[1](c,c)$.
We put minus sign in $\mathfrak{M}_1 = -\delta$ since we need it to show Theorem 9.1 later.

Corollary 7.50. $\delta \circ \delta = 0$. *In other words,* $(\mathfrak{Funk}(\mathcal{F}^1, \mathcal{F}^2), \mathfrak{M}_1)$ *is a chain complex.*

Corollary 7.50 is immediate from Lemma 7.45. We use the symbol $\mathfrak{M}$ to denote the operations on $\mathfrak{Funk}(\mathcal{F}^1, \mathcal{F}^2)$ in order to distinguish it from operations on $\mathcal{C}_1$, $\mathcal{C}_2$.

Remark 7.51. If T is a natural transformation, then $T_0(a) = T_0(a)(1)$ is a closed morphism in $\mathcal{C}_2(\mathcal{F}_0^1(a), \mathcal{F}_0^2(a))$. (Here $1 \in B_0\mathcal{C}_1[1] =$ the coefficient ring R.) Moreover we have

$$\mathfrak{m}_2(T_1(a,b)(x), T_0(b)) \equiv \pm\mathfrak{m}_2(T_0(a), T_1(a,b)(x)) \mod \operatorname{Im}\mathfrak{m}_1.$$

for each $x \in \mathcal{C}_1(a,b)$. Thus natural transformation in our sense defines a natural transformation in the usual sense in homology level.

We next define $\mathfrak{m}_k$, $k \geq 2$. Let $\mathcal{F}^i$, $i = 0, \dots, k$ be A_∞ functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ and $T^i \in \mathfrak{Funk}(\mathcal{F}^{i-1}, \mathcal{F}^i)$ be pre natural transformations. We put $t_i = \mathfrak{deg}' T^i$. Let $\mathbf{x} \in B(\mathcal{C}_1[1])$. We consider

$$\Delta^{2k+1}\mathbf{x} = \sum_a \mathbf{x}_a^{(1)} \otimes \cdots \otimes \mathbf{x}_a^{(2k+1)}.$$

and put

$$\begin{aligned} &\hat{T}(\mathbf{x}) = \\ (7.52)\qquad &-\sum(-1)^{\epsilon_a}\mathfrak{m}\left(\hat{\mathcal{F}}^0(\mathbf{x}_a^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})\right), \end{aligned}$$

where

$$\epsilon_a = \sum_{j=1}^{k}\sum_{i=1}^{2j-1} t_j \deg' \mathbf{x}_a^{(i)}. \tag{7.53}$$

It is easy to see that $\hat{T}$ satisfies (7.46.2). Hence we can use it to define a pre natural transformation T. We also remark that $\mathfrak{deg}T = t_1 + \cdots t_k$ and is $\mathfrak{deg}'T = t_1 + \cdots t_k + 1$.

Definition 7.54. We put $\mathfrak{M}_k(T^1, \dots, T^k) = T$.

We remark that $\mathfrak{M}_k$ is of degree one in the sense of $\mathfrak{deg}'$. Namely we have

$$\mathfrak{deg}'T = \mathfrak{deg}'T^1 + \cdots + \mathfrak{deg}'T^k + 1.$$

We remark that the overall minus sign in (7.52) will be necessary for Theorem 9.1 to hold.

Theorem-Definition 7.55. *There exists an A_∞ category $\mathfrak{Funk}(\mathcal{C}_1, \mathcal{C}_2)$ such that its object is an A_∞ functor : $\mathcal{C}_1 \to \mathcal{C}_2$, the set of morphisms are the set of pre natural transformations, and that (higher) compositions are defined by Definition* 7.54.

Proof. The A_∞ formula we are going to check is:

$$0 = \sum_{1 \le m \le \ell \le k} (-1)^{t_1 + \cdots + t_{m-1}} \mathfrak{M}_{k-\ell+m}(T^1, \dots, T^{m-1}, \mathfrak{M}_{\ell-m+1}(T^m, \dots, T^\ell), T^{\ell+1}, \dots, T^k)(\mathbf{x}), \tag{7.56}$$

where $t_i = \mathfrak{deg}' T^i$. To prove (7.56), we compare it with

$$0 = (-1)^{\epsilon_1(a)} (\mathfrak{m} \circ \hat{d})(\hat{\mathcal{F}}^0(\mathbf{x}_a^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})), \tag{7.57}$$

where

$$\epsilon_1(a) = \sum_{j=1}^{k} \sum_{i=1}^{2j-1} t_j \deg' \mathbf{x}_a^{(i)}. \tag{7.58}$$

The formula (7.57) follows from the A_∞ formula of $\mathcal{C}_2$. We study the terms appearing in (7.56) and (7.57).

We first study the terms appearing in (7.57). There are two types of them. One is:

Type 1: Let $1 \le m < \ell \le k$. Then we have

$$\begin{aligned} \mathfrak{m}\Big(&\hat{\mathcal{F}}^0(\mathbf{x}_m^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, \hat{\mathcal{F}}^{m-1}(\mathbf{x}_{a,b}^{(2m-1),(1)}), \\ &\mathfrak{m}\Big(\hat{\mathcal{F}}^{m-1}(\mathbf{x}_{a,b}^{(2m-1),(2)}), T^m(\mathbf{x}_a^{(2m)}), \dots \\ &\quad \dots, T^{\ell-1}(\mathbf{x}_a^{(2\ell-2)}), \hat{\mathcal{F}}^\ell(\mathbf{x}_{a,b}^{(2\ell-1),(1)})\Big) \\ &\hat{\mathcal{F}}^\ell(\mathbf{x}_{a,b}^{(2\ell-1),(2)}), \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})\Big). \end{aligned} \tag{7.59}$$

Here we put

$$\Delta(\mathbf{x}_a^{(i)}) = \sum_b \mathbf{x}_{a,b}^{(i),(1)} \otimes \mathbf{x}_{a,b}^{(i),(2)}.$$

For a moment we do not put sign in the formula. We will check the sign carefully later.

Type 2: This is an analogue of (7.59) in the case when $1 \le m = \ell \le k$. That is:

$$\begin{aligned}(7.60)\qquad &\mathfrak{m}(\hat{\mathcal{F}}^0(\mathbf{x}_a^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, T^m(\mathbf{x}_a^{(2m)}),\\ &\quad \hat{d}(\hat{\mathcal{F}}^m(\mathbf{x}_a^{(2m+1)})), T^{m+1}(\mathbf{x}_a^{(2m+2)}),\\ &\quad \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})).\end{aligned}$$

Now we turn to the terms appearing in (7.56).

Type 3: This is exactly the same as (7.59).

The other types of terms in (7.56) are the cases when $\mathfrak{M}_1$ appears. We remark that $\mathfrak{M}_1$ appears in (7.56) in case $\ell = m + k - 1$ or $\ell = m$. The terms of Types 4, 5 below correspond to the case either $\ell = m+k-1$ and the terms of Type 6 correspond to the case $\ell = m$.

Let us first consider the case $\ell = m + k - 1$. We recall

$$\begin{aligned}(7.61)\qquad (\mathfrak{M}_1(\mathfrak{M}_k(T^1, \dots, T^k)))(\mathbf{x}) &= (\mathfrak{m} \circ \mathfrak{M}_k(T^1, \dots, T^k)^\wedge)(\mathbf{x})\\ &\quad + (-1)^{t_1+\cdots+t_k}(\mathfrak{M}_k(T^1, \dots, T^k) \circ \hat{d})(\mathbf{x}).\end{aligned}$$

We remark that $\mathfrak{deg}'\mathfrak{M}_k(T^1, \dots, T^k) = t_1 + \cdots + t_k + 1$. Note that the first term of (7.61) is already included in Type 3. So we only need to consider the second term. They are one of the following two types.

Type 4:

$$\begin{aligned}(7.62)\qquad &\mathfrak{m}(\hat{\mathcal{F}}^0(\mathbf{x}_a^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, T^m(\mathbf{x}_a^{(2m)}),\\ &\quad \hat{\mathcal{F}}^m(\hat{d}(\mathbf{x}_a^{(2m+1)})), T^{m+1}(\mathbf{x}_a^{(2m+2)}), \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})).\end{aligned}$$

Type 5:

$$\begin{aligned}(7.63)\qquad &\mathfrak{m}(\hat{\mathcal{F}}^0(\mathbf{x}_a^{(1)}), T^1(\mathbf{x}_a^{(2)}), \dots, \hat{\mathcal{F}}^{m-1}(\mathbf{x}_a^{(2m-1)}),\\ &\quad T^m(\hat{d}(\mathbf{x}_a^{(2m)})), \dots, T^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{F}}^k(\mathbf{x}_a^{(2k+1)})).\end{aligned}$$

We next consider the case $\ell = m$. we recall

$$\begin{aligned}(7.64)\qquad &\mathfrak{M}_k(T^1, \dots, \mathfrak{M}_1(T^m), \dots, T^k)(\mathbf{x})\\ &= \mathfrak{M}_k(T^1, \dots, \mathfrak{m} \circ \hat{T}^m, \dots, T^k)(\mathbf{x})\\ &\quad + (-1)^{t_m}\mathfrak{M}_k(T^1, \dots, T^m \circ \hat{d}, \dots, T^k)(\mathbf{x}).\end{aligned}$$

The first term again is included in Type 3. The second term then gives:

Type 6: This type is actually the same as Type 5.

We have finished describing all the types of the terms in (7.56). We now can prove the case when the coefficient ring is $\mathbb{Z}_2$ immediately. Namely terms of Types 1 and 3 cancel each other and terms of Types 2 and 4 cancel each other, since $\hat{\mathcal{F}}$ is a chain map, and terms of Types 5 and 6 cancel each other.

Let us now check the sign.

The sign in Type 3 is given by $(-1)^{\epsilon_2(a)+t_1+\cdots+t_{m-1}}$ where

$$
\begin{aligned}
\epsilon_2(a) \quad = \quad & \sum_{j=m}^{\ell} t_j \left(\deg' \mathbf{x}_{a,b}^{(2m-1),(2)} + \sum_{i=2m}^{2j-1} \deg' \mathbf{x}_a^i \right) \\
& + \sum_{j=1}^{m-1} \sum_{i=1}^{2j-1} t_j \deg' \mathbf{x}_a^i \\
& + (t_m + \cdots + t_\ell + 1) \times \\
& \quad \times \left(\sum_{i=0}^{2m-2} \deg' \mathbf{x}_a^i + \deg' \mathbf{x}_{a,b}^{(2m-1),(1)} \right) \\
& + \sum_{j=\ell+1}^{k} \sum_{i=1}^{2j-1} t_j \deg' \mathbf{x}_a^i .
\end{aligned}
\tag{7.65}
$$

Note that the first line in (7.65) comes from $\mathfrak{M}_{\ell-m+1}(T^m, \ldots, T^\ell)$ in (7.56) and the rest comes from $\mathfrak{M}_{k-\ell+m}$.

We calculate (7.65) and obtain

$$
\epsilon_2(a) - \epsilon_1(a) = \sum_{i=0}^{2m-2} \deg' \mathbf{x}_a^i + \deg' \mathbf{x}_{a,b}^{(2m-1),(1)} .
$$

On the other hand, the sign in the case of Type 1 in (7.57) is $(-1)^{\epsilon_1(a)+\epsilon_3(a)}$ where

$$
\begin{aligned}
\epsilon_3(a) \quad &= \quad \deg' \mathbf{x}_m^{(1)} + \deg' T^1(\mathbf{x}_a^{(2)}) + \cdots + \deg' \mathbf{x}_{a,b}^{(2m-1),(1)} \\
&\equiv \quad \sum_{i=0}^{2m-2} \deg' \mathbf{x}_a^i + \deg' \mathbf{x}_{a,b}^{(2m-1),(1)} + \sum_{i=1}^{m-1} t_i \quad \mod 2.
\end{aligned}
$$

Thus Types 1 and 3 coincides together with sign.

We next consider Types 2 and 4. We find using (7.61) that the sign of both of them is

$$
\epsilon_1(a) + \sum_{i=1}^{m} t_i + \sum_{j=1}^{2m} \deg' \mathbf{x}_a^j
$$

and coincides to each other.

Next the sign of the terms of Type 5 is $(-1)^{\epsilon_4(a)}$, where

$$\epsilon_4(a) = \epsilon_1(a) + \sum_{i=1}^{m} t_i + \sum_{j=1}^{2m+1} \deg' \mathbf{x}_a^j.$$

We study Type 6. Note that we have sign $(-1)^{t_1+\cdots+t_{m-1}}$ in (7.56). So using (7.64) we find that the sign of the terms of Type 6 is $(-1)^{\epsilon_4(a)+1}$. The proof of A_∞ formula (7.56) is now complete.

Finally we define unit, that is the identity transform $\mathrm{Id}^{\mathcal{F}}$ for each $\mathcal{F} \in \mathfrak{Ob}(\mathcal{C}_1, \mathcal{C}_2)$. Let $\mathbf{e}_c \in \mathcal{C}_2^0(c,c)$ be the unit in $\mathcal{C}_2$. Namely we assume that (1.5) is satisfied for it. We put

$$\mathrm{Id}_0^{\mathcal{F}}(a) = -\mathbf{e}_{\mathcal{F}_0(a)} \in \mathcal{C}_2^0(\mathcal{F}_0(a), \mathcal{F}_0(a)), \tag{7.66.1}$$

$$\mathrm{Id}_k^{\mathcal{F}} = 0, \qquad \text{for } k \geq 1. \tag{7.66.2}$$

(Note that $\mathrm{Id}_1^{\mathcal{F}}(x) \neq x$.) It is easy to see from definition that $\mathrm{Id}^{\mathcal{F}}$ satisfies (1.5) for $\mathfrak{M}$. (We remark that we need minus sign in (7.66) since we put overall minus sign in the definition of $\mathfrak{M}_k$.) □

§8. Homotopy equivalence and A_∞ analogue of J. H. C. Whitehead Theorem

We now define homotopy equivalence between A_∞ categories.

Definition 8.1. Two A_∞ functors $\mathcal{F}^1, \mathcal{F}^2 : \mathcal{C}_1 \to \mathcal{C}_2$ are said to be *homotopic* to each other if they are homotopy equivalent as objects of $\mathfrak{Funk}(\mathcal{C}_1, \mathcal{C}_2)$ in the sense of Definition 6.22.

Definition 8.2. The identity functor $\mathrm{Id}^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined as follows.

(8.3.1) $\mathrm{Id}_0^{\mathcal{C}}(c) = c$.
(8.3.2) $\mathrm{Id}_1^{\mathcal{C}}(x) = x$.
(8.3.3) $\mathrm{Id}_k^{\mathcal{C}} = 0$ for $k \geq 2$.

Remark 8.4. The identity functor is similar to but is different from the identity transformation defined at the end of the last section.

Definition 8.5. An A_∞ functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be a *homotopy equivalence* if there exists an A_∞ functor $\mathcal{F}' : \mathcal{C}_2 \to \mathcal{C}_1$ such that the composition $\mathcal{F} \circ \mathcal{F}'$ is homotopic to $\mathrm{Id}^{\mathcal{C}_2}$ and that $\mathcal{F}' \circ \mathcal{F}$ is homotopic to $\mathrm{Id}^{\mathcal{C}_1}$.

Two A_∞ categories are said to be *homotopy equivalent* to each other if there exists a homotopy equivalence between them.

Now the main result of this section is:

Theorem 8.6. *Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ be an A_∞ functor such that:*

(8.7.1) *$\mathcal{F}_1 : \mathcal{C}_1(c_1, c_1') \to \mathcal{C}_2(\mathcal{F}_0(c_1), \mathcal{F}_0(c_1'))$ induces an isomorphism on $\mathfrak{m}_1$ homology:*

(8.7.2) *For any $c_2 \in \mathfrak{Ob}(\mathcal{C}_2)$ there exists $c_1 \in \mathfrak{Ob}(\mathcal{C}_1)$ such that $\mathcal{F}_0(c_1)$ is homotpy equivalent to c_2.*

Then $\mathcal{F}$ is a homotopy equivalence.

Remark 8.8. In the case of A_∞ algebras, Theorem 8.9 was proved in [FOOO] §A5. (8.7) is used as a *definition* of homotopy equivalence in [KoS1]. It seems that Kontsevich-Soibelman announced in [KoS1] that they will prove in [KoS2] a result similar to Theorem 8.6.

To prove Theorem 8.6, we start with the following special case.

Proposition 8.9. *We assume* (8.7.1) *and that $\mathcal{F}_0 : \mathfrak{Ob}(\mathcal{C}_1) \to \mathfrak{Ob}(\mathcal{C}_2)$ is a bijection. Then $\mathcal{F}$ is a homotopy equivalence.*

Proof. The proof is similar to the argument of [FOOO] §A5. We need to construct an A_∞ functor $\mathcal{F}' : \mathcal{C}_2 \to \mathcal{C}_1$ and a natural transformation $T : \mathcal{F} \circ \mathcal{F}' \to \mathrm{Id}^{\mathcal{C}_2}$. For this purpose, we construct $\mathcal{F}'_k : B_k\mathcal{C}_2[1] \to B_1\mathcal{C}_1[1]$ and $T_k : B_k\mathcal{C}_2[1] \to B_1\mathcal{C}_2[1]$ inductively on k. To describe the induction hypothesis, we define the notions, A_k functor and A_k transformation.

Let $\mathcal{C}$, $\mathcal{C}'$ be A_∞ categories, $\mathcal{G}_0 : B_0\mathcal{C} \to B_0\mathcal{C}'$ be a map and $\mathcal{G}_\ell : B_\ell\mathcal{C}[1](c_1, c_2) \to B_1\mathcal{C}'[1](c_1, c_2)$, $1 \le i \le k$ be R module homomorphisms of degree 0.

We put:

$$B_{a,\dots,b}\mathcal{C}[1] = \frac{\bigoplus_{i=0}^{a} B_i\mathcal{C}[1]}{\bigoplus_{i=0}^{b-1} B_i\mathcal{C}[1]}.$$

The boundary operator of the chain complex $B\mathcal{C}[1]$ induces a boundary operator on $B_{a,\dots,b}\mathcal{C}[1]$. Hence $B_{a,\dots,b}\mathcal{C}[1]$ is a chain complex.

Lemma 8.10. *There exists uniquely a coalgebra homomorphism*

$$\hat{\mathcal{G}} : B\mathcal{C}[1] \to B\mathcal{C}'[1]$$

such that

(8.11.1) *$\hat{\mathcal{G}} = \hat{\mathcal{G}}_{0,\dots,k}$ on $\bigoplus_{i=0}^{k} B_i\mathcal{C}[1]$.*

(8.11.2) *$B_1\mathcal{C}'[1]$ component of $\hat{\mathcal{G}}(B_i\mathcal{C}[1])$ is 0 for $i > k$.*

Proof. We define $\mathcal{G} : B\mathcal{C}[1] \to B_1\mathcal{C}'[1]$ by $\mathcal{G} = \mathcal{G}_\ell$ on $B_\ell\mathcal{C}[1]$ $1 \le \ell \le k$, and $\mathcal{G} = 0$ on $B_\ell\mathcal{C}[1]$ $\ell > k$. We now define $\hat{\mathcal{G}}$ in the same way as (7.3). Namely

$$\hat{\mathcal{G}}(\mathbf{x}) = \sum_{i,a} \mathcal{G}(\mathbf{x}_{i,a}^{(1)}) \otimes \cdots \otimes \mathcal{G}(\mathbf{x}_{i,a}^{(i)}).$$

Here

$$\Delta^i(\mathbf{x}) = \sum_a \mathbf{x}_{i,a}^{(1)} \otimes \cdots \otimes \mathbf{x}_{i,a}^{(i)}.$$

(The definition of $\hat{\mathcal{G}}$ on $B_0\mathcal{C}[1]$ is the same as (7.3.2).) It is easy to check (8.11). □

$\hat{\mathcal{G}}$ induces a homomorphism $:B_{a,\dots,b}\mathcal{C}[1] \to B_{a,\dots,b}\mathcal{C}[1]$ for each $a < b$. We denote it by $\mathcal{G}_{a,\dots,b}$.

Definition 8.12. We say $\mathcal{G}_i$ $i \le k$ is an A_k *functor* if $\mathcal{G}_{0,\dots,k}$ is a chain map.

Lemma 8.13. *If $\mathcal{G}_i$, $i \le k$ is an A_k functor then $\mathcal{G}_{a,\dots,b}$ is a chain map for $b - a \le k - 1$.*

The proof is easy and is left to the reader.

Let us next denfine an A_k transformation. Let $\mathcal{G}$ and $\mathcal{G}'$ be A_k functors from $\mathcal{C}$ to $\mathcal{C}'$. Let $S_\ell : B_\ell\mathcal{C}[1](c_1, c_2) \to B_1\mathcal{C}'[1](\mathcal{G}_0(c_1), \mathcal{G}'_0(c_2))$, $1 \le \ell \le k$ be R module homomorphisms of degree $s+1$. It induces

$$S_{0,\dots,k} : \bigoplus_{i=0}^{k} B_i\mathcal{C}[1] \to \bigoplus_i B_i\mathcal{C}'[1] \tag{8.14}$$

as follows. We define $S : B\mathcal{C}[1] \to B_1\mathcal{C}'[1]$ by $S = S_\ell$ on $B_\ell\mathcal{C}[1]$, $0 \le \ell \le k$ and $S = 0$ on $B_\ell\mathcal{C}$, $\ell > k$. We then put:

$$\begin{aligned} &S_{0,\dots,k}(\mathbf{x}) \\ &= \textstyle\sum_a (-1)^{s \deg' \mathbf{x}_a^{(1)}} \mathcal{G}_{0,\dots,k}(\mathbf{x}_a^{(1)}) \otimes S(\mathbf{x}_a^{(2)}) \otimes \mathcal{G}'_{0,\dots,k}(\mathbf{x}_a^{(3)}). \end{aligned} \tag{8.15}$$

We remark here in (8.15) the case $\mathbf{x}_a^{(2)} = 1 \in B_0\mathcal{C}[1]$ is included. As a consequence, the image of $S_{0,\dots,k}$ is not contained in $\bigoplus_{i=0}^k B_i\mathcal{C}[1]$.

Definition 8.16. We say S_ℓ, $0 \le \ell \le k$, is an A_k *transformation* if

$$\hat{d} \circ S_{0,\dots,k} + (-1)^{s+1} S_{0,\dots,k} \circ \hat{d} = 0,$$

on $\bigoplus_{i=0}^k B_i\mathcal{C}[1]$. We put $\mathfrak{deg}' S = s$.

Now we go back to the proof of Proposition 8.9. We are going to construct an A_k functor $\mathcal{F}'_\ell$ $\ell \le k$, and an A_k transformation T_ℓ $\ell \le k$, from $\mathcal{F} \circ \mathcal{F}'$ to $\mathrm{Id}^{\mathcal{C}_2}$ by induction on k. (We remark that the composition of two A_k functors is well defined and is an A_k functor.)

We start with the case $k = 0$. We put

$$\mathcal{F}'_0 = \mathcal{F}_0^{-1} : \mathfrak{Ob}(\mathcal{C}_2) \to \mathfrak{Ob}(\mathcal{C}_1). \tag{8.17}$$

We remark that

$$B_0\mathcal{C}[1] = \bigoplus_{c \in \mathfrak{Ob}(\mathcal{C})} R$$

hence a map $\mathfrak{Ob}(\mathcal{C}_2) \to \mathfrak{Ob}(\mathcal{C}_1)$ induces a homomorphism $B_0\mathcal{C}_2[1] \to B_0\mathcal{C}_1[1]$.

We next put

$$T_0(c) = \mathbf{e}_c \in \mathcal{C}_1(\mathcal{F}_0(\mathcal{F}'_0(c)), c) = \mathcal{C}_1(c, c). \tag{8.18}$$

Now we assume that we have an A_{k-1} transformation $\mathcal{F}'_\ell$, $0 \le \ell \le k-1$ and an A_{k-1} transformation T_ℓ, $0 \le \ell \le k-1$ and will consider the case of k. It follows from Lemma 8.13 that $\mathcal{F}'_\ell$, $0 \le \ell \le k-1$ determine a chain map

$$\mathcal{F}'_{2,\dots,k} : B_{2,\dots,k}\mathcal{C}_2[1] \to B_{2,\dots,k}\mathcal{C}_1[1]. \tag{8.19}$$

Using the obvious R module isomorphism

$$B_{2,\dots,k}\mathcal{C}[1] = \bigoplus_{i=2}^{k} B_i\mathcal{C}[1]$$

(8.19) induces an R module homomorphism

$$\tilde{\mathcal{F}}'_{2,\dots,k} : \bigoplus_{i=2}^{k} B_i\mathcal{C}[1] \to \bigoplus_{i=1}^{k} B_i\mathcal{C}[1]. \tag{8.20}$$

Note however that (8.20) is *not* a chain map in general.

It is easy to see that (8.20) coincides with $\tilde{\mathcal{F}}'_{1,\dots,k-1}$ on $\bigoplus_{i=2}^{k-1} B_i\mathcal{C}$. Hence we extend (8.20) to

$$\tilde{\mathcal{F}}'_{1,\dots,k} : \bigoplus_{i=1}^{k} B_i\mathcal{C} \to \bigoplus_{i=1}^{k} B_i\mathcal{C}. \tag{8.21}$$

We remark however that (8.21) is *not* a chain map in general. Nevertheless, using the fact that $\mathcal{F}'_{2,\dots,k}$ and $\mathcal{F}'_{1,\dots,k-1}$ are chain maps (Lemma 8.13), we can easily prove the following:

Lemma 8.22. *The image of $\tilde{\mathcal{F}}'_{1,\dots,k} \circ \hat{d} - \hat{d} \circ \tilde{\mathcal{F}}'_{1,\dots,k}$ is contained in $B_1\mathcal{C}_1[1]$. Moreover $\tilde{\mathcal{F}}'_{1,\dots,k} \circ \hat{d} - \hat{d} \circ \tilde{\mathcal{F}}'_{1,\dots,k}$ vanishes on $\bigoplus_{i=1}^{k-1} B_i\mathcal{C}_2[1]$.*

We put

$$\Xi = \tilde{\mathcal{F}}'_{1,\dots,k} \circ \hat{d} - \hat{d} \circ \tilde{\mathcal{F}}'_{1,\dots,k} : B_k\mathcal{C}_2[1] \to B_1\mathcal{C}_1[1].$$

Note that $\mathfrak{m}_1$ defines an R module homomorphism

$$d_1 : B_k\mathcal{C}_2[1] \to B_k\mathcal{C}_2[1],$$

by

$$\begin{aligned} & d_1(x_1 \otimes \cdots \otimes x_k) \\ = & \sum_i (-1)^{\deg' x_1 + \cdots \deg' x_{i+1}} x_1 \otimes \cdots \otimes \mathfrak{m}_1(x_i) \otimes \cdots \otimes x_k. \end{aligned}$$

We can prove $d_1 d_1 = 0$ easily.

We define a boundary operator on $\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_1[1])$ by

$$d\phi = \mathfrak{m}_1 \circ \phi + (-1)^{\deg' \phi} \phi \circ d_1. \tag{8.23}$$

Lemma 8.22 then implies

$$d\Xi = 0.$$

We moreover have the following:

Lemma 8.24. *Ξ is contained in the image of d.*

Proof. We define a boundary operator d on $\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_2[1])$ in a way similar to (8.23). Then $\mathcal{F}_1$ induces a chain homomorphism $\mathcal{F}_{1,*}$: $\mathrm{Hom}(B_k\mathcal{C}_2, B_1\mathcal{C}_1) \to \mathrm{Hom}(B_k\mathcal{C}_2, B_1\mathcal{C}_2)$. By assumption $\mathcal{F}_{1,*}$ induces an isomorphism on homology. So it suffices to show that $\mathcal{F}_1 \circ \Xi$ is a d boundary.

We next rewrite the Definition 8.16 (Sublemma 8.26). We need some notations for it.

We define $T : B\mathcal{C}_1[1] \to B_1\mathcal{C}_1[1]$ by $T = T_\ell$ on $B_\ell\mathcal{C}_1[1]$, $0 \le \ell \le k-1$ and $T = 0$ on $B_\ell\mathcal{C}_1[1]$, $\ell > k$. We define $\overline{T} : B\mathcal{C}_1[1] \to B_1\mathcal{C}_1[1]$ by $\overline{T} = T_\ell$ on $B_\ell\mathcal{C}_1[1]$, $1 \le \ell \le k-1$ and $T = 0$ on $B_\ell\mathcal{C}_1[1]$, $\ell > k$ or $\ell = 0$.

We then define $\hat{T}$ and $\hat{\overline{T}}$ in the same way as (8.17). Namely we put

$$\begin{aligned} \hat{T}(\mathbf{x}) &= \textstyle\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \hat{\mathcal{F}}'(\hat{\mathcal{F}}(\mathbf{x}_a^{(1)})) \otimes T(\mathbf{x}_a^{(2)}) \otimes \mathbf{x}_a^{(3)}, \\ \hat{\overline{T}}(\mathbf{x}) &= \textstyle\sum_a (-1)^{\deg' \mathbf{x}_a^{(1)}} \hat{\mathcal{F}}'(\hat{\mathcal{F}}(\mathbf{x}_a^{(1)})) \otimes \overline{T}(\mathbf{x}_a^{(2)}) \otimes \mathbf{x}_a^{(3)}. \end{aligned} \tag{8.25}$$

We consider a filtration $\mathfrak{F}$ (the number filtration) of $B\mathcal{C}_1[1]$ such that $\mathfrak{F}^k B\mathcal{C}_1[1] = \bigoplus_{i=0}^k B_i\mathcal{C}_1[1]$. Note that $\hat{T}$ does not preserve this filtration since $T_0(B_0\mathcal{C}_1[1]) \subset B_1\mathcal{C}_1[1]$ is nonzero. But $\hat{\hat{T}}$ does preserve this filtration. Therefore $\hat{\hat{T}}$ induces an R module homomorphism $B_{a,\dots,b}\mathcal{C}_1[1] \to B_{a,\dots,b}\mathcal{C}_1[1]$. We write it as $\overline{T}_{a,\dots,b}$.

We next recall that $\mathcal{F}' : B\mathcal{C}_2[1] \to B_1\mathcal{C}_2[1]$ is equal to $\mathcal{F}'_\ell$ on $B_\ell\mathcal{C}_2[1]$ $1 \le \ell \le k-1$, and is equal to 0 on $B_\ell\mathcal{C}_2[1]$ $\ell \ge k$.

Sublemma 8.26. *T_ℓ $0 \le \ell \le k-1$, is an A_{k-1} transformation, if and only if*

$$\mathcal{F} \circ \tilde{\mathcal{F}}'_{1,\dots,k-1} - \mathrm{id} - \mathfrak{m} \circ \overline{T}_{1,\dots,k-1} - \overline{T} \circ \hat{d} = 0. \tag{8.27}$$

Here (8.27) is a formula for an element in

$$\mathrm{Hom}(B_{1,\dots,k-1}\mathcal{C}_2[1], B_1\mathcal{C}_2[1]),$$

and

$$\mathrm{id} \in \mathrm{Hom}(B_1\mathcal{C}_2[1], B_1\mathcal{C}_2[1]) \subset \mathrm{Hom}(B_{1,\dots,k-1}\mathcal{C}_2[1], B_1\mathcal{C}_2[1]).$$

Proof. We consider

$$\tilde{\Psi} = -\hat{d} \circ T_{0,\dots,k-1} + T_{0,\dots,k-1} \circ \hat{d} \in \mathrm{Hom}(B_{0,\dots,k-1}\mathcal{C}_2[1], B\mathcal{C}_2[1]). \tag{8.28}$$

Let Ψ be its $\mathrm{Hom}(B_{0,\dots,k-1}\mathcal{C}_2[1], B_1\mathcal{C}_2[1])$ component. It is easy to see

$$\tilde{\Psi}(\mathbf{x}) = \sum_a \mathbf{x}_a^{(1)} \otimes \Psi(\mathbf{x}_a^{(2)}) \otimes \mathbf{x}_a^{(3)}.$$

Hence $\tilde{\Psi}$ is zero if and only if Ψ is zero. On the other hand $\tilde{\Psi}$ is zero if and only if T_ℓ, $0 \le \ell \le k-1$ is an A_{k-1} transformation, by definition. Hence, to prove Sublemma 8.26, it suffices to show that Ψ is the left hand side of (8.27). This can be easily seen by using (8.18) and the fact that $\mathbf{e}_c$ is the unit as follows.

Let us consider the $\mathrm{Hom}(B_{0,\dots,k-1}\mathcal{C}_2[1], B_1\mathcal{C}_2[1])$ component of (8.28). We find that the sum of the terms of it which contains T_0 is

$$\mathbf{x} \mapsto \sum_a \mathfrak{m}\left((-1)^{\deg' \mathbf{x}_a^{(1)}} \hat{\mathcal{F}}(\hat{\mathcal{F}}'(\mathbf{x}_a^{(1)})) \otimes \mathbf{e} \otimes \mathbf{x}_a^{(2)}\right). \tag{8.29}$$

The terms of the right hand side of (8.29) are zero unless $\mathbf{x}_a^{(2)} = 1$ or $\mathbf{x}_a^{(1)} = 1$. In the first case it is

$$(-1)^{\deg' \mathbf{x}} \mathfrak{m}_2(\hat{\mathcal{F}}(\hat{\mathcal{F}}'(\mathbf{x})), \mathbf{e}) = -\hat{\mathcal{F}}(\hat{\mathcal{F}}'(\mathbf{x})).$$

In the second case, it is

$$\mathfrak{m}(\mathbf{e}\otimes\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{x}\in\mathcal{C}_1[1], \\ 0 & \text{otherwise.} \end{cases}$$

Thus they correspond to the first and second terms of (8.27). The terms of the $\mathrm{Hom}(B_{0,\dots,k-1}\mathcal{C}_2[1], B_1\mathcal{C}_1[1])$ component of (8.28) which do not contain T_0 correspond to the third and fourth terms of (8.27). (Note $\mathfrak{deg} T = 0$.)

The proof of Sublemma 8.26 is complete. □

We go back to the proof of Lemma 8.24. (8.27) implies

$$\mathcal{F}_{a,\dots,b}\circ\mathcal{F}'_{a,\dots,b} - \mathrm{id} - \hat{d}\circ\overline{T}_{a,\dots,b} - \overline{T}_{a,\dots,b}\circ\hat{d} = 0,$$

if $b-a<k-1$. (Here id is the indentity on $B_{a,\dots,b}\mathcal{C}_2[1]$.) We put

$$\Phi = \mathcal{F}_{1,\dots,k}\circ\tilde{\mathcal{F}}'_{1,\dots,k} - \mathrm{id} - \hat{d}\circ\overline{T}_{,\dots,k} - \overline{T}_{1,\dots,k}\circ\hat{d}.$$

Since Φ induces zero on $B_{2,\dots,k}\mathcal{C}_2[1]$ and on $B_{1,\dots,k-1}\mathcal{C}_2[1]$ it follows that

$$\Phi\in\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_2[1]).$$

In other words, $\Phi = 0$ on $B_{1,\dots,k-1}\mathcal{C}_2[1]$ and the image of Φ is contained in $B_1\mathcal{C}_2[1]$.

Now we calculate:

$$\begin{aligned}
\mathcal{F}_1\circ\Xi &= \mathcal{F}_{1,\dots,k}\circ\Xi \\
&= \mathcal{F}_{1,\dots,k}\circ\tilde{\mathcal{F}}'_{1,\dots,k}\circ\hat{d} - \mathcal{F}_{1,\dots,k}\circ\hat{d}\circ\tilde{\mathcal{F}}'_{1,\dots,k} \\
&= \mathcal{F}_{1,\dots,k}\circ\tilde{\mathcal{F}}'_{1,\dots,k}\circ\hat{d} - \mathfrak{m}_1\circ\mathcal{F}_{1,\dots,k}\circ\tilde{\mathcal{F}}'_{1,\dots,k} \\
&= (\mathrm{id} + \hat{d}\circ\overline{T}_{1,\dots,k-1} + \overline{T}_{1,\dots,k-1}\circ\hat{d} + \Phi)\circ\hat{d} \\
&\quad -\hat{d}\circ(\mathrm{id} + \hat{d}\circ\overline{T}_{1,\dots,k-1} + \overline{T}_{1,\dots,k-1}\circ\hat{d} + \Phi) \\
&= -d\Phi.
\end{aligned}$$

The proof of Lemma 8.24 is complete. □

We define $\mathcal{F}'_k\in\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_1[1])$ such that

$$d\mathcal{F}'_k = -\Xi. \tag{8.30}$$

It is now easy to see that $\mathcal{F}'_\ell$, $0\le\ell\le k$ is an A_k functor.

We remark that $\mathcal{F}'_k$ satisfying (8.30) is not unique. Namely we can take any element Ψ in the kernel of $d : \mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_1[1]) \to$

$\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_1[1])$ and can replace $\mathcal{F}'_k$ by $\mathcal{F}'_k + \Psi$. We will use this freedom in the next step.

Now we are going to construct T_k. Using T_ℓ, $\ell = 1, \dots, k-1$ (and $\mathcal{F}'_\ell$) we define

$$\overline{T}_{a,\dots,b} : B_{a,\dots,b}\mathcal{C}_2[1] \to B_{a,\dots,b}\mathcal{C}_2[1].$$

Using T_ℓ, $\ell = 0, 1, \dots, k-1$ (and $\mathcal{F}'_\ell$) we define

$$\hat{T}_{1,\dots,k-1} : \bigoplus_{\ell=0}^{k-1} B_\ell\mathcal{C}_2[1] \to \bigoplus_{\ell=0}^{\infty} B_\ell\mathcal{C}_2[1].$$

$\hat{T}_{1,\dots,k-1}$ is a chain map by induction hypothesis. We consider

$$\overline{T}_{1,\dots,k} : \bigoplus_{i=1}^{k} B_i\mathcal{C}_2[1] \to \bigoplus_{i=1}^{k} B_i\mathcal{C}_2[1].$$

We put

$$\mathfrak{T} = \mathcal{F}_{1,\dots,k} \circ \mathcal{F}'_{1,\dots,k} - \mathrm{id} - \hat{d} \circ \overline{T}_{1,\dots,k} - \overline{T}_{1,\dots,k} \circ \hat{d}. \tag{8.31}$$

In the same way as the proof of Sublemma 8.26, we can use the fact that $\hat{T}_{1,\dots,k-1}$ is a chain map, to show the following:

Lemma 8.32. *The image of $\mathfrak{T}$ is contained in $B_1\mathcal{C}_2[1]$. Moreover $\mathfrak{T}$ vanishes on $\bigoplus_{i=1}^{k-1} B_i\mathcal{C}_2[1]$.*

We may regard $\mathfrak{T} \in \mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_2[1])$ by Lemma 8.32. Then, by definition and Lemma 8.32 we have $d\mathfrak{T} = 0$.

Lemma 8.33. *We can choose $\mathcal{F}'_k$ so that $\mathfrak{T} \in \mathrm{Im}\, d$.*

Proof. If we replace $\mathcal{F}'_k$ by $\mathcal{F}'_k + \Psi$, then, by (8.31), $\mathfrak{T}$ is replaced by $\mathfrak{T} + \mathcal{F}_1 \circ \Psi$. The lemma now follows from the fact that $\mathcal{F}_1$ induces an isomorphism

$$H(\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_1[1]), d) \cong H(\mathrm{Hom}(B_k\mathcal{C}_2[1], B_1\mathcal{C}_2[1]), d).$$

□

Lemma 8.33 and Sublemma 8.26 immediately imply that we can choose T_k such that T_ℓ, $\ell = 0, \dots, k$ is an A_k transformation.

Thus we have constructed $\mathcal{F}'$ and $T : \mathcal{F} \circ \mathcal{F}' \to \mathrm{Id}^{\mathcal{C}_2}$. We next show that T is a homotopy equivalence. We prove it by using the following general result:

Proposition 8.34. *Let $\mathcal{G}$, $\mathcal{G}'$ be A_∞ functors $\mathcal{C} \to \mathcal{C}'$. Let $T : \mathcal{G} \to \mathcal{G}'$ be a natural transformation. We assume that $T_0(c) : \mathcal{G}_0(c) \to \mathcal{G}'_0(c)$ is a homotopy equivalence for any c.*

Then there exits a natural transformation $T' : \mathcal{G}' \to \mathcal{G}$, and pre natural transformations $S : \mathcal{G} \to \mathcal{G}$, $S' : \mathcal{G}' \to \mathcal{G}'$ such that

$$\mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_2(T', T) = \mathfrak{M}_1(S), \quad \mathrm{Id}^{\mathcal{G}} - \mathfrak{M}_2(T, T') = \mathfrak{M}_1(S').$$

Proof. The proof is similar to the construction of $\mathcal{F}'$ and T above. Namely we construct

$$T'_k : B_k\mathcal{C}[1] \to B_1\mathcal{C}'[1], \quad S'_k : B_k\mathcal{C}[1] \to B_1\mathcal{C}'[1]$$

inductively. More precisely we prove the following lemma by induction on k.

Lemma 8.35.k. *Suppose T'_ℓ, for $\ell = 0, \dots, k$ is an A_k transformation and $S'_\ell : B_\ell\mathcal{C}[1] \to B_1\mathcal{C}[1]$ for $\ell = 0, \dots, k$ are R module homomorphisms. We define $T'_{(k)}$, $S'_{(k)}$ by putting $T'_{(k),i} = S'_{(k),i} = 0$ for $i > k$, and $T'_{(k),i} = T'_i$, $S'_{(k),i} = S'_i$ for $i \le k$. Then we have:*

$$\mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_2(T'_{(k)}, T) = \mathfrak{M}_1(S'_{(k)}), \quad \text{on } B_k\mathcal{C}_2[1]. \tag{8.36}$$

Proof. For $k = 0$, we let $T'_0(c) \in \mathcal{C}_2(\mathcal{G}'(c), \mathcal{G}(c))$ be a homotopy inverse to $T_0(c)$. Then there exists $S'_0(c) \in \mathcal{C}_2(\mathcal{G}'(c), \mathcal{G}'(c))$ such that

$$\mathfrak{m}_1(T'_0(c)) = 0, \quad \mathfrak{m}_1(S'_0(c)) = \mathfrak{m}_2(T'_0(c), T_0(c)). \tag{8.37}$$

Thus, we have proved Lemma 8.35.0.

We assume Lemma 8.35.$k-1$ and will prove Lemma 8.35.k.

Sublemma 8.38. *$\mathfrak{M}_1(S'_{(k-1)})$ is zero on $B_{k-1}\mathcal{C}'[1]$. The image of the restriction of $\mathfrak{M}_1(S'_{(k-1)})$ to $B_k\mathcal{C}'[1]$ is in $B_1\mathcal{C}'[1]$.*

The proof is similar to the proof of Lemma 8.22 and is omitted.

We let Ψ be the restriction of $\mathfrak{M}_1(S'_{(k-1)})$ to $B_k\mathcal{C}'[1]$. Sublemma 8.38 implies

$$\Psi \in \mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}'[1]).$$

Sublemma 8.39. $d_1 \circ \Psi - \Psi \circ d_1 = 0$.

Proof. This follows immediately from $\mathfrak{M}_1(\mathfrak{M}_1(S'_{(k-1)})) = 0$ and Sublemma 8.38. □

Sublemma 8.40. *There exists $\Phi \in \mathrm{Hom}(B_k\mathcal{C}_2, B_1\mathcal{C}_2)$. such that:*

$$\Psi = d_1 \circ \Phi + \Phi \circ d_1.$$

Proof. We define

$$T_{0*} : \mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}[1]) \to \mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}'[1])$$

as follows. Let $V \in \mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}[1])$ and $c_0, \dots, c_k \in \mathfrak{Ob}(\mathcal{C}')$, $x_i \in \mathcal{C}'(c_{i-1}, c_i)$. We then put

$$T_{0*}(V)(x_1, \dots, x_k) = \mathfrak{m}_2(V(x_1, \dots, x_k), T_0(c_k)).$$

It follows from assumption that T_{0*} induces an isomorphism on cohomology. Hence, by Sublemma 8.39, it suffices to show that $T_{0*}(\Psi)$ is a boundary. By Sublemma 8.38, we find that $T_{0*}(\Psi)$ is a restriction of $\mathfrak{M}_2(\Psi, T)$ to $B_k\mathcal{C}'[1]$. By A_∞ formula of $\mathfrak{M}_k$ we have

$$\mathfrak{M}_2(\Psi, T) = \mathfrak{M}_2(\mathfrak{M}_1(T'_{(k-1)}), T) = \mathfrak{M}_1(\mathfrak{M}_2(T'_{(k-1)}), T).$$

By induction hypothesis

$$\mathfrak{M}_2(T'_{(k-1)}, T) = \mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_1(S'_{(k-1)})$$

on $B_{k-1}\mathcal{C}'[1]$. Let $-\Phi$ be the restriction of $\mathfrak{M}_2(T'_{(k-1)}, T)$ on $B_k\mathcal{C}'[1]$. Then, by induction hypothesis, $\Phi \in \mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}'[1])$. We now have

$$\mathfrak{M}_1(\mathfrak{M}_2(T'_{(k-1)}), T) = -\mathfrak{M}_1(\Phi) = \hat{d}_1 \circ \Phi + \Phi \circ \hat{d}$$

as required. □

If we put $T'_k = \Phi$ then we have

$$\mathfrak{M}_1(T'_{(k)}) = 0.$$

We remark that Φ satisfying the conclusion of Sublemma 8.39 is not unique. Namely we may change it by any cocycle in $\mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}'[1])$.

We next define S'_k. Using induction hypothesis, we can prove that the restriction of

$$\mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_2(T'_{(k-1)}, T) - \mathfrak{M}_1(S'_{(k-1)})$$

to $B_k\mathcal{C}'[1]$ defines an element of $\mathrm{Hom}(B_k\mathcal{C}'[1], B_1\mathcal{C}'[1])$. We denote it by Ξ. We can prove that Ξ is a cocycle in a way similar to the proof of Sublemma 8.39. Therefore, we may choose T'_k so that

$$\Xi - T_{0*}(T'_k)$$

is a coboundary. Hence we may choose S'_k such that (8.36) is satisfied. The proof of Lemma 8.35.k is now complete. □

By Lemma 8.35, we obtain T' and S'. To construct S, we proceed as follows. We apply the above construction to T' in place of T and obtain T'' and S'' such that

$$\mathrm{Id}^{\mathcal{G}} - \mathfrak{M}_2(T', T'') = \mathfrak{M}_1(S'').$$

Then

$$T'' \equiv \mathfrak{M}_2(\mathfrak{M}_2(T, T'), T'') \equiv \mathfrak{M}_2(T, \mathfrak{M}_2(T', T'')) \equiv T \mod \mathrm{Im}(\mathfrak{M}_1).$$

Therefore

$$\mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_2(T', T) \equiv \mathrm{Id}^{\mathcal{G}'} - \mathfrak{M}_2(T', T'') \equiv 0 \mod \mathrm{Im}(\mathfrak{M}_1).$$

The proof of Proposition 8.34 is complete. □

We need here some elementary properties of homotopy equivalence.

Proposition-Definition 8.41. *An A_∞ functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ induces A_∞ functors $\mathcal{F}_* : \mathfrak{Funk}(\mathcal{C}, \mathcal{C}_1) \to \mathfrak{Funk}(\mathcal{C}, \mathcal{C}_2)$, $\mathcal{F}^* : \mathfrak{Funk}(\mathcal{C}_2, \mathcal{C}) \to \mathfrak{Funk}(\mathcal{C}_1, \mathcal{C})$ such that $(\mathcal{F}_*)_0(\mathcal{G}) = \mathcal{F} \circ \mathcal{G}$, $(\mathcal{F}^*)_0(\mathcal{G}) = \mathcal{G} \circ \mathcal{F}$.*

Proof. Let $T^i \in \mathfrak{Funk}(\mathcal{G}^{i-1}, \mathcal{G}^i)$ be pre natural transformations such that $\mathfrak{deg}' T^i = t_i$. We put

$$\begin{aligned}((\mathcal{F}_*)_k(T^1, \dots, T^k))(\mathbf{x}) = \sum_a (-1)^{\epsilon_a}\, \mathcal{F}(\hat{\mathcal{G}}^0(\mathbf{x}_a^{(1)}), \hat{T}^1(\mathbf{x}_a^{(2)}), \\ \dots, \hat{T}^k(\mathbf{x}_a^{(2k)}), \hat{\mathcal{G}}^k(\mathbf{x}_a^{(2k+1)})),\end{aligned} \tag{8.42}$$

where

$$\epsilon_a = \sum_{j=1}^{k} \sum_{i=1}^{2j-1} t_j \deg' \mathbf{x}_a^{(i)}.$$

We can prove that (8.42) defines an A_∞ functor in a way similar to the proof of Theorem 7.55. (We omit the detail.)

We next define

$$\begin{aligned} ((\mathcal{F}^*)_1(T^1))(\mathbf{x}) &= T^1(\hat{\mathcal{F}}(\mathbf{x})), \\ ((\mathcal{F}^*)_k(T^1, \dots, T^k))(\mathbf{x}) &= 0 \qquad k > 2. \end{aligned}$$

It is easy see that $\mathcal{F}^*$ is an A_∞ functor. □

The following two corollaries are immediate consequences.

Corollary 8.43. *If $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is homotopic to $\mathcal{F}' : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ is homotopic to $\mathcal{G}' : \mathcal{C}_2 \to \mathcal{C}_3$, then $\mathcal{G} \circ \mathcal{F}$ is homotopic to $\mathcal{G}' \circ \mathcal{F}'$.*

Corollary 8.44. *If $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ are homotopy equivalences then $\mathcal{G} \circ \mathcal{F}$ is a homotopy equivalence.*

We are now in the position to complete the proof of Proposition 8.9. We have constructed $\mathcal{F}'$. By using Proposition 8.34, we can prove that $\mathcal{F} \circ \mathcal{F}'$ is homotopic to identity.

We next prove that $\mathcal{F}' \circ \mathcal{F}$ is homotopic to identity. For this purpose, we apply the same argument to $\mathcal{F}'$ and obtain $\mathcal{F}''$ such that $\mathcal{F}' \circ \mathcal{F}''$ is homotopic to identity. It follows that $\mathcal{F}''$ is homotopic to $\mathcal{F}$. Hence by Corollary 8.43, $\mathcal{F}' \circ \mathcal{F}$ is homotopic to identity. The proof of Proposition 8.9 is complete. □

We continue the proof of Theorem 8.6. In this section we complete the proof in the case of differential graded category and postpone the proof of the general case to the next section.

Our next goal is the proof of Lemma 8.45, which is another case of Theorem 8.6. We define the notion of full subcategory of A_∞ category in an obvious way.

Lemma 8.45. *Let $\mathcal{C}_2$ be an A_∞ category such that $\mathfrak{m}_k = 0$ for $k \ge 3$. Let $\mathcal{C}_1$ be a full subcategory of $\mathcal{C}_2$ and $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ be the inclusion. We assume* (8.7.2). *Then $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is a homotopy equivalece.*

Proof. Let $c \in \mathfrak{Ob}(\mathcal{C}_2) - \mathfrak{Ob}(\mathcal{C}_1)$. We choose $\mathcal{F}'_0(c) \in \mathfrak{Ob}(\mathcal{C}_1)$ which is homotopy equivalent to c. If $c \in \mathfrak{Ob}(\mathcal{C}_1)$ we put $\mathcal{F}'_0(c) = c$.

By the choice of $\mathcal{F}'_0(c)$, there exists

$$T'_0(c) \in \mathcal{C}_2(c, \mathcal{F}'_0(c)), \quad T_0(c) \in \mathcal{C}_2(\mathcal{F}'_0(c), c)$$

and

$$S'_0(c) \in \mathcal{C}_2(c, c), \quad S_0(c) \in \mathcal{C}_2(\mathcal{F}'_0(c), \mathcal{F}'_0(c))$$

such that

$$\begin{aligned} \mathbf{e}_{\mathcal{F}_0(c)} - \mathfrak{m}_2(T_0(c), T'_0(c)) &= \mathfrak{m}_1(S_0(c)) \\ \mathbf{e}_{\mathcal{F}'_0(c)} - \mathfrak{m}_2(T'_0(c), T_0(c)) &= \mathfrak{m}_1(S'_0(c)). \end{aligned}$$

For $c \in \mathfrak{Ob}(\mathcal{C}_1)$ we put $T_0(c) = T'_0(c) = \mathrm{Id}^c$ and $S_0(c) = S'_0(c) = 0$.

We next define $\mathcal{F}'_k$. We first put

$$x \circ y = (-1)^{\deg x} \mathfrak{m}_2(x, y), \quad d(x) = \mathfrak{m}_1(x).$$

Then, using the fact that $\mathfrak{m}_k = 0$ for $k \ge 3$, we have

$$\begin{aligned} d(x \circ y) &= dx \circ y + (-1)^{\deg x + 1} x \circ dy, \\ (x \circ y) \circ z &= x \circ (y \circ z), \\ \mathbf{e} \circ x &= x = x \circ \mathbf{e}. \end{aligned}$$

In view of the second formula, we may write

$$x_1 \circ x_2 \circ \cdots \circ x_k$$

etc. Now let $x_i \in \mathcal{C}_2(c_{i-1}, c_i)$. We put

$$\begin{aligned}
&\mathcal{F}'_k(x_1, \dots, x_k) \\
&\quad = \quad T'_0(c_0) \circ x_1 \circ S'_0(c_1) \circ x_2 \circ \cdots \circ S'_0(c_{k-1}) \circ x_k \circ T'_0(c_k).
\end{aligned}$$

We remark that

$$d(S'_0(c)) = \mathbf{e}_{\mathcal{F}_0(c)} - T'_0(c) \circ T_0(c).$$

It follows from definition that

$$\begin{aligned}
&d(\mathcal{F}'_k(x_1, \dots, x_k)) \\
&= \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} \mathcal{F}'_k(x_1, \dots, dx_i, \dots, x_k) \\
&\quad + \sum_i (-1)^{\deg x_1 + \cdots \deg x_i + i} \mathcal{F}'_k(x_1, \dots, x_i \circ x_{i+1}, \dots, x_k) \\
&\quad + \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1 + 1} \mathcal{F}'_i(x_1, \dots, x_i) \circ \mathcal{F}'_{k-i}(x_{i+1}, \dots, x_k).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathfrak{m}_1(\mathcal{F}'_k(x_1, \dots, x_k)) \\
&+ \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} \mathcal{F}'_i(x_1, \dots, x_i) \circ \mathcal{F}'_{k-i}(x_{i+1}, \dots, x_k) \\
&\qquad = \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} \mathcal{F}'_k(x_1, \dots, \mathfrak{m}_1(x_i), \dots, x_k) \\
&\qquad\quad + (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} \mathcal{F}'_k(x_1, \dots, \mathfrak{m}_2(x_i, x_{i+1}), \dots, x_k).
\end{aligned}$$

It follows that $\mathcal{F}'_k$, $k = 0, \dots$ is an A_∞ functor.

The composition $\mathcal{F}' \circ \mathcal{F}$ is an identity functor. We are going to show that $\mathcal{F} \circ \mathcal{F}'$ is homotopic to identity. For this purpose, we are going to construct $T : \mathcal{F} \circ \mathcal{F}' \to 1_{\mathcal{C}_2}$ satisfying the assumption of Proposition 8.34. We already constructed T_0. We remark that

$$(\mathcal{F} \circ \mathcal{F}')(x_1, \dots, x_k) = \mathcal{F}'_k(x_1, \dots, x_k).$$

Hence the condition $T : \mathcal{F} \circ \mathcal{F}' \to \mathrm{Id}^{\mathcal{C}_2}$ to be an A_∞ functor can be written as

$$\begin{aligned}
&\sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1 + 1} T_0(c_0) \circ x_1 \circ S_0'(c_1) \circ x_2 \circ S_0'(c_2) \circ \cdots \\
&\qquad \cdots \circ x_i \circ T_0'(c_i) \circ T_{k-i}(x_{i+1}, \ldots, x_k) \\
&= \mathfrak{m}_1(T_k(x_1, \ldots, x_k)) \\
&\quad + \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} T_k(x_1, \ldots, \mathfrak{m}_1(x_i), \ldots, x_k) \\
&\quad + \sum_i (-1)^{\deg x_1 + \cdots \deg x_{i-1} + i - 1} T_k(x_1, \ldots, \mathfrak{m}_2(x_i, x_{i+1}), \ldots, x_k).
\end{aligned} \tag{8.46}$$

We put

$$T_k(x_1, \ldots, x_k) = T_0(c_0) \circ x_1 \circ S_0'(c_1) \circ x_2 \circ \cdots \circ x_k \circ S_0'(c_k).$$

(8.46) can be checked easily.

Lemma 8.45 now follows from Proposition 8.34. □

Now we show:

Proposition 8.47. *Theorem* 8.6 *holds if* $\mathfrak{m}_k = 0$ *for* $k > 2$ *in* $\mathcal{C}_1$, $\mathcal{C}_2$.

Proof. Let $\mathcal{C}_2'$ be the full subcategory such that $\mathfrak{Ob}(\mathcal{C}_2')$ is the image of $\mathcal{F}_0$. Lemma 8.41 implies that the inclusion $\mathcal{C}_2' \to \mathcal{C}_2$ is a homotopy equivalence.

For each $c \in \mathfrak{Ob}(\mathcal{C}_2')$ we take and fix $\alpha(c) \in \mathfrak{Ob}(\mathcal{C}_1)$ such that $\mathcal{F}_0(\alpha(c)) = c$. Let $\mathcal{C}_1'$ be the full subcategory of $\mathcal{C}_1$ such that $\mathfrak{Ob}(\mathcal{C}_1')$ is the image of α. Proposition 8.9 implies that the restriction of $\mathcal{F}$ to $\mathcal{C}'$ induces a homotopy equivalence $\mathcal{C}_1' \to \mathcal{C}_2'$. Therefore, using Corollary 8.43 and 8.44, it suffices to show that $\mathcal{C}_1' \to \mathcal{C}_1$ is a homotopy equivalence to complete the proof of Proposition 8.47. This follows from Lemma 8.45 and the following Lemma 8.48. □

Lemma 8.48. *We assume* (8.7). *If* $\mathcal{F}_0(b) = \mathcal{F}_0(b')$ *then* b *is homotopy equivalent to* b'.

Proof. By (8.7) we have

$$\mathcal{F}_{1*} : H(\mathcal{C}_1(b, b'); \mathfrak{m}_1) \to H(\mathcal{C}_2(\mathcal{F}_0(b), \mathcal{F}_0(b')); \mathfrak{m}_1)$$

is an isomorphism. We take $[f] \in H(\mathcal{C}_1(b, b'); \mathfrak{m}_1)$ which is mapped to $[\mathbf{e}_{\mathcal{F}_0(b)}]$ by $\mathcal{F}_{1*}$. It is easy to see that

$$\begin{aligned}
\mathfrak{m}_2(f, \cdot) &: \mathcal{C}_1(b', a) \to \mathcal{C}_1(b, a) \\
\mathfrak{m}_2(\cdot, f) &: \mathcal{C}_1(a, b) \to \mathcal{C}_1(a, b')
\end{aligned}$$

induces isomorphism on homology. The lemma then follows from Lemma 6.24. □

We need the following proposition in the next section.

Proposition 8.49. *If $\mathcal{C}_1$ is homotopy equivalent to $\mathcal{C}_2$, then $\mathfrak{Funk}(\mathcal{C},\mathcal{C}_1)$ is homotopy equivalent to $\mathfrak{Funk}(\mathcal{C},\mathcal{C}_2)$ and $\mathfrak{Funk}(\mathcal{C}_1,\mathcal{C})$ is homotopy equivalent to $\mathfrak{Funk}(\mathcal{C}_2,\mathcal{C})$.*

Proof. Let $\mathcal{F}:\mathcal{C}_1\to\mathcal{C}_2$, $\mathcal{F}':\mathcal{C}_2\to\mathcal{C}_1$ be as in Definition 8.5 and let H, H' be natural transformations from $\mathcal{F}\circ\mathcal{F}'$ to identity functor and from identity functor to $\mathcal{F}\circ\mathcal{F}'$ respectively. Let $\mathcal{G}:\mathcal{C}\to\mathcal{C}_2$ be an A_∞ functor. We put

$$\begin{aligned}\mathfrak{H}_0(\mathcal{G}) &= (\mathcal{G}^*)_1(H)\in\mathfrak{Funk}((\mathcal{F}_*\circ\mathcal{F}'_*)_0(\mathcal{G}),\mathcal{G})\\ \mathfrak{H}_k &= 0,\quad k>0.\quad (\text{Note } (\mathcal{G}^*)_k=0 \text{ for } k>1.)\end{aligned}$$

It is easy to check that $\mathfrak{H}$ is a natural transform from $\mathcal{F}_*\circ\mathcal{F}'_*$ to the identity functor $\mathrm{Id}_{\mathfrak{Funk}(\mathcal{C},\mathcal{C}_2)}$. We define in a similar way a natural transformation $\mathfrak{H}'$ from $\mathrm{Id}_{\mathfrak{Funk}(\mathcal{C},\mathcal{C}_2)}$ to $\mathcal{F}_*\circ\mathcal{F}'_*$. We assume that

$$\mathfrak{M}_2(H,H')-\mathrm{Id}_{\mathrm{Id}_{\mathfrak{Funk}(\mathcal{C}_1,\mathcal{C}_1)}}=\mathfrak{M}_1(T).$$

(Note that the confusing symbol $\mathrm{Id}_{\mathrm{Id}_{\mathfrak{Funk}(\mathcal{C}_1,\mathcal{C}_1)}}$ denotes the identity *transform* from the identity *functor* : $\mathcal{C}_1\to\mathcal{C}_1$ to itself.) We assume also that

$$\mathfrak{M}_2(H',H)-\mathrm{Id}_{\mathcal{F}\circ\mathcal{F}'}=\mathfrak{M}_1(T').$$

We put

$$\begin{aligned}\mathfrak{T}_0(\mathcal{G}) &= (\mathcal{G}^*)_1(T)\in\mathfrak{Funk}((\mathcal{F}_*\circ\mathcal{F}'_*)_0(\mathcal{G}),\mathcal{G})\\ \mathfrak{T}_k &= 0,\quad k>0.\end{aligned}$$

And we define $\mathfrak{T}'$ in a similar way. Then we find

$$\begin{aligned}\mathrm{Id}_{\mathrm{Id}_{\mathfrak{Funk}(\mathcal{C},\mathcal{C}_1)}}-\hat{\mathfrak{M}}_2(\mathfrak{H},\mathfrak{H}') &= \hat{\mathfrak{M}}_1(\mathfrak{T}),\\ \mathrm{Id}_{(\mathcal{F}\circ\mathcal{F}')_*}-\hat{\mathfrak{M}}_2(\mathfrak{H}',\mathfrak{H}) &= \hat{\mathfrak{M}}_1(\mathfrak{T}').\end{aligned}$$

Here $\hat{\mathfrak{M}}$ is the A_∞ structure on $\mathfrak{Funk}(\mathfrak{Funk}(\mathcal{C},\mathcal{C}_1),\mathfrak{Funk}(\mathcal{C},\mathcal{C}_1))$.

Thus we proved that $(\mathcal{F}\circ\mathcal{F}')_*$ is homotopic to identity. This complete the proof of the first half of the Proposition 8.49. The second half can be proved in a similar way. □

§9. An A_∞ analogue of Yoneda's lemma

In §7 we constructed an A_∞ functor $\mathfrak{Rep}_0(c) : \mathcal{C}^o \to \mathcal{CH}$ for each object c of $\mathcal{C}$. The purpose of this section is to make it functorial. Moreover we prove that $c \mapsto \mathfrak{Rep}_0(c)$ defines a homotopy equivalence between the A_∞ category $\mathcal{C}$ and one of representable A_∞ functors : $\mathcal{C}^o \to \mathcal{CH}$. Namely we prove the following Theorem 9.1. We let $\mathfrak{Rep}(\mathcal{C}^o, \mathcal{CH})$ be the full subcategory of $\mathfrak{Funk}(\mathcal{C}^o, \mathcal{CH})$ such that $\mathfrak{Ob}(\mathfrak{Rep}(\mathcal{C}^o, \mathcal{CH}))$ is the set of all representable A_∞ functors. We define a full subcategory $\mathfrak{DRep}(\mathcal{C}^o, \mathcal{CH})$ of $\mathfrak{Funk}(\mathcal{C}^o, \mathcal{CH})$ such that $\mathfrak{Ob}(\mathfrak{DRep}(\mathcal{C}^o, \mathcal{CH}))$ is the set of all derived representable A_∞ functors.

Theorem 9.1. *There exists a homotopy equivalences of A_∞ functors $\mathfrak{Rep} : \mathcal{C} \cong \mathfrak{Rep}(\mathcal{C}^o, \mathcal{CH})$, $\mathfrak{DRep} : \mathcal{DC} \cong \mathfrak{DRep}(\mathcal{C}^o, \mathcal{CH})$, such that $\mathfrak{Rep}_0(c)$ is an in Definition* 7.28.

Remark 9.2. The first half of Theorem 9.1 was proved in [Fu4] §12, over $\mathbb{Z}_2$ coefficient. In this article we are going to put precise sign in its proof. We also improve the presentation of the proof in [Fu4].

Remark 9.3. In the case of usual category, the first half of Theorem 9.1, which is Yoneda's lemma, is well known.

We remark that actually $\mathfrak{Rep}(\mathcal{C}^o, \mathcal{CH})$ is a differential graded category, since all the higher compositions are zero. Therefore Theorem 9.1 implies the following:

Corollary 9.4. *Any A_∞ category is homotopy equivalent to a differential graded category.*

Proof of Theorem 9.1. We already defined $\mathfrak{Rep}_0$. We will define $\mathfrak{Rep}_k : B_k(\mathcal{C}) \to B_1(\mathfrak{Rep}(\mathcal{C}^o, \mathcal{CH}))$. Let

$$c_0, \dots, c_k \in \mathfrak{Ob}(\mathcal{C}), \ x_i \in \mathcal{C}(c_{i-1}, c_i).$$

We need to define a natural transformation

$$\mathfrak{Rep}_k(x_1, \dots, x_k) : \mathfrak{Rep}_0(c_0) \to \mathfrak{Rep}_0(c_k). \tag{9.5}$$

Let $b_0, \dots, b_\ell \in \mathfrak{Ob}(\mathcal{C})$, $y_i \in \mathcal{C}^o(b_{i-1}, b_i) = \mathcal{C}(b_i, b_{i-1})$. To define (9.5) we need to define

$$\mathfrak{Rep}_k(x_1, \dots, x_k)_\ell(y_1, \dots, y_\ell) \in \mathrm{Hom}(\mathfrak{Rep}_0(c_0)_0(b_0), \mathfrak{Rep}_0(c_k)_0(b_\ell)).$$

Let $z \in \mathfrak{Rep}_0(c_0)_0(b_\ell) = \mathcal{C}(b_0, c_0)$. We put $\mathbf{x} = x_1 \otimes \cdots \otimes x_k$, $\mathbf{y} = y_1 \otimes \cdots \otimes y_\ell$. We use the notations $\mathbf{x}^{op} = x_k \otimes \cdots \otimes x_1$, $\epsilon(\mathbf{x}) = \sum_{i<j} (\deg x_i + 1)(\deg x_j + 1)$ etc. introduced in §7. Now we define

Definition 9.6.

$$(9.7)\ (\mathfrak{Rep}_k(\mathbf{x}))_\ell(\mathbf{y})(z) = (-1)^{\epsilon(\mathbf{y})+(\deg'\mathbf{x})(\deg'\mathbf{y}+\deg' z)}\mathfrak{m}_{k+\ell+1}(\mathbf{y}^{op}, z, \mathbf{x}).$$

We remark that the right hand side of (9.7) is in $\mathfrak{Rep}_0(c_k)(b_\ell) = \mathcal{C}(b_\ell, c_k)$. We also remark that

$$\mathfrak{deg}'(\mathfrak{Rep}_k(\mathbf{x}))_\ell(\mathbf{y}) = \deg'\mathbf{x} + \deg'\mathbf{y}.$$

Lemma 9.8. *(9.7) defines an A_∞ functor.*

Proof. By A_∞ formula of $\mathfrak{m}$ we have

$$\begin{aligned} 0 &= \sum_a (-1)^{\deg'\mathbf{y}_a^{(3)}}\mathfrak{m}(\mathbf{y}_a^{(3)op}, \mathfrak{m}(\mathbf{y}_a^{(2)op}), \mathbf{y}_1^{(1)op}, z, \mathbf{x}) \\ &\quad + \sum_{a,b} (-1)^{\deg'\mathbf{y}_a^{\prime\prime op}}\mathfrak{m}(\mathbf{y}_a^{\prime\prime op}, \mathfrak{m}(\mathbf{y}_a^{\prime op}, z, \mathbf{x}'_b), \mathbf{x}''_b) \\ &\quad + (-1)^{\deg'\mathbf{y}+\deg' z}\mathfrak{m}(\mathbf{y}^{op}, z, \hat{d}\mathbf{x}). \end{aligned} \tag{9.9}$$

We will rewrite each term of (9.9). The first term is equal to:

$$(9.10)\quad (-1)^{\epsilon_1}\mathfrak{Rep}(\mathbf{x})(\mathbf{y}_a^{(1)}, \mathfrak{m}^o(\mathbf{y}_a^{(2)}), \mathbf{y}_a^{(3)})(z) = (-1)^{\epsilon_2}\mathfrak{Rep}(\mathbf{x})(\hat{d}^o\mathbf{y})(z)$$

where $\hat{d}^{op}$ is the coderivation induced by $\mathfrak{m}^o$ on $\mathcal{C}^o$ and

$$\begin{aligned} \epsilon_1 &= \deg'\mathbf{y}_a^{(3)} + \epsilon(\mathbf{y}_a^{(2)}) + 1 + (\deg'\mathbf{x})(\deg'\mathbf{y} + \deg' z + 1) \\ &\quad + \epsilon(\mathbf{y}_a^{(1)}, \mathfrak{m}^o(\mathbf{y}_a^{(2)}), \mathbf{y}_a^{(3)}) \\ &= \deg'\mathbf{y}_a^{(1)} + (\deg'\mathbf{x})(\deg'\mathbf{y} + \deg' z + 1) + \epsilon(\mathbf{y}). \end{aligned}$$

Hence

$$\epsilon_2 = (\deg'\mathbf{x})(\deg'\mathbf{y} + \deg' z + 1) + \epsilon(\mathbf{y}). \tag{9.11}$$

We divide the second terms of (9.9) into the following five cases.
Case 1: $\mathbf{x}'_b \ne 1 \in B_0\mathcal{C}$, $\mathbf{x}''_b \ne 1 \in B_0\mathcal{C}$:
The second term of (9.9) of this case is

$$\begin{aligned} &\sum_{a,b} (-1)^{\epsilon_3(a,b)}\mathfrak{m}(\mathbf{y}_a^{\prime\prime op}, \mathfrak{Rep}(\mathbf{x}'_b)(\mathbf{y}'_a)(z), \mathbf{x}''_b) \\ &= \sum_{a,b} (-1)^{\epsilon_4(a,b)}\mathfrak{Rep}(\mathbf{x}''_b)(\mathbf{y}''_a)(\mathfrak{Rep}(\mathbf{x}'_b)(\mathbf{y}'_a)(z)) \\ &= \sum_{a,b} (-1)^{\epsilon_5(a,b)}\mathfrak{m}_2(\mathfrak{Rep}(\mathbf{x}'_b)(\mathbf{y}'_a), \mathfrak{Rep}(\mathbf{x}''_b)(\mathbf{y}''_a))(z) \end{aligned} \tag{9.12}$$

where $\mathfrak{m}_2$ in the third line is an operation in $\mathcal{CH}$ and

$$\begin{aligned}
\epsilon_3(a,b) &= \deg' \mathbf{y}''_a + (\deg' \mathbf{x}'_b)(\deg' \mathbf{y}'_a + \deg' z) + \epsilon(\mathbf{y}'_a) \\
\epsilon_4(a,b) &= \epsilon_3(a,b) + (\deg' \mathbf{x}''_b)(1 + \deg' \mathbf{y}''_a + \deg' z + \deg' \mathbf{x}'_b \\
&\quad + \deg' \mathbf{y}'_a) + \epsilon(\mathbf{y}''_a) \\
\epsilon_5(a,b) &= \epsilon_4(a,b) + (\deg' \mathbf{x}'_b + \deg' \mathbf{y}'_a + 1)(\deg' \mathbf{x}''_b + \deg' \mathbf{y}''_a) + 1.
\end{aligned}$$

We calculate (using $\mathfrak{deg}'\mathfrak{Rep}(\mathbf{x}'_b)(\mathbf{y}'_a) = \deg' \mathbf{x}'_b + \deg' \mathbf{y}'_a + 1$) that

$$\epsilon_5(a,b) = \epsilon(\mathbf{y}) + \deg' \mathbf{x} \deg' z + \deg' \mathbf{x} \deg' \mathbf{y} + \deg' \mathbf{x}''_b \deg' \mathbf{y}'_a.$$

Therefore (9.12) is equal to

$$\sum_b (-1)^{\epsilon_6} \mathfrak{M}_2(\mathfrak{Rep}(\mathbf{x}'_b), \mathfrak{Rep}(\mathbf{x}''_b))(\mathbf{y})(z) \tag{9.13}$$

where

$$\epsilon_6 = \epsilon(\mathbf{y}) + (\deg' \mathbf{x})(\deg' z + \deg' \mathbf{y}) + 1. \tag{9.14}$$

Case 2: $\mathbf{x}'_b = 1 \in B_0\mathcal{C}$, $\mathbf{y}'_a \neq 1 \in B_0\mathcal{C}$:
In this case the second term of (9.9) is:

$$\begin{aligned}
&\sum_a (-1)^{\epsilon_7(a)} \mathfrak{m}(\mathbf{y}''^{op}_a, \mathfrak{Rep}_0(c_0)(\mathbf{y}'_a)(z), \mathbf{x}) \\
&= \sum_{a,b} (-1)^{\epsilon_8(a)} \mathfrak{Rep}(\mathbf{x})(\mathbf{y}''_a)(\mathfrak{Rep}_0(c_0)(\mathbf{y}'_a)(z)) \\
&= \sum_a (-1)^{\epsilon_9(a)} \mathfrak{m}_2(\mathfrak{Rep}_0(c_0)(\mathbf{y}'_a), \mathfrak{Rep}(\mathbf{x})(\mathbf{y}''_a))(z) \\
&= (-1)^{\epsilon_6} \mathfrak{M}_2(\mathfrak{Rep}_0(c_0), \mathfrak{Rep}(\mathbf{x}))(\mathbf{y})(z)
\end{aligned} \tag{9.15}$$

where ϵ_6 is as in (9.14). In fact

$$\begin{aligned}
\epsilon_7(a) &= \deg' \mathbf{y}''_a + \epsilon(\mathbf{y}'_a), \\
\epsilon_8(a) &= \epsilon_7(a) + (\deg' \mathbf{x})(\mathbf{y}''_a + \mathbf{y}'_a + \deg' z + 1) + \epsilon(\mathbf{y}''_a) \\
\epsilon_9(a) &= \epsilon_8(a) + (\deg' \mathbf{y}'_a + 1)(\deg' \mathbf{x} + \deg' \mathbf{y}''_a) \\
&\equiv \epsilon_6 + \deg' \mathbf{x} \deg' \mathbf{y}'_a \mod 2.
\end{aligned}$$

Case 3: $\mathbf{x}'_b = 1 \in B_0\mathcal{C}$, $\mathbf{y}'_a = 1 \in B_0\mathcal{C}$:
In a similar way, we have

$$(-1)^{\epsilon_6 + \deg' \mathbf{x} + \deg' \mathbf{y}} \mathfrak{Rep}(\mathbf{x})(\mathbf{y})(\mathfrak{m}_1(z)). \tag{9.16}$$

In face we have

$$\deg' \mathbf{y} + (\deg' \mathbf{x})(\deg' z + \deg' \mathbf{y}) + \epsilon(\mathbf{y}) = \epsilon_6 + \deg' \mathbf{x} + \deg' \mathbf{y}.$$

Case 4: $\mathbf{x}_b'' = 1 \in B_0\mathcal{C}$, $\mathbf{y}_a'' \neq 1 \in B_0\mathcal{C}$:
In a similar way we have

$$(-1)^{\epsilon_6}\mathfrak{M}_2(\mathfrak{Rep}(\mathbf{x}), \mathfrak{Rep}_0(c_k))(\mathbf{y})(z). \tag{9.17}$$

Case 5: $\mathbf{x}_b'' = 1 \in B_0\mathcal{C}$, $\mathbf{y}_a'' = 1 \in B_0\mathcal{C}$:
In a similar way we have

$$(-1)^{\epsilon_6}\mathfrak{m}_1(\mathfrak{Rep}(\mathbf{x})(\mathbf{y})(z)). \tag{9.18}$$

We remark that the sum of (9.10), (9.16) and (9.18) is

$$(-1)^{\epsilon_6}\mathfrak{M}_1(\mathfrak{Rep}(\mathbf{x}))(\mathbf{y})(z) \tag{9.19}$$

Finally the third term of (9.9) is:

$$(-1)^{\epsilon_6+1}\mathfrak{Rep}(\hat{d}\mathbf{x})(\mathbf{y})(z). \tag{9.20}$$

Thus $(-1)^{\epsilon_6}$ times (9.9) implies

$$\begin{aligned}
\sum_{\mathbf{x}_b' \neq 1,\ \mathbf{x}_b'' \neq 1} & \mathfrak{M}_2(\mathfrak{Rep}(\mathbf{x}_b'), \mathfrak{Rep}(\mathbf{x}_b''))(\mathbf{y}) + \mathfrak{M}_2(\mathfrak{Rep}_0(c_0), \mathfrak{Rep}(\mathbf{x}))(\mathbf{y}) \\
& + \mathfrak{M}_2(\mathfrak{Rep}(\mathbf{x}), \mathfrak{Rep}_0(c_k))(\mathbf{y}) + \mathfrak{M}_1(\mathfrak{Rep}(\mathbf{x}))(\mathbf{y}) \\
= {} & \mathfrak{Rep}(\hat{d}\mathbf{x})(\mathbf{y}).
\end{aligned}$$

(We remark that we need over all minus sign in the definition of $\mathfrak{M}_k$ to show this formula.) The proof of Lemma 9.8 is complete. $\square$

Using Lemma 9.8 and Proposition 8.9, it suffices to check (8.7.1) for $\mathfrak{Rep}$ to complete the proof of Theorem 9.1. Namely we need to show

$$\mathfrak{Rep}_1 : \mathcal{C}(c_1, c_2) \to \mathfrak{Func}(\mathfrak{Rep}_0(c_1), \mathfrak{Rep}_0(c_2))$$

induces an isomorphism on homology. We define

$$\Pi : \mathfrak{Func}(\mathfrak{Rep}_0(c_1), \mathfrak{Rep}_0(c_2)) \to \mathcal{C}(c_1, c_2)$$

by

$$\Pi(T) = (-1)^{\mathfrak{deg}' T} T_0(c_1)(\mathbf{e}_{c_1}).$$

(Note $\mathbf{e}_{c_1} \in \mathcal{C}(c_1, c_1) = \mathfrak{Rep}_0(c_1)_0(c_1)$. Hence $T_0(c_1)(\mathbf{e}_{c_1}) \in \mathfrak{Rep}_0(c_2)_0(c_1)$ $= \mathcal{C}(c_1, c_2)$.)

Lemma 9.21. Π *is a chain map.*

Proof. Let $T \in \mathfrak{Func}(\mathfrak{Rep}_0(c_1), \mathfrak{Rep}_0(c_2))$. We have

$$\begin{aligned}\Pi(\mathfrak{M}_1(T)) &= (-1)^{\mathfrak{deg}'T}(\mathfrak{M}_1(T))_0(c_1)(\mathbf{e}_{c_1}) \\ &= (-1)^{\mathfrak{deg}'T+1}\mathfrak{m}_1(T_0(c_1)(\mathbf{e}_{c_1})) = \mathfrak{m}_1(\Pi(T)).\end{aligned}$$

(We remark that overall minus sign in the definition of $\mathfrak{M}_1$ is essential here.) □

We have

$$(\Pi \circ \mathfrak{Rep}_1)(x) = (-1)^{\deg' x}(\mathfrak{Rep}_1(x)_0(c_1))(\mathbf{e}_{c_1}) = \mathfrak{m}_2(\mathbf{e}_{c_1}, x) = x.$$

So, to complete the proof of Theorem 9.1, it suffices to show that $\mathfrak{Rep}_1 \circ \Pi$ is homotopic to identity. We define an operator

$$\mathcal{H} : \mathfrak{Func}(\mathfrak{Rep}_0(c_1), \mathfrak{Rep}_0(c_2)) \to \mathfrak{Func}(\mathfrak{Rep}_0(c_1), \mathfrak{Rep}_0(c_2))$$

of degree $+1$ by

$$(\mathcal{H}(T))_k(\mathbf{y})(z) = (-1)^{\deg' z \deg' \mathbf{y} + \deg' z + \deg' \mathbf{y} + \mathfrak{deg}'T} T_{k+1}(z, \mathbf{y})(\mathbf{e}_{c_1}).$$

Here $\mathbf{y} \in B_k\mathcal{C}^j(b_1, b_k)$, $z \in \mathcal{C}(b_1, c_1) = \mathfrak{Rep}_0(c_1)(b_1)$. (Then $T_{k+1}(\mathbf{y}, z)$ $(\mathbf{e}_{c_1}) \in \mathfrak{Rep}_0(c_2)(b_k)$.)

Lemma 9.22.

$$T - (\mathfrak{Rep}_1 \circ \Pi)(T) = (\mathfrak{M}_1 \circ \mathcal{H} + \mathcal{H} \circ \mathfrak{M}_1)(T). \tag{9.23}$$

Proof. We have

$$\begin{aligned}&(-1)^{\mathfrak{deg}'T}\mathfrak{M}_1(\mathcal{H}(T)(\mathbf{y})(z)) \\ &\quad= \mathfrak{m}_1(\mathcal{H}(T)(\mathbf{y})(z)) + (-1)^{\mathfrak{deg}'T}\mathcal{H}(T)(\hat{d}^{op}\mathbf{y})(z) \\ &\qquad + (-1)^{\mathfrak{deg}'T + \deg' \mathbf{y}}\mathcal{H}(T)(\mathbf{y})(\mathfrak{m}_1(z)) \\ &\qquad + \sum_a \mathfrak{m}_2(\mathcal{H}(T)(\mathbf{y}'_a), \mathfrak{Rep}_0(\mathbf{y}''_a))(z) \\ &\qquad + \sum_a (-1)^{(\mathfrak{deg}'T+1)\deg' \mathbf{y}'_a}\mathfrak{m}_2(\mathfrak{Rep}_0(\mathbf{y}'_a), \mathcal{H}(T)(\mathbf{y}''_a))(z) \\ &\quad= (-1)^{\epsilon_1}\mathfrak{m}_1(T(z, \mathbf{y})(\mathbf{e})) \\ &\qquad + \sum_a (-1)^{\epsilon_2} T(z, \mathbf{y}^{(1)}_a, \mathfrak{m}(\mathbf{y}^{(2)op}_a), \mathbf{y}^{(3)}_a)(\mathbf{e}) \\ &\qquad + (-1)^{\epsilon_3} T(\mathfrak{m}_1(z), \mathbf{y})(\mathbf{e}) \\ &\qquad + \sum_a (-1)^{\epsilon_4}\mathfrak{m}(\mathbf{y}''^{op}_a, T(z, \mathbf{y}'_a)(\mathbf{e})) \\ &\qquad + \sum_a (-1)^{\epsilon_5} T(\mathfrak{m}(\mathbf{y}'^{op}_a, z), \mathbf{y}''_a)(\mathbf{e}).\end{aligned}$$

Here

$$\begin{aligned}
\epsilon_1 &= \deg' \mathbf{y} \deg' z + \deg' \mathbf{y} + \deg' z, \\
\epsilon_2 &= \mathfrak{deg}' T + (\deg' \mathbf{y} + 1)(\deg' z) + \deg' \mathbf{y} + 1 + \deg' z + \deg' \mathbf{y}_a^{(1)} \\
&\quad + \epsilon(\mathbf{y}_a^{(2)}), \\
\epsilon_3 &= \mathfrak{deg}' T + \deg' \mathbf{y} + (\deg' \mathbf{y})(\deg' z + 1) + \deg' \mathbf{y} + \deg' z + 1, \\
\epsilon_4 &= (\mathfrak{deg}' T + 1 + \deg' \mathbf{y}'_a)(\deg' \mathbf{y}''_a) + \deg' z \deg' \mathbf{y}'_a + \epsilon(\mathbf{y}''_a) + \deg' \mathbf{y}'_a \\
&\quad + \deg' z, \\
\epsilon_5 &= (\mathfrak{deg}' T + 1)(\deg' \mathbf{y}'_a) + (\deg' \mathbf{y}'_a + 1)(\mathfrak{deg}' T + \deg' \mathbf{y}''_a + 1) + \epsilon(\mathbf{y}'_a) \\
&\quad + \deg' \mathbf{y}''_a(\deg' z + \deg' \mathbf{y}'_a + 1) + \deg' \mathbf{y}''_a + \deg' z + \deg' \mathbf{y}'_a + 1.
\end{aligned}$$

We also have

$$\begin{aligned}
&(-1)^{\mathfrak{deg}' T + 1} \mathcal{H}(\mathfrak{M}_1(T))(\mathbf{y})(z) \\
&= \quad (-1)^{\deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y}} \mathfrak{m}_1(T(z, \mathbf{y})(\mathbf{e})) \\
&\quad + \sum_a (-1)^{\epsilon_6} T(\mathfrak{m}(\mathbf{y}_a'^{op}, z), \mathbf{y}''_a)(\mathbf{e}) \\
&\quad + \sum_a (-1)^{\epsilon_7} T(z, \mathbf{y}_a^{(1)}, \mathfrak{m}(\mathbf{y}_a^{(2)op}), \mathbf{y}_a^{(3)})(\mathbf{e}) \qquad (9.24) \\
&\quad + (-1)^{\deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y}} \sum_{\deg \mathbf{y}''_a \neq 0} \mathfrak{m}_2(T(z, \mathbf{y}'_a), \mathfrak{Rep}(\mathbf{y}''_a))(\mathbf{e}) \\
&\quad + (-1)^{\deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y}} \mathfrak{m}_2(T_0, \mathfrak{Rep}_0(z, \mathbf{y}))(\mathbf{e}) \\
&\quad + (-1)^{\deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + \mathfrak{deg}' T \deg' z} \mathfrak{m}_2(\mathfrak{Rep}_0(z), T(\mathbf{y}))(\mathbf{e}).
\end{aligned}$$

Here

$$\begin{aligned}
\epsilon_6 &= \deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + \deg' z \deg' \mathbf{y}'_a + \epsilon(\mathbf{y}'_a) \\
&\quad + \mathfrak{deg}' T + 1 = \epsilon_5, \\
\epsilon_7 &= \deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + \mathfrak{deg}' T + 1 + \deg' z + \deg' \mathbf{y}_a^{(1)} \\
&\quad + \epsilon(\mathbf{y}^{(2)}) = \epsilon_2.
\end{aligned}$$

The fourth term of (9.24) is:

$$(-1)^{\epsilon_8} \mathfrak{m}(\mathbf{y}_a''^{op}, T(z, \mathbf{y}'_a)(\mathbf{e})),$$

where

$$\begin{aligned}
\epsilon_8 &= \deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + (\mathfrak{deg}' T + \deg' z + \deg' \mathbf{y}'_a) \deg' \mathbf{y}''_a \\
&\quad + \epsilon(\mathbf{y}''_a) = \epsilon_4.
\end{aligned}$$

The fifth term of (9.24) is

$$\begin{aligned}
(-1)^{\epsilon_9} \mathfrak{m}(\mathbf{y}^{op}, z, T(\mathbf{e})) &= \mathfrak{Rep}_1(T(\mathbf{e}))(\mathbf{y})(z) \\
&= (-1)^{\mathfrak{deg}' T} ((\mathfrak{Rep}_1 \circ \Pi)(T))(\mathbf{y})(z),
\end{aligned}$$

where

$$\begin{aligned}\epsilon_9 &= \deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + (\mathfrak{deg}' T)(\deg' \mathbf{y} + \deg' z) \\ &\quad + \deg' \mathbf{y} \deg' z + \epsilon(\mathbf{y}).\end{aligned}$$

Finally the sixth term of (9.24) is:

$$\begin{aligned}&(-1)^{\deg' \mathbf{y} + \deg' z + \deg' z \deg' \mathbf{y} + \mathfrak{deg}' T \deg' z + (\deg' z + 1)(\mathfrak{deg}' T + \deg' \mathbf{y})} \\ &\qquad\qquad T(\mathbf{y})(\mathfrak{m}_2(\mathbf{e}, z)), \\ &= (-1)^{\mathfrak{deg}' T + 1} T(\mathbf{y})(z).\end{aligned}$$

The lemma follows. □

The proof of the first half of Theorem 9.1 is now complete. We omit the proof of the second half. (Which is analogous to the argument of the last section of [Fu7].) □

We are now in the position to complete the proof of Theorem 8.6. We consider the following Diagram. We can easily see that the diagram

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{C}_2 \arrow[d] \\ \mathcal{Rep}(\mathcal{C}_1^o, \mathcal{CH}) \arrow[r, "\mathcal{F}_*"] & \mathcal{Rep}(\mathcal{C}_2^o, \mathcal{CH}) \end{tikzcd}$$

Diagram 9.1.

commutes. Moreover

$$\mathfrak{F}_* : \mathfrak{Rep}(\mathcal{C}_1^o, \mathcal{CH}) \to \mathfrak{Rep}(\mathcal{C}_2^o, \mathcal{CH})$$

satisfies the assumption of Theorem 8.6, by the proof of Proposition 8.49. Hence by Proposition 8.47 it is a homotopy equivalence. Moreover by Theorem 9.1 the vertical allows are homotopy equivalence. Therefore by Corollary 8.44 $\mathfrak{F}$ is a homotopy equivalence. The proof of Theorem 8.6 is now complete. □

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